AEA 2012 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. (a) Notice that $f(x)$ is a quadratic form in x with a positive leading coefficient. Therefore its range, which is the set $\{f(x) : x \in [0, \infty)\}$, will be the interval $[a, \infty)$, with a being the minimum of f (on $[0, \infty)$) – draw a sketch if you don't immediately see why! To find the minimum, either complete the square:

$$f(x) = x^2 - 2x + 6 = (x-1)^2 - 1 + 6 = (x-1)^2 + 5,$$

or else differentiate:

$$f'(x) = 2x - 2.$$

From either approach, we find that the function f attains its minimum at 1, with $f(1) = 5$. Thus $a = 5$, and the range of f is the interval $[5, \infty)$.

(b) We are to asked to find the composition $g \circ f =: gf$. (The notation gf is actually slightly ambiguous, as it could also be interpreted as meaning the product, rather than the composition of the functions g and f.) Notice that the range of f is a subset of the domain of g, and hence the domain of $g \circ f$ is that of f, namely $[0, \infty)$. For all x in this range, we have

$$(g \circ f)(x) = g(f(x)) = 3 + \sqrt{f(x) + 4} = 3 + \sqrt{x^2 - 2x + 6 + 4} = 3 + \sqrt{x^2 - 2x + 10},$$

as requested.

(c) As remarked in the solution to (1b), the domain of $g \circ f$ is $[0, \infty)$. To find the range, we observe that g is a continuous strictly increasing function, and $g(x)$ tends to ∞ as x tends to ∞. From this we conclude that g maps the range of f, $[5, \infty)$, onto $[g(5), \infty)$. Since $g(5) = 3 + \sqrt{5+4} = 3 + \sqrt{9} = 3 + 3 = 6$, the range of $g \circ f$ is the interval $[6, \infty)$.

2. (a) We need to convert the $\sin(3x)$ into trigonometric functions involving only x. To do this, we start by using the addition formula $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$ with $\alpha = 2x$ and $\beta = x$ to deduce

$$\begin{aligned} \sin(3x) &= \sin(2x + x) \\ &= \sin(2x)\cos(x) + \cos(2x)\sin(x) \end{aligned}$$

Now we can use the double angle and addition formulae, $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = \cos^2(x) - \sin^2(x)$, to obtain

$$\begin{aligned} \sin(3x) &= 2\sin(x)\cos^2(x) + (\cos^2(x) - \sin^2(x))\sin(x) \\ &= 2\sin(x)(1 - \sin^2(x)) + (1 - 2\sin^2(x))\sin(x), \\ &= 3\sin(x) - 4\sin^3(x), \end{aligned}$$

where we have also applied $\sin^2(x) + \cos^2(x) = 1$ in the second line. Alternatively, using DeMoivre's theorem, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we have (for $\theta = 3x$)

$$e^{3ix} = \cos(3x) + i\sin(3x). \tag{1}$$

On the other hand

$$e^{3ix} = (e^{ix})^3 = (\cos(x) + i\sin(x))^3,$$

again by DeMoivre's theorem (with $\theta = x$). Finally, applying the identity $(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$, we expand, to get:

$$e^{3ix} = \cos^3(x) - i\sin^3(x) + 3i\cos^2(x)\sin(x) - 3\cos(x)\sin(x)^2. \tag{2}$$

Comparing (1) and (2) (and using the fact that if two complex numbers are equal, then so are their imaginary parts), we find

$$\sin(3x) = 3\cos^2(x)\sin(x) - \sin^3(x) = 3(1-\sin^2(x))\sin(x) - \sin^3(x).$$

which easily yields the required identity.

(b) We observe that by (2a),

$$6\sin(x) - 2\sin(3x) = 6\sin(x) - 2(3\sin(x) - 4\sin^3(x)) = 8\sin^3(x).$$

It follows that

$$(6\sin(x) - 2\sin(3x))^{2/3} = 8^{2/3}(\sin^3(x))^{2/3} = 2^2\sin^2(x) = 4\sin^2(x).$$

Thus we are to compute $I_1 := 4\int \sin^2(x)\cos(x)dx$. Now, a substitution $v = \sin(x)$, $dv = \cos(x)dx$, yields immediately

$$I_1 = 4\int v^2 dv = \frac{4}{3}v^3 + c = \frac{4}{3}\sin^3(x) + c,$$

where c is a constant of integration.

(c) Again we first simplify the integrand (by using the double angle formula for sine, and (2a)):

$$\begin{aligned} 3\sin(2x) - 2\sin(3x)\cos(x) &= 6\sin(x)\cos(x) - 2(3\sin(x) - 4\sin^3(x))\cos(x) \\ &= 8\sin^3(x)\cos(x), \end{aligned}$$

and hence

$$(3\sin(2x) - 2\sin(3x)\cos(x))^{1/3} = 2\sin(x)\cos^{1/3}(x).$$

Thus we are to compute $I_2 := 2\int \cos^{1/3}(x)\sin(x)dx$. Substitute $u = \cos(x)$, $du = -\sin(x)dx$, to obtain

$$I_2 = -2\int u^{1/3}du = -\frac{2}{4/3}u^{4/3} + c = -\frac{3}{2}\cos^{4/3}(x) + c,$$

where c is a constant of integration.

3. (a) First note that θ needs to be distinct from $\pi/4$ in order for $\tan(2\theta)$ to be well-defined! We next observe that the right-hand side of the equation is a geometric progression with common ratio $\rho = \cos(2\theta)$ and scale factor $a = 2$. It follows that

$$\sum_{r=0}^{\infty} 2\cos^r(2\theta) = \frac{a}{1-\rho} = \frac{2}{1-\cos(2\theta)},$$

where, thanks to $\theta \in (0, \pi/2)\backslash\{\pi/4\}$, $|\rho| = |\cos(2\theta)| < 1$. So, the equation becomes:

$$\tan\theta\tan(2\theta) = \frac{2}{1-\cos(2\theta)}.$$

We now use the double angle formulae, $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$ and $\tan(2\theta) = 2\tan(\theta)/(1-\tan^2(\theta))$, to express everything in terms of trigonometric functions of θ (guided by the question, wherein computing $\tan(\theta)$ is ultimately desired). These give

$$\frac{2\tan^2(\theta)}{1-\tan^2(\theta)} = \frac{1}{\sin^2(\theta)}.$$

We still need to express the right-hand side in terms of $\tan(\theta)$. To this end, we have:

$$\tan^2(\theta) = \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{\sin^2(\theta)}{1-\sin^2(\theta)} = \frac{1}{\operatorname{cosec}^2(\theta) - 1}.$$

It follows that

$$\operatorname{cosec}^2(\theta) = 1 + \frac{1}{\tan^2(\theta)}.$$

So we must solve:

$$\frac{2\tan^2(\theta)}{1-\tan^2(\theta)} = \frac{1+\tan^2(\theta)}{\tan^2(\theta)}.$$

(Note that since $\theta \in (0, \pi/2)\backslash\{\pi/4\}$, then $\tan(\theta) \notin \{0, 1\}$). Cross-multiplying yields:

$$2\tan^4(\theta) = 1 - \tan^4(\theta).$$

Hence $\tan^4(\theta) = 1/3$, and so $\tan(\theta) = 3^{-1/4}$, i.e. $p = -1/4$.

(b) On the interval $(0, \pi/2)$, tan is a strictly increasing bijection from $(0, \pi/2)$ onto $(0, \infty)$. Thus, the equation $\tan(\theta) = 3^{-1/4}$ has a unique solution, and this will be in the interval $(\pi/6, \pi/4)$ if and only if $\tan(\pi/6) < 3^{-1/4} < \tan(\pi/4)$. Since $\tan(\pi/6) = 3^{-1/2} < 3^{-1/4}$ and $\tan(\pi/4) = 1 > 3^{-1/4}$, the value of θ must indeed lie in the interval $(\pi/6, \pi/4)$.

4. (a) We first observe that if X and Y are two vertices of a cube, side length a, then the length of the line joining them, $|XY|$, can be one of a, $a\sqrt{2}$ or $a\sqrt{3}$, according to whether XY is an edge, a face diagonal, or a diagonal of the cube, respectively. Thus if $|AB|$, $|BC|$ and $|AC|$ are all different, we shall be able to conclude that the smallest is the length of the edge a. We calculate:

$$\begin{aligned} |AC| &= |c-a| = |(11, -5, -1)| = \sqrt{147}; \\ |BC| &= |c-b| = |(3, -2, -6)| = \sqrt{49} = 7; \\ |AB| &= |(8, -3, 5)| = \sqrt{98}. \end{aligned}$$

It follows that $a = 7$, and the volume of the cube is $a^3 = 7^3 = 343$.

(b) We will aim to find the value of α from the relation:

$$\vec{PQ} \cdot \vec{PR} = |\vec{PQ}||\vec{PR}| \cos(QPR).$$

We can compute $\vec{PQ} \cdot \vec{PR} = 3 \cdot 7 + 4 \cdot 1 + \alpha \cdot 0 = 25$ and also

$$\begin{aligned} |\vec{PQ}| &= \sqrt{3^2 + 4^2 + \alpha^2} = \sqrt{25 + \alpha^2}, \\ |\vec{PR}| &= \sqrt{7^2 + 1^2 + 0^2} = \sqrt{50} = 5\sqrt{2}, \\ \cos 60^\circ &= 1/2. \end{aligned}$$

So

$$50 = 5\sqrt{2}\sqrt{25 + \alpha^2},$$

from which $\alpha = 5$ is readily obtained.

(c) From the first observation of (4a), it follows immediately that $|\vec{PQ}| = |\vec{PR}|$ must be lengths of face diagonals. (Indeed, were PR and PQ edges, $QPR = 90°$; and since $PR \neq PQ$, these two certainly can't be space diagonals.) Thus the length of the side is

$$a = |\vec{PQ}|/\sqrt{2} = 5\sqrt{2}/\sqrt{2} = 5,$$

and the length of the space diagonal $a\sqrt{3} = 5\sqrt{3}$ follows. (The term diagonal is slightly ambiguous, and it would have been better to distinguish between the face diagonals and diagonals of the cube explicitly in the instruction.)

5. (a) The correct was to reexpress $\log_a x^n$ is $n \log_a x$. This means the equation we are trying to solve is $nz = z^n$, where $z = \log_a x$. Since we are told $x \neq 1$, we know that $z \neq 0$, and we can divide by z to obtain $n = z^{n-1}$. We can take roots to get

$$\log_a x = z = n^{1/(n-1)},$$

where we note that $n-1 > 0$ (since $n > 1$). Finally, by the definition of the logarithm,

$$x = a^{n^{1/(n-1)}}.$$

(b) (i) To solve this equation, we first simplify the left-hand side, by using $\log_a x^m = m \log_a x$, $m \in \{1, 2, 3\}$. This yields:

$$6 \log_a x = \log_a x + \log_a x^2 + \log_a x^3.$$

Hence, we are now trying to solve

$$z^3 + z^2 - 5z = 0,$$

where we again write $z = \log_a x$. As in our previous answer, we know that $z \neq 0$, and so we can divide by z to obtain $z^2 + z - 5 = 0$. This has solutions

$$z_1 = \frac{-1 + \sqrt{21}}{2}, \qquad z_2 = \frac{-1 - \sqrt{21}}{2}.$$

Now, since $a > 1$, it must be that $a^{z_1} > 1 > a^{z_2}$. We conclude

$$x_1 = a^{z_1} = a^{\frac{-1+\sqrt{21}}{2}}, \qquad x_2 = a^{z_2} = a^{\frac{-1-\sqrt{21}}{2}}.$$

(ii) It follows from (5b) that

$$\log_a \left(\frac{x_1}{x_2}\right) = \log_a x_1 - \log_a x_2 = z_1 - z_2 = \sqrt{21}.$$

(c) The left-hand side can be simplified in a similar way as in (5b). Specifically:

$$\log_a x + \cdots + \log_a x^n = (1 + \cdots + n) \log_a x = \frac{n(n+1)}{2} \log_a x, \tag{3}$$

where we have used $\log_a x^m = m \log_a x$ for $m \in \{1, \ldots, n\}$, and also the identity

$$1 + \cdots + n = \frac{n(n+1)}{2}.$$

To simplify the right-hand side, we use the geometric progression formula

$$1 + \cdots + z^m = \frac{z^{m+1} - 1}{z - 1}$$

for $m \in \mathbb{N}$, $z \neq 1$. This gives (with $z = \log_a x$, $m = n - 1$):

$$(\log_a x)^1 + \cdots + (\log_a x)^n = \log_a x \left(1 + \cdots + (\log_a x)^{n-1}\right) = \log_a x \frac{(\log_a x)^n - 1}{\log_a x - 1}, \quad (4)$$

where we have benefited from $x \neq a$, hence $\log_a x \neq 1$. Equating (3) and (4), we are thus able to simplify the equation to

$$\frac{n(n+1)}{2}\cancel{\log_a x} = \cancel{\log_a x}\frac{(\log_a x)^n - 1}{\log_a x - 1},$$

where the cancellation is allowed because $\log_a x \neq 0$ (as $x \neq 1$). Rearranging gives,

$$n(n+1)\log_a x - n(n+1) = 2(\log_a x)^n - 2,$$

from which the desired identity follows.

6. (a) The points P and Q have y-coordinates 0, and the x-coordinates are obtained by solving $y = (x+a)(x-b)^2$ with $y = 0$. This yields that $x_P = -a$ and $x_Q = b$, i.e. P has coordinates $(-a, 0)$ and Q has coordinates $(b, 0)$.

   (b) To calculate the area of the shaded region between the curve PSQ and the x-axis, we must compute the integral:

$$G := \int_{-a}^{b} (x+a)(x-b)^2 dx.$$

   To this end, first make a change of variables, $y = x + a$, $dy = dx$, to get

$$G = \int_0^{(a+b)} y(y - (a+b))^2 dy.$$

   (The motivation for this shift is that the limits of integration, 0 and $a + b$, seem convenient given the answer we are trying to obtain.) We now expand the square to give

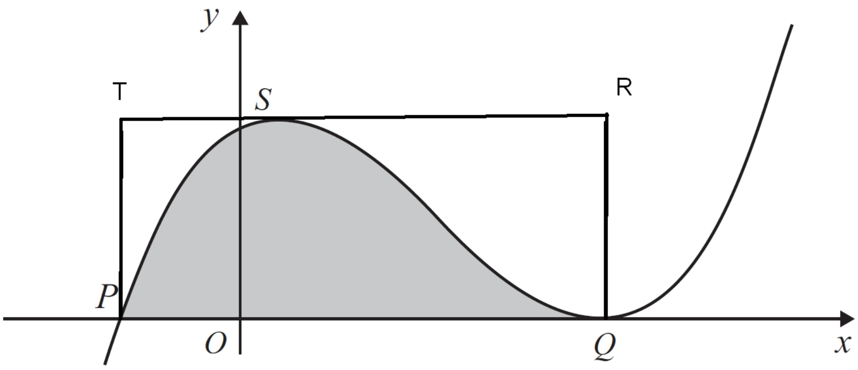
$$G = \int_0^{(a+b)} y^3 dy - 2(a+b)\int_0^{(a+b)} y^2 dy + (a+b)^2 \int_0^{(a+b)} y dy.$$

   Since $\int_0^{(a+b)} y^3 dy = (a+b)^4/4$, $\int_0^{(a+b)} y^2 dy = (a+b)^3/3$ and $\int_0^{(a+b)} y dy = (a+b)^2/2$, it follows that

$$G = \frac{(a+b)^4}{4} - \frac{2(a+b)^4}{3} + \frac{(a+b)^4}{2} = \frac{(a+b)^4}{12}.$$

   Of course, an alternative would be to conduct a direct expansion of $(x+a)(x-b)^2$ into a polynomial in x, followed by term-by-term integration, though this is more cumbersome.

   (c) We first sketch the rectangle $PQRST$ we are trying to find the area of:



We know that the horizontal edges of $PQRST$ have length $(b-(-a)) = (a+b)$. Thus, to find the area of $PQRST$, it will suffice to find the length of its vertical edges. This quantity is given by the y-coordinate of S, y_S say. Since S is a local maximum of the curve $y(x)$, we first try to identify its x-coordinate by looking for solutions to the equation $y'(x) = 0$. Using the product rule $(fg)' = f'g + fg'$ with $f(x) = x + a$ and $g(x) = (x-b)^2$:

$$y'(x) = \frac{d}{dx}\left((x+a)(x-b)^2\right) = 1 \cdot (x-b)^2 + (x+a) \cdot 2(x-b).$$

Setting the derivative equal to zero yields that either $x = b$ (x-coordinate of point Q, a local minimum), or else $x - b + 2(x+a) = 0$, hence $x = (b-2a)/3$. Since the point $(b, y(b)) = (b, 0)$ corresponds to Q, we deduce that $x_S = (b-2a)/3$. We then obtain y_S by computing

$$y_S = y(x_S) = y((b-2a)/3) = \frac{a+b}{3}\left(\frac{-2a-2b}{3}\right)^2 = \frac{4}{27}(a+b)^3.$$

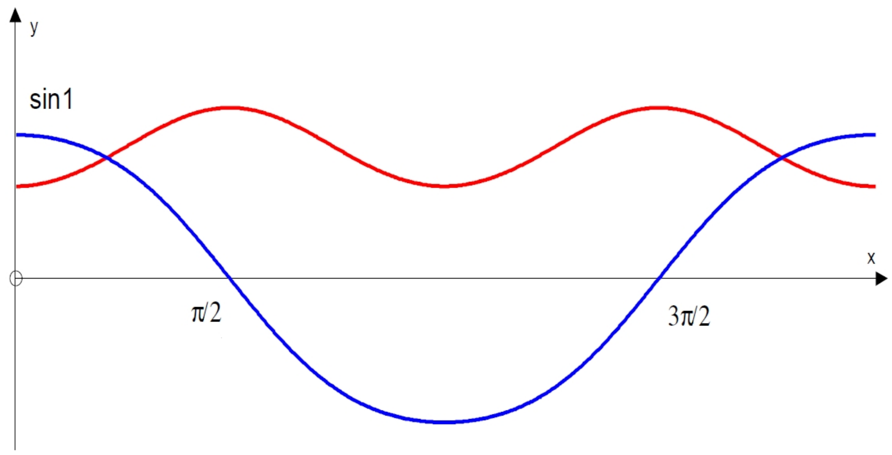
Finally, the area of the rectangle $PQRST$ is $(a+b)y_S = \frac{4}{27}(a+b)^4$, and so:

$$k = \frac{\frac{1}{12}(a+b)^4}{\frac{4}{27}(a+b)^4} = \frac{27}{48} = \frac{9}{16},$$

where we have taken into account that $G = (a+b)^4/12$, as shown in (6b).

7. (a) The cosine function attains its maximum 1 at $2k\pi$, $k \in \mathbb{Z}$. Since its range is the interval $[-1,1]$ and $1 < 2\pi$, it follows that $\cos(\cos(x))$ attains its maximum of 1 at values x, for which $\cos(x) = 0$, i.e. $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. Thus we obtain that $P = (\pi/2, 1)$ and $R = (3\pi/2, 1)$. On the other hand, since the range of cos is $[-1,1]$, cos is even and decreasing on $[0, \pi/2]$, and $1 \leq \pi/2$, it must be the case that $\cos(\cos(x))$ attains its minimum of $\cos(1) = \cos(-1)$ at values x for which $\cos(x) = \pm 1$, i.e. $x = k\pi$, $k \in \mathbb{Z}$. Thus Q has coordinates $(\pi, \cos(1))$.

   (b) As x increases over the interval $[0, 2\pi)$, $\cos(x)$ decreases from 1 to -1 and then increases back to 1 again. The function $\sin(\cos(x))$ thus starts at $\sin(1)$, decreases to 0 at $x = \pi/2$, continues to decrease to $\sin(-1)$ at $x = \pi$, and then makes a return, crossing 0 again at $x = 3\pi/2$. From this it is clear that the minimum point of C_2 has coordinates $(\pi, -\sin(1))$, whereas the points of intersection of C_2 with the x-axis are $(\pi/2, 0)$ and $(3\pi/2, 0)$. Moreover, $\cos(1) < \sin(1) < 1$, since $\pi/4 < 1 < \pi/2$. The sketch of C_2 follows:



(c) The point S is an intersection of curves C_1 and C_2, and this means that

$$\cos(\cos(\alpha)) = \sin(\cos(\alpha)),$$

where α is the x-coordinate of S. Thus

$$\tan(\cos(\alpha)) = 1$$

(there is no risk of dividing by zero, since $\cos(\cos(\alpha))$ is no smaller than $\cos(1) > 0$). This implies $\cos(\alpha) = \pi/4 + k\pi$, $k \in \mathbb{Z}$. Since $\cos(\alpha) \in [-1, 1]$, we conclude $\cos(\alpha) = \pi/4$, and so $\alpha = \arccos(\pi/4)$, as desired.

(d) The number d is the y-coordinate value of S, so we need only to plug in α into the equation for C_2 (or C_1):

$$d = \sin(\cos(\arccos(\pi/4))) = \sin(\pi/4) = 1/\sqrt{2}.$$

Since $\cos(x) = \cos(2\pi - x)$, the two curves C_1 and C_2 are symmetric in a reflection about the line $x = \pi$. Thus, the x-coordinate of T is given by $2\pi - \arccos(\pi/4)$ and the y-coordinate by $1/\sqrt{2}$.

(e) To obtain the gradient $m = \tan(\beta)$ of the tangent to C_1 at S, we should evaluate the derivative of $\cos(\cos(x))$ with respect to x at $x = \arccos(\pi/4)$. Using the chain rule $(g \circ f)' = (g' \circ f) \cdot f'$ with $g = f = \cos$, we obtain

$$\frac{d}{dx}(\cos(\cos(x))) = -\sin(\cos(x))(-\sin(x)) = \sin(x)\sin(\cos(x)).$$

Plugging in $x = \arccos(\pi/4)$, we get

$$m = \sin(\arccos(\pi/4))\sin(\pi/4).$$

Now, $\sin(\pi/4) = 1/\sqrt{2}$ and

$$\sin(\arccos(\pi/4)) = +\sqrt{1 - \cos^2(\arccos(\pi/4))} = \sqrt{1 - (\pi/4)^2},$$

where it is clear that $\sin(\arccos(\pi/4)) \geq 0$ (since the gradient at the point S is clearly positive). Finally,

$$m = \frac{1}{\sqrt{2}}\sqrt{1 - (\pi/4)^2} = \sqrt{\frac{16 - \pi^2}{32}},$$

as required.

(f) Just as we have found the gradient of the tangent to C_1 at S in (7e), we find the gradient $m' = \tan\beta'$ of the tangent to C_2 at S. First,

$$\frac{d}{dx}\left(\sin(\cos(x))\right) = \cos(\cos(x))(-\sin(x)) = -\sin(x)\cos(\cos(x)).$$

Plugging in $x = \arccos(\pi/4))$, we get

$$m' = -m = -\sqrt{\frac{16-\pi^2}{32}}.$$

We now remark that $\sqrt{\frac{16-\pi^2}{32}} < 1$, hence $\beta < \pi/4$, and $\beta' = -\beta > -\pi/4$. The acute angle between the two tangents is thus $\beta - \beta' = \beta - (-\beta) = 2\beta < \pi/2$ and the obtuse angle ($\in (\pi/2, \pi)$) is $\pi - 2\beta$.