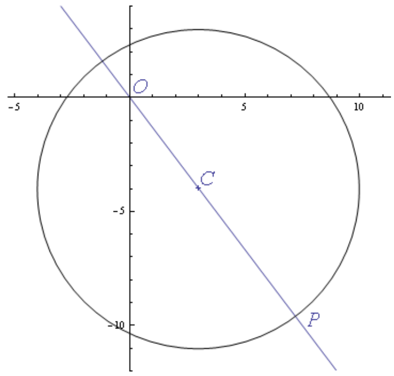
## AEA 2005 Extended Solutions

*These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.*

1. In many cases, finding maxima and minima require differentiation. However, because the the equation in this question is of a form that, after some basic manipulation, is readily recognisable as describing a simple geometric object, this turns out not to be the best approach. Indeed, by completing the square (twice), we can rewrite the equation as

$$(x-3)^2 + (y+4)^2 = 9 + 16 + 24 = 49.$$

Hence the point $P$ lies on a circle with centre $(3, -4)$ and radius 7.




An easy consequence of this is that the greatest and least values of the length $OP$ are attained when $P$, as well as sitting on the circle, lies on the line that runs through $O$ and $C$, where $C$ is the centre of the circle. (The figure shows the case when the length $OP$ is the greatest.) In particular, the greatest value of the length $OP$ is given by $7 + |OC|$, and the least is given by $7 - |OC|$. Since $|OC| = (3^2 + 4^2)^{1/2} = 5$, these two values are equal to 12 and 2, respectively.

2. To simplify the expression so that all of the trigonometric functions are in terms of $\theta$, rather than $\theta$ and $2\theta$, we will start by transforming $\sin 2\theta$ and $\cos 2\theta$ using the double-angle formulae

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = 2\cos^2\theta - 1.$$

In particular, these show that the equality of the question is equivalent to

$$2\sin\theta\cos\theta + 2\cos^2\theta = \sqrt{6}\cos\theta.$$

What is immediately noticeable is that all the terms incorporate a factor of $\cos\theta$. This means, after we move all the terms to the same side, we can factorise as follows:

$$\cos\theta\left(2\sin\theta + 2\cos\theta - \sqrt{6}\right) = 0.$$

The solutions of this in the range $0 < \theta < 2\pi$ are hence given by the solutions of $\cos\theta = 0$ AND the solutions of $2\sin\theta + 2\cos\theta - \sqrt{6} = 0$. Firstly, $\cos\theta = 0$ at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$. Secondly, $2\sin\theta + 2\cos\theta - \sqrt{6} = 0$ is equivalent to

$$\sin\theta + \cos\theta = \frac{\sqrt{6}}{2}.$$

The left-hand side here can be rewritten using the identity

$$\sin\left(\theta+\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)\cos\theta+\cos\left(\frac{\pi}{4}\right)\sin\theta=\frac{1}{\sqrt{2}}\left(\cos\theta+\sin\theta\right).$$

Hence, we are trying to solve

$$\sin\left(\theta+\frac{\pi}{4}\right)=\frac{\sqrt{6}}{2}\times\frac{1}{\sqrt{2}}=\frac{\sqrt{3}}{2}.$$

This gives $\theta+\frac{\pi}{4}=\frac{\pi}{3},\frac{2\pi}{3}$, and so $\theta=\frac{\pi}{12},\frac{5\pi}{12}$. In conclusion, the solutions of

$$\sin 2\theta+\cos 2\theta+1=\sqrt{6}\cos\theta$$

in the range $\theta\in(0,2\pi)$ are $\theta=\frac{\pi}{12},\frac{5\pi}{12},\frac{\pi}{2},\frac{3\pi}{2}$.

3. At first glance the equation looks quite awkward. However, if we write $v=\sqrt{x}$, then it is clear the product rule can be applied to the left-hand side as follows:

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}=u\frac{d\sqrt{x}}{dx}+\sqrt{x}\frac{du}{dx}.$$

Since

$$\frac{d\sqrt{x}}{dx}=\frac{1}{2\sqrt{x}},$$

it follows that the equation given in the question is equivalent to

$$\frac{u}{2\sqrt{x}}+\sqrt{x}\frac{du}{dx}=\frac{1}{2\sqrt{x}}\frac{du}{dx}.$$

Rearranging so as to collect all the terms involving $du/dx$ on one side of the equation, we obtain

$$\left(\frac{1}{2\sqrt{x}}-\sqrt{x}\right)\frac{du}{dx}=\frac{u}{2\sqrt{x}},$$

and consequently, separating the variables,

$$\frac{1}{u}\frac{du}{dx}=\frac{1}{1-2x}.$$

This is readily integrated to give

$$\ln u=-\frac{1}{2}\ln(1-2x)+c.$$

Note here that $x\in(0,\frac{1}{2})$, and so $\ln(1-2x)$ is well-defined. To compute the constant of integration, we insert the condition that at $u=4$, we have $x=\frac{3}{8}$. In particular, this implies

$$c=\ln 4+\frac{1}{2}\ln(1-2\times\frac{3}{8})=\ln 4+\frac{1}{2}\ln\left(\frac{1}{4}\right)=\ln 4-\ln 2=\ln 2.$$

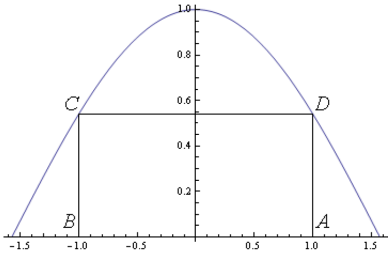
(The rules for logarithms we are applying here are that $a\ln x=\ln(x^a)$ and also $\ln x-\ln y=\ln(x/y)$.) Hence

$$\ln u=-\frac{1}{2}\ln(1-2x)+\ln 2,$$

and taking the exponential of both sides yields

$$u=\frac{2}{\sqrt{1-2x}}.$$

4. (a) On a question like this, a quick sketch helps to clarify the problem. (The curve shown is $\cos x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.)




As this illustrates, the rectangle has base length equal to $2p$ and height equal to $\cos p$. Thus its area is given by $2p\cos p$.

(b) We need to estimate where the maximum of $\mathcal{A} = 2p\cos p$ over $p \in (0, \frac{\pi}{2})$ lies. We start by investigating the derivative of $\mathcal{A}$ to find the stationary points of $\mathcal{A}$. In particular,

$$\frac{d\mathcal{A}}{dp} = 2\cos p - 2p\sin p = 2\cos p(1 - p\tan p).$$

Since $\cos p > 0$ in the range we are considering, this function is $< 0$, $= 0$ or $> 0$ according to whether $p\tan p$ is $> 1$, $= 1$ or $< 1$. Now, $p\tan p$ is strictly increasing on $(0, \pi/2)$, and satisfies

$$\frac{\pi}{4}\tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} < 1,$$

$$1\tan(1) = \tan(1) > \tan\left(\frac{\pi}{4}\right) = 1.$$

Hence, there is a unique value $\alpha \in (\frac{\pi}{4}, 1)$ such that

$$p\tan p \begin{cases} < 1, & \text{for } 0 < p < \alpha, \\ = 1 & \text{for } p = \alpha, \\ > 1 & \text{for } \alpha < p < \frac{\pi}{2}. \end{cases}$$

We have therefore proved that $\mathcal{A}$ is increasing on $(0, \alpha)$, stationary at $\alpha$, and decreasing on $(\alpha, \frac{\pi}{2})$. Thus $p = \alpha$ is where $\mathcal{A}$ is maximised, and we have already shown that $\alpha \in (\frac{\pi}{4}, 1)$.

(c) The maximum area of the rectangle satisfies $S = 2\alpha\cos\alpha$. To evaluate $\cos\alpha$, we recall from the previous part of the question that $\alpha$ satisfies $\alpha\tan\alpha = 1$, i.e. $\tan\alpha = \alpha^{-1}$. By rearranging the identity $\cos^2\alpha + \sin^2\alpha = 1$, it is thus possible to check that

$$\cos^2\alpha = \frac{1}{\tan^2\alpha + 1} = \frac{\alpha^2}{1 + \alpha^2}.$$

Since $\cos\alpha > 0$, this implies that

$$S = 2\alpha\cos\alpha = \frac{2\alpha^2}{\sqrt{1 + \alpha^2}}.$$

(d) We notice that the two bounds in the question are simply the expression $2\alpha^2/\sqrt{1 + \alpha^2}$ evaluated at $\alpha = \frac{\pi}{4}$ and $\alpha = 1$. More specifically,

$$\frac{2(\pi/4)^2}{\sqrt{1 + (\pi/4)^2}} = \frac{\pi^2}{2\sqrt{16 + \pi^2}},$$

$$\frac{2 \times 1^2}{\sqrt{1+1^2}} = \sqrt{2}.$$

Hence, to deduce the result, it will be enough to show that the expression for $S$ is strictly increasing in the range $\alpha \in (\frac{\pi}{4}, 1)$. To do this, we will differentiate:

$$\begin{aligned} \frac{dS}{d\alpha} &= \frac{4\alpha}{\sqrt{1+\alpha^2}} - \frac{2\alpha^3}{(1+\alpha^2)^{3/2}} \\ &= \frac{4\alpha + 2\alpha^3}{(1+\alpha^2)^{3/2}} \\ &> 0. \end{aligned}$$

Thus $S$ is indeed strictly increasing, and the inequality

$$\frac{\pi^2}{2\sqrt{16+\pi^2}} < S < \sqrt{2}$$

follows.

5. (a) The vector $\overrightarrow{AB}$ can be computed as a difference:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 5\mathbf{i} + \mathbf{j} - 8\mathbf{k}. \qquad (1)$$

Thus the line $L_1$ can be expressed as

$$\mathbf{r} = 7\mathbf{i} + 2\mathbf{j} - 7\mathbf{k} + \lambda\,(5\mathbf{i} + \mathbf{j} - 8\mathbf{k}), \qquad \lambda \in \mathbb{R}.$$

(b) To show that the line $L_2$ passes through the origin, we need to find $\mu \in \mathbb{R}$ such that

$$\mathbf{0} = -4\mathbf{i} + 12\mathbf{k} + \mu(\mathbf{i} - 3\mathbf{k}).$$

It is easy to see that this is the case with $\mu = 4$.

(c) We will have proved that the lines $L_1$ and $L_2$ intersect at a point $C$ if we can find $\lambda, \mu \in \mathbb{R}$ such that

$$7\mathbf{i} + 2\mathbf{j} - 7\mathbf{k} + \lambda\,(5\mathbf{i} + \mathbf{j} - 8\mathbf{k}) = -4\mathbf{i} + 12\mathbf{k} + \mu(\mathbf{i} - 3\mathbf{k}).$$

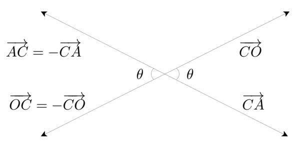
Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, this requires

$$\begin{aligned} 5\lambda - \mu &= -11, \\ \lambda &= -2, \\ -8\lambda + 3\mu &= 19. \end{aligned}$$

Since these equations are solved by $\lambda = -2$ and $\mu = 1$, the lines $L_1$ and $L_2$ intersect. Moreover, the point of intersection $C$ has position vector

$$\overrightarrow{OC} = -4\mathbf{i} + 12\mathbf{k} + 1 \times (\mathbf{i} - 3\mathbf{k}) = -3\mathbf{i} + 9\mathbf{k}.$$

(d) The $\angle OCA = \theta$ is that between the vectors $\overrightarrow{CO}$ and $\overrightarrow{CA}$, or equivalently the angle between $\overrightarrow{OC}$ and $\overrightarrow{AC}$.



In general, the cosine of the angle between two vectors is most easily computed using their scalar product, and we will use this approach here. In particular,

$$\overrightarrow{OC} \cdot \overrightarrow{AC} = |\overrightarrow{OC}||\overrightarrow{AC}| \cos\theta.$$

We already know that $\overrightarrow{OC} = -3\mathbf{i} + 9\mathbf{k}$ from the previous part of the question. We can also check that

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -10\mathbf{i} - 2\mathbf{j} + 16\mathbf{k}.$$

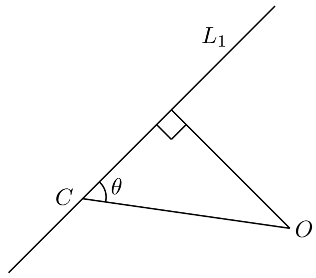
It follows that

$$\begin{aligned} |\overrightarrow{OC}| &= (3^2 + 9^2)^{1/2} = \sqrt{90} = 3\sqrt{10}, \\ |\overrightarrow{AC}| &= (10^2 + 2^2 + 16^2)^{1/2} = \sqrt{360} = 6\sqrt{10}, \\ \overrightarrow{OC} \cdot \overrightarrow{AC} &= -3 \times (-10) + 0 \times (-2) + 9 \times 16 = 174, \end{aligned}$$

and hence

$$\cos\theta = \frac{\overrightarrow{OC} \cdot \overrightarrow{AC}}{|\overrightarrow{OC}||\overrightarrow{AC}|} = \frac{174}{180} = \frac{29}{30}.$$

(e) We are asked to find the shortest distance from 0 to $L_1$, and to do so, it is suggested that it will be helpful to apply the conclusion of part (c) regarding the value of $\cos\theta$, where $\theta = \angle OCA$. Let us start by drawing a sketch that includes the relevant quantities. (Recall that $C$ and $A$ are on the line $L_1$.)




Clearly the point that minimises the distance from 0 to $L_1$ is that at the root of the perpendicular shown in the figure; let us call this point $D$. Since we know that $|\overrightarrow{OC}| = 3\sqrt{10}$ and $\cos\theta = 29/30$, it follows that

$$|\overrightarrow{OD}| = |\overrightarrow{OC}| \sin\theta = 3\sqrt{10}\sqrt{1 - \left(\frac{29}{30}\right)^2} = \sqrt{\frac{59}{10}},$$

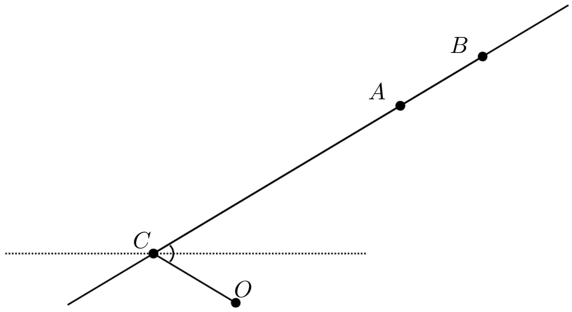
where we have used the fact that $\sin^\theta + \cos^2\theta = 1$.

(f) We already checked in part (d) that $|\overrightarrow{CO}| = |\overrightarrow{OC}| = 3\sqrt{10}$. Moreover, we know from (1) that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 5\mathbf{i} + \mathbf{j} - 8\mathbf{k}$, and so

$$|\overrightarrow{AB}| = (5^2 + 1^2 + 8^2)^{1/2} = \sqrt{90} = 3\sqrt{10}.$$

Thus $|\overrightarrow{CO}| = |\overrightarrow{AB}|$, as desired.

(g) That the previous part of the question was relatively easy is a hint that the conclusion could be useful in this part. Before we consider exactly how, let us sketch the situation. In particular, $A$, $B$ and $C$ all lie in a line, and so we can draw these three points and $O$ in a common plane as follows.



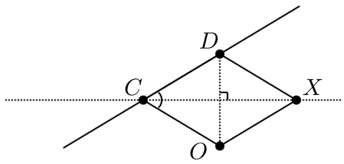

Note that $A$ lies between $C$ and $B$ on the line. The dotted line shows the bisector of $\angle OCA$, the equation of which we are asked to find. Now, if we define $D$ by setting

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{AB},$$

then $D$ lies on the line $L_1$ and also $|\overrightarrow{CD}| = |\overrightarrow{AB}| = |\overrightarrow{OC}|$. Thus, if we define $X$ by

$$\overrightarrow{OX} = \overrightarrow{AB},$$

then it holds that $OCDX$ forms a rhombus, and the bisector of $\angle OCA$ passes through $C$ and $X$:




Since

$$\overrightarrow{CX} = \overrightarrow{CO} + \overrightarrow{OX} = 8\mathbf{i} + \mathbf{j} - 17\mathbf{k},$$

this means that the vector equation of the relevant line is given by

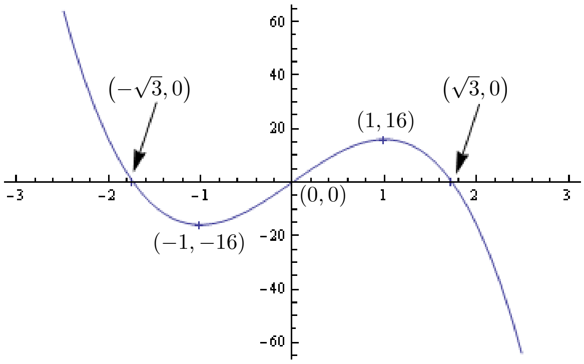
$$\mathbf{r} = -3ii + 9\mathbf{k} + \lambda\,(8\mathbf{i} + \mathbf{j} - 17\mathbf{k})\,, \qquad \lambda \in \mathbb{R}.$$

6. (a) The function $f(x) = x(12 - x^2)$ has roots at 0 and $\pm\sqrt{12}$. Thus $P = (-\sqrt{12}, 0)$ and $R = (\sqrt{12}, 0)$. The derivative of $f$ is given by

$$f'(x) = 12 - 36x^2,$$

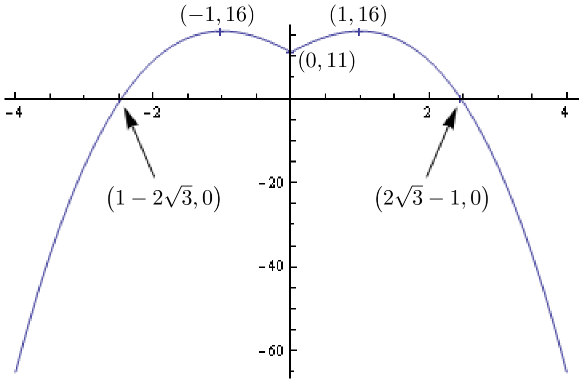
and so the stationary points of $f$ (i.e. where $f'(x) = 0$) are at $x = \pm 2$. Since $Q$ has a positive $x$-coordinate, it must therefore have coordinates $(2, 16)$.

(b) (i) Whilst it would of course be possible to compute $f(2x)$ explicitly and thereby derive the asked-for quantities directly, it is easier to recognise that $f(2x)$ is obtained by a simple transformation of $f(x)$. In particular, $f(2x)$ has the same shape as $f(x)$, but is 'compressed' in the $x$-coordinate by a factor of 2. Thus, from the quantities obtained in part (a), we can easily sketch $f(2x)$ as follows:



Note that the roots of $f(2x)$ are at 0 and $\pm\sqrt{12}/2 = \pm\sqrt{3}$. The local maximum has $x$-coordinate $2/2 = 1$, but the $y$-coordinate of 16 is the same as for $Q$. That local minimum is at $(-1, -16)$ is a simple result of the symmetry $f(x) = -f(-x)$.

(ii) For this part of the question, we have to consider the change in the behaviour of $|x|$ at $x = 0$. For $x \geq 0$, we have that $|x| = x$, and so $f(|x| + 1) = f(x + 1)$, which is simply a unit shift of $f(x)$ along the $x$-axis (to the left). For $x \leq 0$, we have that $f(|-x| + 1) = f(|x| + 1)$, and so the function $f(|x| + 1)$ is symmetric about the $x$-axis. In particular, using these facts we obtain the following sketch.




Note that we now only have two roots for the function, which are at $\sqrt{12} - 1 = 2\sqrt{3} - 1$ and $1 - 2\sqrt{3}$. One local maximum has $x$-coordinate equal to $2 - 1 = 1$ and $y$-coordinate 16. The other, by symmetry is at $(-1, 16)$. Finally, although not asked for, we note that the function has a local minimum at $(0, 11)$, with the function having a sharp point there, rather than a smooth join.

(c) It would be possible to compute $f(x - v) + w$ directly, solve for $v$ and $w$ using the given constraints, and then compute its roots. However, this ignores the useful observation that $f(x - v) + w$ simply represents the function $f(x)$ being shifted to the right by $v$ and up by $w$. In particular, to move the local minimum that was originally at $(-2, -16)$ to $T = (-2 + v, 0)$, we need to shift $f(x)$ up by 16. Hence we straightaway find that $w = 16$. Now, we are told that $S = (0, f(0 - v) + 16)$ has the same $y$-coordinate as $U$, which is a local maximum. Since the local maximum of $f$ had coordinates $(2, 16)$, we deduce that $U = (2 + v, 16 + 16 = 32)$ (from the fact the new graph has been shifted up by 16 units), and therefore $f(-v) + 16 = 32$. Hence we can find $v$ by solving $f(-v) = 16$. This means solving

$$-v(12 - v^2) = 16,$$

or equivalently

$$(v + 2)^2(v - 4) = 0,$$

which implies that $v = -2$ or $v = 4$. Since the minimum of $f$ has moved to a positive value, we know that $v$ is positive. Thus $v = 4$, and we can conclude that the graph shows $f(x - 4) + 16$.

To find the roots of $f(x - 4) + 16$, we again need to do no detailed calculations. Indeed, the first root is at $T$, and so this has $x$-coordinate $-2 + v = 2$. For the second, we observe that the symmetry $f(x) = -f(-x)$ implies that the $x$-coordinate of the root is greater than the $x$-coordinate of $U$ by precisely the same amount as the $x$-coordinate of $T$ is greater than that of $S$, i.e. 2. Hence the second root is at $6 + 2 = 8$.

7. (a) If $x = \sec\theta$, then $dx = \sec\theta\tan\theta d\theta$. Thus, making the suggested substitution,

$$\begin{aligned}\int \sqrt{x^2-1}dx &= \int \sqrt{\sec^2\theta - 1}\sec\theta\tan\theta d\theta \\ &= \int \sec\theta\tan^2\theta d\theta,\end{aligned}$$

where the second inequality holds because $\sec^2\theta - 1 = \tan^2\theta$.

(b) We are told to integrate by parts, and so the difficulty here is choosing how to decompose $\sec\theta\tan^2\theta$ into a part we can integrate and a part we can differentiate. Recall that integration by parts states that

$$\int u(\theta)v'(\theta)d\theta = u(\theta)v(\theta) - \int u'(\theta)v(\theta)d\theta.$$

Inspecting the form of the solution, this suggests we should take $u(\theta) = \tan\theta$ and $v(\theta) = \sec\theta$. In particular, with this choice $u'(\theta) = \sec^2\theta$ and $v'(\theta) = \sec\theta\tan\theta$, and so

$$\begin{aligned}\int \sec\theta\tan^2\theta d\theta &= \int u(\theta)v'(\theta)d\theta \\ &= u(\theta)v(\theta) - \int u'(\theta)v(\theta)d\theta \\ &= \sec\theta\tan\theta - \int \sec^3\theta d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta(1+\tan^2\theta)d\theta,\end{aligned}$$

where for the final equality, we again apply that $\sec^2\theta - 1 = \tan^2\theta$. We can not solve the remaining integral. However, if we break it into two terms, then these are: $\int \sec\theta d\theta$, which is known to be equal to $\ln|\sec\theta + \tan\theta|$; and $\int \sec\theta\tan^2\theta d\theta$, which is equal to the left-hand side. Thus a rearrangement gives

$$2\int \sec\theta\tan^2\theta d\theta = \sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|.$$

Of course, in the above calculation we have omitted to include the constant of integration. Adding this on, we obtain

$$\int \sec\theta\tan^2\theta d\theta = \frac{1}{2}\left[\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right] + c.$$

(c) The obvious first thought on a question like this should be: How do I use the previous conclusions to help solve this integral? The clue is in the square root that appears here and in part (a). In particular, if we apply the identity $\cos 2x = 2\cos^2 x - 1$, then we can write

$$\mathcal{I} = \int_0^{\pi/4} \sin x\sqrt{\cos 2x}dx = \int_0^{\pi/4} \sin x\sqrt{2\cos^2 x - 1}dx.$$

To get an integrand similar to part (a), this suggests we should set $u = \sqrt{2}\cos x$. Indeed, if we do this, then $\sqrt{2\cos^2 x - 1} = \sqrt{u^2-1}$ and $du = -\sqrt{2}\sin x dx$, from which it follows that

$$\mathcal{I} = -\frac{1}{\sqrt{2}}\int_{\sqrt{2}}^{1} \sqrt{u^2-1}du = \frac{1}{\sqrt{2}}\int_1^{\sqrt{2}} \sqrt{u^2-1}du.$$

(Note the change of sign when we reversed the limits of integration.) Now, by applying parts (a) and (b), we obtain that

$$
\begin{aligned}
\mathcal{I} &= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \sec\theta \tan^2\theta d\theta \\
&= \frac{1}{2\sqrt{2}} \left[\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} \\
&= \frac{1}{2\sqrt{2}} \left(\sqrt{2} - \ln(1+\sqrt{2})\right).
\end{aligned}
$$