AEA 2008 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. The terms of an arithmetic sequence $(u_n)_{n\geq 1}$ can be written in the form $u_n = a+d(n-1)$, where $a = u_1$ is the value of the first term and $d = u_n - u_{n-1}$ is the common difference. Given that the first and second terms of the arithmetic series are $u_1 = 200$ and $u_2 = 197.5$, it is immediate that $a = 200$ and $d = 197.5 - 200 = -\frac{5}{2}$. Thus, the nth-term is given by

$$u_n = 200 - \frac{5}{2}(n-1).$$

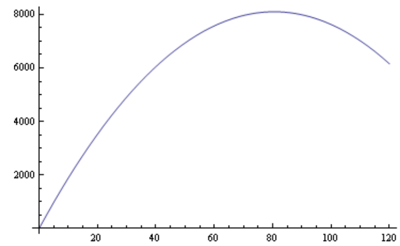
Moreover, for any arithmetic series, the sum of the first n terms satisfies

$$S_n = u_1 + u_2 + \ldots + u_n = \frac{n}{2}(u_1 + u_n).$$

In this case, this means that

$$\begin{aligned} S_n &= \frac{n}{2}\left(200 + 200 - \frac{5}{2}(n-1)\right) \\ &= \frac{1}{4}(805n - 5n^2). \end{aligned}$$

To find the largest positive value of S_n, we therefore need to study the function $s(x) = \frac{1}{4}(805x - 5x^2)$. This is a polynomial of degree 2 and the maximum value is achieved at the value of x_0 satisfying $s'(x_0) = 0$.



Since $s'(x) = \frac{1}{4}(805 - 10x)$, it follows that $x_0 = 80.5$. However, the number of terms of S_n is necessarily a whole number, and so the maximum value of S_n is obtained either when $n = 80$ or $n = 81$. One can check that $S_{80} = S_{81} = 8,100$. In conclusion, the largest positive value of S_n equals $8,100$.

2. (a) To compute the coordinates of P, we will try to put together the information given in the question about the coordinates of this point to arrive at an equation we can solve. First, the question tells us that the point P lies both on the curve C and the line $y = 2x + 5$. Hence, substituting the latter equation into the former, it follows that

$$(x+1)(x+2)\frac{dy}{dx} = x(2x+5).$$

Moreover, because at P the line $y = 2x + 5$ is a tangent to C, we know that $\frac{dy}{dx} = 2$ at P, and so it follows that

$$2(x+1)(x+2) = x(2x+5). \tag{1}$$

Note that it was important to use all the information about the point P that was given to derive this equation that involves only one variable. Equation (1) is equivalent to $2x^2 + 6x + 4 = 2x^2 + 5x$, which simplifies to $x = -4$. This gives us the x-coordinate of the point P. To find the y-coordinate, it suffices to substitute $x = -4$ into the equation $y = 2x + 5$. This gives $y = -3$. Therefore $P = (-4, -3)$.

(b) In trying to solve an equation like $(x+1)(x+2)\frac{dy}{dx} = xy$, it is often helpful to separate the variables, i.e. put all the terms involving x on one side, and all the terms involving y on the other. In this case, doing this and then integrating yields

$$\int \frac{dy}{y} = \int \frac{x}{(x+1)(x+2)}\,dx. \tag{2}$$

The right-hand side is a little tricky, but using partial fractions helps simplify the integrand. In particular, we can check that

$$\frac{x}{(x+1)(x+2)} = \frac{2}{x+2} - \frac{1}{x+1}.$$

Consequently, (2) can be integrated:

$$\ln|y| = 2\ln|x+2| - \ln|x+1| + C.$$

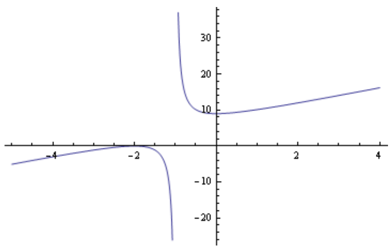
It is important not to forget the constant C! Using the standard rules for logarithms, $\ln(a) + \ln(b) = \ln(a\,b)$ and $p\ln(a) = \ln(a^p)$, one can see that there exists a constant A satisfying

$$y = A\,\frac{(x+2)^2}{x+1}.$$

At this point, in order to find the value of A, one has to use the extra information given by the first question: $P = (-4, -3)$ belongs to the curve C. This means that $-3 = A\frac{(-4+2)^2}{-4+1}$ which gives $A = \frac{9}{4}$. The curve C has equation

$$y = \frac{9(x+2)^2}{4(x+1)}.$$

A quick sketch of C using this equation shows it looks like this:



3. (a) Values of trigonometric functions at 15° are not often quoted, but those at 30° are. In particular, you will hopefully have seen that $\tan(30°) = \frac{1}{\sqrt{3}}$. We will use this fact and the trigonometric identity

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

to find the value of $\tan(15°)$. First, by defining $t = \tan(15°)$, the trigonometric identity implies that t satisfies $\frac{1}{\sqrt{3}} = \frac{2t}{1-t^2}$. This simplifies to the quadratic equation $t^2 + 2\sqrt{3}t - 1 = 0$, which is easily solved to yield

$$t = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = -\sqrt{3} \pm 2.$$

Clearly $\tan(15°)$ can not be equal to both of these values, and so we need to decide which of the roots is the correct. This is easy to do though, because $\tan(15°)$ is positive. Hence it must be the case that $\tan(15°) = -\sqrt{3} + 2$.

Finally, note that it is often the case that there are multiple ways to compute trigonometric quantities. Here, for example, one could have instead used the identities $\tan(15°) = \tan(45° - 30°)$ or $\tan(15°) = \tan(60° - 45°)$. Both of these would have been appropriate, and led to similar computations. As an exercise you could try solving the question using one of these alternative approaches.

(b) First, we will use the identity $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ on the terms $\sin(\theta+60°)$ and $\sin(\theta-60°)$ to simplify the arguments of the trigonometric functions in the equation. In particular, applying the known values $\sin(60°) = \frac{\sqrt{3}}{2}$ and $\cos(60°) = \frac{1}{2}$, we obtain that the equation is equivalent to

$$\left(\sin(\theta)\frac{1}{2} + \cos(\theta)\frac{\sqrt{3}}{2}\right)\left(\sin(\theta)\frac{1}{2} - \cos(\theta)\frac{\sqrt{3}}{2}\right) = (1 - \sqrt{3})\cos^2(\theta). \tag{3}$$

Subsequently, using the identity $(a+b)(a-b) = a^2 - b^2$, equation (3) can be written as

$$\sin^2(\theta)\frac{1}{4} - \cos^2(\theta)\frac{3}{4} = (1 - \sqrt{3})\cos^2(\theta). \tag{4}$$

When dealing with an equation involving terms such as $\cos^2(\theta)$ and $\sin^2(\theta)$, several identities can be used,

$$\cos^2(\theta) + \sin^2(\theta) = 1 \quad \text{or} \quad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \quad \text{or} \quad \tan^2(\theta) = \frac{\sin^2(\theta)}{\cos^2(\theta)}.$$

The first question asked to compute the exact value of $\tan(15°)$: this is a hint that it might be best to use the third identity. By dividing by $\cos^2(\theta)$, it easily seen that equation (4) also reads

$$\tan^2(\theta) = 7 - 4\sqrt{3}. \tag{5}$$

To find the square roots of $7 - 4\sqrt{3}$ (there are 2 distinct square roots!), one can expand $(a + b\sqrt{3})^2 = (a^2 + 3b^2) + 2ab\sqrt{3}$. Thus, we need to find a and b such that $a^2 + 3b^2 = 7$ and $ab = -2$: the choice $a = 2$ and $b = -1$ gives a solution. This shows that the equation (4) is equivalent to

$$\tan(\theta) = 2 - \sqrt{3} \qquad \text{or} \qquad \tan(\theta) = -2 + \sqrt{3}. \tag{6}$$

Now, remember that if $\tan(\alpha) = C$ then all the solutions of the equation $\tan(x) = C$ are given by $x = \alpha + 180°k$ for $k \in \mathbb{Z}$. Since $\tan(15°) = 2 - \sqrt{3}$, the solutions $0° \leq \theta < 360°$ of the equation $\tan(\theta) = 2 - \sqrt{3}$ are $\theta = 15°, 195°$. Also, since $\tan(-15°) = -\tan(15°)$, the solutions $0° \leq \theta < 360°$ of the equation $\tan(\theta) = -2 + \sqrt{3}$ are $\theta = 165°, 345°$. In summary, the solutions of the original equation are

$$x = 15°, 165°, 195°, 345°.$$

4. (a) To find the maximum or minimum of a function, the general approach consists in computing its derivative. To differentiate $y = \cos(x)\ln(\sec x)$, use the product rule and the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \left(\frac{d}{dx}\cos(x)\right)\ln(\sec x) + \cos(x)\left(\frac{d}{dx}\ln(\sec x)\right)\\ &= -\sin(x)\ \ln(\sec x) + \cos(x)\frac{\frac{d}{dx}\sec(x)}{\sec(x)}\\ &= -\sin(x)\ \ln(\sec x) + \cos(x)\tan(x)\\ &= -\sin(x)\ \ln(\sec x) + \sin(x).\end{aligned}$$

Here, we have applied that $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$, and also that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. The equation $\frac{dy}{dx} = 0$ is thus equivalent to

$$\sin(x)\,(1 - \ln(\sec x)) = 0. \tag{7}$$

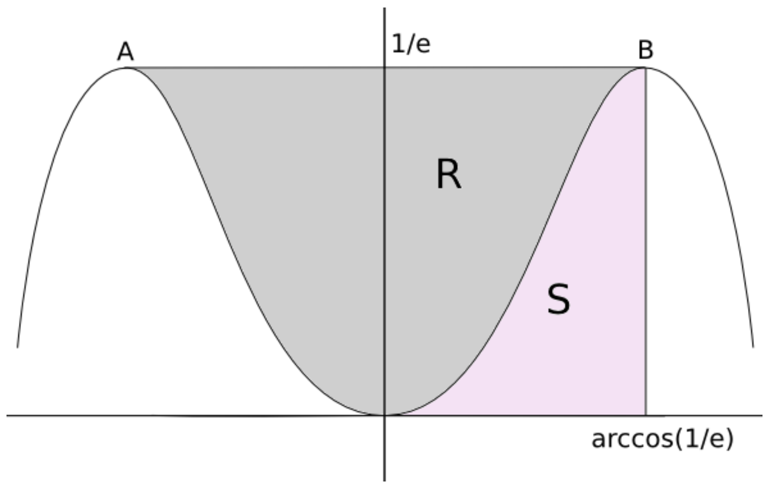
Remember that the solutions of an equation of the type $f(x)g(x) = 0$ are given by the solutions of $f(x) = 0$ **and** the solutions of $g(x) = 0$. Therefore, the solutions of (7) are given by the solutions of equation $\sin(x) = 0$ and the solutions of equation $\ln(\sec x) = 1$.

- The only solution of $\sin(x) = 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is $x = 0$. This clearly corresponds to the unique minimum, as indicated by Figure 1.
- The equation $\ln(\sec x) = 1$ is equivalent to $\sec(x) = e$, or alternatively $\cos(x) = 1/e$. The two solutions between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ of this equation are given by $x = -\arccos\left(\frac{1}{e}\right)$ and $x = \arccos\left(\frac{1}{e}\right)$. Figure 1 indicates that the x-coordinate of B is positive. Consequently, the x-coordinate of B is equal to $x = \arccos\left(\frac{1}{e}\right)$. Therefore

$$B = \left(\arccos\left(\frac{1}{e}\right), \frac{1}{e}\right).$$

(For the y-coordinate, we used the formula $y = \cos(x)\ln(\sec x)$.)

(b) The region R is shown in the following figure, along with the region S enclosed by the x-axis, the curve C and the line $x = \arccos(\frac{1}{e})$.



The symmetry of $\cos(x)$ tells us that $y(x) = y(-x)$. Thus by adding the area of R to twice the area of S we obtain the area of the rectangle with corners at

$$\left(\arccos\left(\frac{1}{e}\right), 0\right), B, A = \left(-\arccos\left(\frac{1}{e}\right), \frac{1}{e}\right), \left(-\arccos\left(\frac{1}{e}\right), 0\right).$$

Since the area of the relevant rectangle is given by $\frac{2}{e}\arccos\left(\frac{1}{e}\right)$, this implies that

$$\text{Area}(R) = \frac{2}{e}\arccos(\frac{1}{e}) - 2 \times \text{Area}(S).$$

Consequently, it suffices to find the area of the region S. To do this, we can simply integrate under the curve C as follows

$$\text{Area}(S) = \int_0^{\arccos(1/e)} \cos(x)\ \ln(\sec x)\ dx$$

A term involving ln often suggests integration by parts is a good technique to try, and so that is what we will do here. Since the derivative of $\ln(\sec(x))$ is given by $\tan(x)$, we obtain that

$$\begin{aligned} I &= \int \cos(x)\ \ln(\sec x)\ dx \\ &= \sin(x)\ln(\sec x) - \int \sin(x)\ \tan(x)\ dx \\ &= \sin(x)\ln(\sec x) - \int \frac{\sin^2(x)}{\cos(x)}\ dx. \end{aligned}$$

This is still not quite in a form we can integrate, but by applying the identity $\sin^2(x)+\cos^2(x) = 1$, we obtain

$$\begin{aligned} I &= \sin(x)\ln(\sec x) - \int \sec(x) - \cos(x)\ dx \\ &= \sin(x)\ln(\sec x) - \ln|\sec(x) + \tan(x)| + \sin(x). \end{aligned}$$

(We have omitted the constant of integration here.) It follows from this calculation that

$$\begin{aligned} &\text{Area}(S) \\ &= [\sin(x)\ln(\sec x) - \ln|\sec(x) + \tan(x)| + \sin(x)]_0^{\arccos(1/e)} \\ &= 2\sin\left(\arccos(1/e)\right) - \ln\left|e + \tan\left(\arccos(1/e)\right)\right|. \end{aligned}$$

To complete the question, we need to express $\sin(\arccos(1/e))$ and $\tan(\arccos(1/e))$ as a function of e. To do this, we will again use the identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$. In particular, because $\arccos(1/e) \in (0, \frac{\pi}{2})$,

$$\sin(\arccos(1/e)) = \sqrt{1-(1/e)^2} = \frac{\sqrt{e^2-1}}{e}.$$

Furthermore,

$$\tan(\arccos(1/e)) = \frac{\sin(\arccos(1/e))}{\cos(\arccos(1/e))} = \sqrt{e^2-1}.$$

This means that $\text{Area}(S) = 2\sqrt{e^2-1}/e - \ln(e + \sqrt{e^2-1})$, and so

$$\text{Area}(R) = \frac{2}{e}\arccos\left(\frac{1}{e}\right) + 2\ln\left(e + \sqrt{e^2-1}\right) - \frac{4\sqrt{e^2-1}}{e}.$$

5. (a) The correct rules for combining logarithms are given by

$$\log_3(p) + \log_3(q) = \log_3(pq) \qquad \text{and} \qquad q\log_3(p) = \log_3(p^q) \tag{8}$$

for all real numbers $p, q > 0$. Applying the second of these, if we write $a = \log_3 p$, then the first of Anna's equations can be written as $a^2 = 2a$, which is equivalent to $a(a-2) = 0$. This quadratic equation has solutions $a = 0, 2$, which means that $p = 3^0 = 1$ or $p = 3^2 = 9$. Now, from Anna's second statement and the first of the logarithm rules above, we also have that

$$\log_3(p+q) = \log_3 p + \log_3 q = \log_3(pq),$$

which is equivalent to $p + q = pq$. If $p = 1$, then there is no solution to this equation. If $p = 9$, then we must have that $q = \frac{p}{p-1} = \frac{9}{8}$. In conclusion, the only values of p and q for which the two assertions of Anna are correct are $p = 9$ and $q = \frac{9}{8}$.

(b) Since $\log_3(9) = 2$, we can rewrite the equation as

$$\begin{aligned}
&\log_3(3x^3 - 23x^2 + 40x) \\
&= \log_3(9)\,(0.5 + \log_3(3x-8)) \\
&= 1 + 2\log_3(3x-8) \\
&= 1 + \log_3\left((3x-8)^2\right),
\end{aligned}$$

where for the final line we use the rule that $q\log_3(p) = \log_3(p^q)$, as stated at (8). To simplify the equation, we proceed by trying to combine the two terms involving logarithms as follows

$$\begin{aligned}
1 &= \log_3(3x^3 - 23x^2 + 40x) - \log_3\left((3x-8)^2\right) \\
&= \log_3(3x^3 - 23x^2 + 40x) + \log_3\left((3x-8)^{-2}\right) \\
&= \log_3\left(\frac{3x^3 - 23x^2 + 40x}{(3x-8)^2}\right).
\end{aligned}$$

This is the same as

$$\frac{3x^3 - 23x^2 + 40x}{(3x-8)^2} = 3. \tag{9}$$

At this point, one could be tempted to re-write equation (9) as $3x^3 - 23x^2 + 40x = 3(3x-8)^2$ and to expand both sides, but this leads to an equation of degree 3 which is not very easily solved. Instead, the fact that x divides the numerator suggests that we could try factorising it, and if we do this we obtain

$$3x^3 - 23x^2 + 40x = x(3x^2 - 23x + 40) = x(3x-8)(x-5).$$

Therefore equation (9) also reads $x(x-5) = 3(3x-8)$, which simplifies to $(x-12)(x-2) = 0$. Clearly the solutions to this are $x = 2$ and $x = 12$.

Note that in both parts of the question, all that was needed to deduce the answer were the two rules for combining logarithms and a bit of careful manipulation.

6. (a) If we define $z = f^{-1}(x)$, then it must be the case that $f(z) = f(f^{-1}(x)) = x$. This also reads

$$\frac{az+b}{z+2} = x.$$

By multiplying both sides by $z+2$ and expressing z as a function of x it follows that

$$z = \frac{2x-b}{a-x}.$$

Consequently, the inverse function f^{-1} is defined by

$$f^{-1}(x) = \frac{2x-b}{a-x} \qquad x \in \mathbb{R}, \quad x \neq a.$$

(b) To find the value of a such that $ff(x) = x$, one could try to expand

$$ff(x) = \frac{a\left(\frac{ax+b}{x+2}\right)+b}{\left(\frac{ax+b}{x+2}\right)+2},$$

but this leads to a complicated equation. Instead, we will try to apply the previous part of the question. In particular, by applying f^{-1} to both sides of the equation $ff(x) = x$, we obtain that $f(x) = f^{-1}(x)$, i.e.

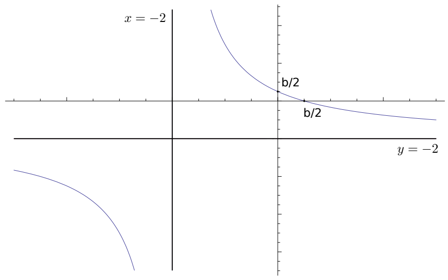
$$\frac{ax+b}{x+2} = \frac{2x-b}{a-x} \qquad \forall x \in \mathbb{R}, \quad x \neq -2, a,$$

so that $(ax+b)(a-x) - (x+2)(2x-b) = 0$ for all $x \neq -2, a$. This is equivalent to $-(a+2)x^2 + (a^2-4)x + b(a+2) = 0$, and it therefore follows that $ff(x) = x$ if and only if $a = -2$.

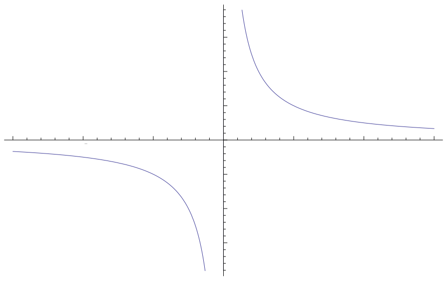
(c) (i) By the previous part of the question, we will sketch the curve $y = f(x)$, where

$$f(x) = \frac{-2x+b}{x+2} \qquad x \in \mathbb{R}, \quad x \neq -2.$$

First we establish where the asymptotes are. One of these is at $x = -2$, as this is where the function explodes. We also note that the function converges to the line $y = -2$ as $x \to \pm\infty$. To decide where the function crosses the x-axis, we need to solve $f(x) = 0$. This is clearly the case when $x = b/2$. Similarly, the function crosses the y-axis at $f(0) = b/2$.



(ii) For this part of the question, we do not need to do any complicated calculations, but simply note that the curve $y = f(x)+2$ is the same as $y = f(x)$, apart from being shifted 'up' by two. Moreover, the curve $y = f(x-2)+2$ is the same as $y = f(x)+2$, apart from shifted 'to the right' by two. Hence, the sketch looks like:



(d) The normal to a curve at a point is a line that is perpendicular to the curve at that point. Thus, because $y = f(x-2)+2$ is obtained by a simple shift from C, the image of the normal to the curve C at P on the curve $y = f(x-2)+2$ has equation $y = 4(x-2) - 39 + 2 = 4x - 45$.

Now, from the sketch in the previous part of the question, we can see that there are exactly two points where the normal to the curve $y = f(x-2)+2$ has gradient 4. Moreover, by the symmetry of the graph, if the point $P' = (a, b)$ has normal $y = 4x - 45$, then the other is at $Q' = (-a, -b)$ and has normal $y = 4x + 45$.

Clearly Q' is the image of Q under the transformation that takes $y = f(x)$ to $y = f(x-2)+2$, and so reversing this transformation yields that the normal to the curve C at Q is given by $y = 4(x+2) + 45 - 2 = 4x + 51$, i.e. $k = 51$.

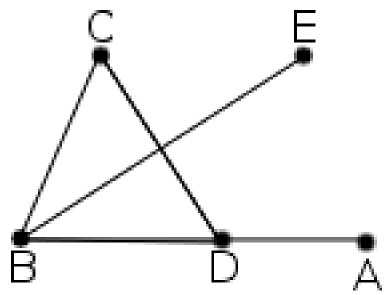
7. (a) The angle ABC is that between the vectors $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = -4\mathbf{i} + \mathbf{j} - 8\mathbf{k}$ and $\overrightarrow{BC} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. In general, to find the cosine of the angle θ between two vectors $\vec{u}$ and $\vec{v}$, we can use the formula

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta), \tag{10}$$

where the norm of a vector $\vec{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by $\|\vec{u}\| = \sqrt{a^2 + b^2 + c^2}$. In the particular case with $\vec{u} = \overrightarrow{BA}$ and $\vec{v} = \overrightarrow{BC}$, so that $\|\overrightarrow{BA}\| = \sqrt{4^2 + 1^2 + 8^2} = 9$, $\|\overrightarrow{BC}\| = 6$ and $\overrightarrow{BA} \cdot \overrightarrow{BC} = -16 + 2 + 32 = 18$, formula (10) shows that

$$\cos(\theta) = \frac{18}{9 \times 6} = \frac{1}{3}.$$

(b) To find the equation of the line L, the simplest idea is to construct a rhombus $CBDE$ as described by the following figure.



To be precise, D is chosen on the line passing through B and A such that $\|\overrightarrow{BD}\| =$

$\|\overrightarrow{BC}\| = 6$. In particular, this implies that $\overrightarrow{BD} = \frac{2}{3}\overrightarrow{BA}$, and so

$$\overrightarrow{BE} = \overrightarrow{BC} + \overrightarrow{BD} = \overrightarrow{BC} + \frac{2}{3}\overrightarrow{BA} = \frac{4}{3}\left(\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}\right).$$

Consequently $\vec{v} = \mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$ is a gradient of the line L. Since $B \in L$, an equation of L is given by

$$L: \qquad \mathbf{r} = \mathbf{i} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}), \qquad t \in \mathbb{R}. \tag{11}$$

Note that L is <u>not</u> the line that goes through B and the midpoint of AC.

(c) Since $\overrightarrow{AC} = 8\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ we have $\|\overrightarrow{AC}\| = \sqrt{8^2 + 1^2 + 4^2} = 9 = \|\overrightarrow{BA}\|$

(d) The centre of the inscribed circle S is the intersection of the 3 angle bisectors of the triangle ABC. In fact, it is always enough to find the intersection of just two of these. Since we already known that the equation of the bisector L of angle ABC is given by (11), we only need to find one more of the bisectors. A hint for doing this was given in the previous part of the question. In particular, since $\|\overrightarrow{AC}\| = \|\overrightarrow{AB}\|$, a gradient of the bisector of angle BAC is $\vec{u} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) = 6(\mathbf{i} + \mathbf{k})$. Consequently, an equation of the bisector of angle BAC is

$$\mathbf{r} = -3\mathbf{i} + \mathbf{j} - 9\mathbf{k} + s(\mathbf{i} + \mathbf{k}), \qquad s \in \mathbb{R}.$$

The centre N of the circle S can thus be found by solving the equation

$$\mathbf{i} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}) = -3\mathbf{i} + \mathbf{j} - 9\mathbf{k} + s(\mathbf{i} + \mathbf{k}). \tag{12}$$

By matching the coefficients of the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, we find that this is equivalent to the system of linear equations

$$\begin{aligned} 1 + t &= -3 + s \\ 2t &= 1 \\ -1 - 7t &= -9 + s \end{aligned}$$

The second equation gives $t = \frac{1}{2}$ and the first equation then gives $s = \frac{9}{2}$. One can check that $(s,t) = (\frac{9}{2}, \frac{1}{2})$ is indeed solution of the third equation, which confirms that $(s,t) = (\frac{9}{2}, \frac{1}{2})$ is the unique solution of equation (12). Finally, this shows that the vector $\overrightarrow{ON} = \frac{3}{2}\mathbf{i} + \mathbf{j} - \frac{9}{2}\mathbf{k}$.

(e) Since $\|\overrightarrow{AC}\| = \|\overrightarrow{AB}\|$ the intersection between the circle S and the side BC is precisely located at the mid-point X of the segment BC. The radius r of S is thus equal to the length of the segment NX. Since $\overrightarrow{OX} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}) = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\overrightarrow{ON} = \frac{3}{2}\mathbf{i} + \mathbf{j} - \frac{9}{2}\mathbf{k}$ it follows that

$$r = \|\overrightarrow{NX}\| = \sqrt{\left(3 - \frac{3}{2}\right)^2 + (1-1)^2 + \left(-3 + \frac{9}{2}\right)^2} = \frac{3\sqrt{2}}{2}.$$