SESSION AIMS

This session will cover the following:

- ◁ Reexamine ideas of probability and averages.
- ◁ Introduce new concepts that you will meet later in your degree.
- ◁ Provide the opportunity for you to explore these ideas.

BEFORE YOU START YOU SHOULD KNOW

- ◁ How to compute the probability of events associated with coins flips.
- ◁ How to solve simultaneous equations.

1.1.1 INTRODUCTION

- ◁ Consider a fair coin, if we flip the coin then we should recall that $\mathbf{P}(H) = \mathbf{P}(T) = \frac{1}{2}$.
- ◁ If we flip the coin (independently) twice, then $\mathbf{P}(HH) = \mathbf{P}(TT) = \frac{1}{4}$.

Motivating problem Consider two players, let's call them P1 and P2. Both players select a (different) sequence from HH, TT, HT, TH. Now flip a fair coin observing the sequence until we see one of the two selected sequences. The player whose sequence occurs first is the winner. For example, if P1 selects HH and P2 selects HT and the sequence of flips is

$$T, T, T, H, H$$

then P1 wins, while if the sequence of flips is

$$T, T, T, T, T, T, T, H, T$$

then P2 wins.

QUESTIONS

1) Which pattern is better, HH or HT?
2) We might ask, what is the average number of flips until we see HH or HT—call this the *average waiting time*. Do we think they will be the same?

QUESTION FOR YOU

Can you show that the average number of flips needed to reach HT from the start is 4?

In summary,

- ⊲ The waiting time for HT is 4 and for HH is 6. Does this mean that this mean that HT is better than HH?

QUESTION

What is the probability that HT beats HH? (That is, HT occurs before HH in the sequence.)

QUESTION FOR YOU TO TRY

Show that the probability that TH beats HH is $\frac{3}{4}$

QUESTIONS

- ◁ Is it true that two sequences with the same average waiting time should be equally likely to occur?
- ◁ Will a sequence with a lower average waiting time be better than a sequence with a higher average waiting time?

1.1.2 GENERAL SEQUENCES

It turns out that the situation is more complex than we might imagine. We introduce the ideas using sequences of three coin flips.

Example 1.1.1.

Consider the sequence of flips HHT and HTT. Determine
1) The probability that HHT beats HTT.
2) The average waiting time for HHT and HTT.

We now have all the techniques necessary to show the general strategy.

GENERAL STRATEGY

The best strategy for P1 is to let P2 select their sequence first, for example, P1, choose ABC, where A, B, C are one of H or T. Then P2 chooses $\overline{B}AB$, where $\overline{B}$ means the opposite of B. For example, if P1 chooses HHH, then $A = H$, $B = H$, $C = H$, P2 chooses $\overline{B}AB = \overline{H}HH = THH$.

Using the techniques available we can now show that

P1	P2	$\mathbf{P}$(P2 wins)
HHH	THH	$\frac{7}{8}$
HHT	THH	$\frac{3}{4}$
HTH	HHT	$\frac{2}{3}$
HTT	HHT	$\frac{2}{3}$
THH	TTH	$\frac{2}{3}$
THT	TTH	$\frac{2}{3}$
TTH	HTT	$\frac{3}{4}$
TTT	HTT	$\frac{7}{8}$

Notice that

- ⊲ HHT beats HTT
- ⊲ HTT beats TTH
- ⊲ TTH beats THH
- ⊲ THH beats HHT

So it appears to show that HHT beats HHT, but this is not true. We have show that the idea of one sequence beating another is *nontransitive*.

CHALLENGE

- ⊲ What is the average waiting time of HTHH and THTH?
- ⊲ What is the probability that THTH beats HTHH?
- ⊲ Is it true that a sequence with a shorter expected waiting time must be likely to occur before a sequence with a longer expected waiting time?