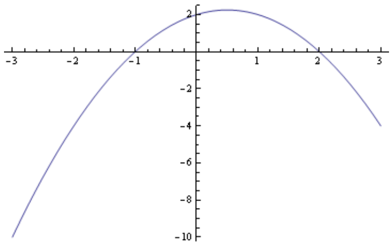
AEA 2009 Extended Solutions

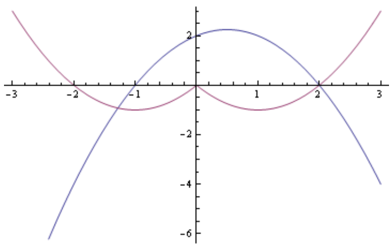
These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. (a) The quadratic curve $y = (x+1)(2-x)$ crosses the x-axis at $x = -1$ and $x = 2$, and the y-axis at $y = 2$. From these facts, it is easy to draw an accurate sketch of function:



To sketch the function $y = x^2 - 2|x|$, it is necessary to consider the change in the behaviour of $|x|$ at $x = 0$. In particular, for $x \geq 0$ it holds that $|x| = x$, whereas for $x \leq 0$ it is the case that $|x| = -x$.

Thus, when $x \geq 0$, the given function looks like $y = x^2 - 2x = x(x-2)$, which is a quadratic function that crosses the x-axis at $x = 0$ and $x = 2$, and the y-axis at $y = 0$. Moreover, in the case $x \leq 0$, the given function looks like $y = x^2 + 2x = x(x+2)$, which is a quadratic function that crosses the x-axis at $x = -2$ and $x = 0$, and the y-axis at $y = 0$. Combining these observations, the sketch can be completed as follows:



Note that, because the two parts of the curve that have been used to draw the graph for $y = x^2 - 2|x|$ are both *U-shaped*, there must be a sharp point at $x = 0$ rather than a smooth join there.

(b) It can be seen from the sketch that there are two points of intersection of the two curves. One of these occurs when $x = 2$. (Since it was already noted when drawing the sketch that this is the point where both curves cross the axis, no more calculations are needed to check this.) The other can be seen to occur in the range $x \in (-2, -1)$. Since in this range it holds that $x^2 - 2|x| = x^2 + 2x$, it is necessary to solve

$$(x+1)(2-x) = x^2 + 2x.$$

By rearranging, this is equivalent to solving $2x^2 + x - 2 = 0$, which yields

$$x = \frac{-1 \pm \sqrt{17}}{4}.$$

To select which of these roots is the one of interest, observe that

$$x = \frac{-1+\sqrt{17}}{4} > -\frac{1}{4} > -1,$$

and so this root is outside the range $(-2, -1)$. This means that the correct choice is given by taking

$$x = \frac{-1 - \sqrt{17}}{4}.$$

2. (a) To work out what the tangent to the curve $y = x^{\sin x}$ is at the point where $x = \frac{\pi}{2}$, it is first necessary to work out the value of y there. This is given by $y = (\frac{\pi}{2})^{\sin(\pi/2)} = \frac{\pi}{2}$. Next, one must calculate the value of the slope at this point. Although it is not immediately clear how to differentiate $y = x^{\sin x}$, if logs are applied to both sides, then one obtains

$$\ln y = \sin x \ln x,$$

upon which the chain rule and the product rule can be used. In particular, by differentiating using these rules, it follows that

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{\sin x}{x},$$

and multiplying both sides by $y = x^{\sin x}$ gives the derivative of interest:

$$\frac{dy}{dx} = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right). \tag{1}$$

In particular, at $x = \frac{\pi}{2}$, this takes the value

$$\frac{\pi}{2}\left(\cos\left(\frac{\pi}{2}\right)\ln\left(\frac{\pi}{2}\right) + \frac{\sin\left(\frac{\pi}{2}\right)}{\left(\frac{\pi}{2}\right)}\right) = 1.$$

If a line passes through (a, b) with slope m, then it has equation $y - b = m(x - a)$. Hence, since the tangent to $y = x^{\sin x}$ at $x = \frac{\pi}{2}$ passes through $(\frac{\pi}{2}, \frac{\pi}{2})$ and has slope 1, it has equation $y - \frac{\pi}{2} = x - \frac{\pi}{2}$, which can be simplified to $y = x$. The diagram on the left-hand side of Figure 1 shows the curve and this tangent.

Finally for this part, note that the above argument for differentiating $y = x^{\sin x}$ could be used to differentiate any function of the form $y = f(x)^{g(x)}$ where $f(x) > 0$. You might like to check as an exercise that, for such a function,

$$\frac{dy}{dx} = f(x)^{g(x)}\left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right).$$

(b) The curves $y = x^{\sin x}$ and $y = x$ intersect at x if and only if $x^{\sin x} = x$. Clearly this holds whenever $\sin x = 1$, which occurs for $x = \frac{\pi}{2} + 2n\pi$ for any $n = 0, 1, 2, \ldots$. To show that the line $y = x$ is only *touching* at these points, it will be enough to show that $x^{\sin x} < x$ for every $x > 1$ which is not of the form $\frac{\pi}{2} + 2n\pi$, because this means that the curve $y = x^{\sin x}$ must rise up to hit the line $y = x$ from below at $x = \frac{\pi}{2} + 2n\pi$ and then immediately fall below again. To do this, first observe that for those $x \in (1, \infty)$ that are not equal to $\frac{\pi}{2} + 2n\pi$ for some $n = 0, 1, 2, \ldots$, it holds that $\sin x < 1$ and $\ln x > 0$. Hence $\sin x \ln x < \ln x$, and so, because e^x is a strictly increasing function, $x^{\sin x} < x$, as desired. The first five points where the two graphs touch are shown on the right-hand side of Figure 1. Observe that $y = x^{\sin x}$ and $y = x$ also intersect at $x = 1$, but at this point the gradient of the curve $y = x^{\sin x}$ is given by $\sin 1 \neq 1$, and so the two graphs actually cross there.

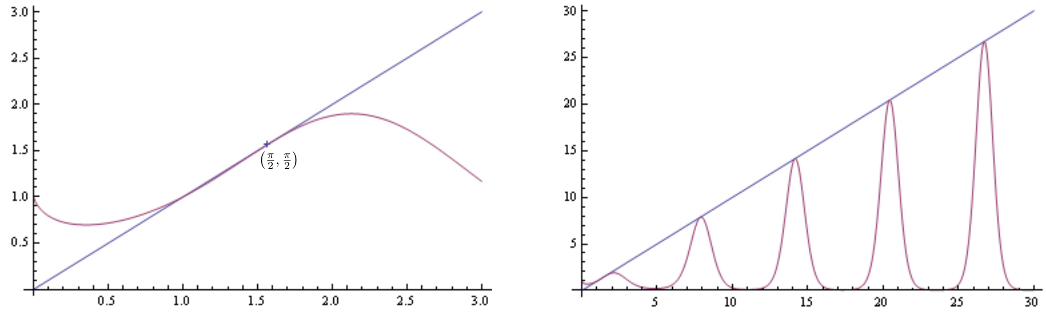


Figure 1: Graphs of the functions $y = x^{\sin x}$ and $y = x$.

Finally, let it be noted that the above argument implies that $y = x$ is a tangent to $y = x^{\sin x}$ at each of the points $x = \frac{\pi}{2} + 2n\pi$ for any $n = 0, 1, 2, \ldots$. Checking this can also be done directly by first using the formula at (1) to deduce that the gradient of $y = x^{\sin x}$ at $x = \frac{\pi}{2} + 2n\pi$ is given by

$$\left(\frac{\pi}{2} + 2n\pi\right)\left(\cos\left(\frac{\pi}{2} + 2n\pi\right)\ln\left(\frac{\pi}{2} + 2n\pi\right) + \frac{\sin\left(\frac{\pi}{2} + 2n\pi\right)}{\left(\frac{\pi}{2} + 2n\pi\right)}\right) = 1.$$

Since the tangent to the curve $y = x^{\sin x}$ at the point where $x = \frac{\pi}{2} + 2n\pi$ passes through $(\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi)$ and has gradient 1, it follows that its equation is given by $y = x$, as claimed.

3. (a) To start with, use the trigonometric identity

$$\sin(a - b) = \sin(a)\cos(b) - \cos(a)\sin(b) \tag{2}$$

to rewrite the given equation as

$$\sin\left(\frac{\pi}{3}\right)\cos(\theta) - \cos\left(\frac{\pi}{3}\right)\sin(\theta) = \frac{1}{\sqrt{3}}\cos(\theta).$$

Such an argument should be the first thing to spring to mind when confronted with a similar type of equation. Since $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$, the above equation is equivalent to

$$\sin(\theta) = \left(\sqrt{3} - \frac{2}{\sqrt{3}}\right)\cos(\theta) = \frac{1}{\sqrt{3}}\cos(\theta).$$

Now, the places where $\cos(\theta) = 0$, i.e. $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ are not solutions of the equation (because at these places $\sin(\theta) \neq 0$), so it is possible to ignore these values of θ and divide through by $\cos(\theta)$ to obtain

$$\tan(\theta) = \frac{1}{\sqrt{3}}.$$

(Note that if the values $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ were not excluded, then it would mean dividing by 0 at these points, which is a bad thing to do!) At this point, it is important not to forget that in general a trigonometric equation has more than one solution. In this case, the equation has 2 solutions between 0 and 2π given by $\theta = \frac{\pi}{6}$ and $\theta = \frac{7\pi}{6}$.

As a final remark, we recall that to solve an equation of the type $a\cos(\theta)+b\sin(\theta)=c$, one may use the fact that the left-hand side can always be written as $r\cos(\theta+\alpha)$, where $r>0$ and $0\le\alpha<2\pi$ are well-chosen constants. Can you work out what r and α are?

(b) Suppose x solves the given equation. To get rid of the arcsin, it will be useful to take sin of both sides, which yields

$$\sin\left(\arcsin\left(1-2x\right)\right)=\sin\left(\frac{\pi}{3}-\arcsin x\right).$$

For any $x\in[-1,1]$ we have $\sin(\arcsin(x))=x$, and so the left-hand side is equal to $1-2x$. For the right-hand side, the trigonometric identity at (2) again seems suitable. In particular, it implies

$$\begin{aligned}1-2x &= \sin\left(\frac{\pi}{3}\right)\cos\left(\arcsin x\right)-\cos\left(\frac{\pi}{3}\right)\sin\left(\arcsin x\right)\\ &= \frac{\sqrt{3}}{2}\cos\left(\arcsin x\right)-\frac{x}{2}.\end{aligned}$$

To compute $\cos(\arcsin(x))$ one can use the fact that

$$\cos(\arcsin(x))^2+\sin(\arcsin(x))^2=1.$$

This give $\cos(\arcsin(x))=\pm\sqrt{1-x^2}$. However, by assumption $0<x<0.5$, which implies $0<\arcsin(x)<\pi/6$, and consequently $\cos(\arcsin(x))$ must be positive, i.e. $\cos(\arcsin(x))=\sqrt{1-x^2}$. Thus we can rewrite our equation as

$$1-2x=\frac{\sqrt{3}}{2}\sqrt{1-x^2}-\frac{x}{2},$$

or equivalently

$$2-3x=\sqrt{3}\sqrt{1-x^2}.$$

To solve this equation, one needs to square both sides, which leads to the quadratic equation

$$12x^2-12x+1=0$$

whose solutions are given by $x=\frac{3\pm\sqrt{6}}{6}$. The only solution that lies in the interval $(0,\frac{1}{2})$ is $x=\frac{3-\sqrt{6}}{6}$.

4. (a) The function $f(x)$ satisfies $f'(x)=\frac{u(x)}{v(x)}$. To find $f''(x)$, we need to differentiate, which we can do by applying the quotient rule, which is just a combination of the chain rule and the product rule. In particular,

$$\begin{aligned}f''(x) &= \left(\frac{d}{dx}u(x)\right)\frac{1}{v(x)}+u(x)\frac{d}{dx}\left(\frac{1}{v(x)}\right)\\ &= \frac{u'(x)}{v(x)}-\frac{u(x)v'(x)}{v(x)^2}\\ &= \frac{u'(x)v(x)-u(x)v'(x)}{v(x)^2}.\end{aligned}$$

Since $f'(k)=0$, it must hold that $u(k)=0$, and therefore

$$f''(k)=\frac{u'(k)v(k)}{v(k)^2}=\frac{u'(k)}{v(k)}.$$

(b) (i) The point A is the place where the curve C crosses the y-axis. Since the y-axis is simply the line $x = 0$, A must have coordinates $(0, y_A)$, where $y_A = \frac{2\times 0^2+3}{0^2-1} = -3$.

(ii) It is important to remember that, in general, curves can have horizontal and/or vertical asymptotes. As is shown on the sketch, the curve C has both. To find the vertical asymptotes of C, we have to find where the denominator of $\frac{2x^2+3}{x^2-1}$ vanishes. This is the case when $x^2-1 = 0$, meaning that the equations of the two vertical asymptotes are given by $x = -1$ and $x = 1$. The horizontal asymptote is found by computing the limit of $\frac{2x^2+3}{x^2-1}$ as $|x|$ becomes large (note that, by the symmetry of C about the y-axis, it does not matter if we look in the positive or negative direction). To find this, we rewrite the expression as follows:

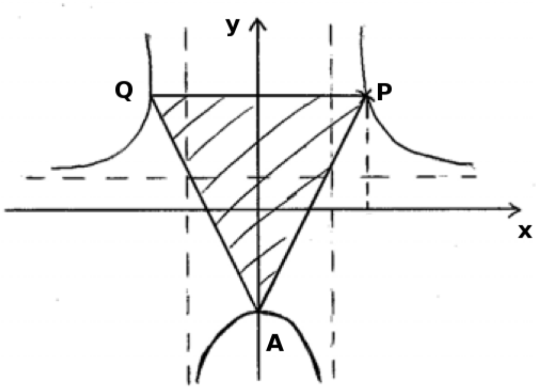
$$\frac{2x^2+3}{x^2-1} = \frac{2+\frac{3}{x^2}}{1-\frac{1}{x^2}}.$$

Since the terms involving x^{-2} disappear as $|x|$ goes to infinity, the limit is equal to 2, meaning that the equation of the horizontal asymptote is $y = 2$.

(iii) The point $P = (a, b)$, where $a > 0$ and $b > 0$, lies on C; we note for later that this means

$$b = \frac{2a^2+3}{a^2-1}. \tag{3}$$

The point Q also lies on C with PQ parallel to the x-axis and $AP = AQ$. Again using the symmetry of the curve C about the y-axis, this means that $Q = (-a, b)$.



As with any triangle, the area $\mathcal{A}(PAQ)$ of the triangle PAQ equals

$$\frac{1}{2} \times (\text{base}) \times (\text{height}).$$

In this case, we can take

$$\begin{aligned}(\text{base}) &= |PQ| = 2a \\ (\text{height}) &= b + |y_A| = b + 3 = \frac{2a^2+3}{a^2-1} + 3,\end{aligned}$$

where we have applied (3) to obtain the final expression. Therefore

$$\mathcal{A}(PAQ) = a\left(\frac{2a^2+3}{a^2-1} + 3\right) = \frac{5a^3}{a^2-1}.$$

(iv) By the previous part of the question, in order to find the minimum area of triangle PAQ as the variable $a > 0$ varies, we need to find the minimum of the function $f(a) = 5a^3/(a^2-1)$. By applying the quotient rule, the derivative of this function can be checked to be equal to

$$f'(a) = \frac{5a^2(a^2-3)}{(a^2-1)^2}.$$

This is equal to 0 if $a = 0$ or $a = \pm\sqrt{3}$. Thus the only stationary point of $f(a)$ with $a > 0$ is given by $a = \sqrt{3}$. To check that this point is a minimum, we study the second derivative of f there. We could do this directly, but it will be quicker to use part (a) of the question. In particular, since we have that $f'(a)$ is of the form $u(a)/v(a)$, with $u(a) := 5a^2(a^2-3)$ and $v(a) = (a^2-1)^2$, and also $u(\sqrt{3}) = 0$, part (a) of the question implies that

$$f''(\sqrt{3}) = \frac{u'(\sqrt{3})}{v(\sqrt{3})}.$$

Now, $u'(a) = 20a^3 - 30a$, and therefore

$$f''(\sqrt{3}) = \frac{20\sqrt{3}^3 - 30\sqrt{3}}{(\sqrt{3}^2-1)^2} = \frac{(60-30)\sqrt{3}}{4} = \frac{15\sqrt{3}}{2} > 0,$$

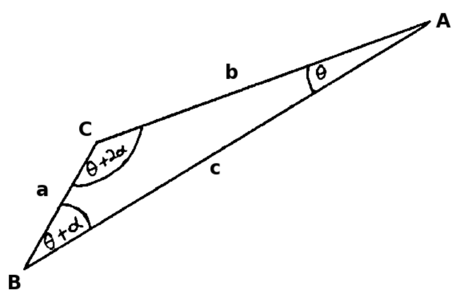
which implies that $a = \sqrt{3}$ is a minimum. In conclusion, the minimum area of PAQ is equal to

$$f(\sqrt{3}) = \frac{5\sqrt{3}^3}{\sqrt{3}^2 - 1} = \frac{15\sqrt{3}}{2}.$$

5. (a) (i) To compute the area $\mathcal{A}(ABC)$ of the triangle ABC, knowing the angles A, B, C and the side lengths a,b and c one can use the formula

$$\mathcal{A}(ABC) = \frac{1}{2}ab\sin(C) = \frac{1}{2}bc\sin(A) = \frac{1}{2}ca\sin(B). \tag{4}$$

The final expression here looks a little bit like the answer we are given. In particular, it will be enough to check that $\sin B = \frac{\sqrt{3}}{2}$. To check this, we need to find a way to apply the assumption that the sizes of the angles A, B and C form an arithmetic sequence, i.e. $A = \theta$, $B = \theta + \alpha$ and $C = \theta + 2\alpha$ for some $\theta, \alpha > 0$.



A fact that holds for all triangles is that the internal angles sum to π. In our case, this means

$$\pi = A + B + C = 3\theta + 3\alpha.$$

Or equivalently, $\pi/3 = \theta + \alpha = B$. Hence, $\sin B = \frac{\sqrt{3}}{2}$, and so formula (4) shows that

$$\mathcal{A}(ABC) = \frac{1}{2} ca \sin(B) = \frac{ac\sqrt{3}}{4},$$

as desired. Note that we did not need to work out the exact values of θ or α.

(ii) To exploit the relationships between the angles and the side lengths of a triangle one can use the sine rule,

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}.$$

Since $B = \frac{\pi}{3}$ and $a = 2$ and $\sin(A) = \frac{\sqrt{15}}{5}$, it follows that $b = \frac{a \sin(B)}{\sin(A)} = \sqrt{5}$.

(iii) Here, it will be easiest to use the cosine rule,

$$b^2 = a^2 + c^2 - 2ac\cos(B),$$

because we already know the values of a, b and B. In particular, replacing these variables by their numerical values leads to the quadratic equation

$$c^2 - 2c - 1 = 0.$$

The solutions of this equation are $c = 1 \pm \sqrt{2}$, but since c is clearly positive, it must be the case that $c = 1 + \sqrt{2}$.
Alternate solution: Again using the sine rule,

$$c = a\frac{\sin(C)}{\sin(A)} = \frac{10\sin(C)}{\sqrt{15}}.$$

We do need to compute the value of $\sin(C)$, though. To do this, we again use that $A + B + C = \pi$, and also that $\sin(\pi - x) = \sin(x)$, to deduce

$$\begin{aligned}\sin(C) &= \sin(\pi - (A + B)) = \sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)\\ &= \frac{\sqrt{15}}{5}\frac{1}{2} + \cos(A)\frac{\sqrt{3}}{2}.\end{aligned}$$

Since $\sin(A) = \frac{\sqrt{15}}{5}$ and $\sin(A)^2 + \cos(A)^2 = 1$, we have that $\cos(A) = \frac{\sqrt{2}}{\sqrt{5}}$ (recall that $0 < A < B = \pi/3$, and so $\cos(A)$ is positive). This shows that

$$\sin(C) = \frac{\sqrt{3}(1 + \sqrt{2})}{2\sqrt{5}}$$

and gives

$$c = \frac{10\sin(C)}{\sqrt{15}} = 1 + \sqrt{2}.$$

(b) For any n-sided polygon, the sum of the internal angles equals $(n-2) \times 180°$. We will use this fact to help us work out the value of n. In particular, we are told that the internal angles form an arithmetic sequence with first term $u_1 = 143°$ and common difference $2°$, i.e. the k-th internal angle u_k is given by $u_k = 143° + (k-1) \times 2°$. Since a polynomial with n sides has n internal angles, it follows that the sum of internal angles is given by

$$\begin{aligned} S &= u_1 + \cdots + u_n \\ &= \frac{n}{2}(u_1 + u_n) \\ &= \frac{n}{2}(143^\circ + 143^\circ + (n-1) \times 2^\circ) \\ &= (n^2 + 142n)^\circ. \end{aligned}$$

By equating the two different expressions that we have for the sum of internal angles, we obtain that

$$n(142 + n) = 180(n - 2). \tag{5}$$

The quadratic equation (5) is readily solved, with its two solutions being given by $n = 18$ and $n = 20$. The solution $n = 20$ is not satisfying because $u_{20} = 143^\circ + 19 \times 2^\circ = 181^\circ > 180^\circ$. Hence, the polygon has $n = 18$ sides.

6. (a) The point $P = (x_P, y_P)$ corresponds to the point on the curve C where $t = \frac{\pi}{3}$. Therefore

$$x_P = 2\sin(t_p) = \sqrt{3}, \qquad y_P = \ln(\sec(t_p)) = \ln(2).$$

We also need to find the gradient of C at P. Since x and y are functions of the variable t, we start by rewriting $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and $\frac{dx}{dt}$ as follows:

$$\frac{dy}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} = \frac{\tan(t)}{2\cos(t)}.$$

Thus, the gradient of C at the point P, is given by

$$m = \left.\frac{dy}{dx}\right|_{t_p} = \frac{\tan(\pi/3)}{2\cos(\pi/3)} = \sqrt{3}.$$

Hence, because the equation of the tangent to C at P is given by $y = y_P + m(x - x_P)$, we obtain

$$y = \ln(2) + \sqrt{3}(x - \sqrt{3}).$$

This line intersects the x-axis at $A = (x_A, 0)$. The x-coordinate x_A satisfies

$$0 = \ln(2) + \sqrt{3}(x_A - \sqrt{3}).$$

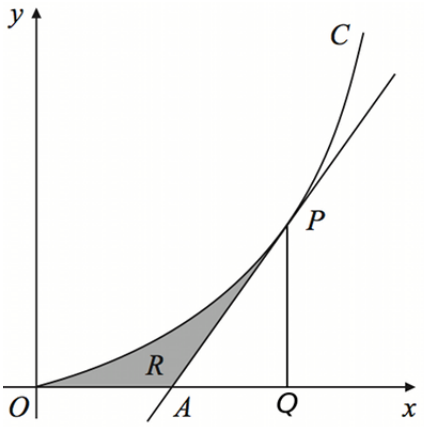
This shows that

$$A = \left(\frac{3 - \ln 2}{\sqrt{3}}, 0\right).$$

(b) The area R is difficult to compute directly. However, if we let $Q = (x_P, 0)$ (see figure below), then one can compute the area of R as the difference

$$\text{Area}(R) = \text{Area(under C)} - \text{Area}(APQ),$$

where by Area(under C) we mean the area enclosed by C, the x-axis and the line PQ.



We have Area$(APQ) = \frac{1}{2} \times$ (base) $\times$ (height) with (base) $= |AQ| = \sqrt{3} - \frac{3-\ln 2}{\sqrt{3}}$ and (height) $= y_P = \ln(2)$, therefore

$$\text{Area}(APQ) = \frac{(\ln 2)^2}{2\sqrt{3}}. \tag{6}$$

The area under C between O and P can be obtained by integration,

$$\begin{aligned}
\text{Area(under C)} &= \int_{t=0}^{\frac{\pi}{3}} y\,dx \\
&= \int_{t=0}^{\frac{\pi}{3}} y\,\frac{dx}{dt}\,dt \\
&= \int_{t=0}^{\frac{\pi}{3}} \ln(\sec t)\,2\cos(t)\,dt.
\end{aligned}$$

The term $\ln(\sec t)$ is difficult to integrate but has a simple derivative,

$$\frac{d}{dt}\ln(\sec t) = \tan t.$$

This motivates an integration by parts,

$$\begin{aligned}
\text{Area(under C)} &= \int_{t=0}^{\frac{\pi}{3}} \ln(\sec t)\,2\cos(t)\,dt \\
&= \Big[2\tan(t)\ \cos(t)\Big]_0^{\frac{\pi}{3}} - \int_{t=0}^{\frac{\pi}{3}} 2\ \tan(t)\sin(t)\,dt.
\end{aligned}$$

To integrate the term $\tan(t)\cos(t)$, one can try first to simplify it and express it in terms of the usual trigonometric functions,

$$\tan(t)\sin(t) = \frac{\sin(x)^2}{\cos(x)} = \frac{1-\cos(x)^2}{\cos(x)} = \sec(x) - \cos(x).$$

Consequently $\int \tan(t)\sin(t)\,dt = \ln\big|\sec(t) + \sin(t)\big| - \sin(t)$ and

$$\begin{aligned}
\text{Area(under C)} &= \Big[2\tan(t)\ \cos(t) - 2\ln\big|\sec(t) + \sin(t)\big| + 2\sin(t)\Big]_0^{\frac{\pi}{3}} \\
&= \sqrt{3}\big(\ln(2) + 1\big) - 2\ln\big(2 + \sqrt{3}\big).
\end{aligned} \tag{7}$$

By combining (6) and (7) it follows that

$$\begin{aligned}\text{Area}(R) &= \text{Area(under C)} - \text{Area}(APQ)\\ &= \sqrt{3}\big(\ln(2)+1\big) - 2\ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{6}(\ln 2)^2,\end{aligned}$$

which is what we were asked to prove.

7. (a) To find the cosine of the angle θ between two vectors $\vec{u}$ and $\vec{v}$, the most obvious thing to do is to use the formula

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{u}\|\cos(\theta). \tag{8}$$

Consequently, $\theta = \text{angle}(ABC)$ satisfies $\overrightarrow{BA}\cdot\overrightarrow{BC} = \|\overrightarrow{BA}\|\|\overrightarrow{BC}\|\cos(\theta)$. We have

$$\begin{aligned}\overrightarrow{BA} &= \mathbf{a}-\mathbf{b} = -5\mathbf{i}+5\mathbf{k}\\ \overrightarrow{BC} &= \mathbf{c}-\mathbf{b} = 2\mathbf{i}+4\mathbf{j}.\end{aligned}$$

Thus $\overrightarrow{BA}\cdot\overrightarrow{BC} = -10$ and $\|\overrightarrow{BA}\| = 5\sqrt{2}$ and $\|\overrightarrow{BC}\| = 2\sqrt{5}$. This gives

$$\cos(\theta) = -\frac{1}{\sqrt{10}}.$$

(b) The area of a triangle ABC is equal to $\frac{1}{2}\times|BC|\times|BA|\times\sin(\theta)$ with $\theta = \text{angle}(ABC)$. Since

$$\sin(\theta) = \sqrt{1-\cos(\theta)^2} = \sqrt{1-\frac{1}{10}}$$

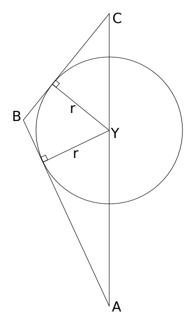
(note that we must take the positive root, because the angle is less than π), it follows that the area of the triangle ABC equals

$$\frac{1}{2}\times 2\sqrt{5}\times 5\sqrt{2}\times\sqrt{1-\frac{1}{10}}.$$

By symmetry, the area of the kite K is equal to twice the area of ABC,

$$\text{Area(K)} = 2\,\text{Area}(ABC) = 2\times\frac{1}{2}\times 2\sqrt{5}\times 5\sqrt{2}\times\sqrt{1-\frac{1}{10}} = 30.$$

(c) The circle is drawn inside K and touches each of the 4 sides of K. Notice that, since K is a kite, the centre Y of the circle lies on the segment AC. Moreover, the two radii shown in the following figure are perpendicular to BC and AB, respectively.



To find the radius r of the inscribed circle, we start by observing that

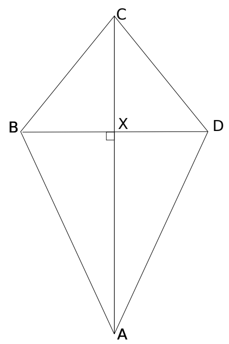
$$\text{Area}(K) = 2 \times \Big[\text{Area}(ABY) + \text{Area}(BCY)\Big].$$

This formula is useful because the triangles can be considered as each having height r. In particular, $\text{Area}(ABY) = \frac{1}{2}r \times |AB|$ and $\text{Area}(BCY) = \frac{1}{2}r \times |BC|$. Since $|AB| = 5\sqrt{2}$ and $|BC| = 2\sqrt{5}$ it follows that

$$30 = \text{Area}(K) = 2 \times \left[\frac{1}{2}r \times 5\sqrt{2} + \frac{1}{2}r \times 2\sqrt{5}\right],$$

where we have recalled the value of Area(K) from part (b). This equation is readily solved and gives $r = 5\sqrt{2} - 2\sqrt{5}$.

(d) The point D is the symmetric to B with respect to the line AC, and there are many different approaches to compute it. One can, for example, compute first the coordinate of the middle point X of the segment BD.



Since K is a kite, the point X belongs to AC and vector $\overrightarrow{BX}$ is perpendicular to $\overrightarrow{AC}$. The first of these observations means that $\overrightarrow{AX} = t \times \overrightarrow{AC}$ for a certain real number $t \in [0,1]$, which is equivalent to $\overrightarrow{AX} = (7t, 4t, -5t)$. Since $\overrightarrow{BX} = \overrightarrow{BA} + \overrightarrow{AX} = (-5+7t, 4t, 5-5t)$ and $\overrightarrow{AC} = (7,4,-5)$, it follows from the second observation that

$$0 = \overrightarrow{BX} \cdot \overrightarrow{AC} = 7 \times (-5+7t) + 4 \times 4t - 5 \times (5-5t).$$

This gives $t = \frac{2}{3}$. To obtain D one can use the equation $\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BX}$ with $\overrightarrow{OB} = (4, \frac{4}{3}, 2)$ and $\overrightarrow{BX} = (-5+7t, 4t, 5-5t)$ and $t = \frac{2}{3}$. This shows that

$$\overrightarrow{OD} = \left(\frac{10}{3}, \frac{20}{3}, \frac{16}{3}\right).$$