AEA 2011 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. The first guess here might be to use the angle-sum formulae for tan (i.e. $\tan(x + y) = (\tan x + \tan y)/(1 - \tan x \tan y)$). Proceeding in this way, however, means having to deal with $\tan 35°$ and $\tan 53°$, for which we do not know explicit expressions.

   Instead, we try writing out the tan and cot $= 1/\tan$ in terms of sin and cos. In particular, by the definition of tan and cot, $\tan(\theta + 35°) = \cot(\theta - 53°)$ is equivalent to

   $$\frac{\sin(\theta + 35°)}{\cos(\theta + 35°)} = \frac{\cos(\theta - 53°)}{\sin(\theta - 53°)}.$$

   Rather than expanding let us cross-multiply and carry over the terms to one side:

   $$\cos(\theta - 53°)\cos(\theta + 35°) - \sin(\theta + 35°)\sin(\theta - 53°) = 0.$$

   The right-hand side here is reminiscent of the addition theorem for cosine, i.e. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. Applying this identity, we arrive at

   $$\cos(2\theta - 53° + 35°) = 0.$$

   This implies that $2\theta - 18° = 90° + k \cdot 180°$, where $k \in \mathbb{Z}$. It follows that $\theta = 54° + k \cdot 90°$. Since $0 \leq \theta \leq 180°$, the solutions of the equation given must be either $\theta = 54°$ or $\theta = 144°$. Finally, we note that for either of these choices of θ, neither $\cos(\theta + 35°)$ nor $\sin(\theta - 53°)$ are 0, and so the cross-multiplication by these earlier in the argument did not create any extra solutions. Thus $54°$ and $144°$ are both solutions of the equation given.

   For an alternative argument, one could use $\tan(90° - \alpha) = \cot\alpha$ (if you are unfamiliar with this, try checking it from the simple relationships that connect sine and cosine). This implies

   $$\cot(\theta - 53°) = \tan(90° - (\theta - 53°)),$$

   and hence the given equation is equivalent to

   $$\tan(\theta + 35°) = \tan(90° - (\theta - 53°)) = \tan(143° - \theta).$$

   Since tan is periodic with period $180°$, it follows that $\theta + 35° = 143° - \theta + k \cdot 180°$, which is the same as above.

2. We do not immediately recognise the integrand $I(x) := (1 + \tan(x/2))^2$ as something we can integrate, so we multiply it out to see where it takes us. In particular, expanding the square, we obtain

   $$I(x) = 1 + 2\tan(x/2) + \tan^2(x/2).$$

   Now, we see that for the first and last terms it might be helpful to apply that $1 + \tan^2\alpha = \sec^2\alpha$, which gives

   $$I(x) = \sec^2(x/2) + 2\tan(x/2).$$

   This is promising because we now know how to integrate both of the terms. Indeed, by making the change of variables $u = x/2$, one can compute

   $$\int I(x)dx = 2 \cdot \tan(x/2) - 2 \cdot 2\ln\cos(x/2),$$

where the extra factor of 2 arises from the change of variables. Hence, plugging in the limits of integration,

$$\begin{aligned}\int_0^{\pi/2} I(x)dx &= [2\tan(x/2) - 4\ln\cos(x/2)]_0^{\pi/2} \\ &= 2\tan(\pi/4) - 4\ln\cos(\pi/4) - (0) \\ &= 2 + 4\ln\sqrt{2}.\end{aligned}$$

Making the final simplification using $\alpha\ln\beta = \ln\beta^\alpha$, we arrive at the result $2+\ln 4$. Therefore $a = 2$ and $b = 4$.

We note here that if we allow a and b to be any real numbers, then the given equation does not determine them uniquely. However, if we restrict our choice to the integers (which is what the question seems implicitly to require), then there can only be one possible choice for them, as a little thought reveals. The details, for those interested, are as follows. Suppose one were to have $a_1 + \ln b_1 = a_2 + \ln b_2$ with a_1, b_1, a_2, b_2 all integers and $a_1 > a_2$. It would then follow that $b_1 e^{a_1 - a_2} - b_2 = 0$. But e is transcendental and therefore not the root of any polynomial with rational coefficients (ask your maths teacher if you want more details!), and so we have a contradiction. Similarly, we can not have $a_1 < a_2$. Thus $a_1 = a_2$, which also implies $b_1 = b_2$.

3. (a) Direct computation yields, alternately multiplying by p (for even n) and by q (for odd n),

$$\begin{aligned}u_1 &= k, \\ u_2 &= kp, \\ u_3 &= kpq, \\ u_4 &= kp^2q, \\ u_5 &= kp^2q^2, \\ u_6 &= kp^3q^2.\end{aligned}$$

(b) The trick here is to notice that we can split the given sum into two geometric progressions (the clue for this is how the individual terms are defined in an alternating way). More specifically, in the sum $S_{2n} := \sum_{r=1}^{2n} u_r$, one has a total of n odd terms and n even terms. The odd terms form a geometric progression with initial term $a_{odd} := k$ and ratio $r := pq \neq 1$. Their sum is therefore

$$S_{2n}^{odd} = \frac{a_{odd}(1-r^n)}{1-r}.$$

Similarly the even terms form a geometric progression with initial term $a_{even} := kp$ and the same ratio r. Their sum is

$$S_{2n}^{even} = \frac{a_{even}(1-r^n)}{1-r}.$$

Putting the two together we have:

$$S_{2n} = S_{2n}^{odd} + S_{2n}^{even} = \frac{k(1+p)(1-(pq)^n)}{1-pq}. \tag{1}$$

Alternatively, one could obtain the above formula directly by pairing up $u_1 + u_2$, $u_3 + u_4$ and so on, yielding a geometric progression with initial term $a = k(1+p)$ and again the same ratio r.

(c) To help understand the terms in the sum, we start by writing it out

$$\sum_{r=1}^{\infty} 6 \times \left(\frac{4}{3}\right)^{\left[\frac{r}{2}\right]} \times \left(\frac{3}{5}\right)^{\left[\frac{r-1}{2}\right]} = 6 + 6 \times \left(\frac{4}{3}\right) + 6 \times \left(\frac{4}{3}\right) \times \left(\frac{3}{5}\right) + \dots.$$

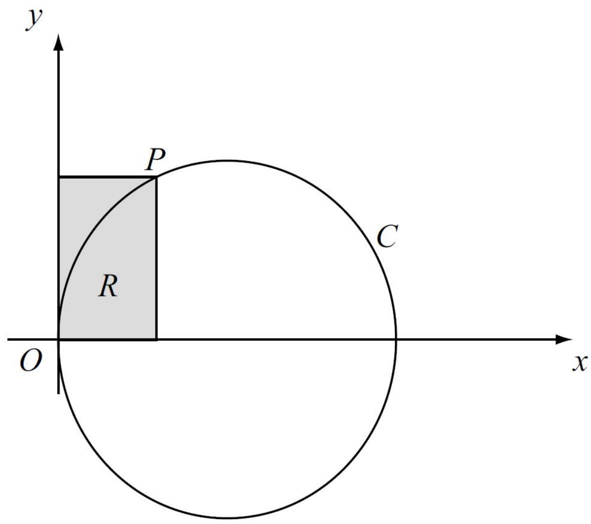
From this, we notice that the k-th term of the infinite series is in exactly the same form as u_k above with $k = 6$, $p = 4/3$ and $q = 3/5$. (In a question like this, it is always a good idea to try and find a link with the earlier parts of the question.) Moreover, we have that $r = pq = 4/5$, so that $r \neq 1$. Hence, from (1), we obtain

$$\sum_{r=1}^{2n} 6 \times \left(\frac{4}{3}\right)^{\left[\frac{r}{2}\right]} \times \left(\frac{3}{5}\right)^{\left[\frac{r-1}{2}\right]} = \frac{6 \times \frac{7}{3}\left(1 - \left(\frac{4}{5}\right)^n\right)}{1 - \frac{4}{5}} = 70\left(1 - \left(\frac{4}{5}\right)^n\right).$$

Taking into account that $\lim_{n\to\infty}(4/5)^n = 0$, it follows that

$$\sum_{r=1}^{\infty} 6 \times \left(\frac{4}{3}\right)^{\left[\frac{r}{2}\right]} \times \left(\frac{3}{5}\right)^{\left[\frac{r-1}{2}\right]} = \lim_{n\to\infty} \sum_{r=1}^{2n} 6 \times \left(\frac{4}{3}\right)^{\left[\frac{r}{2}\right]} \times \left(\frac{3}{5}\right)^{\left[\frac{r-1}{2}\right]} = 70.$$

4. (a) The sketch provided gives some indication that the circle under question will have its centre at a point $(a, 0)$ for some $a > 0$ and with a certain radius $b > 0$.



To see if this is so, we need to do the following.

- Determine whether $(x(t) - a)^2 + y(t)^2 = r^2$ for all $t \in [0, \pi)$ for some positive constants a and r.
- Determine whether or not the points $(x(t), y(t))$, $t \in [0, \pi)$, actually exhaust the circle (they could form only part of a circle).

To deal with the first of these issues, we start by expanding $(x(t) - a)^2 + y(t)^2 = r^2$ to obtain the requirement

$$\cos^4 t + a^2 - 2a\cos^2 t + \cos^2 t \sin^2 t = r^2.$$

We now convert the sine into a cosine using $\sin^2 t = 1 - \cos^2 t$, which allows us to cancel the $\cos^4 t$ term and leads to the equivalent demand

$$a^2 - (2a - 1)\cos^2 t = r^2.$$

Noting that the $\cos t$ term varies but that a and r should be constant, the only possible choice of a that could satisfy this equation is $a = 1/2$. Taking this value for a implies that $r = 1/2$ (note that r is a radius, and so should satisfy $r \geq 0$). In particular, we have shown that

$$(x(t) - 1/2)^2 + y^2(t) = (1/2)^2$$

for every $t \in [0, \pi)$, meaning that all the relevant points lie on a circle with centre $(1/2, 0)$ and radius $1/2$.

Now, by the double angle formulas, we can write

$$x(t) - 1/2 = \frac{2\cos^2 t - 1}{2} = \frac{\cos(2t)}{2}$$

and

$$y(t) = \sin t \cos t = \frac{\sin(2t)}{2}.$$

Apart from confirming that all the points lie on a circle with radius and centre as described above, this also proves that the points $(x(t), y(t))$, $t \in [0, \pi)$, exhaust it.

Alternatively, one could proceed to eliminate t (say, via, $\cos t = \pm\sqrt{x}$ from which we get $\sin t$ in terms of x and carry this into $y = \cos(t)\sin(t)$, whence we square both sides) in order to obtain $x^2 + y^2 = x$. If we now complete the square to obtain $(x - 1/2)^2 + y^2 = (1/2)^2$, we may then proceed as above.

(b) The area of R is simply the product of its sides, i.e.

$$A := \cos^2\alpha \times \cos\alpha \sin\alpha = \sin\alpha \cos^3\alpha.$$

(c) Note that as α increases from 0 to $\pi/2$, the value of A is zero at the endpoints and positive otherwise. It follows that the maximum of A is attained at some $\alpha_* \in (0, \pi/2)$ satisfying

$$\frac{dA}{d\alpha}(\alpha_*) = 0.$$

Differentiating A using the product and the chain rule, one has

$$\frac{dA}{d\alpha} = \cos\alpha \cos^3\alpha - 3\cos^2\alpha \sin^2\alpha,$$

and so we are looking for $\alpha_* \in (0, \pi/2)$ that solves

$$\cos^4\alpha_* - 3\cos^2\alpha_* \sin^2\alpha_* = 0.$$

Factorising this gives

$$\cos^2\alpha_*(\cos^2\alpha_* - 3\sin^2\alpha_*) = 0.$$

Since $\alpha_* \in (0, \pi/2)$, $\cos^4\alpha_* \neq 0$ and so we can divide by this to obtain

$$1 - 3\tan^2\alpha_* = 0,$$

i.e. $\tan\alpha_* = \pm 1/\sqrt{3}$. The only solution of this in the desired region is $\alpha_* = \pi/6$, which means that the maximum area is:

$$\cos^3(\pi/6)\sin(\pi/6) = \left(\frac{\sqrt{3}}{2}\right)^3 \left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{16}.$$

5. (a) By the definition of U, the x-coordinate is 0. The y-coordinate is obtained by plugging 0 into the equation for the curve C, $y(x)$, which gives $y(0) = (0-2)/(0-4) = 1/2$. Thus the coordinates of U are $(0, 1/2)$.

(b) Since P lies on C, it must have coordinates $(x_P, y_P) = (a, \frac{a^2-2}{a^2-4})$ (we are told that $a \neq 0$, and necessarily $a \neq -2, 2$). To compute the normal to C at P, we will start by establishing the derivative of y at a. By the quotient rule, this is given by

$$k := \left.\frac{dy}{dx}\right|_a = \left.\frac{2x(x^2-4) - 2x(x^2-2)}{(x^2-4)^2}\right|_a = \frac{-4a}{(a^2-4)^2}.$$

Now, the normal to C at P is perpendicular to the tangent, which has the above slope. Thus the gradient of the normal to C at P is equal to

$$k' := -\frac{1}{k} = \frac{(a^2-4)^2}{4a}.$$

Since the point $P = (x_P, y_P)$ lies on said normal, it follows that the equation of the normal is given by $y - y_P = k'(x - x_P)$. Substituting in the values for x_P and y_P identified above thus yields

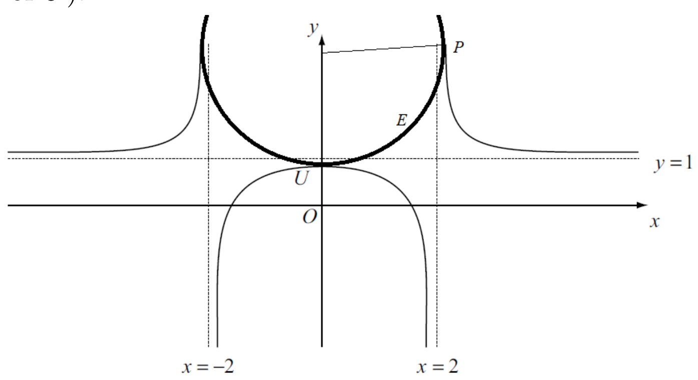
$$y - \frac{a^2-2}{a^2-4} = \frac{(a^2-4)^2}{4a}(x-a).$$

Finally, the intersection with the y-axis is obtained by setting $x = 0$ in this equation, which yields

$$y = \frac{a^2-2}{a^2-4} - \frac{(a^2-4)^2}{4},$$

as required.

(c) (i) Let the centre of the circle E be $(0, \alpha)$ with $\alpha > 0$. As is clear from the figure below, its radius is then given by $r_1 = \alpha - y_U = \alpha - 1/2$ (here, y_U is the y-coordinate of U).



Moreover, if P is on the curve C, then the distance from the centre of E to P is

$$r_2 := \sqrt{(a-0)^2 + (\frac{a^2-2}{a^2-4} - \alpha)}.$$

Now, if P is one of the points on both on the curve C and the circle E, not equal to U (this seems to be assumed in the question and mark scheme, even though it is never stated explicitly), the normal to C at P is the same as the normal to

E at P. It therefore must be the case that the centre of the circle, $(0, \alpha)$, is in fact equal to

$$\left(0, \left[\frac{a^2-2}{a^2-4} - \frac{(a^2-4)^2}{4}\right]\right),$$

and also that $r_1 = r_2$. Hence, equating the expressions we have for r_1 and r_2, and substituting in the above value for α yields

$$\left[\frac{a^2-2}{a^2-4} - \frac{(a^2-4)^2}{4} - \frac{1}{2}\right]^2 = a^2 + \frac{(a^2-4)^4}{16}.$$

Combining the first and the last term on the left-hand side then gives

$$\left[\frac{a^2}{2(a^2-4)} - \frac{(a^2-4)^2}{4}\right]^2 = a^2 + \frac{(a^2-4)^4}{16}, \tag{2}$$

which was the equality asked for in the question. (Note that this equality justifies our assuming that P should lie on both E and C, as without doing this, it would not be true in general! We think that this was an omission on the part of the exam setters.)

(ii) Expand the left-hand side of (2) to obtain (after cancellations):

$$\frac{a^4}{4(a^2-4)^2} - \frac{a^2(a^2-4)}{4} + \cancel{\frac{(a^2-4)^4}{16}} = a^2 + \cancel{\frac{(a^2-4)^4}{16}}.$$

Now divide by a^2 to get:

$$\frac{a^2}{4(a^2-4)^2} = 1 + \frac{a^2-4}{4} \left(= a^2\right),$$

and hence:

$$(a^2-4)^2 = 1, \tag{3}$$

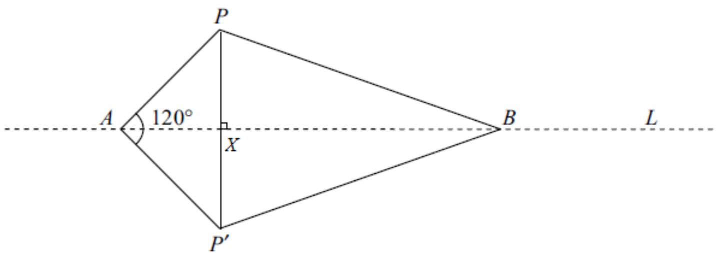
as desired. (Again, this is clearly not true in general. To deduce it, it is essential to use that P is an element of E.)

(iii) The equation at (3) yields $a^2 - 4 = \pm 1$, i.e. $a = \pm\sqrt{3}$ or $a = \pm\sqrt{5}$. It can not be the case that $a = \pm\sqrt{3}$, however, because the vertical asymptotes of C lie at $x = \pm 2$ and $\sqrt{3} < 2$. Thus $a = \pm\sqrt{5}$ – the two choices correspond to the two possible locations of P. Consequently, using the notation from 5(c)(i), we have that the y-coordinate of the centre of E is given by

$$\alpha = \frac{a^2-2}{a^2-4} - \frac{(a^2-4)^2}{4} = \frac{5-2}{1} - \frac{1}{4} = \frac{11}{4},$$

i.e. the centre of E is $(0, \frac{11}{4})$. Moreover, the radius of E is $\alpha - 1/2 = \frac{9}{4}$.

6. (a) To find the position of P', we will start by finding the point X on the line L such that the vector $\vec{PX}$ is perpendicular to L (see the figure below).



Let $R = R(t) = (13-5t, -3+3t, -8+4t)$ be a generic point on the line L. If $\vec{PR}$ is perpendicular to L, then we must have that $\vec{PR} \cdot (-5,3,4) = 0$. Since

$$\vec{PR} = (13-5t-(-7), -3+3t-2, -8+4t-7) = (20-5t, -5+3t, -15+4t), \quad (4)$$

it follows that we must have $-100+25t-15+9t-60+16t = 0$. Hence $50t = 175$, i.e. $t = 7/2$. Thus the midpoint between P and P' is $X = R(7/2)$. Consequently, the position vector of P' is given by adding twice $\vec{PX}$ to the position vector of P, that is

$$(-7,2,7) + 2(5/2, 11/2, -1) = (-2, 13, 5),$$

where we have used (4) to compute $\vec{PX}$.

(b) To determine whether or not A lies on L, we have to establish whether or not there is a $t \in \mathbb{R}$ with $R(t) = A$. This a system of three linear equations for the one unknown, or more specifically

$$(13-5t, -3+3t, -8+4t) = (-7, 9, 8).$$

Solving any one of these components gives $t = 4$, which establishes that $A = R(4)$, and so lies on L.

(c) We know that $\vec{AP} \cdot \vec{AP'} = \|\vec{AP}\|\|\vec{AP'}\| \cos(\angle PAP')$. Direct calculation yields

$$\vec{AP} = (-7-(-7), 2-9, 7-8) = (0, -7, -1)$$

and

$$\vec{AP'} = (-2+7, 13-9, 5-8) = (5, 4, -3).$$

Therefore,

$$\cos(\angle PAP') = \frac{0-28+3}{\sqrt{50}\sqrt{50}} = -\frac{25}{50} = -\frac{1}{2}.$$

(Here we have used that the norm $\|(x_1, x_2, x_3)\|$ is equal to $\sqrt{x_1^2 + x_2^2 + x_3^2}$.) It follows that $\angle PAP' = 120°$ (since one always considers the angle as lying in $[0, 180°]$).

(d) The area of the kite is equal to

$$\frac{1}{2} \times (\text{length of diagonal 1}) \times (\text{length of diagonal 2}).$$

The first diagonal has length:

$$\|\vec{PP'}\| = \sqrt{(-7-(-2))^2 + (2-13)^2 + (7-5)^2} = \sqrt{25+121+4} = \sqrt{150} = 5\sqrt{6}.$$

We are told the area of the kite is $50\sqrt{3}$, and hence the length of the second diagonal, $\|\vec{AB}\|$, is given by

$$\frac{2 \cdot 50\sqrt{3}}{5\sqrt{6}} = 10\sqrt{2}.$$

Since B lies on L it must be equal to $R(t)$ for some t. This t will solve

$$\|\vec{AR}(t)\|^2 = \|\vec{AB}\|^2 = 200.$$

Expanding the left-hand side, this is equivalent to

$$(20-5t)^2 + (-12+3t)^2 + (-16+4t)^2 = 200,$$

which simplifies to $50t^2 - 400t + 600 = 0$. Solving this quadratic then gives $t = 2$ or $t = 6$. To choose which root gives B, observe that A corresponds to $t = 4$ and X to $7/2$, hence t is decreasing as we go from A through X to B. This means that we must take $t = 2$, yielding that the position vector of B is $R(2) = (3, 3, 0)$.

(e) Calculating the scalar product gives $\vec{PA} \cdot \vec{PB} = (0, 7, 1) \cdot (10, 1, -7) = 0$. Since neither of the vectors $\vec{PA}$ and $\vec{PB}$ have 0 length, it follows that $\angle BPA = 90°$.

(f) For this part of the question, we will use Thales' theorem, which states that if a, b and c are points on a circle where the line ac is a diameter of the circle, then the angle abc is a right angle. In particular, since we have that $\angle BPA = 90°$, it follows that AB is the diameter of the circle C. As a consequence, the centre of the circle is the midpoint

$$\frac{1}{2}\left((-7, 9, 8) + (3, 3, 0)\right) = (-2, 6, 4).$$

7. (a) A and B are (local) extrema of the smooth function f. So to find their location, we differentiate (using the quotient rule) and equate to 0. In particular, $f'(x) = 0$ implies that

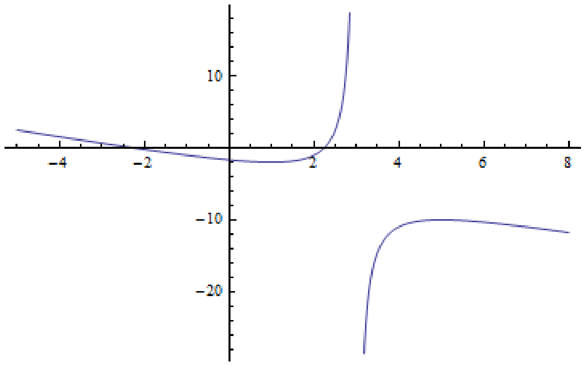
$$\frac{2x(3 - x) - (-1)(x^2 - 5)}{(3 - x)^2} = 0.$$

For this to hold true, the numerator $-x^2 + 6x - 5$ must equal 0, and factorising yields

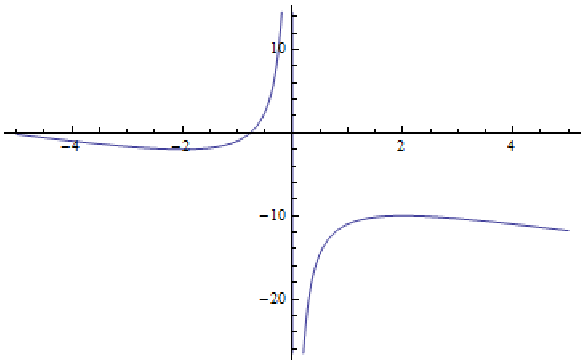
$$(x - 1)(x - 5) = 0.$$

The first solution is $x = 1$, which gives the local minimum $A = (1, f(1)) = (1, -2)$. The second solution is $x = 5$, which gives the local maximum $B = (5, f(5)) = (5, -10)$.

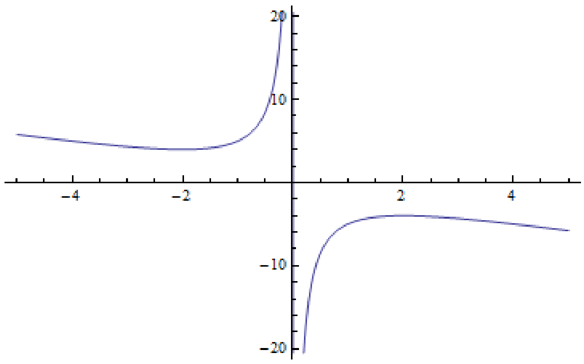
(b) (i) To calculate the values of p and q, we will consider what effect each transformation has on the function f, which is sketched as follows.



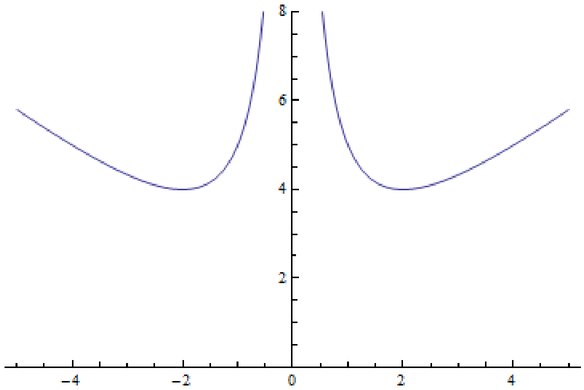
Firstly, $f(g(x)) = f(x + p)$ is a shift of f by p units to the left. To move the asymptote of f to $x = 0$, which is where the asymptote of y is, we need to take $p = 3$. The following figure shows the function $f(g(x))$ with this choice of p.



Secondly, $f(g(x)) + q$ moves $f(g(x))$ up by q units, and from this we obtain $y(x)$ by taking absolute values. To ensure the symmetry of the function y, we need that C and D have the same y coordinate. This means that we should choose q so that the absolute values of the y coordinate of the local extrema of $f(g(x))+q$ are the same, i.e. we need to choose it to satisfy $-2+q = -(-10+q)$. In particular, this implies that $q = 6$. Here is the graph of $f(g(x)) + q$ for this choice of q.



Finally taking absolute values gives us the following picture.



(ii) The coordinates of D, following the construction of $y(x)$ from $y = f(x)$ in 7(b)(i), are obtained from the coordinates of B by first subtracting $(3, 0)$, then adding $(0, 6)$, and then taking the absolute value of the y coordinate. Hence $D = (5-3, |-10+6|) = (2, 4)$.

(c) Recall from part (a) that $\alpha = 1$.

(i) To find m^{-1} we must solve for x in the equation $y = (x^2 - 5)/(3 - x)$. To do this, multiply both sides by $3 - x$ (note that this is never 0 in the range of x considered), and carry the terms over to one side to get a quadratic equation in x:

$$x^2 + xy - (3y + 5) = 0.$$

By the usual quadratic formula (i.e. $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ for the solution of $ax^2 + bx + c = 0$), one obtains

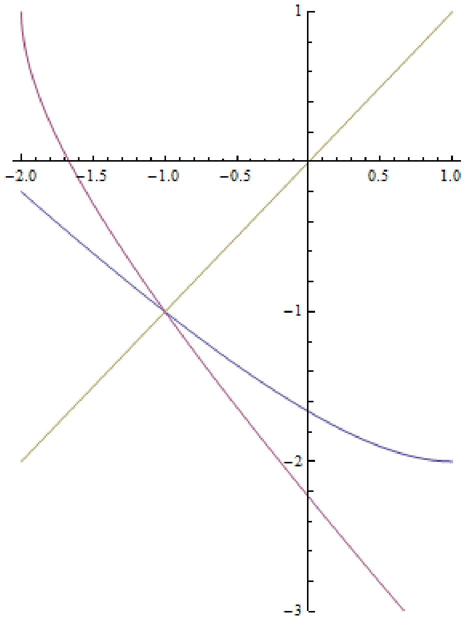
$$x = \frac{-y \pm \sqrt{y^2 + 12y + 20}}{2} = \frac{-y \pm \sqrt{(y+10)(y+2)}}{2}.$$

Now, for $x \leq \alpha$, the function m is continuous and strictly decreasing (we already calculated the relevant derivative in part (a)). Hence the inverse of m is also continuous and strictly decreasing. Moreover, the expression under the square root has -10 and -2 as its only zeros. These facts imply that the $\pm$ can change signs only at these points. Now, one sees easily that $(0, m(0)) = (0, -5/3)$, which forces us to choose a minus on $y \in (-2, \infty)$. Since the y coordinate of $(x, m(x))$ is in this range for every $x < \alpha$, it follows that one can retain the choice of minus throughout, yielding

$$m^{-1}(y) = \frac{-y - \sqrt{y^2 + 12y + 20}}{2}.$$

(ii) The domain of m^{-1} is equal to the range of $m(x)$, $x \leq \alpha$, which is the set $[-2, \infty)$.

(iii) Note that m^{-1} is obtained from m by reflecting in the line $y = x$:



(the blue curve is m, the purple curve is m^{-1}, and the grey line is $y = x$). Thus, because the graph of m crosses the line $y = x$, the intersection point of m and m^{-1} must be on this line, i.e. the t solving $m(t) = m^{-1}(t)$ will actually satisfy $m(t) = t$. Thus, rather than using the formula for m^{-1} we deduced in part (c)(i), we need only solve for t in

$$\frac{t^2 - 5}{3 - t} = t.$$

Multiplying by $3 - t$ one has after rearranging $2t^2 - 3t - 5 = 0$. The left-hand side can be factorised to yield $(2t - 5)(t + 1) = 0$. The root $5/2$ is outside the range $x \leq 1$, and therefore $t = -1$.

Finally, we note that $f(x) = f^{-1}(x)$ does not require $f(x) = x$ in general. Take, for example, the function $f(x) = \sqrt{1 - x^2}$ with $x \in [0, 1]$, which describes the unit circle in the first quadrant. Manifestly $f^{-1} = f$, and the latter equation is satisfied identically on the whole of the interval $[0, 1]$, whereas $f(x) = x$ only at $x = 1/\sqrt{2}$!