Mathematical Appendix

Here some of the mathematical methods that will be used for the course will be outlined.

• The Dirac Delta Function

The delta function $\delta(t-t_0)$ can be thought of as a brief impulse centred at t_0. The function has zero width but infinite height, such that its integral is equal to one and therefore it has units that are the reciprocal of its argument: if t is time then $\delta(t-t_0)$ has units of 1/time.

$$\int_{-\infty}^{\infty} ds\, \delta(s-t_0) = 1 \quad \text{and} \quad \int_{-\infty}^{t} ds\, \delta(s-t_0) = \theta(t-t_0) \tag{1}$$

where $\theta(t-t_0)$, which is 1 if $t > t_0$ and zero otherwise is called the Heaviside step function. It provides a convenient method for representing events that are much faster than other timescales in the system. To keep matters general, consider the equation

$$\frac{dx}{dt} = -\frac{x}{\tau} + a_0\, \delta(t-t_0). \tag{2}$$

Integrating between $t = t_0 - \epsilon$ and $t_0 + \epsilon$, where ϵ is small and positive:

$$\Delta x(t_0) = x(t_0+\epsilon) - x(t_0-\epsilon) = -\int_{t_0-\epsilon}^{t_0+\epsilon} \frac{dt}{\tau}\, x(t) \;+\; a_0 \tag{3}$$

because the integrand on the RHS is finite, when the limit $\epsilon \to 0$ is taken, its contribution vanishes, yielding a jump in x at t_0 of $\Delta x(t_0) = a_0$. With this construction, the effect of a discrete event, or initial condition (i.e. a pulse at t_0) can be encoded into a continuous differential equation. A concrete example would be a short current pulse $I(t) = Q_0\delta(t-t_0)$ delivering a total charge Q_0 at time t_0 which would cause the voltage to jump instantaneously by $\Delta V = Q_0/C$ before slowly relaxing back to baseline once the charge leaks out.

Another important property of the delta function, which follows directly from its zero width, is its behaviour in convolutions

$$\int_{-\infty}^{\infty} ds\, f(s)\delta(s-t_0) = f(t_0) \quad \text{and} \quad \int_{-\infty}^{t} ds\, f(s)\delta(s-t_0) = \theta(t-t_0)f(t_0) \tag{4}$$

for any smooth (i.e. differentiable) $f(t)$. These results can be easily derived if one considers a delta function as a rectangular pulse of width Δ and height $1/\Delta$. If Δ is shorter than any structure in the function $f(t)$ (which in the limit $\Delta \to 0$ will always be the case) it is fair to replace $f(s)$ in the integral by its value $f(t_0)$. With the result (4) in mind we will now calculate the full solution of (2). The equation can be written

$$e^{-t/\tau}\frac{d}{dt}\left(xe^{t/\tau}\right) = a_0\delta(t-t_0) \quad \text{so that} \quad xe^{t/\tau}\Big|_{-\infty}^{t} = a_0\int_{-\infty}^{t} ds\, e^{s/\tau}\delta(s-t_0). \tag{5}$$

On evaluating the LHS and multiplying by an exponential factor this yields

$$x(t) = a_0\int_{-\infty}^{t} dse^{-(t-s)/\tau}\delta(s-t_0) = a_0\theta(t-t_0)e^{-(t-t_0)/\tau}. \tag{6}$$

where the second result of equation set (4) has been used. For an equation of the form (2) for which the complimentary function is an exponential, the response to a short pulse at t_0 is a jump and then exponential decay. By using the delta function, the initial conditions and relaxation can all be contained in equation (2). However, the convenience comes when we consider the effect of many pulses. Imagine that pulses arrive at times t_k, each with an amplitude of a_k. The corresponding differential equation is now

$$\frac{dx}{dt} = -\frac{x}{\tau} + \sum_k a_k \, \delta(t - t_k) \tag{7}$$

and because of the linearity of the equation we can, on comparing the steps (5) and (6), write down the solution directly

$$x(t) = \sum_k a_k \theta(t - t_k) e^{-(t-t_k)/\tau}. \tag{8}$$

The steps just taken can be interpreted as the differential equation having "pasted on" exponential templates, of amplitude a_k, on each delta pulse of the pulse train.

Before closing this section, we will identify some common errors in using delta functions. First, in equations like

$$\frac{dx}{dt} = -x + x a_0 \delta(t - t_0) \tag{9}$$

some care must be taken. This is because it contains the *multiplicative* form $x(t)\delta(t - t_0)$ and $x(t)$ will be discontinuous at t_0. There are two ways around this: (i) either one considers that we really meant $x(t - \epsilon)\delta(t - t_0)$; or (ii) we divide the entire equation by x (to remove the prefactor of the delta function) then write the differential as $d\log(x)/dt$ and integrate. These two approaches give different results

$$(i) \quad \Delta(x) = x_0 a_0 \qquad (ii) \quad \Delta(x) = x_0(e^{a_0} - 1). \tag{10}$$

where x_0 is the value of x just before the pulse. The second result is the one that obeys the usual rules of calculus whereas result (i) is more convenient for numerical integration schemes. It can be noted that in the limit of small a_0 these results become identical.

Another common error is when the argument of the delta function is not the same as the integration variable. Note that

$$\int_{-\infty}^{\infty} dt \; f(t) \; \delta(h(t) - h(t_0)) \neq f(t_0) \tag{11}$$

unless the function $h(t) = t$. In general, such forms must be converted using the result

$$\left|\frac{dh}{dt}\right| \delta\left(h(t) - h(t_0)\right) = \delta(t - t_0) \tag{12}$$

and inserted into the integral to give

$$\int_{-\infty}^{\infty} dt \; f(t) \; \delta(h(t) - h(t_0)) = \int_{-\infty}^{\infty} dt \; f(t) \; \frac{\delta(t - t_0)}{\left|\frac{dh}{dt}\right|} = \frac{f(t_0)}{\left|\frac{dh(t_0)}{dt}\right|}. \tag{13}$$

From result (12) we also get $\delta(t - s) = \delta(s - t)$.

• Green's functions

This is a method of solving linear differential equations that is closely related to the discussion on delta functions above. Consider the two differential equations

$$\mathcal{D}x = f(t) \quad \text{and} \quad \mathcal{D}g = \delta(t-s) \tag{14}$$

where $\mathcal{D}$ is some linear differential operator with respect to t and $f(t)$ is a "driving" term. Because we can always write any function as an integral of delta functions, as shown in equation (4), we can build the solution x out of the solution g:

$$f(t) = \int_{-\infty}^{\infty} ds\, f(s)\delta(t-s) \quad \text{so the full solution for } x \text{ is} \quad x(t) = \int_{-\infty}^{\infty} ds\, f(s)g(s,t). \tag{15}$$

The solution $g(s,t)$ is called the Green's function for the operator $\mathcal{D}$. The advantage of this formalism is that once we know g we can write down the solution x to any driving force $f(t)$.

We now illustrate the use with an example. We would like to solve the equation

$$\tau \frac{dx}{dt} = -x + \theta(t)e^{-t/\tau_s}. \tag{16}$$

So here the operator is $\mathcal{D} = \tau d/dt + 1$ and $f(t) = \theta(t)e^{-t/\tau_s}$. We have already derived the Green's function for this equation in the previous section (see Eqs 2 and 6). So using equation (15) the solution for equation (16) is

$$x(t) = \int_{-\infty}^{\infty} \frac{ds}{\tau}\, \theta(s)e^{-s/\tau_s}\theta(t-s)e^{-(t-s)/\tau} = \theta(t)\int_0^t \frac{ds}{\tau} e^{-s/\tau_s}e^{-(t-s)/\tau} \tag{17}$$

which has solution

$$x(t) = \theta(t)\frac{\tau_s}{\tau - \tau_s}\left(e^{-t/\tau} - e^{-t/\tau_s}\right). \tag{18}$$