1. Expanding Base I

Recall that the number of ways n distinct objects can be put in order is $n! = 1 \times 2 \times \dots n$.

Let

$$a_k = \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k-1}{k!}.$$

What is a_k as a simple fraction?

Hint Consider $b_k = a_k + \frac{1}{k!} = \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k-2}{(k-1)!} + \frac{k}{k!}$ and cancel the obvious common factor in the last fraction.

Answer Following the hint, $b_k = \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k-2}{(k-1)!} + \frac{k}{k!} = \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{k-1}{(k-1)!} = \dots = \frac{1}{2!} + \frac{3}{3!} = \frac{1}{2!} + \frac{1}{2!} = 1$, so

$$a_k = 1 - \frac{1}{k!}.$$

Extensions

(1) Find N_k, the number of sequences $x_2, \dots x_k$ with $x_2 = 0$ or 1, $x_3 = 0$ or 1 or 2, ..., $x_k = 0, 1, \dots$ or $k-1$.

Hint How many values can x_2 take? For each of these, how many values can x_3 take?

Answer Using the hint, for each choice of the sequence $x_2, \dots, x_k$, $x_k + 1$ can take $k+1$ values, so, by induction

$$N_k = k!$$

(2) Deduce that every fraction in the interval $[0, 1)$ of the form $x = \frac{m}{n!}$ (with m and n integers) can be written as

$$\frac{x_2}{2!} + \dots \frac{x_n}{n!}, \tag{1.1}$$

with $x_2 = 0$ or 1, $x_3 = 0$ or 1 or 2, ..., $x_n = 0, 1, \dots$ or $n-1$.

Hint Use (1).

Answer Suppose x has a representation as in (2.1). Since $m!$ is a factor of $n!$ for every choice of $m < n$, we know we can write $x = \frac{m}{n!}$ for some non-negative integer m. Then by the original question, we know that, since $0 \leq x \leq a_n$, $0 \leq m \leq n! - 1$, so x is a fraction in $[0, 1)$ of the right form. There are $n!$ of these and there are $n!$ distinct sequences of the right form so we must be able to represent all of the required fractions in this way.

(3) Remember that a number x is said to be *rational* if it can be written as a (possibly improper) fraction $\frac{m}{n}$ for some pair of integers m and n.

Show that every rational, x, in $[0, 1)$ can be written

$$\frac{x_2}{2!} + \dots + \frac{x_n}{n!},$$

with $x_2 = 0$ or 1, $x_3 = 0$ or 1 or 2, …, $x_n = 0, 1, \ldots$ or $n-1$ for some value of n.

Hint Use the previous part!

Answer If x is a rational in [0,1) then it can be written as $\frac{a}{b}$ for two integers with $0 \le a < b$. Taking $n = b$ and $m = a \times (b-1)!$ we see that $x = \frac{m}{n!}$ and so the representation holds.