AEA 2004 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. First, one needs to eliminate the square root. To do this, we move it to the other side of the equation and square. The equation then reads

$$\cos^2 x = 1 - \frac{1}{2}\sin 2x.$$

To continue, one can get rid of the term $\sin 2x$ by using the trigonometric identity $\sin 2x = 2\sin x\cos x$. This leads to the equation

$$\cos^2 x = 1 - \sin x\cos x.$$

In order to remove the constant, one can use the identity $\cos^2 x + \sin^2 x = 1$ to obtain

$$\sin^2 - \sin x\cos x = 0.$$

One must then be careful not to miss any solution. In particular, one could divide both sides by $\sin x$, but should before be careful to ensure that $\sin x \neq 0$. A cleaner way of tackling the problem is to factorise by $\sin x$. The equation $\sin^2 - \sin x\cos x = 0$ is equivalent to

$$\sin x\,(\sin x - \cos x) = 0. \tag{1}$$

The solutions of (1) are given by the solutions of $\sin x = 0$ AND the solutions of $\sin x = \cos x$.

- The solutions $0° \le x < 360°$ of the equation $\sin x = 0$ are $x = 0°$ and $x = 180°$.
- The equation $\sin x = \cos x$ also reads $\tan x = 1$. The solutions $0° \le x < 360°$ are $x = 45°$ and $x = 45 + 180 = 225°$.

To sum up, the solutions of $\cos x + \sqrt{1 - \frac{1}{2}\sin 2x} = 0$ in the interval $0° \le x < 360°$ must be amongst the values $x = 0°, 45°, 180°, 225°$. Finally, one should substitute these into the original equation to check that we did not create any extra solutions when squaring. This eliminates $x = 0°, 45°$, and demonstrates that the only actual solutions are $x = 180°, 225°$.

2. (a) (i) We will apply here the binomial expansion of $1/(1-x)$. In particular,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k$$

for $|x| < 1$. Since

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

it thus follows that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(1 + x + x^2 + x^3 + \ldots\right) = 1 + 2x + 3x^2 + 4x^3 + \ldots \tag{2}$$

(ii) Equation (2) also reads $\sum_{n=0}^{\infty}(n+1)x^n$. The coefficient of x^n is $(n+1)$.

(b) One can notice that $nx^n = (n+1)x^n - x^n$. Therefore, being careful with the initial value $n = 0$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} nx^n &= \sum_{n=0}^{\infty} nx^n \\ &= \sum_{n=0}^{\infty}(n+1)x^n - \sum_{n=0}^{\infty} x^n \\ &= \frac{1}{(1-x)^2} - \frac{1}{1-x} \\ &= \frac{x}{(1-x)^2}.\end{aligned}$$

This identity is valid for $|x| < 1$.

(c) Again, we will look to build on the answers to previous parts of the question. Since $(an+1)x^n = anx^n + x^n$ it follows that

$$\sum_{n=1}^{\infty}(an+1)x^n = a\sum_{n=1}^{\infty} nx^n + \sum_{n=1}^{\infty} x^n.$$

One needs to be careful because the sum starts at $n = 1$. We have

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1$$

and therefore

$$\sum_{n=1}^{\infty}(an+1)x^n = a\,\frac{x}{(1-x)^2} + \frac{1}{1-x} - 1 = \frac{(a+1)x - x^2}{(1-x)^2}.$$

(d) To evaluate $\sum_{n=1}^{\infty} \frac{5n+1}{2^{3n}}$ it suffices to notice that this also equals $\sum_{n=1}^{\infty}(an+1)x^n = \frac{(a+1)x-x^2}{(1-x)^2}$ with $a = 5$ and $x = \frac{1}{2^3} = \frac{1}{8}$. This proof is valid since $|x| < 1$. As a conclusion

$$\sum_{n=1}^{\infty} \frac{5n+1}{2^{3n}} = \frac{47}{49}.$$

3. (a) To check that the curve with equation $y = f(x)$ passes through the point $(2, 0)$, we need to verify that $0 = f(2)$. Since $f(2) = 2^3 - (k-4) \times 2 + 2k = 0$ for any value of k, the conclusion follows.

(b) Since $x_0 = 2$ is a root of the polynomial equation $f(x) = 2$, the polynomial $f(x) = x^3 - (k+4)x + 2k$ can be factorised as $f(x) = (x-2)(ax^2 + bx + c)$ for some constants $a, b, c \in \mathbb{R}$. To find the values of a, b, c, we expand $(x-2)(ax^2+bx+c)$ as $ax^3 + (b-2a)x^2 + (c-2b)x - 2c$, and then equate the coefficients with those appearing in the definition of $f(x)$ given in the question. Specifically, a, b, c must satisfy

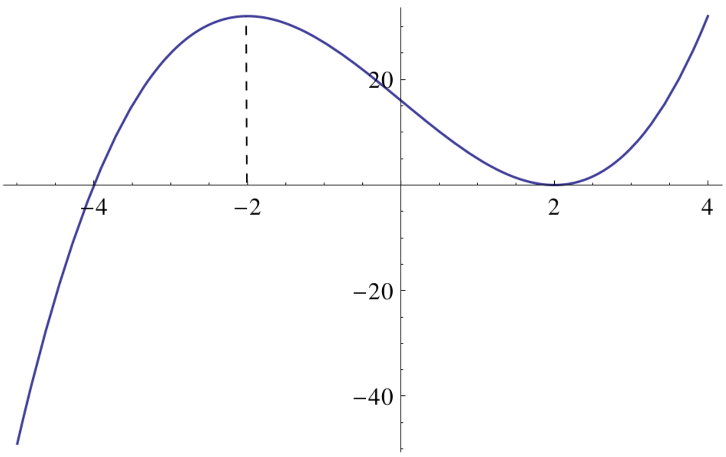
$$a = 1 \qquad b - 2a = 0 \qquad c - 2b = -(k+4) \qquad -2c = 2k.$$

This leads to $a = 1$, $b = 2$, $c = -k$ and $f(x) = (x+2)(x^2 + 2x - k)$. In order for the equation $f(x) = 0$ to have exactly two distinct solutions, one must ensure that either:

- the equation $x^2 + 2x - k = 0$ has only one real solution d and that this solution is different from 2. This means that there exists $d \neq 2$ such that $x^2 + 2x - k = (x-d)^2$. The only possible d that could satisfy this condition is $d = -1$. This subsequently implies $k = -1$.
- the equation $x^2 + 2x - k = 0$ has two distinct solutions, one of which equals 2. As a consequence one can factorise the polynomial $x^2 + 2x - k$ as $x^2 + 2x - k = (x-2)(x-d)$ with $d \neq 2$. Expanding and equating coefficients, this leads to $d = -4$ and $k = 8$.

To sum up, precisely when $k = -1$ or $k = 8$ does the equation $f(x) = 0$ have exactly two distinct roots.

(c) Since f is a cubic, the conditions that x is a tangent to the curve and there is a line $y = p$ intersecting the curve in three distinct points mean that $f(x) = 0$ has exactly two distinct roots (see sketch below – note that it is often a good idea to draw a sketch to work out what the conditions in a question really mean). From our answer to question 3b, this means that k is equal to either -1 or 8. As we are told in the question that $k > 0$, k must take the latter of these two values, and $f(x) = (x+4)(x-2)^2$.



First note that there exists an $\alpha \in (-4, 2)$ such that the curve $y = f(x)$ is strictly increasing for $-\infty < x < \alpha$ and strictly decreasing for $\alpha < x < 2$ (and then strictly increasing for $2 < x < \infty$). Consequently, the equation $f(x) = p$ has exactly 3 solutions if, and only if, the value of p satisfies $0 < p < f(\alpha)$. To find $\alpha \in (-4, 2)$ one must solve the equation $f'(\alpha) = 0$, which also reads $3x^2 - 12 = 0$. Therefore $\alpha = -2$ and the equation $f(x) = p$ has exactly three solutions if, and only if, we have $0 < p < f(-2) = 32$.

4. (a) (i) Since the circle has radius r and touches the y-axis at $(0, 4)$, its centre C must have coordinates $(r, 4)$. The circle touches the line $4y - 3x = 0$ at the point $A = (4k, 3k)$, where $k > 0$. Note that $|OA| = \sqrt{(4k)^2 + (3k)^2} = 5k$. Moreover, by symmetry about the line OC, we must have that $|OA| = 4$. Equating these two expressions for $|OA|$ yields $k = \frac{4}{5}$ and $A = (x_A, y_A) = (\frac{16}{5}, \frac{12}{5})$. There are then several approaches to compute the exact value of the radius r.

- One can exploit the fact that the distance $|CA|$ equals r. This leads to the equation $(x_A - r)^2 + (y_A - 4)^2 = r^2$ with $(x_A, y_A) = (\frac{16}{5}, \frac{12}{5})$. This quadratic equation is easily solved and the positive solution is given by $r = 2$.

- One can notice that the vector $\overrightarrow{CA}$ is perpendicular to the line $4y - 3x = 0$. This leads to the equation $4(x_A - r) + 3(y_A - 4) = 0$. The solution is indeed $r = 2$.

(ii) The most obvious right-angle in the picture is that between the x and y axes. We can obtain this as follows:

$$\angle(XOA) + \angle(AOC) + \angle(COY) = \frac{\pi}{2},$$

where we set $X = (1, 0)$ and $Y = (0, 1)$. Furthermore, $\angle(XOA) = \arctan\left(\frac{3}{4}\right)$ and $\angle(AOC) = \angle(COY) = \arctan\left(\frac{r}{4}\right)$ with $r = 2$. In other words,

$$\arctan\left(\frac{4}{3}\right) + 2\arctan\left(\frac{1}{2}\right) = \frac{\pi}{2},$$

as desired.

(b) Let $B = (c, d)$ be the point on the circle where the line $4x + 3y = q$ is a tangent (for some $q > 12$). We must have that $|CB| = r = 2$. Moreover, the tangent condition means that $\overrightarrow{CB}$ is perpendicular to the line $4x+3y = q$. Let us now gives the details of the computations that these observations imply, starting with the latter.

- The point $B = (c, d)$, which satisfies $4c + 3d = q$ (since it lies on the line), also satisfies $3(c - 2) - 4(d - 4) = 0$. The solution to these equations is given by

$$B = \left(\frac{4q - 30}{25}, \frac{3q + 40}{25}\right).$$

- The square of the distance from $C = (2, 4)$ to the line is

$$\left(\frac{4q - 30}{25} - 2\right)^2 + \left(\frac{3q + 40}{25} - 4\right)^2.$$

This quantity equals $r^2 = 4$ only if

$$(4q - 80)^2 + (3q - 60)^2 = 50^2.$$

The solutions of this quadratic are $q = 10, 30$. Since it is the only solution in the desired range, the solution we are looking for is therefore $q = 30$.

5. (a) One can use the chain rule applied to $y = \ln u$ where $u = t + \sqrt{1 + t^2}$,

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = \frac{1}{u}\left(1 + \frac{t}{\sqrt{1+t^2}}\right) = \frac{1}{t + \sqrt{1+t^2}}\left(1 + \frac{t}{\sqrt{1+t^2}}\right) = \frac{1}{\sqrt{1+t^2}}.$$

Note that we also used the chain rule to compute $\frac{du}{dt}$.

(b) (i) The student made a mistake when computing the quantity $\frac{dy}{dx}$. Indeed, the variable t is not independent from the variable x since they satisfy the relation $x = (1 + t^2)^{-\frac{1}{2}}$. The student should have written instead

$$\frac{dy}{dx} = \frac{t + x\frac{dt}{dx}}{tx + 1} - \frac{1}{x}.$$

(The term $t + x\frac{dt}{dx}$ comes from differentiating tx according to the product rule.)

(ii) When confronted with parametric equations, the usual approach for computing the gradient $\frac{dy}{dx}$ is to compute separately $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and then use the relation

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)}.$$

We have $\frac{dx}{dt} = -t(1+t^2)^{-3/2}$ and it was shown in part 5a that $\frac{dy}{dt} = (1+t^2)^{-1/2}$. This leads to

$$\frac{dy}{dx} = \frac{(1+t^2)^{-1/2}}{-t(1+t^2)^{-3/2}} = -\frac{1+t^2}{t}.$$

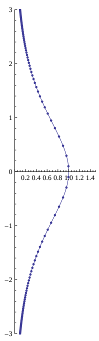
(c) The key here is to recall that $-\ln u = \ln(1/u)$. In particular, we have

$$-\ln\left(t+\sqrt{1+t^2}\right) = \ln\left(\frac{1}{t+\sqrt{1+t^2}}\right).$$

One can then multiply the numerator and the denominator of the fraction inside the logarithm on the right-hand side by the conjugate quantity $t-\sqrt{1+t^2}$, yielding that

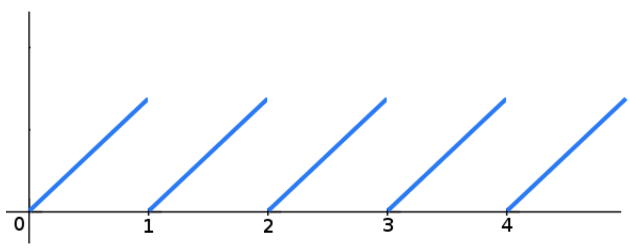
$$\begin{aligned} -\ln\left(t+\sqrt{1+t^2}\right) &= \ln\left(\frac{1}{t+\sqrt{1+t^2}}\right) = \ln\left(\frac{t-\sqrt{1+t^2}}{(t-\sqrt{1+t^2})(t+\sqrt{1+t^2})}\right) \\ &= \ln\left(\frac{t-\sqrt{1+t^2}}{t^2-(1+t^2)}\right) = \ln\left(-t+\sqrt{1+t^2}\right). \end{aligned}$$

(d) Using question 5c it follows that $(x(-t), y(-t)) = (x(t), -y(t))$. It follows that the curve is symmetric about the x-axis. It can be sketched as follows:



The key points to drawing this are that at $t = 0$, the curve passes through the point $(1, 0)$. Moreover, the derivative there is infinite (see the answer to part 5b(ii)), and so there is no cusp at this point. As t goes to infinity, x tends to 0 and y tends to infinity (i.e. the y-axis is an asymptote). Finally, the curve should be symmetric about the x-axis, as we just observed!

6. (a) The function $f(x) = x - [x]$ is the fractional part of x, and can be sketched as follows:



(b) Since

$$\int_2^3 f(x)\,dx = \int_2^3 (x-2)\,dx = \frac{1}{2} > 0.18,$$

it follows that $2 < p < 3$. Consequently,

$$0.18 = \int_2^p f(x)\,dx = \int_2^p (x-2)\,dx = \frac{1}{2}(p-2)^2,$$

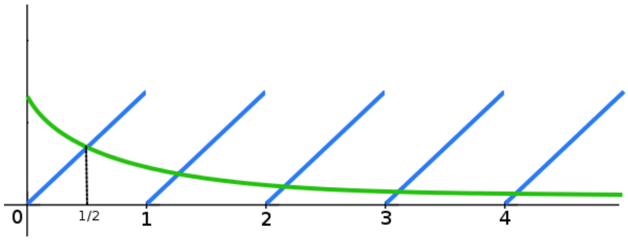
which yields $p = 2 + \sqrt{2 \times 0.18} = 2.6$.

(c) The value $x_0 = \frac{1}{2}$ is a root of the equation $x - [x] = \frac{1}{1+kx}$, i.e. we have

$$\frac{1}{2} = \frac{1}{1+\frac{k}{2}}.$$

Solving for k gives $k = 2$.

(d) The curves of the function $y = \frac{1}{1+2x}$ and the function $y = x - [x]$ intersect at $x_0 = \frac{1}{2}$. Using this fact, we can draw the following sketch:



(e) The solution $n < x_n < n+1$ of the equation $x - [x] = \frac{1}{1+2x}$ satisfies

$$x_n - n = \frac{1}{1+2x_n},$$

since the integral part of x_n is n. This also reads

$$2x_n^2 - (2n-1)x_n - (n+1) = 0.$$

(f) The smallest integer $n \in \mathbb{N}$ such that $x_n - n < 0.05$ is the smallest integer such that

$$\frac{1}{1+2x_n} = x_n - n < 0.05.$$

Therefore, this is the smallest integer $n \in \mathbb{N}$ such that

$$1 + 2x_n > \frac{1}{0.05} = 20,$$

which also reads $x_n > 9.5$. Noting that $n < x_n \leq n + \frac{1}{2}$ for every n, we deduce that $n = 10$.

7. (a) Since $c = AB$ is a diameter of the circumcircle of the triangle ABC, the angle ACB must be a right-angle. It follows that $a^2 + b^2 = c^2$ (draw a sketch to see this!). It is assumed that a^2, b^2 and c^2 are three consecutive terms of an arithmetic progression. The common difference $d > 0$ of this arithmetic progression verifies $b^2 = a^2 + d$ and $c^2 = a^2 + 2d$. The relation $a^2 + b^2 = c^2$ is thus equivalent to $2a^2 + d = a^2 + 2d$. This shows that $d = a^2$ and

$$b = \sqrt{2}a \qquad \text{and} \qquad c = \sqrt{3}a.$$

(b) The triangle ABC has a right angle at the vertex C. This is why

$$\cot A = \frac{b}{a} = \sqrt{2},$$

$$\cot B = \frac{a}{b} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

and

$$\cot C = 0.$$

These are three consecutive terms of an arithmetic progression with common difference $-\frac{\sqrt{2}}{2}$.

(c) There are several approaches for proving the 'sine rule', and we will give just one. For example, the area of the triangle PQR can be expressed in three different ways,

$$\text{Area(triangle } PQR) \;=\; \frac{1}{2}p\,q\,\sin R \;=\; \frac{1}{2}q\,r\,\sin P \;=\; \frac{1}{2}r\,p\,\sin Q.$$

The equation $\frac{1}{2}p\,q\,\sin R = \frac{1}{2}q\,r\,\sin P$ leads to the identity $\frac{\sin R}{r} = \frac{\sin P}{p}$. Similarly, the equation $\frac{1}{2}q\,r\,\sin P = \frac{1}{2}r\,p\,\sin Q$ leads to the identity $\frac{\sin Q}{q} = \frac{\sin P}{p}$. The result follows by putting these two equalities together.

(d) The cosine rule states that

$$p^2 = q^2 + r^2 - 2qr\cos P,$$

which also reads

$$\frac{\cos P}{p} = \frac{q^2 + r^2 - p^2}{2pqr}.$$

Similarly, we have

$$\frac{\cos Q}{q} = \frac{r^2 + p^2 - q^2}{2pqr}$$

and

$$\frac{\cos R}{r} = \frac{p^2 + q^2 - r^2}{2pqr}.$$

It is assumed that p^2, q^2 and r^2 are three consecutive terms of an arithmetic progression with common difference d. It follows that

$$2\left(r^2 + p^2 - q^2\right) = 2q^2 = \left(q^2 + r^2 - p^2\right) + \left(p^2 + q^2 - r^2\right).$$

Dividing both sides of this equation by $2pqr$ we have

$$\frac{2(r^2 + p^2 - q^2)}{2pqr} = \frac{q^2 + r^2 - p^2}{2pqr} + \frac{p^2 + q^2 - r^2}{2pqr},$$

which is equivalent to

$$\frac{2\cos Q}{q} = \frac{\cos P}{p} + \frac{\cos R}{r}. \tag{3}$$

(e) The sine rule states that

$$\alpha = \frac{\sin P}{p} = \frac{\sin P}{q} = \frac{\sin R}{r}.$$

Dividing both sides of equation (3) by $\alpha \neq 0$, we have

$$2\,\frac{1}{\alpha}\,\frac{\cos Q}{q} = \frac{1}{\alpha}\,\frac{\cos P}{p} + \frac{1}{\alpha}\,\frac{\cos R}{r}.$$

This is equivalent to

$$2\,\frac{q}{\sin Q}\,\frac{\cos Q}{q} = \frac{p}{\sin P}\,\frac{\cos P}{p} + \frac{r}{\sin R}\,\frac{\cos R}{r},$$

which also reads $2\cot Q = \cot P + \cot R$ or $\cot Q - \cot P = \cot R - \cot Q$. This expresses the fact that $\cot P$, $\cot Q$ and $\cot R$ are three consecutive terms of an arithmetic progression.