AEA 2013 Extended Solutions

These extended solutions for Advanced Extension Awards in Mathematics are intended to supplement the original mark schemes, which are available on the Edexcel website.

1. (a) Formally, the binomial expansion of $(1+a)^n$ is given by

$$(1+a)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k,$$

where the generalized binomial coefficients are given by

$$\binom{n}{k} := \frac{n(n-1)\dots(n-k+1)}{k!}.$$

Note that n need not be an integer for this to make sense. Probably, though, the question should have included more information about the possible range of n and x, because otherwise there might be an issue with the (non-)convergence of the infinite sum (as is illustrated in the later part of the question). Putting these considerations aside, however, taking $a = \frac{12n}{5}x$, we see that the 'x^2 term' in this binomial expansion will come from $k = 2$, whilst the 'x^3 term' from $k = 3$. Moreover, the coefficient in front of x^2 is

$$a_2(n) := \binom{n}{2}\left(\frac{12n}{5}\right)^2 = \frac{n(n-1)}{2!}\left(\frac{12n}{5}\right)^2,$$

while the coefficient in front of x^3 is

$$a_3(n) := \binom{n}{3}\left(\frac{12n}{5}\right)^2 = \frac{n(n-1)(n-2)}{3!}\left(\frac{12n}{5}\right)^3.$$

The possible values of n, then, are given precisely by the solutions to the equation $a_2(n) = a_3(n)$ for which $a_2(n)$ (hence $a_3(n)$) does not equal zero. We see from this last requirement that n must not be equal to either 0 or 1. Hence, if we start from the equation $a_2(n) = a_3(n)$, then we can cancel the non-zero term $\left(\frac{12n}{5}\right)^2 n(n-1)$ appearing on both sides to obtain the equivalent statement:

$$\frac{1}{2!} = \frac{(n-2)}{3!}\left(\frac{12n}{5}\right).$$

(This is a nicer way to proceed than multiplying everything out, which would yield an algebraic equation in n of degree 6.) This equation is easily rearranged to give

$$4n^2 - 8n - 5 = 0,$$

which can be factorised to

$$(2n+1)(2n-5) = 0.$$

Hence we find that

$$n = -\frac{1}{2} \text{ or } n = \frac{5}{2}.$$

(We note that for these two values of n both $a_2(n)$ and $a_3(n)$ are indeed non-zero.)

(b) We now know $x = 1/2$ and we wish to establish for which of the two values of n from part (1a) the binomial series *converges*. Now, with $n = -1/2$, we obtain

$$a = \frac{12n}{5}x = -\frac{3}{5}.$$

Since $|a| < 1$, it follows that the series converges, and hence gives a valid expansion. (If you have not seen a proof of this, then you can look forward to doing so by taking a maths course at university!) On the other hand, for $n = 5/2$,

$$a = \frac{12n}{5}x = 3.$$

consecutive terms in the series ultimately grow larger and larger in absolute value, meaning the series cannot converge. Indeed, in this case, the quotient of the absolute values of the $(k+1)$-th and k-th term of the binomial expansion is equal to

$$\left| \frac{\binom{5/2}{k+1} 3^{k+1}}{\binom{5/2}{k} 3^k} \right| = 3\frac{|\frac{5}{2} - k|}{k+1},$$

which is strictly larger than 2 for all $k \geq 10$. In conclusion, a valid expansion (i.e. a convergent series) transpires when $n = -1/2$, but not when $n = 5/2$.

2. (a) We are told to use the formula

$$\sin(A - B) = \sin(A)\cos(-B) + \cos(A)\sin(-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

Applying this with $A = 90°$ and $B = x$, we have that

$$\sin(90° - x) = \sin(90°)\cos(x) - \cos(90°)\sin(x).$$

Since $\sin(90°) = 1$ and $\cos(90°) = 0$, we obtain

$$\sin(90° - x) = \cos(x).$$

(b) Note that for the question to make sense we must have $\cos(\theta + 17°) \neq 0$ and we shall have to be careful in the end to check this condition. Under this provision, however, for $0° < \theta < 360°$, the given equation is equivalent to

$$2\sin(\theta + 17°)\cos(\theta + 17°) = \cos(\theta + 8°).$$

We now recognise that the left-hand side looks like what appears in the double-angle formula $\sin(2u) = 2\sin(u)\cos(u)$ with $u = \theta + 17°$. In particular, applying this observation we can write

$$\sin(2\theta + 34°) = \cos(\theta + 8°).$$

This was the crucial step. We may now use the result of part (2a) (remembering what we proved in earlier parts of questions is often helpful in exams!), so as to make both sides of the equation be evaluations of the sine function:

$$\sin(2\theta + 34°) = \sin(82° - \theta).$$

There are various ways to solve this equation, but we will use that

$$\sin(\alpha) - \sin(\beta) = 2\sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right),$$

taking $\alpha = 2\theta + 34°$ and $\beta = 82° - \theta$. In particular, this implies that

$$2\sin\left(\frac{3\theta - 48°}{2}\right)\cos\left(\frac{\theta + 116°}{2}\right) = 0.$$

The latter is then further equivalent to:

$$\sin\left(\frac{3\theta - 48°}{2}\right) = 0 \text{ or } \cos\left(\frac{\theta + 116°}{2}\right) = 0.$$

Finally,

$$\sin\left(\frac{3\theta - 48°}{2}\right) = 0 \Leftrightarrow \frac{3\theta - 48°}{2} = k180°,\ k \in \mathbb{Z} \Leftrightarrow \theta = 16° + k120°,\ k \in \mathbb{Z};$$

whilst

$$\cos\left(\frac{\theta + 116°}{2}\right) = 0 \Leftrightarrow \frac{\theta + 116°}{2} = 90° + l180°,\ l \in \mathbb{Z} \Leftrightarrow \theta = 64° + l360°,\ l \in \mathbb{Z}.$$

Together with the requirement that $0° < \theta < 360°$ we obtain solutions for $k = 0$, $k = 1$, $k = 2$ and $l = 0$ of $\theta = 16°$, $\theta = 136°$, $\theta = 256°$ and $\theta = 64°$, respectively. We note that for each of these values, $\cos(\theta + 17°) \neq 0$.

3. For typographical convenience we will write vectors as rows. Let

$$\mathbf{r}_{L_1}(\lambda) = (-7, 7, 1) + \lambda(2, 0, -3), \qquad \lambda \in \mathbb{R},$$

and

$$\mathbf{r}_{L_2}(\mu) = (7, p, -6) + \mu(10, -4, -1), \qquad \mu \in \mathbb{R},$$

be equations of the two lines. Further, for a point X, we will let $\mathbf{r}_X$ denote its position vector.

(a) (i) For the lines to intersect at the point $\mathbf{r}_{L_1}(\lambda) = \mathbf{r}_{L_2}(\mu)$, we need each of the components to be equal, i.e.

$$\begin{aligned} -7 + 2\lambda &= 7 + 10\mu \\ 7 &= p - 4\mu \\ 1 - 3\lambda &= -6 - \mu \end{aligned}$$

We note that the first and last equations only feature λ and μ, and so we will try to use them to find the values of these parameters. In particular, adding to the first equation a multiple of 10 of the third, we conclude $3 - 28\lambda = -53$, which implies $\lambda = 2$. From either the first or third equation, we then deduce that $\mu = -1$. Finally, from the second one, we find $p = 3$.

(ii) The position vector of C is given by $\mathbf{r}_{L_1}(2) = \mathbf{r}_{L_2}(-1) = (-3, 7, -5)$.

(b) In order to show that B lies on L_2, we need to show that there exists a $\mu \in \mathbb{R}$ with $(-13, 11, -4) = \mathbf{r}_{L_2}(\mu)$, i.e. to find a solution to the system:

$$\begin{aligned} -13 &= 7 + 10\mu \\ 11 &= 3 - 4\mu \\ -4 &= -6 - \mu \end{aligned}$$

(where we have taken into account $p = 3$ from part (3a)). We conclude from the third equation, say, that such a μ must equal -2, and then check the remaining equations are indeed satisfied with $\mu = -2$.

(c) To compute the cosine, we will use the scalar product via the formula:

$$\cos(\angle ACB) = \frac{\vec{CA} \cdot \vec{CB}}{|CA||CB|}.$$

The relevant vectors are given by

$$\vec{CA} = \mathbf{r}_A - \mathbf{r}_C = (-7, 7, 1) - (-3, 7, -5) = (-4, 0, 6)$$

and

$$\vec{CB} = \mathbf{r}_B - \mathbf{r}_C = (-13, 11, -4) - (-3, 7, -5) = (-10, 4, 1).$$

Thus

$$\vec{CA} \cdot \vec{CB} = (-4) \cdot (-10) + 0 \cdot 4 + 6 \cdot 1 = 46.$$

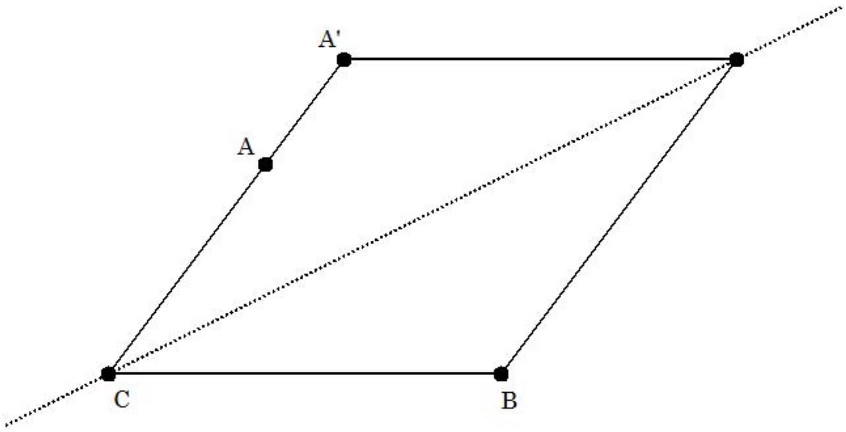
Furthermore,

$$|CA| = \sqrt{(-4)^2 + 0^2 + 6^2} = \sqrt{52} \text{ and } |CB| = \sqrt{(-10)^2 + 4^2 + 1^2} = \sqrt{117}.$$

Altogether this yields:

$$\cos(\angle ACB) = \frac{46}{\sqrt{52}\sqrt{117}} = \frac{46}{2 \cdot \sqrt{13} \cdot 3 \cdot \sqrt{13}} = \frac{23}{39}.$$

(d) To start a question like this, a sketch always helps. To begin with, we note from the computation in (3c) that $|CA| = 2\sqrt{13} = \frac{2}{3}|CB|$. So, if we let A' be the point with position vector $\mathbf{r}_{A'} = \mathbf{r}_C + \frac{3}{2}\vec{CA}$, then C, B and A' will form three of the four vertices of a rhombus:



If D is the fourth vertex of the rhombus, then the line through C and D bisects the angle $\angle ACB$. Since

$$\vec{CD} = \vec{CB} + \vec{CA'} = \vec{CB} + \frac{3}{2}\vec{CA} = (-10, 4, 1) + \frac{3}{2}(-4, 0, 6) = (-16, 4, 10),$$

a vector equation for L_3 is given by

$$\mathbf{r}_{L_3}(t) = \mathbf{r}_C + t\vec{CD} = (-3, 7, -5) + t(-16, 4, 10), \qquad t \in \mathbb{R}.$$

4. (a) One calculates in a straightforward way

$$a_1 = 1,\ a_2 = 3,\ a_3 = 7,\ a_4 = 15,\ a_5 = 31,\ a_6 = 63.$$

(b) We have

$$a_{r+1} = 2^{r+1} - 1 = 2 \cdot (2^r - 1 + 1) - 1 = 2 \cdot (2^r - 1) + 2 - 1 = 2a_r + 1.$$

(c) For each natural number n:

$$\begin{aligned} \sum_{r=1}^{n} a_r &= \sum_{r=1}^{n} (2^r - 1) \\ &= \sum_{r=1}^{n} 2^r - \sum_{r=1}^{n} 1. \end{aligned}$$

Using the geometric series formula (i.e. $\sum_{k=1}^{n} x^k = x \sum_{k=0}^{n-1} x^k = x \frac{x^n - 1}{x-1}$) for the first term, we find that

$$\begin{aligned} \sum_{r=1}^{n} a_r &= 2\frac{2^n - 1}{2 - 1} - n \\ &= 2(2^n - 1) - n. \end{aligned}$$

(d) From part (4b), we know that

$$2a_r < a_{r+1}.$$

Hence dividing through by $2a_r$ and a_{r+1} (note that $(a_r)_{r=1}^{\infty}$ is a sequence of positive terms and so doing this does not change the direction of the inequality) yields

$$\frac{1}{a_{r+1}} < \frac{1}{2} \times \frac{1}{a_r}.$$

(e) By a repeated use of the inequality found in part (4d), we find that, for all $k \in \mathbb{N}$,

$$\frac{1}{a_{3+k}} < \frac{\frac{1}{2}}{a_{3+(k-1)}} < \frac{(\frac{1}{2})^2}{a_{3+(k-2)}} < \cdots < \frac{(\frac{1}{2})^k}{a_3}.$$

(This can be rigorously argued via mathematical induction.) Applying this bound term-by-term, it follows that

$$S_\infty := \sum_{r=1}^{\infty} \frac{1}{a_r} < \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \left(\sum_{k=0}^{\infty} \frac{1}{2^k} \right).$$

Plugging in the values from part (4a), we obtain the desired conclusion. (Note that it is essential to make this argument with *infinite* series, since this is how the question is set up and moreover this is what is later used in part (4f). It is *not* sufficient to make the argument just for truncated series as was indicated in the mark scheme.)

(f) To find a strict lower limit we need only sum a sufficient number of the (positive!) terms of the series. Were this number to be very large then this would of course not be a viable strategy, however three terms suffice;

$$S_\infty > 1 + \frac{1}{3} + \frac{1}{7} = \frac{31}{21}.$$

For the strict upper bound use the conclusion of (4e),

$$S_\infty < 1 + \frac{1}{3} + \frac{1}{7} \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{3} + \frac{1}{7} \frac{1}{1 - \frac{1}{2}} = \frac{34}{21},$$

where now we have benefited from the geometric series progression formula $\sum_{k=0}^{\infty} x^k = (1-x)^{-1}$ (with $x = 1/2$).

5. Write $\int u = \int u dx$, $\int v = \int v dx$, $\int uv = \int uv dx$ for short. As directed in the question, we will assume the relation $\int uv = \int u \int v$ holds for the *indefinite* integrals appearing therein. (In fact, it should have been made clearer in the question what the range of integration is, as the following computations are extremely sensitive to this.)

   (a) To recover a function from its integral, we differentiate. By differentiating both sides of $\int uv = \int u \int v$, using the product rule for the right-hand side ($(fg)' = f'g + fg'$, where $f = \int u$ and $g = \int v$), we obtain

   $$uv = u \int v + v \int u$$

   (since $(\int uv)' = uv$, $(\int u)' = u$ and $(\int v)' = v$). Since $uv \neq 0$ by assumption, we can divide by it to obtain the desired equality.

   (b) From part (5a),

   $$\frac{\int v}{v} = 1 - \frac{\int u}{u} = 1 - \sin^2(x) = \cos^2(x).$$

   (c) We could try differentiating $\frac{\int u}{u} = \sin^2(x)$ directly, but this involves the quotient rule. Instead we will try to use the easier product rule on the right-hand side of $\int u = u \sin^2(x)$. Doing this, we conclude

   $$u(x) = u'(x)\sin^2(x) + u(x)2\sin(x)\cos(x). \tag{1}$$

   Hence

   $$u(x)(1 - 2\sin(x)\cos(x)) = u'(x)\sin^2(x).$$

   Now from (1) and the assumption that $u(x) \neq 0$, we know that $\sin(x) \neq 0$. (Note that this puts a restriction on the domain of the functions u and v!) Thus we can divide by the non-zero $\sin^2(x)u(x)$ to obtain

   $$\frac{u'}{u} = \frac{1 - 2\sin(x)\cos(x)}{\sin^2(x)}. \tag{2}$$

   (d) Separation of variables in (2) gives up to an additive constant,

   $$\int \frac{du}{u} = \int \frac{1 - 2\sin(x)\cos(x)}{\sin^2(x)} dx.$$

   Now the left-hand side equals (up to an additive constant) $\ln(u)$, while the right hand-side (again, up to an additive constant) is

   $$\int \frac{dx}{\sin^2(x)} - \int \frac{(\sin^2(x))'}{\sin^2(x)} dx = -\cot(x) - \ln \sin^2(x) = -\cot(x) - 2\ln\sin(x).$$

   Taking exponentials of both sides gives

   $$u(x) = Ae^{-\cot(x)}\mathrm{cosec}^2(x),$$

   where A is a constant.

(e) We have, using the chain rule for the derivative of a composition of two functions $(g \circ f)' = (g' \circ f)f'$,

$$\frac{d}{dx}e^{\tan(x)} = e^{\tan(x)}\frac{1}{\cos^2(x)},$$

i.e.

$$\frac{e^{\tan(x)}}{\frac{d}{dx}e^{\tan(x)}} = \cos^2(x).$$

Thus we recognize at once from part (5b) it is possible to obtain a solution for v by setting $\int v = Be^{\tan(x)}$ for some multiplicative constant B. This implies that $v(x) = Be^{\tan(x)}\sec^2(x)$.

6. (a) Expanding $(f(x) - \lambda g(x))^2$ (which is clearly non-negative) gives

$$g(x)^2\lambda^2 - 2f(x)g(x)\lambda + f(x)^2 \geq 0.$$

Upon definite integration, this becomes:

$$\left(\int_a^b g(x)^2 dx\right)\lambda^2 - 2\left(\int_a^b f(x)g(x)dx\right)\lambda + \int_a^b f(x)^2 dx \geq 0.$$

The two facts we have used here are that:

- for a nonnegative function h, $\int_a^b h(x)dx \geq 0$ for every $a \leq b$;
- for any functions h_1, h_2, constants c_1, c_2, and $a \leq b$,

$$\int_a^b (c_1 h_1(x) + c_2 h_2(x))dx = c_1 \int_a^b h_1(x)dx + c_2 \int_a^b h_2(x)dx.$$

(b) In part (6a) we have produced a quadratic in λ, which is always nonnegative. If its discriminant (recall, if $a\lambda^2 + b\lambda + c$ is a quadratic, its discriminant is $b^2 - 4ac$) was strictly positive, then it would have two real roots. This clearly contradicts the nonnegativity of the quadratic function – draw a sketch if you are unsure why! Thus the discriminant must be less than or equal to 0. Hence

$$\left(-2\left(\int_a^b f(x)g(x)dx\right)\right)^2 - 4\left(\int_a^b g(x)^2 dx\right)\left(\int_a^b f(x)^2 dx\right) \leq 0,$$

from which the desired conclusion easily follows.

(c) Let

$$E := \int_{-1}^2 (1 + x^3)^{1/3} \cdot 1dx.$$

From part (6b) (as applied to the functions $f(x) = (1+x^3)^{1/3}$ and $g(x) = 1$, $a = -1$, $b = 2$), we have

$$E^2 \leq \int_{-1}^2 1^2 dx \int_{-1}^2 ((1 + x^3)^{1/2})^2 dx.$$

Integration yields

$$\int_{-1}^2 (1 + x^3)dx = \left[x + \frac{x^4}{4}\right]_{-1}^2 = (2 + 4) - \left(-1 + \frac{1}{4}\right) = \frac{27}{4},$$

while, of course, $\int_{-1}^{2} 1dx = 2-(-1) = 3$. It follows that

$$E^2 \leq \frac{81}{4},$$

and so

$$E \leq \frac{9}{2}.$$

(d) Here it is beneficial to 'spot' x^2 as the derivative of x^3 (up to a multiplicative constant), which then leads us to use the change of variables $u = 1 + x^3$, $du = 3x^2dx$. Using this approach, we find that

$$\int_{-1}^{2} x^2(1+x^3)^{1/4}dx = \frac{1}{3}\int_{0}^{9} u^{1/4}du = \frac{1}{3}\left[\frac{u^{5/4}}{5/4}\right]_0^9 = \frac{4}{15}9^{5/4} = \frac{4}{15} \times 9 \times \sqrt{3} = \frac{12\sqrt{3}}{5},$$

as requested.

(e) A sensible thing to be doing here is thinking about how to use the previous parts of the question to deduce the required inequality. We will start by applying the conclusion of part (6b) to the functions $f(x) = (1+x^3)^{1/4}$ and $g(x) = x^2$ on the interval $[-1, 2]$, which appeared in (6d). This gives

$$\left(\int_{-1}^{2} x^2(1+x^3)^{1/4}dx\right)^2 \leq \int_{-1}^{2} x^4dx \int_{-1}^{2}(1+x^3)^{1/2}dx.$$

From (6d), we know that the left-hand side is equal to

$$\left(\frac{12\sqrt{3}}{5}\right)^2 = \frac{144 \times 3}{25}.$$

Calculating

$$\int_{-1}^{2} x^4dx = \left[\frac{x^5}{5}\right]_{-1}^{2} = \frac{33}{5},$$

we conclude

$$\int_{-1}^{2}(1+x^3)^{1/2}dx \geq \frac{144 \times 3}{25} \Big/ \frac{33}{5} = \frac{144}{55}.$$

7. (a) A and B are a local minimum and maximum of C_1, respectively. (Note that the 'local' part of this description was missed in the instructions to the problem.) Their x-coordinates may be obtained by differentiating f and setting the derivative equal to 0. In particular,

$$f'(x) = \frac{1}{3} - \frac{12}{x^2} = 0 \qquad \Rightarrow \qquad x^2 = 36 \qquad \Rightarrow \qquad x \in \{-6, 6\}.$$

We conclude from the sketch (or by investigating the second derivative) that A has coordinates

$$(6, f(6)) = (6, 4)$$

and B has coordinates

$$(-6, f(-6)) = (-6, -4).$$

(b) The equation $x = k$ gives a line that is perpendicular to the x-axis. For the normal to be perpendicular to the x-axis, the tangent at that point must be parallel to it, i.e. have a gradient of 0. It thus follows from part (7a) that $k = 6$ (since $k > 0$).

(c) Investigating the sketch, we see that the normal to C_1 at the point $(x, f(x))$ for $x > 6$ will intersect C_1 at some point $(x', f(x'))$ with $x' \in (0, 6)$. Similarly, if $x \in (0, 6)$ and the gradient of the normal at $(x, f(x))$ is different from $1/3$, then the normal from this point will intersect C_1 at some point $(x', f(x'))$ with $x' > 6$.

On the other hand the normal to the point $(x, f(x))$ on C_1 with x-coordinate in $(0, 6)$ and gradient of the normal $1/3$, will never meet C_1 again. This is equivalent to the gradient of tangent being equal to -3. Therefore the point P has x-coordinate given by the solution to

$$f'(x) = -3,$$

i.e.

$$\frac{1}{3} - \frac{12}{\alpha^2} = -3,$$

so

$$\alpha = \frac{6}{\sqrt{10}}$$

(since $\alpha > 0$).

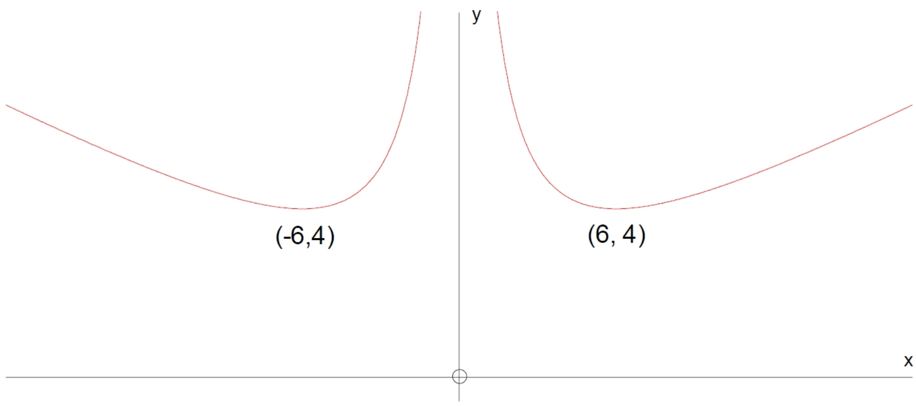
(d) First,

$$\beta = f(\alpha) = f(6/\sqrt{10}) = \frac{\sqrt{10}}{5} + 2\sqrt{10} = \frac{11}{5}\sqrt{10}.$$

Also, we know the gradient of the normal is $p := 1/3$. Therefore the equation for the normal is given by $y - \beta = p(x - \alpha)$, which after some rearrangement yields

$$y = \frac{1}{3}x + 2\sqrt{10}.$$

(e) We first observe that f is positive (resp. negative) on the positive (resp. negative) half-line. It follows that $|f|$ is given by mirroring the negative-valued branch of f over the x-axis, as is shown in the following sketch:



The two turning points of C_2 are clearly its two global minima, and their coordinates are then $(6, 4)$ and $(-6, 4)$. (These facts may be rigorously verified by inspecting the first and second derivatives of $|f|$, or else by inferring them from the properties of the curve C_1.) Finally, as regards asymptotes, the vertical axis, given by the equation

$x = 0$, is one. The other two are 'inherited' from the $y = x/3$ asymptote of C_1 and have equations $y = x/3$ and $y = -x/3$.

(f) The condition that the line with equation $y = mx + 1$ does not intersect or touch C_2 is equivalent to there not being a real-valued solution to the equation

$$mx + 1 = |f(x)|.$$

For positive x, the above equation is equivalent to

$$mx + 1 = \frac{x}{3} + \frac{12}{x}.$$

After multiplying on both sides by $3x$ and rearranging, this is in turn equivalent to

$$(3m - 1)x^2 + 3x - 36 = 0.$$

Similarly to the observation made in the answer to (6b), a necessary and sufficient condition for there not being a real-valued solution to the latter equation is that the discriminant is negative, i.e. $3^2 - 4(3m - 1)(-36) < 0$. This is equivalent to

$$48m < 15 \qquad \Leftrightarrow \qquad m < \frac{5}{16}.$$

Now, we could repeat a similar analytic argument for the negative half-line, but it is easier to use symmetry to deduce that, for there not to be any solutions to

$$mx + 1 = |f(x)|$$

with $x < 0$, it is necessary and sufficient that

$$-\frac{5}{16} < m.$$

In conclusion, the set of possible values for m is the open symmetric interval

$$\left(-\frac{5}{16}, \frac{5}{16}\right).$$