2. Expanding Base II

What is

$$c_n = \sum_{k=n+1}^{\infty} \frac{k-1}{k!}?$$

Hint What is $c_1 = \sum_{k=2}^{\infty} \frac{k-1}{k!}$?

Extensions

(1) Show that every rational x in (0,1) can be written as

$$\sum_{k=2}^{\infty} \frac{y_k}{k!},$$

with $y_k \in \{0, 1, \ldots, k-1\}$ for each k, in exactly *two* ways: one in which all but finitely many of the y_k's are 0 and the other in which all but finitely many of the y_k's take the value $k-1$.

Hint Prove this by contradiction: so suppose that $x = \frac{m}{n!} = \sum_{k=2}^{\infty} \frac{y_k}{k!}$, with infinitely many of the y_k's not being zero and infinitely many not being $k-1$ and deduce that x cannot be of the form $\frac{m}{n!}$

Answer Clearly we can write each x in the two ways described. The terminating version follows from our previous results so

$$x = \frac{x_2}{2!} + \ldots + \frac{x_n}{n!}, \tag{2.1}$$

with x_n strictly positive (if not, then $x = \frac{m'}{(n-1)!}$ so just replace n by $n-1$ in the argument and repeat if necessary).

Then define y_k as follows:

$$y_k = \begin{cases} x_k & : k < n \\ x_n - 1 & : k = n \\ k-1 & : k > n. \end{cases} \tag{2.2}$$

This is a non-terminating representation of x.

Uniqueness of these representations follows from uniqueness of the terminating versions: as we saw in Expanding Base I, there are precisely $n!$ representations of the form (2.1) and $n!$ distinct rationals in [0,1) of the form $\frac{m}{n!}$.

For the contradiction, suppose that $x = \frac{m}{n!} = \sum_{k=2}^{\infty} \frac{y_k}{k!}$, with infinitely many of the y_k's not being 0 and infinitely many not being $k-1$. Now define x' as follows:

$$x' = \sum_{k=2}^{n} \frac{y_k}{k!}.$$

We see that x' is a multiple of $\frac{1}{n!}$ in the interval $[0,1)$ (each y_k is non-negative so $0 \leq x'$, and $x' \leq x \leq a_n < 1$). Moreover, since $y_k > 0$ for some $k > n$ we see that $x' < x$. It follows that

$$x' = \frac{m'}{n!}$$

for some $m' \leq n! - 2$. But

$$\sum_{k=n+1}^{\infty} \frac{y_k}{k!} \in (0, \frac{1}{n!})$$

since there are is a $y_k < k-1$ for some $k > n$ by assumption. So $\sum_{k=n+1}^{\infty} \frac{y_k}{k!} < c_k = \frac{1}{n!}$ and thus

$$x - x' \in (0, \frac{1}{n!}),$$

which contradicts the assumption that $x = \frac{m}{n!}$.

(2) Show that $e - 2$ is not a rational number (and hence e is not rational).

Hint What is the power series for e^x?

Answer Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$,

$$e - 2 = \sum_{k=2}^{\infty} \frac{1}{k!}$$

and is in [0,1) (since $e \approx 2.71828$), so, using the previous part we can see that it is not a fraction of the form $\frac{m}{n!}$ *for any* n, so cannot be rational. If e were rational then $e - 2$ would be, so e cannot be rational either.