1. Inverse roots

Suppose that $Q(x) = ax^2 + bx + c$ satisfies $ac \neq 0$ and has roots (i.e. solutions of $Q(x) = 0$) α and β.

Show that the quadratic $\tilde{Q}(x) = cx^2 + bx + a$ has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Hint Let $x = \frac{1}{t}$ in the definition of $Q(x)$.

Extensions

(1) Show that if $\alpha_1, \ldots, \alpha_n$ are the roots of the polynomial P, where
$$P(x) = a_0x^n + \ldots + a_{n-1}x + a_n \text{ with } a_0a_n \neq 0,$$
then the roots of $\tilde{P}$ given by
$$\tilde{P}(x) = a_nx^n + \ldots + a_0$$
are $\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}$.

(2) Show that the roots of
$$P_e(x) = a_0x^{2n} + a_1x^{2n-1} + \ldots + a_{n-1}x^{n+1} + a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0 \ (a_0 \neq 0)$$
are of the form $\alpha_1, \ldots, \alpha_n, \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}$.

(3) What can you say about the roots of
$$P_+(x) = a_0x^{2n+1} + a_1x^{2n} + \ldots + a_nx^{n+1} + a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$$
and the roots of
$$P_-(x) = a_0x^{2n+1} + a_1x^{2n} + \ldots + a_nx^{n+1} - a_nx^n - a_{n-1}x^{n-1} - \ldots - a_0?$$

2. Trigonometric polynomials

The angle sum formula tells us that
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1.$$
Find a similar expression involving powers of $\cos\theta$ for $\cos 3\theta$.

Hint Write $3x = 2x + x$!

Extensions

(1) Find the roots of $4\sqrt{2}x^3 - 3\sqrt{2}x = 1$.

Hint Set $x = \cos\theta$. What is $\cos\frac{3\pi}{4}$?

(2) What is $\cos\frac{\pi}{12}$?