Mathematical Techniques : Differentiation SAMPLE

The total number of marks is 30. The marks available for each question are indicated in parentheses. The pass mark is 24 or above. Calculators must not be used.

Section A. Differentiate the following expressions with respect to x.

1. $\dfrac{x^3}{4} + \dfrac{1}{x^2}$ (2)

2. $\dfrac{1}{1 + \dfrac{1}{x}}$ (2)

3. $x^2 e^x \cos(2x)$ (3)

4. $\log\left(xe^x\right)$ (2)

Section B.

5. Find the gradient of the curve $y = x^2 - 6x - 7$ at the points where the curve cuts the x-axis. (3)

6. Find the x-coordinates and nature of any stationary points on the curve $y = xe^{-x}$. (3)

7. Let $y = e^{ax} + b(x+1)^3$. When $x = 0$, suppose that $dy/dx = 0$ and $d^2y/dx^2 = 0$. Find the possible values of a and b. (3)

Section C. Find the derivatives of the following functions at the indicated point.

8. $\sin(\sin x)$, $x = \pi$ (3)

9. $\dfrac{(3x-8)^3}{x}$, $x = 3$ (3)

10. $\cos x . \log(3x)$, $x = \pi$ (3)

11. $\log\left(\dfrac{x^3}{\sqrt{x}}\right)$, $x = 5/2$ (3)

Differentiation: solutions.

(A) 1. $\frac{3}{4}x^2 - \frac{2}{x^3}$ 2. $\frac{1}{(1+x)^2}$

3. $2xe^x\cos(2x) + x^2e^x\cos(2x) - 2x^2e^x\sin(2x)$

4. $1 + \frac{1}{x}$ $\left(\frac{(1+x)e^x}{xe^x}\right)$

(B) 5. $\frac{dy}{dx} = 2x - 6$ curve cuts when $x^2 - 6x - 7 = 0$

at $x = 7$, $\frac{dy}{dx} = 8$

at $x = -1$, $dy/dx = -8$

6. $\frac{dy}{dx} = (1-x)e^{-x}$, $\frac{d^2y}{dx^2} = (x-2)e^{-x}$

$\frac{dy}{dx} = 0$ when $x = 1$, at which point $\frac{d^2y}{dx^2} = -e^{-1} < 0$

so is a maximum.

7. $\frac{dy}{dx} = ae^{ax} + 3b(x+1)^2$, $\frac{d^2y}{dx^2} = a^2e^{ax} + 6b(x+1)$

at $x = 0$ $\frac{dy}{dx} = a + 3b$, $\frac{d^2y}{dx^2} = a^2 + 6b$.

so ($a = 0$ and $b = 0$) or ($a = 2$ and $b = -2/3$)

(C) 8. -1 $(\cos(x) \times \cos(\sin(x)))$

9. $26/9$ $\left(\frac{9x(3x-8)^2 - (3x-8)^3}{x^2}\right)$

10. $-\frac{1}{\pi}$ $\left(-\sin(x)\log(3x) + \frac{1}{x}\cos(x)\right)$

11. 1 $(5/2x)$

Mathematical Techniques : Integration SAMPLE

The total number of marks is 30. The marks available for each question are indicated in parentheses. The pass mark is 24 or above. Calculators must not be used.

Section A. Integrate the following expressions with respect to x.

1. $\dfrac{1}{4+3x}$ (3)

2. $\sin x \cos x$ (3)

3. $\sin^2(3x) + \cos^2(3x)$ (3)

4. $(4x-2)^6$ (3)

Section B. Evaluate the following definite integrals.

5. $\displaystyle\int_0^1 x\, e^{x^2}\, dx$ (3)

6. $\displaystyle\int_0^{\pi/2} \cos x \sin^5 x\, dx$ (3)

7. $\displaystyle\int_0^{\pi/2} x \cos x\, dx$ (3)

8. $\displaystyle\int_0^2 \frac{dx}{2x+5}$ (3)

Section C.

9. Calculate the area of the finite region enclosed by the curve $y = x^2 - 3x + 2$ and the x-axis. (3)

10. Calculate the area of the finite region enclosed by the curves $y = \frac{2|x|}{\pi}$ and $y = \sin x$. (3)

Integration: solutions

A

1. $\frac{1}{3}\log(4+3x)+c$
2. $\frac{1}{2}\sin^2 x+c$ or $-\frac{1}{2}\cos^2 x+c$
3. $x+c$ (NB $\sin^2(3x)+\cos^2(3x)=1$!)
4. $\frac{1}{28}(4x-2)^7+c$

B

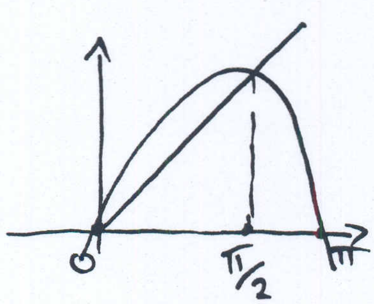
5. $\left[\frac{1}{2}e^{x^2}\right]_0^1 = \frac{1}{2}(e^1-1)$
6. $\left[\frac{1}{6}\sin^6 x\right]_0^{\pi/2} = \frac{1}{6}$
7. $[x\sin x]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx = \pi/2 - 1$
8. $\left[\frac{1}{2}\log(2x+5)\right]_0^2 = \frac{1}{2}(\log 9 - \log 5)$
$$= \frac{1}{2}\log(9/5)$$
$$= \log(3/\sqrt{5})$$

C

9. Require $-\int_1^2 (x^3-3x+2)\,dx = \underline{1/6}$

(minus sign since area is <u>under</u> the x-axis).

10.



(not to scale!!)

require $\int_0^{\pi/2} \sin x - \frac{2x}{\pi}\,dx$

$$= \underline{1-\pi/4}$$

Mathematical Techniques : Trigonometry SAMPLE

The total number of marks is 30. The marks available for each question are indicated in parentheses. The pass mark is 24 or above. Calculators must not be used.

Section A. Determine the exact value of the following numbers.

1. $\cos\pi$ (2) 2. $\sin(5\pi/6)$ (2) 3. $\cos(5\pi/4)$ (2)

4. $\tan(3\pi/4)$ (2) 5. $\sin(7\pi/2)$ (2) 6. $\sin(11\pi/3)$ (2)

Section B.

7. Find all values of x in the range $\pi \le x \le 2\pi$ for which $\sin(x - 5\pi/6) = 1/2$. (3)

8. Find all values of x in the range $0 \le x \le 2\pi$ for which $4\sin^2 x = 3$. (3)

9. Find all values of x in the range $\pi \le x \le 2\pi$ for which $2\sin 2x \cos 2x + 1 = 0$ (3)

Section C.

10. Find all values of x in the range $-\pi \le x \le \pi$ for which $\sin x = \cos 2x$. (3)

11. If $\cos\left(x + \frac{\pi}{6}\right) = \sin x$, find the exact value of $\tan x$. (3)

12. Express $\sqrt{3}\sin\theta + \cos\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 \le \alpha \le \pi/2$. (3)

Trigonometry: solutions

(A) 1. -1 2. $\frac{1}{2}$ 3. $-\frac{1}{\sqrt{2}}$

4. -1 5. -1 6. $-\sqrt{3}/2$

(B) 7. $\sin(x - 5\pi/6) = \frac{1}{2}$ for $\pi \leq x \leq 2\pi$

means $x - 5\pi/6 = \pi/6 + 2n\pi$ so $\underline{x = \pi}$

or $x - 5\pi/6 = 5\pi/6 + 2n\pi$ so $\underline{x = 5\pi/3}$

8. $4\sin^2 x = 3$ for $0 \leq x \leq 2\pi$ means $\sin x = \pm\frac{\sqrt{3}}{2}$

so $x = \underline{\pi/3,\ 2\pi/3,\ 4\pi/3,\ 5\pi/3}$

9. $2\sin 2x \cos 2x + 1 = 0$ for $\pi \leq x \leq 2\pi$

means $\sin 4x + 1 = 0$ so $\underline{x = \frac{11\pi}{8}, \frac{15\pi}{8}}$

(C) 10. $\sin x = \cos 2x$ for $-\pi \leq x \leq \pi$

means $\sin x = 1 - 2\sin^2 x$ so $2\sin^2 x + \sin x - 1 = 0$

$2u^2 + u - 1 = 0 \Rightarrow u = -1$ or $\frac{1}{2}$

$\sin x = -1 \Rightarrow \underline{x = -\pi/2}$ $\sin x = \frac{1}{2} \Rightarrow \underline{x = \pi/6 \text{ or } \frac{5\pi}{6}}$

11. $\cos(x + \pi/6) = \sin x \Rightarrow$

$\sin x = \cos x \cos \pi/6 - \sin x \sin \pi/6$

i.e. $\sin x = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$

i.e. $\underline{\tan x = \frac{1}{\sqrt{3}}}$ PTO

12. $\sqrt{3}\sin\theta + \cos\theta = R\cos(\theta-\alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

so $R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$

so $R^2 = 4 \Rightarrow \underline{R = 2}$

$\tan\alpha = \sqrt{3} \Rightarrow \alpha = \underline{\pi/3}$