Populations of Neurons: Questions

• Q1. Steady-State Distribution for the Subthreshold Voltage

The current equation for a leaky IF model

$$J = \frac{(E_0 - V)}{\tau_0} P - \frac{\sigma_V^2}{\tau_0} \frac{\partial P}{\partial V}. \tag{1}$$

Consider a population of neurons that do not have spike-generating currents, and so there is no threshold for action potential generation. The population is in the steady state.
[Q] What form does the function for the current J take?
[A] $J = 0$.
[Q] Show that in this case the probability distribution P is Gaussian.
[A] Equation (1) simplifies to

$$0 = \frac{\partial}{\partial V}\left(e^{(V-E_0)^2/2\sigma_V^2} P\right) \quad \text{so that} \quad P(V) = \frac{1}{\sqrt{2\pi\sigma_V^2}} \exp\left(-\frac{(V-E_0)^2}{2\sigma_V^2}\right) \tag{2}$$

• Q2. Firing-Rate of the Non-Leaky IF Neuron

The Non-Leaky IF model has a voltage dynamics

$$\tau_0 \frac{\partial V}{\partial t} = E_0 + \sqrt{2\tau_0}\sigma_V \xi(t) \tag{3}$$

where $E_0 > 0$ and so has a corresponding Fokker-Planck equation

$$\tau_0 \frac{\partial P}{\partial t} = \sigma_V^2 \frac{\partial^2 P}{\partial V^2} - E_0 \frac{\partial P}{\partial V} \tag{4}$$

[Q] Use the continuity equation to show that the current J obeys

$$\tau_0 J = E_0 P - \sigma_V^2 \frac{\partial P}{\partial V} \tag{5}$$

We now impose a threshold V_{th} and reset V_{re} such that the current obeys $J = r\theta(V - V_{re})$ for $V < V_{th}$ where r is the firing rate. The density vanishes at threshold $P(V_{th}) = 0$. Rescale the voltage by $x = V/\sigma_V$ and define $\mu = E_0/\sigma_V$, $x_{re} = V_{re}/\sigma_V$ and $x_{th} = V_{th}/\sigma_V$.
[Q] Show that this rescaling yields:

$$-r\tau_0\theta(x - x_{re}) = \frac{\partial p}{\partial x} - \mu p \tag{6}$$

where $p(x)dx = P(V)dV$.
[A] Straightforward substitution.
[Q] Solve this equation to give the probability density as a function of the unknown firing rate. You may write the answer in terms of an integral - the full solution is required in the next question.
[A]

$$p(x) = r\tau_0 e^{\mu x} \int_x^{x_{th}} dy e^{-\mu y}\theta(y - x_{re}). \tag{7}$$

[Q] Use the fact that the probability density integrates to one to show that

$$1 = r\tau_0 \int_{-\infty}^{x_{th}} dx e^{\mu x} \int_{x}^{x_{th}} dy e^{-\mu y} \theta(y - x_{re}). \tag{8}$$

[A] Straightforward integration of $p(x)$.
[Q] Reverse the order of integrals to drop the θ function and then perform the integrations to show that

$$r = \frac{1}{\tau_0} \frac{\mu}{x_{th} - x_{re}} \tag{9}$$

and therefore that for this simple model the firing rate is independent of the noise strength σ_V.
[A] With the integrals reversed we have

$$1 = r\tau_0 \int_{x_{re}}^{x_{th}} dy \int_{-\infty}^{y} dx e^{\mu x - \mu y}. \tag{10}$$

Now set $x = y - z$ to yield two simple integrations. On putting the original forms for the variables for the resulting firing rate is

$$r = \frac{1}{\tau_0} \frac{E_0}{V_{th} - V_{re}}. \tag{11}$$

As σ_V does not feature in this equation the rate is independent of the noise strength.

• Q3. Probability Density for the Non-Leaky IF Neuron

We continue the analysis of the NLIF model now by calculating the form of the distribution.
[Q] Using the form of the probability density and the current calculated for the previous question show that

$$p(x) = \frac{1}{x_{th} - x_{re}} \left(1 - e^{-\mu(x_{th} - x)}\right) \quad \text{for } x > x_{re} \tag{12}$$

$$p(x) = \frac{1}{x_{th} - x_{re}} \left(e^{-\mu(x_{re} - x)} - e^{-\mu(x_{th} - x)}\right) \quad \text{for } x < x_{re} \tag{13}$$

The current J given in equation (5) evaluated at threshold gives the firing rate r.
[Q] Verify this using equation (12).
[A] Use the fact that $p(v_{th}) = 0$. The other term in the current comes from the differential of equation (12) evaluated at x_{th}.
[Q] By considering the currents at the reset, show that the following is true:

$$J(V_{re-}) + r = J(V_{re+}) \tag{14}$$

[A] The current of trajectories taken out at threshold is re-inserted at the reset. At this voltage the total currents in must balance the total currents out.
[Q] Given that $P(V_{re-}) = P(V_{re+})$ show that the gradient discontinuity in $P(V)$ is proportional to the firing rate.
[A] Insert the form for the current into the (14) relation to yield

$$\left.\frac{\partial p}{\partial x}\right|_{re-} - \left.\frac{\partial p}{\partial x}\right|_{re+} = r\tau_0. \tag{15}$$

[Q] Sketch the probability density. What happens when the noise goes to zero? (i.e. $\mu \to \infty$).