Alice and Bob

You play a game with Alice and Bob:

1. They pick one number each, with the only constraint that $A \neq B$, meaning that they are not allowed to pick the same number. (However, they can use any strategy: communicate with each other, use randomness to pick their numbers, etc.)
2. You toss a coin to choose who reveals their number, Alice or Bob.
3. After seeing the revealed number, you are to guess who has the bigger number.

Find a strategy so that, if playing this game repeatedly, you win more often than you lose.

Hint

Assume first that they pick numbers from the faces of a die, that is, from the set $S = \{1, 2, 3, 4, 5, 6\}$ and that your strategy is deterministic. For example, if the revealed number was > 3, you always declared that it was the bigger of the two numbers. Would your strategy always work? What if Alice and Bob always picked 4 and 5? What if S is a different set, e.g. the real numbers? Can you think of a more abstract description of the strategy that enables you to introduce a dependency on what you see?

Answer

For your strategy you first pick a function ψ on S that is strictly increasing, that goes to 0 for X to $-\infty$ and that goes to 1 for X to ∞.

Then you declare the revealed number X (where X may be A or B) bigger with probability $\psi(A)$, where ψ is a cumulative distribution function. (It can be any such function.

Why it works:
Without loss of generality, assume $A < B$. After tossing the coin, the chance that you see A or B is the same and equals $1/2$. However, what is the chance your answer is correct?

$$\begin{aligned}
P(\text{win}) &= P(\text{seen } B)P(\text{declared } B \text{ as bigger given } B \text{ was seen }) \\
&\quad + \; P(\text{seen } A)P(\text{declared } B \text{ as bigger given } A \text{ was seen }) \\
&= \frac{1}{2}\psi(B) + \frac{1}{2}(1 - \psi(A)) \\
&= \frac{1}{2} + \frac{1}{2}\Big(\psi(B) - \psi(A)\Big) \\
&> \frac{1}{2},
\end{aligned}$$

because ψ is strictly monotone and $A < B$.

Notes

- An interpretation of this is that it is a probabilistic strategy and ψ is a cumulative distribution function. An examples for a cumulative functions if S is the real numbers is the normal distribution. In the case where S is a discrete set, the function ψ does not have to be strictly increasing, but could be piecewise constant and jump up by a value larger than 0 in each values of S, e.g. $\psi(k) = k/6$ $(k = 1, 2, \ldots, 6)$ for the die. (For more details about such functions see `https://en.wikipedia.org/wiki/Cumulative_distribution_function`.)

- The strategy always works no matter whether the set S is finite or infinite, discrete or continuous.

- What does "without loss of generality" means? It means that the argument works in the same way without that assumption. Here, the assumption was $A < B$. The case of equality was excluded. If $A > B$ then the argument can just be rewritten with the roles of A and B exchanged.