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Tolerance Analysis for Design of Multistage Manufacturing Processes Using Number-Theoretical Net Method (NT-net)

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Abstract. Recent developments in modeling stream of variation in multistage manufacturing system along with the urgent need for yield enhancement in the semiconductor industry has led to complex large scale simulation problems in design and performance prediction, thus challenging current Monte Carlo (MC) based simulation techniques. MC method prevails in statistical simulation approaches for multi-dimensional cases with general (i.e., non-Gaussian) distributions and/or complex response functions. A method is proposed based on number theory (NT-net) to reduce computing effort and the variability of MC's results in tolerance design and circuit performance simulation. The sampling strategy is improved by introducing NT-net that can provide better convergent rate over MC. The new method is presented and verified using several case studies, including analytical and industrial cases of a filter design and analyses of a four-bar mechanism. Results indicate a 90–95% reduction of computation effort with significant improvement in accuracy that can be achieved by the proposed technique.

Key Words: tolerance analysis, Monte Carlo simulation

1. Introduction

Modeling and predicting stream-of-variation (SOVA or SoV) in complex manufacturing systems is essential for both product and process development (Ceglarek, Shi, and Wu, 1994; Hu, 1997; Koren et al., 1999; NSF-ERC, 1999). Some of the most critical factors and barriers in the competitive development of modern production systems (especially in automotive, aerospace, appliance and semiconductor industry, etc.) lay in (1) large number of design/process alterations resulting in 67–70% of all changes related to product dimensional variation in aerospace and automotive industries (Shalon, Gossard, Ulrich, and Fitzpatrick, 1992; Ceglarek and Shi, 1995); (2) long ramp-up time wherein, major efforts during ramp-up focus on identifying root causes of process variation, calibration, and readjustment of fixtures and tooling; and (3) downtime of production system caused by dimensional variation problems, as has been reported that more than 70% of faults were caused by maintenance, design, and part errors (Ceglarek and Shi, 1995). Thus, a critical research area focuses on the variation prediction model of product quality and high efficient simulation techniques.

In the semiconductor industry, parametric performance of integrated circuits (IC) depends on both the circuit design and fabrication process. The ability to predict this performance is essential for product and process design (Spence and Soin, 1997; Gibson, Poddar, and May, 1997). Integrated circuit manufacturing complexities have resulted in decreasing product yields and reliabilities. This process has been accelerated with the advent of newly developed product/process technologies, such as deep submicron technologies coupled with the introduction of newer materials and techniques as copper interconnects, silicon-on-insulator, and increased wafer sizes (Koren and Koren, 1998). The need to improve product yields has challenged current circuit simulation techniques, which are typically Monte Carlo based (Luigi, 1990). Still, significant increase in problem size implies that considerable time and effort can be saved if the designer could efficiently predict the yield of each design stage.

Recent research in modeling and simulation of dimensional variation at systems level (Mantripragada and Whitney, 1999; Jin and Shi, 1999; Ding, Ceglarek, and Shi, 2000a, 2002a, 2002b; Camelio, Hu, and Ceglarek, 2003), as well as the urgent need for high efficiency circuit performance simulation tools in semiconductor industry, have raised new questions on the efficiency and effectiveness of Monte Carlo (MC) based simulation techniques in the following areas:

- Larger scale problem (this involves substantial numbers of product/process variables: 150–250 sheet metal parts, 1700–2500 fixture locators, VLSI circuit simulation, and analogy circuit simulation on continuous frequency range);
- Non-Gaussian distribution of product and process characteristics (for example, mean shifts and various data trends and patterns caused by tooling degradation, defect distribution (for example, Poisson) on a given area of a IC chip etc.);
- Nonlinear tolerance stack up model due to geometrical nonlinearities in real 2D and 3D assemblies; analyses/synthesis of robot mechanisms envelope tolerances; and/or highly nonlinear frequency response function of circuits in semiconductor products (or nonlinear objective function with constraint functions in yield optimization).

The main drawbacks of MC-based methods have been widely recognized as:

- very time consuming, especially for large scale problems;
- poor computational accuracy at small to median sample sizes.

A number theory based technique (NT-net) is proposed in this paper to replace the commonly used the Monte Carlo method. The new method improved sampling strategy in parameter domain, therefore, the computational effort needed in simulation is expected to be significantly reduced with higher accuracy yields. The outline of the paper is as follows: literature review is presented in Section 2; in Sections 3 and 4 the proposed NT-net algorithm and simulation examples are given to provide a detailed procedure and to evaluate the efficiency and effectiveness of the NT-net method; and Section 5 summarizes the conclusions. To provide the necessary theoretical background, some important concepts and theorems associated with number-theoretical method (NT-net) are given at the end of the paper in the Appendix.

2. Background and related work

Analysis models and simulation algorithms have been recognized as two critical aspects within statistical performance simulation techniques. However, recent advances both in dimensional variation modeling and in CAD/simulation techniques in the semiconductor industry pose major challenges to current statistical simulation techniques.

The last few decades have seen the development of assembly tolerance modeling techniques such as 2D vector-loop model (Fortini, 1967; Chase and Parkinson, 1991), mechanistic model (Liu, Hu, and Woo, 1996), kinematics adjustments (Whitney, Gilbert, and Jastrzebski, 1994), and functional model (Voelcker, 1998), to name a few. These techniques focus mainly on an assembly that is built up through numerous mating features of individual components or so called product-oriented tolerancing, since the inclusion of process information in the models is very limited. To simplify the analyses these tolerancing techniques involve two assumptions: (1) individual variations are normally distributed; and (2) explicit linear stack up model (or the stack up model which can be linearized). Based on these assumptions, several simplified algorithms have been developed such as worst case (WC) and root square sum (RSS). For more complex models (i.e., the implicit or nonlinear models) and random distribution of variables (non-normal), Monte Carlo-based simulation techniques prevail.

However, researchers have begun to question the traditional product-oriented approach as it overlooks the impact of process variables. State transition/state-space model (Mantripragada and Whitney, 1999; Jin and Shi, 1999; Ding, Ceglarek, and Shi, 2000a) have been proposed to integrate the process/product parameters into a variation propagation model at a system level. These models are especially desirable for products whose dimensional variation is significantly affected by both product and process variables (the state transition/state-space model is also called stream-of-variation: SoV or SOVA). Process variables are not only rather diverse but also cover very broad areas associated with manufacturing processes. The development of state transition/state space model for multistage manufacturing processes led to further research in developing tolerance models (Ding, Ceglarek, Jin, and Shi, 2000b; Shiu, Apley, Ceglarek, and Shi, 2003), design evaluation techniques (Ceglarek and Shi, 1998; Ding et al., 2002b) and methods for diagnosing root causes of product and process variability (Ceglarek and Shi, 1996; Rong, Ceglarek, and Shi, 2000; Apley and Shi, 2001; Ding et al., 2002a; Ding, Shi, and Ceglarek, 2002c; Zhou, Ding, Chen, Shi, 2003). Camelio et al. (2003) extended state-space model to include parts compliance and process variables such as errors caused by fixtures and welding guns. In addition, these factors are in some cases nonstationary in nature, for example, the tooling/fixtures errors may change over time due to their degradation. Attempt has also been made to represent form or profile errors of compliant parts by developing mode-based error decomposition model (Huang and Ceglarek, 2002). The mode-based error decomposition model allows for compact representation and integration of compliant parts errors within assembly variation simulation framework. Increased model complexity and model scale are involved in all the above developments. This is one of the main reasons that the current Monte Carlo based techniques pose a major hurdle for variation simulation.

Similar modeling and simulation techniques have also been developed for large-scale circuit design in the semiconductor industry (Spence and Soin, 1997). Current VLSI technology allows manufacturing large-area IC with millions of devices by deep submicron techniques. However, the yield-reducing defects in manufacturing process increase proportionally with the feature size and density. Variations of product/process parameters have been recognized as major contributors of low yield and reliability. Koren and Koren (1998) presented a comprehensive review of defect tolerance modeling and simulation techniques for VLSI circuits design. The Poisson distribution yield model and Monte Carlo techniques are used for their yield analyses models. A number of methods to improve computational efficiency of simulations have been developed by Gyvez and Di (1992), Gyvez (1998), and Wagner and Koren (1995). They use geometrical techniques to accommodate the chip's area shape complexity. While Monte Carlo has long been applied it is widely recognized as too time consuming to be applied on large IC designs.

Monte Carlo based techniques are used often in cases of non-Gaussian distribution and nonlinear (yield) response function as a last resort. MC has proved to be a versatile and simple method, making it as one of the most commonly used techniques in statistical tolerance analyses/synthesis (Nigam and Turner, 1995) and in circuit design (Spence and Soin, 1997). MC-based techniques are widely used to calculate design function values in synthesis problems as well (Ashiagobor, Liu, and Nnaji, 1998).

In recent years, the development of the optimization algorithm has become one of the most important aspects in tolerancing research. Tolerance design is actually a problem of determining the optimal allotment of the components tolerances under the constraints of function requirement and acceptance probability (Wu, ElMaraghy, and ElMaraghy, 1988; Lee and Woo, 1989, 1990). The need for MC simulations for system/tolerance optimization means that a large number of simulations have to be conducted each time an array of trial tolerances is constructed. Thus, a large number of iterations are commonly required in the optimization procedure.

Current research in statistical tolerance synthesis has focused on a class of optimization algorithm called direct search techniques. These algorithms do not use gradient or derivative information for tolerance synthesis since gradient is usually difficult to obtain. The genetic algorithm (GA) and simulated annealing algorithm (SA) was introduced in tolerance synthesis by Lee and Johnson (1993), Zhang and Wang (1998), and Ashiagobor et al. (1998), respectively. However, these require a large number of assembly analyses (GA: 30 samples per case, 100 cases per generation, and 300 generations results into 900,000 design samples (Lee and Johnson, 1993)). If MC simulations were used for each optimization iteration for assembly response analyses, the computational effort would be extremely high (in case of using 100,000 runs for tolerance analyses of one design sample, the total number of simulations will be as high as 9×10^{10}).

In general, all of the above applications require high accuracy of the obtained results, which further increases the required number of sampling points using MC-based approaches. In fact, the accuracy of the basic MC technique is proportional to the square root of the number of samples used. Thus, to increase the accuracy of Monte-Carlo calculations by a factor of 10, the number of samples must be increased by a factor of 100. The requirement of having very large sample sizes to run simulations makes MC method extremely time consuming and computation intensive, especially for large scale 2D or 3D complex assembly models in multistage manufacturing systems and large scale IC circuit yield analyses and defect tolerance design. Thus, due to the enormous amount of computational effort required, MC-based approaches are not easy to use for simulations of large-scale optimization algorithms in tolerance design.

Two techniques have been proposed in MC-based tolerancing to increase computational efficiency, *correlation and approximation function;* and *important sampling* (Skowronski and Turner, 1997). Similar techniques have also been applied in circuit yield prediction (Hocevar, Lightner, and Trick, 1983). The *importance sampling* technique is especially useful when there are regions of the probability space that are more important than others. In tolerance synthesis, this region would be located in the vicinity of the acceptance zone boundaries. However, constructing an importance function for such a region is usually difficult and needs a priori knowledge of the function.

An alternative approach for tolerance analyses based on number theory is proposed in this paper. The method is based on research in the area of number theory done by Fang and Wang (1994). To provide the necessary theoretical background, some relevant definitions and theorems are briefly reviewed in the Appendix. The proposed methodology: NT-net tolerance analyses will be presented in a few steps. First, the concept of discrepancy, which provides a measure for uniformity of a point set on a specific variable space domain, is introduced. The discrepancy is used to compare the convergence rate of various point sets from different sampling strategies (i.e., Monte Carlo etc.). Next, the good-lattice point (glp) and NT-net of points are defined followed by the method of generating glp and NT-net from the so-called generating vector. There are two theorems, which provide the milestones in the development. Theorem 1 is about the existence of glp and NT-net. It also provides a way to create a glp or NT-net in a unit cube domain. For the problems defined in a simplex variable domain, Theorem 2 provides a way to create glp in the simplex domain through a mapping process.

3. NT-net algorithm for tolerance design

The problem of reducing computational efforts in statistical simulation can be described as the process of exploring an optimum sampling strategy which can provide sampling point set uniformly scattered in a s-dimensional unite cube C^s. Dimensional variables fall in a feasible tolerance region, R^s, which is usually a simplex or a superrectangular domain. The first step in simulation is to generate sampling point set in a closed variable domain, R^s, which implies that the set of random points generated should be uniformly scattered or have good representation for the model (function) to be simulated.

It has been proved that the MC method is not the best one, and it has discrepancy of $O(n^{1/2}(\log\log n)^{1/2})$, or the variance of estimation will converge with an average rate of $1/\sqrt{n}$ (Kiefer, 1961). In contrast, the proposed NT-net method provides a better convergent rate of $O(1/n^{1-\varepsilon})$ for any given $\varepsilon > 0$ which results in a smaller discrepancy than that of MC. Geometrically, this implies that the MC random points are not uniformly scattered in a given variable domain, which is one of the major reasons as to why MC has a lower computational efficiency. The uniformly scattered set of points on C^s obtained by NT-net is called a number-theoretical net (NT-net).

The following steps are proposed to conduct tolerance analyses using the NT-net based approach:

Step 1. Select sample size n and generating vector $(n, h_1, \ldots, h_s)$.

Theorem 1 proved the existence of the generating vectors to create glp for NT-net. The generating vectors of the glp sets have been produced up to $s = 18$ (in case of $s > 18$ an alternative is described in step 2). Given n and s, the generating vector $(n, h_1, \ldots, h_s)$ can be found in the appendix presented in Fang and Wang (1994). Sample size n is the number of simulation points which should be selected according to the convergent requirement, i.e., it is suggested that n is 10% of sample size used in MC simulation, s is the number of design variables.

Step 2. Generate glp (or NT-net) on a unite cube C^s: $\{c_k\}$.

The definition of glp provides a simple method to generate the glp set when the related generating vector is known $(n, h_1, \ldots, h_s)$, where the set

$$x_{ki} = \frac{2q_{ki} - 1}{2n}, \quad q_{ki} = kh_i, (\text{if } kh_i > n, \text{ then let } q_{ki} = \text{mod}(kh_i, n), \\ k = 1, \ldots, n, i = 1, \ldots, s$$

forms the NT-net used in the tolerance simulation, i.e., the point set $\{(x_1, x_2, \ldots x_s)_k, k = 1, \ldots, n\} \in C^s$. It should be noted that there is a limitation on the number of variables: $s \leq 18$. To date, the generating vector for $s > 18$ is not available in the literature. A proposed alternative method for $s > 18$ is based on directly generating points set $\mathbf{x}$, also called NT-net, by using the circle division method (Fang and Wang, 1994):

$$\begin{aligned} \gamma &= (\gamma_1, \gamma_2, \ldots, \gamma_s) = \left(\left\{2\cos\frac{2\pi}{p}\right\}, \left\{2\cos\frac{4\pi}{p}\right\}, \ldots, \left\{2\cos\frac{2\pi s}{p}\right\}\right) \in C^s \\ \boldsymbol{x} &= (\{\boldsymbol{k}\gamma_1\}, \{\boldsymbol{k}\gamma_2\}, \ldots, \{\boldsymbol{k}\gamma_s\}) \\ &= \left(\left\{2\boldsymbol{k}\cos\frac{2\pi}{\boldsymbol{p}}\right\}, \left\{2\boldsymbol{k}\cos\frac{4\pi}{\boldsymbol{p}}\right\}, \ldots, \left\{2\boldsymbol{k}\cos\frac{2\pi \boldsymbol{s}}{\boldsymbol{p}}\right\}\right) \in C^s \\ &\qquad \boldsymbol{k} = 1, 2, \ldots, \boldsymbol{n}, \boldsymbol{s} > 18 \end{aligned}$$

where $\{x\}$ means taking the fixed point part of x, and where p is a selected arbitrary prime number such as $p > 2s + 3$. $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$ is called good point (gp), and $\mathbf{x}$ is called good point set.

Step 3. Map C^s to simplex A_s to generate NT-net $\{x_k\} = \{x_{k1}, \ldots, x_{ks}\}$ on A_s.

Assume that the tolerance range of all control variables forms the superrectangular simplex A_s as $A_s = \{(x_1, \ldots, x_s) : 0 \leq x_1 \leq \cdots \leq x_s \leq 1\}$. Let $x = (X_1, \ldots, X_s)$ be a random vector which is uniformly distributed on the simplex A_s. Let

$$\begin{cases} X_1 = \phi_1\phi_2 \cdots \phi_s \\ X_2 = \phi_2\phi_3 \cdots \phi_s \\ \cdots \\ X_{s-1} = \phi_{s-1}\phi_s \\ X_s = \phi_s \end{cases}$$

where $\phi = (\phi_1, \ldots, \phi_s) \in C^s$. This transformation maps C^s into A_s. It was proved by Fang and Wang (1994) that

(1) $\phi_1, \ldots, \phi_s$ are mutually independent;
(2) ϕ_j has pdf equal to $j\phi_j$ and cdf equal to $F_j(\phi) = \phi^j$, for $j - 1, \ldots, s$, where $0 \leq \phi \leq 1$; and
(3) The inverse function of $F_j(\phi)$ is $F_j^{-1}(\phi) = \phi^{1/j}$.

Suppose that $\{c_k\}$ is a NT-net on C^s, then $\{\mathbf{x}_k\}$ is a NT-net on A_s, where $\{\mathbf{x}_k\} = \{x_{k1}, \ldots, x_{ks}\}$, it can be expressed as

$$\begin{cases} \boldsymbol{x}_{k1} = \boldsymbol{c}_{k1}\boldsymbol{c}_{k2}^{1/2} \cdots \boldsymbol{c}_{ks}^{1/s}, \\ \boldsymbol{x}_{k2} = \boldsymbol{c}_{k2}^{1/2} \cdots \boldsymbol{c}_{ks}^{1/s}, \\ \cdots \quad \cdots \\ \boldsymbol{x}_{ks} = \boldsymbol{c}_{ks}^{1/s}, \end{cases} \qquad \text{where } k = 1, 2, \ldots, n.$$

Step 4. Generate the random samples of dimensional variables by their cdf.

Using the above NT-net and the inverse functions of cdf of dimensional variables, sample points set $\tilde{\boldsymbol{X}} = (\tilde{\boldsymbol{x}}_1, \tilde{\boldsymbol{x}}_2, \ldots, \tilde{\boldsymbol{x}}_s)$ in the design parameter domain can be generated following a similar procedure used in the Monte Carlo simulation.

Step 5. Conduct tolerance analyses by using the design (assembly) response function.

The random sample point set obtained in step 4 is entered into the design response function and is used to generate the data set of targeted dimensional responses. The population mean, variance as well as higher moments and the probability distribution function can be estimated. The whole tolerance analyses method is illustrated in Figure 1.

In the case of tolerance synthesis, additional steps are necessary to estimate the final yield and cost of allocated tolerances. When the assigned tolerances for each design variables

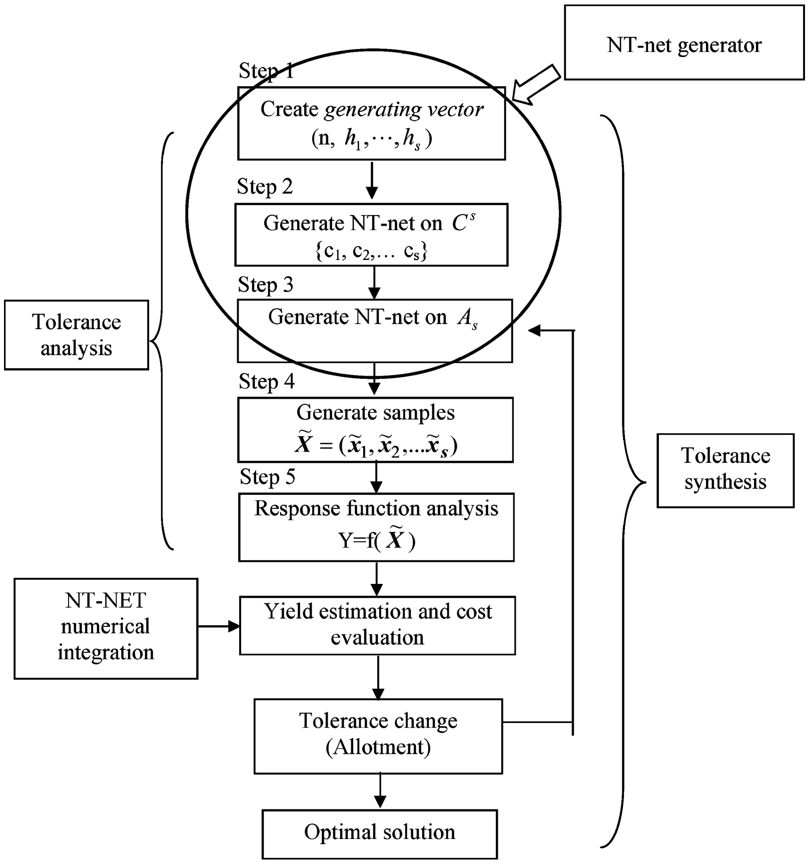

Figure 1. Flow chart of tolerance analyses/synthesis and simulation (the proposed NT-net replaces the Monte Carlo as the sample generator).

are predetermined, the probability that design variables x fall in the feasible region—the so-called yield can be represented as a multidimensional integral:

$$Y = \int_{x_{1l}}^{x_{1u}} \cdots \int_{x_{sl}}^{x_{sl}} k(x_1, \dots, x_s)\varphi(x_1 \cdots x_s)dx_1 \cdots dx_s.$$

Here $k(x_1, x_2, \dots, x_s)$ is defined as an indicator function, $k(\cdot) = 1$ when all constraints are satisfied otherwise $k(\cdot) = 0$. $\varphi(x_1, x_2, \dots, x_s)$ is a multivariate probability function of $(x_1, x_2, \dots, x_s)$. It is worth pointing out that this multiple integration will be very computational intensive or a solution cannot be reached when $s > 10$ by classical numerical integration techniques. An alternative option is to use the Monte Carlo integration algorithm. The basic idea of this algorithm is to transform the integral into a random simulation problem. The same idea has been utilized here by the NT-net method based simulation, which provides much better computational efficiency, and accuracy when compared with Monte Carlo based methods or other direct optimization algorithm (Zhou, Huang, and Zhang, 2001). The yield can thus be given simply by $Y = \frac{1}{n}\sum_1^n k(x_{i1} \ldots x_{is})\varphi(x_{i1} \ldots x_{is})$, where the sample points $x_i = (x_{i1} \ldots x_{is}),\ i = 1, 2, \ldots, n$ are NT-net generated by NT-net in step 3.

4. Verification of the effectiveness of NT-net in simulation

For verification purposes of the NT-net tolerance analyses method, the following five types of design response functions are considered: (a) polynomials; (b) hybrid of polynomial and trigonometric functions; (c) inverse trigonometric functions (for example, used in clutch design); (d) rational functions (for example, most of the frequency response functions used in circuit design); and (e) implicit nonlinear function (for example, function used for position variation control in design of mechanisms). Analyses by both the MC-based simulation and the proposed NT-net based approach are conducted for the aforementioned five types of design response functions. Normal distributions of design variables are assumed. The standard deviation of each variable is defined as 1/6 of the assigned tolerance range. The standard deviation and relative error by both MC and NT-net tolerancing approaches are compared. The relative errors are defined as $\sigma_R = \frac{|\sigma_S - \sigma_T|}{\sigma_T}$, the convergent rate of σ_R can be utilized to represent the efficiency of these two methods. In the presented cases σ_S is the simulated standard deviation by NT-net or MC. σ_T is the accurately estimated standard deviation by using very large sample size (n = 300,000).

4.1. Design response function of polynomial type

The polynomials have been selected to represent general nonlinear design response function. As shown in Table 1, polynomials up to 3rd order are studied. For convenience we assume

Table 1. Results of Monte Carlo simulations for polynomial type of design response functions.

Sample size n	Estimated $\hat{\mu}/\hat{\sigma}$	Design response functions					
		$a+b+c$	$a^2+b^2+c^2$	$a^3+b^3+c^3$	$ab+bc+ac$	abc	$a+ab+abc$
100	$\hat{\mu}$	11.0969	45.3818	201.8134	38.8797	43.2202	53.7783
	$\hat{\sigma}$	0.0320	0.2547	1.7697	0.2368	0.4230	0.5055
1,000	$\hat{\mu}$	11.1011	45.4201	202.0770	38.9073	43.2596	53.8211
	$\hat{\sigma}$	0.0352	0.2690	1.8329	0.2659	0.4852	0.5842
10,000	$\hat{\mu}$	11.0998	45.4087	201.9975	38.8988	43.2463	53.8063
	$\hat{\sigma}$	0.0351	0.2741	1.8850	0.2621	0.4729	0.5671
50,000	$\hat{\mu}$	11.1003	45.4131	202.0309	38.9019	43.2517	53.8124
	$\hat{\sigma}$	0.0349	0.2707	1.8587	0.2617	0.4741	0.5694
100,000	$\hat{\mu}$	11.1002	45.4122	202.0231	38.9012	43.2502	53.8107
	$\hat{\sigma}$	0.0347	0.2698	1.8551	0.2604	0.4718	0.5667
Accurate	μ	11.1000	45.4112	202.0183	38.9000	43.2480	53.8080
Solution	$\sigma(=\sigma_T)$	0.03461	0.26955	1.85509	0.25971	0.47037	0.56494

Table 2. Results by NT-net simulations for polynomial type of design response functions.

Sample size n	Estimated $\hat{\mu}/\hat{\sigma}$	Design response functions					
		$a+b+c$	$a^2+b^2+c^2$	$a^3+b^3+c^3$	$ab+bc+ac$	abc	$a+ab+abc$
101	$\hat{\mu}$	11.1003	45.4134	202.0317	38.9022	43.2518	53.8124
	$\hat{\sigma}$	0.0339	0.2649	1.8329	0.2547	0.4627	0.5558
597	$\hat{\mu}$	11.1003	45.4130	202.0277	38.9020	43.2514	53.8121
	$\hat{\sigma}$	0.0358	0.2764	1.8853	0.2685	0.4856	0.5827
1,010	$\hat{\mu}$	11.1006	45.4153	202.0393	38.9046	43.2557	53.8172
	$\hat{\sigma}$	0.0352	0.2737	1.8761	0.2634	0.4769	0.5723
5,037	$\hat{\mu}$	11.100	45.4112	202.0186	38.9000	43.2481	53.8081
	$\hat{\sigma}$	0.0347	0.2697	1.8558	0.2598	0.4705	0.5651
8,191	$\hat{\mu}$	11.1000	45.4112	202.0185	38.9000	43.2480	53.8081
	$\hat{\sigma}$	0.0346	0.2695	1.8547	0.2596	0.4703	0.5648
Accurate	μ	11.1000	45.4112	202.0183	38.9000	43.2480	53.8080
Solution	$\sigma(=\sigma_T)$	0.03461	0.26955	1.85509	0.25971	0.47037	0.56494

Table 3. Generating vectors ($s = 3$) for NT-net simulations.

n	h_1	h_2	h_3
101	1	40	85
597	1	63	169
1,010	1	140	237
5,037	1	580	1997
8,191	1	739	5515

that the three variables a, b and c follow normal distributions: $a \sim N(2.4, 0.02)$, $b \sim N(3.4, 0.02)$, and $c \sim N(5.3, 0.02)$. The NT-net on C^3 and A^3 are then generated following the steps proposed in Section 3. The polynomials used and the true values of mean and standard deviation obtained by moment method are given as the benchmark in the last row of Tables 1 and 2.

Different sample sizes n are used by the NT-net and MC tolerance method, since n in the NT-net method has to be a prime number. Yet, it does not affect the comparison results as shown in Tables 1 and 2. The generating vectors (n, h_1, h_2, h_3) used for NT-net are given in Table 3.

Estimation results of the standard deviation for different sample sizes n (100–100,000) and different polynomial functions by NT-net and MC techniques are shown in Tables 1 and 2. The convergent rate comparison of relative errors of standard deviations is given in Figures 2–7. Figures 2–7 show that the relative errors of NT-net converge much faster than that of MC simulation for all of the analyzed cases. For example, the results with the same accuracy, i.e., relative error of estimated standard deviation $\sigma_R \leq 0.1\%$, are achieved with

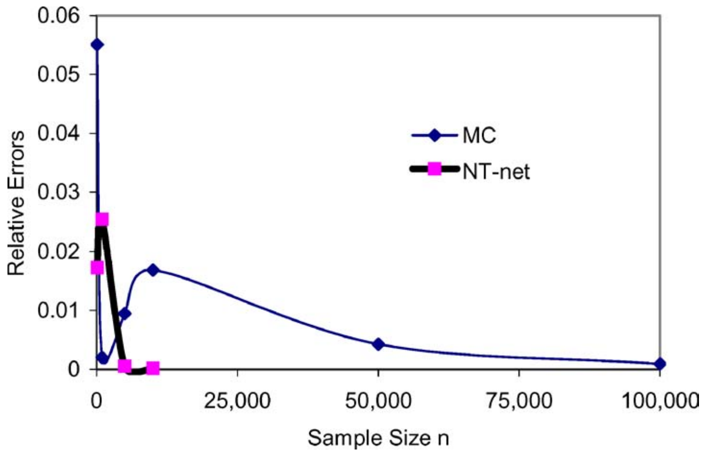

Figure 2. Relative errors σ_R of standard deviation for tolerance analyses of design response function: $a^2+b^2+c^2$.

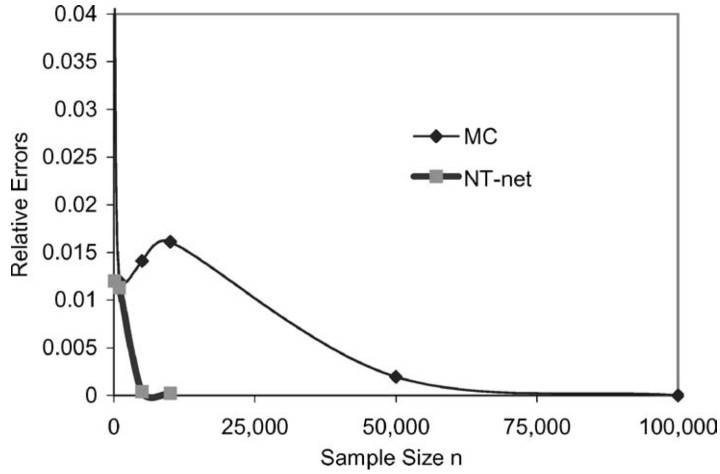

Figure 3. Relative errors σ_R of standard deviation for tolerance analyses of design response function: $a^3+b^3+c^3$.

a sample size of $n = 5{,}037$ by using NT-net and with sample size of $n = 100{,}000$ by using the MC-based method. The NT-net method used only about 5% of the computational efforts needed by the MC-based simulations.

4.2. Design response function of hybrid type

A hybrid-type design response function of combined polynomial and trigonometric functions are defined as follows:

$$F(x, y, z, t) = \sin x + \cos y + \sin z * \sin t + x^2 + xy + y^3 + zt.$$

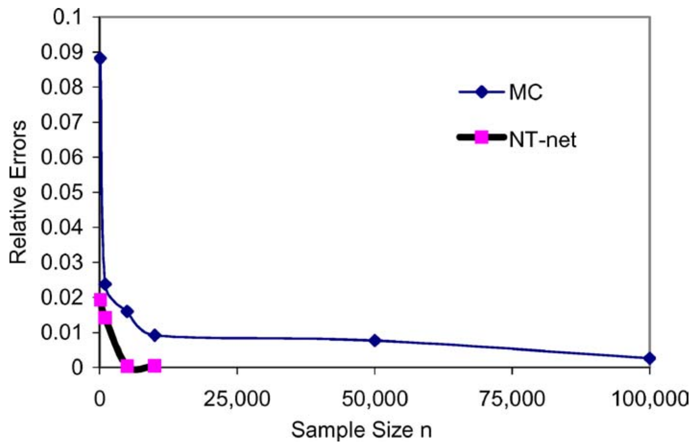

Figure 4. Relative errors σ_R of standard deviation for tolerance analyses of design response function: $ab + bc + ac$.

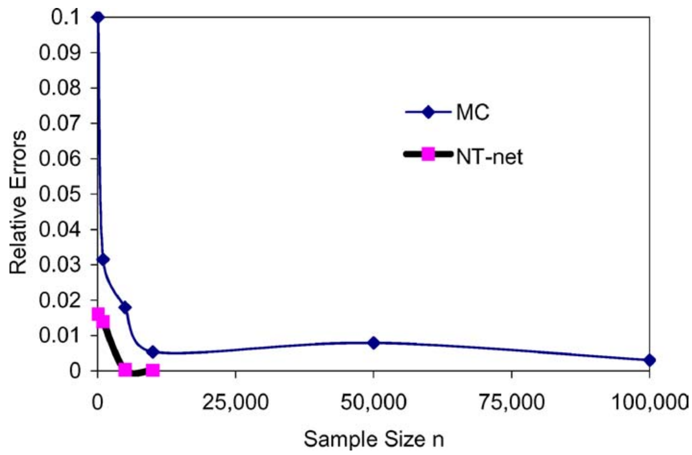

Figure 5. Relative errors σ_R of standard deviation for tolerance analyses of design response function: abc.

All design variables are assumed to follow normal distribution: $x \sim N(2.5, 0.02)$, $y \sim N(3.0, 0.03)$, $z \sim N(4.2, 0.035)$, $t \sim N(2.8, 0.025)$. The generating vectors are given in Table 4.

For hybrid functions the comparisons have been made between the simulation results (mean and standard deviation of assembly/design response function) obtained by the NT-net and Monte Carlo method with the same sample sizes. It is difficult to obtain the theoretical values for the generic hybrid function. In this case we use the average of the estimates of standard deviation obtained by the MC and by the NT-net methods at $n = 10{,}007$ (for NT-net) and $n = 100{,}000$ (for MC). Since the results of MC-based simulation are unstable, we have used three runs to demonstrate the fluctuations as shown in Figures 8–10. The

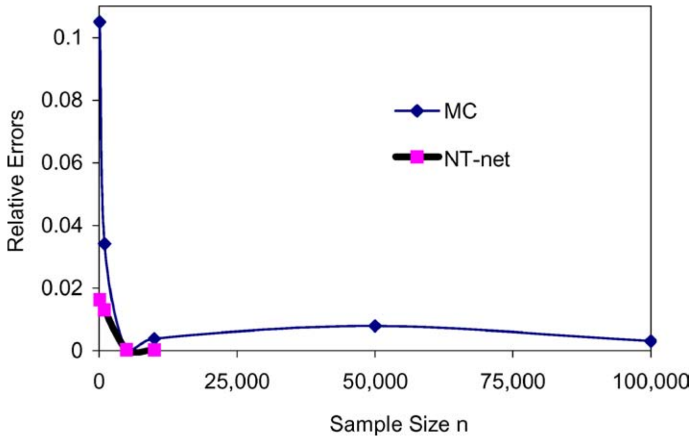

Figure 6. Relative errors σ_R of standard deviation for tolerance analyses of design response function: $a + ab + abc$.

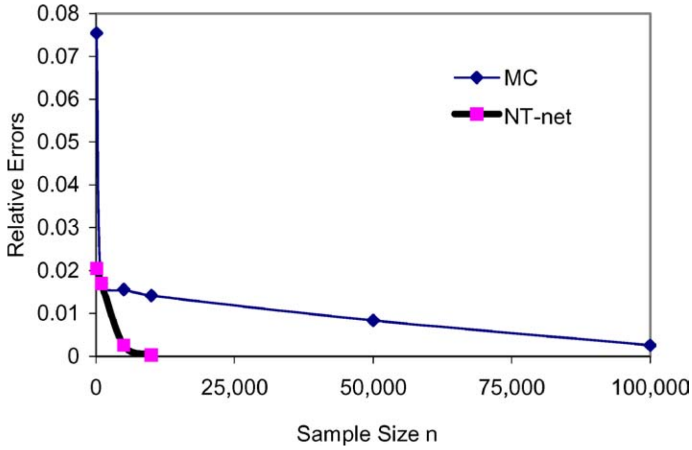

Figure 7. Relative errors σ_R of standard deviation for tolerance analyses of design response function: $a + b + c$.

NT-net method shows the similar high computational efficiency as presented in Section 4.1, i.e., only 5% of computational efforts is needed to achieve the same accuracy (relative error of estimated standard deviation $\sigma_R \leq 0.1\%$) and the obtained results show much higher stability. In comparison the fluctuation of the results obtained by using the MC-based method can be as large as 1% even for a sample size equal to $n = 100{,}000$ (Figure 10).

4.3. Design response function of inverse trigonometric type

The inverse trigonometric function is often used as a design response function, for example, in design of mechanical clutches and other power train systems. The following arccosine

Table 4. Generating vectors ($s = 4$) for NT-net simulations.

n	h_1	h_2	h_3	h_4
562	1	53	89	221
3,001	1	174	266	1,269
5,003	1	792	1,889	191
10,007	1	1,206	3,421	2,842

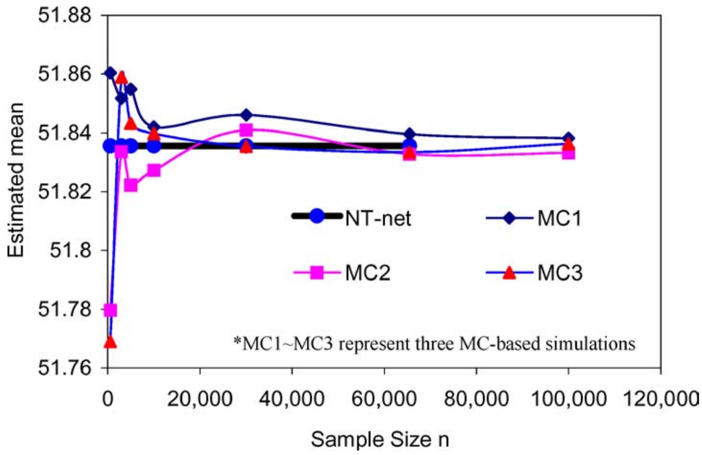

Figure 8. Comparison of estimated mean for hybrid type design response function: $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) = \sin \boldsymbol{x} + \cos \boldsymbol{y} + \sin \boldsymbol{z} * \sin \boldsymbol{t} + \boldsymbol{x}^2 + \boldsymbol{xy} + \boldsymbol{y}^3 + \boldsymbol{zt}$.

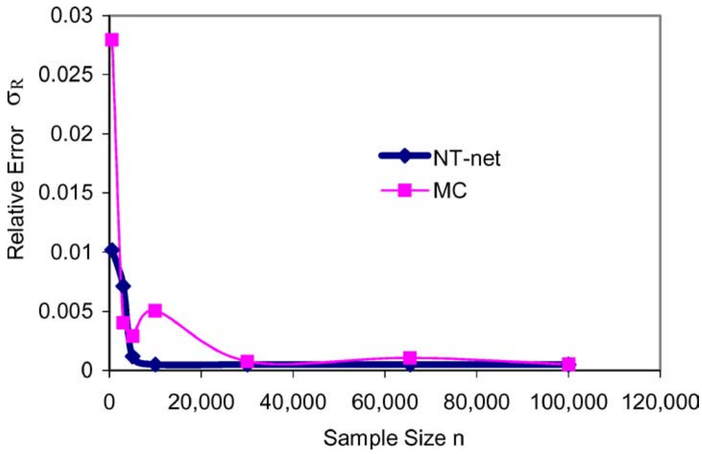

Figure 9. Comparison of relative errors σ_R of standard deviation for hybrid type design response function: $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) = \sin \boldsymbol{x} + \cos \boldsymbol{y} + \sin \boldsymbol{z} * \sin \boldsymbol{t} + \boldsymbol{x}^2 + \boldsymbol{xy} + \boldsymbol{y}^3 + \boldsymbol{zt}$.

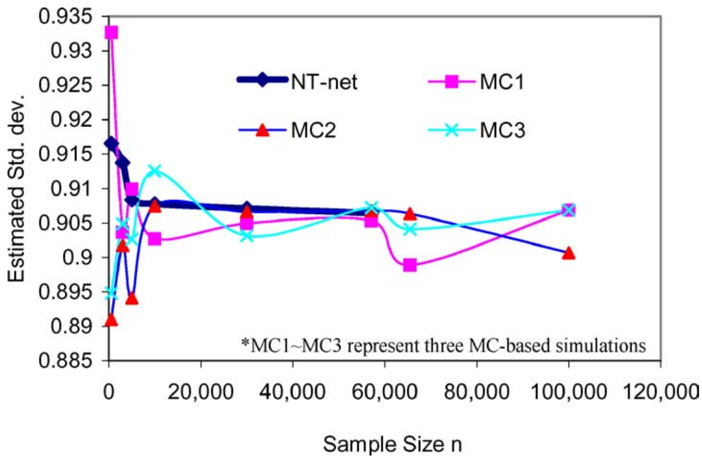

Figure 10. Comparison of standard deviation $\hat{\sigma}$ for hybrid type design response function: $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t}) = \sin \boldsymbol{x} + \cos \boldsymbol{y} + \sin \boldsymbol{z} * \sin \boldsymbol{t} + \boldsymbol{x}^2 + \boldsymbol{xy} + \boldsymbol{y}^3 + \boldsymbol{zt}$.

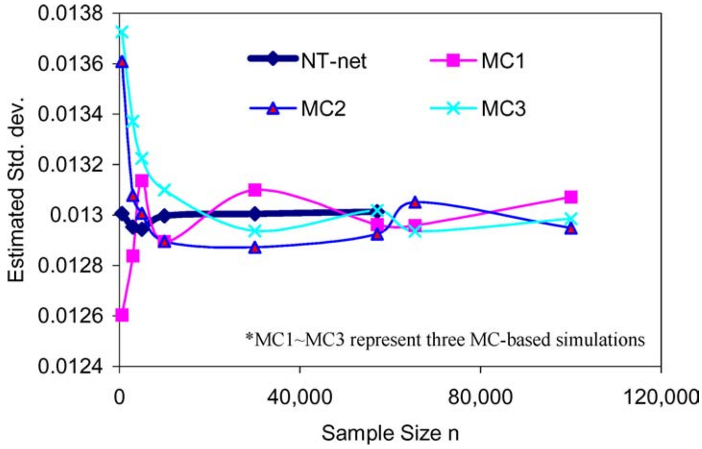

Figure 11. Comparison of standard deviation $\hat{\sigma}$ for the design response function: $F(x, y, z, t) = \arccos(\frac{x+y+z}{t+y+z})$.

function is selected for comparative simulation of NT-net and MC-based method:

$$F(x, y, z, t) = \arccos\left(\frac{x + y + z}{t + y + z}\right).$$

The variables distribution and generating vector are the same as in Section 4.2, the generating vectors are shown in Table 4. Both the mean and the standard deviation estimation results are used for comparison. The simulations show that computational efficiency of NT-net is about 10–20 times better than the MC-based simulation to achieve the same accuracy. The detailed results are presented in Figures 11 and 12.

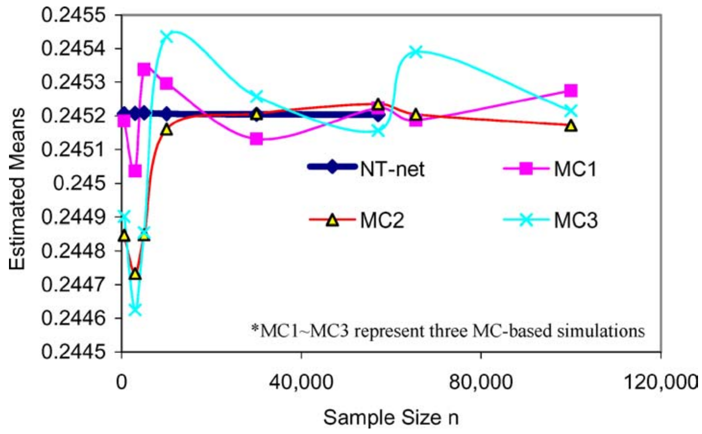

Figure 12. Comparison of means $\hat{\mu}$ for the design response function: $F(x, y, z, t) = \arccos(\frac{x+y+z}{t+y+z})$.

4.4. *Case study 1: Tolerance analyses for dual-amplifier bandpass filter (DABF) design*

In analog circuit design, circuits are designed to ensure that their frequency response functions (design response function in electronic circuit design) meet the specifications in frequency domain. Given the tolerances of all inputs and their corresponding distributions, the yield can be obtained by using random simulation techniques. In general, tremendous computational efforts are needed to conduct simulations on a number of discrete points in the frequency domain with a highly nonlinear (frequency response) function, which are usually rational functions.

The dual-amplifier bandpass filter design is used for comparative analyses of the NT-net and MC-based methods. The studied circuit is shown in Figure 13 with the following parameters values:

$$R_1 = 377.7 \pm 3.777 \text{ k}\Omega, \quad R_2 = R_3 = 5343 \pm 53.43 \ \Omega, \quad C = 0.01 \pm 0.0001 \ \mu\text{F},$$
$$f_r = 2978.9 \text{ Hz}.$$

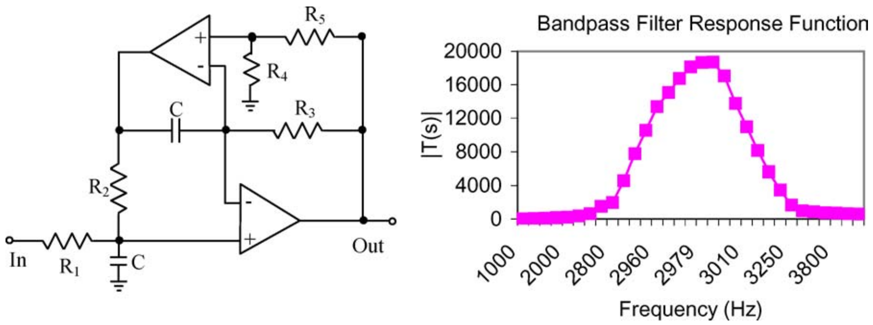

Figure 13. Dual-amplifier bandpass filter.

The frequency response function of the circuit is the following complex rational polynomial:

$$T(s) = \frac{s^2/R_1C}{s^2 + s/R_1C + 1/R_2R_3C^2}, \quad s = j2\pi f.$$

For comparison purposes, only the amplitude of $T(s)$ is considered on a single frequency point $f_r = 2978.9$ Hz. The natural logarithm of the norm of the complex function, i.e., $\ln(|T(s)|)$ is taken as the design response function. All parameters R_1–R_3 and C are assumed to be normally distributed, the 3σ tolerance ranges of the parameters are given above. The generating vectors are shown in Table 4. Estimated mean and standard diviation are obtained by both MC and proposed NT-net method. The more stable results are obtained and much less runs are needed by NT-net, i.e., about 1/10–1/20 of runs needed for the MC-based method, as shown in Figures 4–17.

4.5. Case study 2: Tolerance analyses for four-bar mechanism

This is a typical tolerance analyses problem for a kinematics path calculation of a mechanism. The goal is to analyze the position variation of a specified point P on the linkage given the dimensional tolerances of the four components of the analyzed mechanism (bar lengths and position). By continuously changing the position of the mechanism the entire tolerance envelope around the nominal path can be obtained. An example of the four-bar mechanism used in the simulations is shown in Figure 18. The position of point P(P_x, P_y) located on linkage r_3 depends on the dimensions of bars r_1–r_4 and its location on the linkage r_3 (denoted in local coordinate system as (t_3, t_4)). The length of linkages r_1–r_4 is assumed

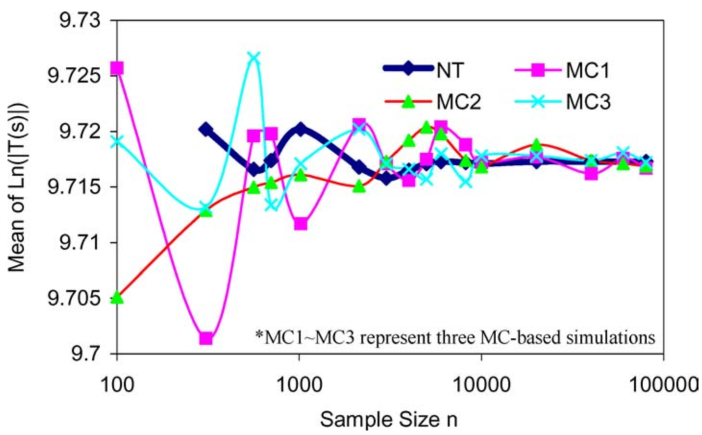

Figure 14. Comparison of means $\hat{\mu}_{\ln(|T(s)|)}$ of the design response function (complex rational polynomial) of a dual-amplifier bandpass filter.

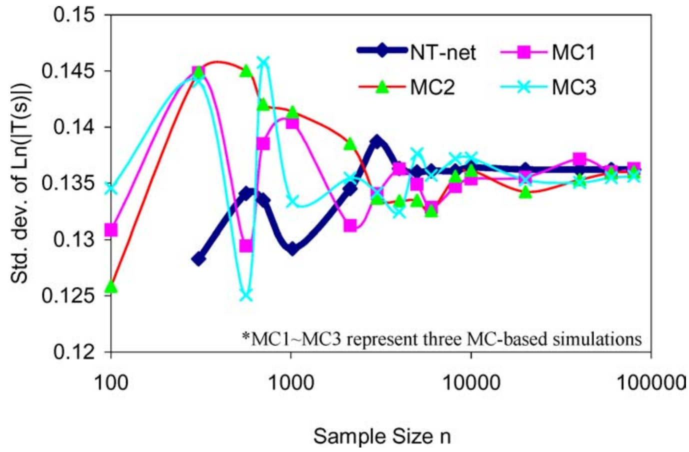

Figure 15. Comparison of standard deviations of $\ln(|T(s)|)$ of the design response function (rational polynomial) for a dual-amplifier bandpass filter.

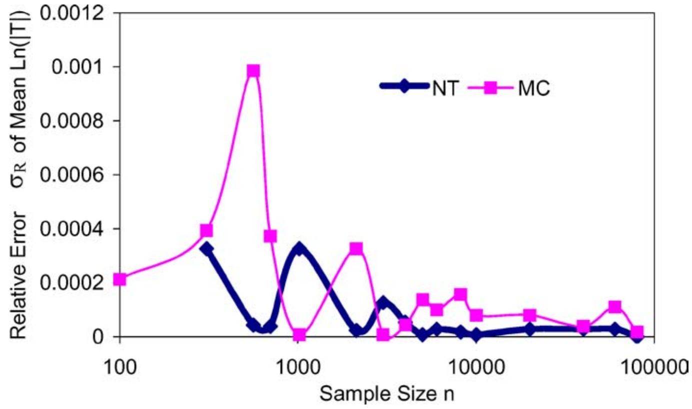

Figure 16. Comparison of relative errors σ_R of means of the design response function of (rational polynomial) a dual-amplifier bandpass filter.

to be independent and follows normal distributions with nominal values equal to their mean length and 1/6 of tolerance range equal to one sigma (standard deviation). The variations of point P in the global coordinate system have been obtained by using simulation.

The location of point P can be described by the vector loop equation: $\vec{r}_2 + \vec{r}_3 - \vec{r}_1 - \vec{r}_4 = 0$, which can be written as

$$
\begin{aligned}
r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_1 \cos\theta_1 - r_4 \cos\theta_4 = 0 \\
r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_1 \sin\theta_1 - r_4 \sin\theta_4 = 0
\end{aligned}
$$

Table 5. Performance comparison of Monte Carlo and NT-net simulation.

	Performance indices			
	Computational efforts[a]		Numerical stability	
Types of design response function	Monte Carlo ε	NT-net ε	Monte Carlo	NT-net
Polynomials	100%	$\varepsilon \approx 5\%$	Poor	Very good
Hybrid (trigonometric and polynomial)	100%	$\varepsilon \approx 5\%$	Poor	Very good
Inverse trigonometric	100%	$\varepsilon < 5\%$	Poor	Very good
Complex rational polynomial	100%	$\varepsilon < 5\%$	Poor	Very good
Mechanism (implicit function)	100%	$\varepsilon < 10\%$	Poor	Very good

[a]Percentage of number of simulation runs to achieve the same accuracy (σ_R) by taking MC as a benchmark ($\varepsilon = 100\%$).

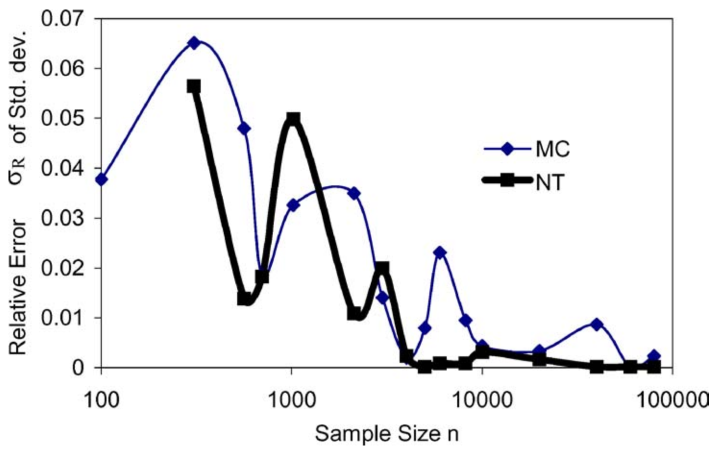

Figure 17. Comparison of relative error σ_R of the standard deviations of $\ln(|T(s)|)$ for the design response function (rational polynomial) of a dual-amplifier bandpass filter.

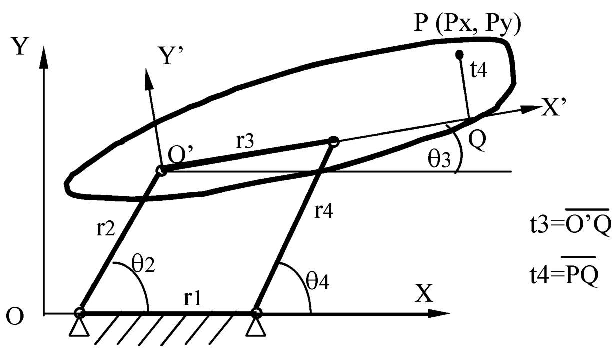

Figure 18. Four-bar mechanism.

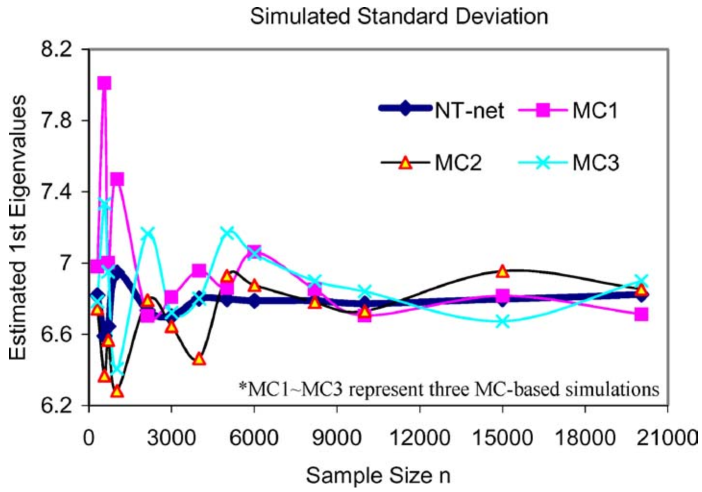

Figure 19. Principal component analyses of four-bar mechanism (the 1st eigenvalues of P(P_x, P_y) for different sample size n).

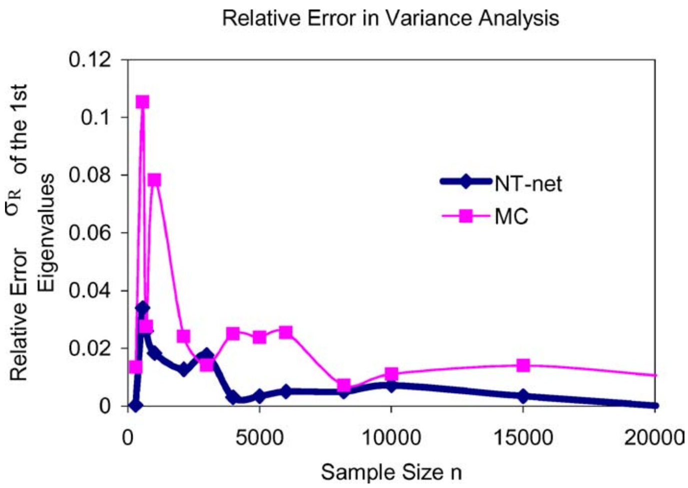

Figure 20. Relative error σ_R of the 1st eigenvalues of the position P(P_x, P_y) for a four-bar mechanism.

Let $\theta_1 = 0, \theta_2 = 50$ (degrees), $t_3 = 135, t_4 = 50, r_1 = 60 \pm 0.5, r_2 = 100 \pm 1, r_3 = 120 \pm 0.5, r_4 = 160 \pm 1.5$. Then, the θ_3, θ_4 can be obtained from the above equations. The generating vectors for NT-net generation are given in Table 4. And the location of P is

$$P_x = r_2 \cos\theta_2 + t_3 \cos\theta_3 + t_4 \cos(\theta_3 + \pi/2)$$
$$P_y = r_2 \sin\theta_2 + t_3 \sin\theta_3 + t_4 \cos(\theta_3 + \pi/2).$$

The point P(P_x, P_y) forms a bivariate population. The principal component analyses has been conducted to characterize the variation in both coordinates P_x and P_y. The 1st and 2nd eigenvalues of covariance matrices are estimated by using NT-net and Monte Carlo simulations to represent the variations in principal directions. The results show that the computational effort needed by NT-net for the same accuracy (relative error of estimated standard deviation $\sigma_R \leq 1\%$) is about 1/10 of that needed by the MC-based method. Additionally, the obtained results are more stable (fluctuation of estimated standard deviation by NT-net method for sample size of $n \geq 3000$ is about 1/5 of that produced by using MC-based simulations), as shown in Figures 19–22. The summary of simulation results for all case studies is presented in Table 5.

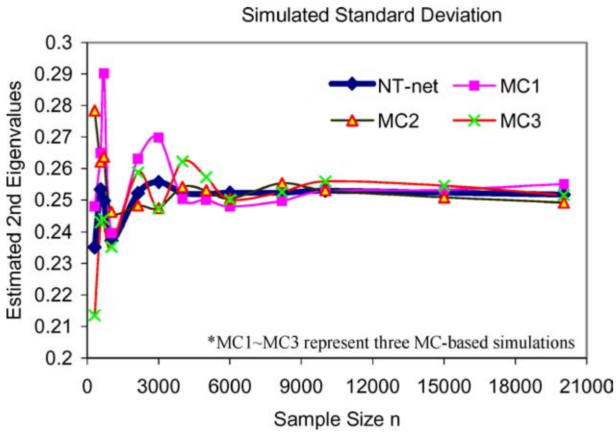

Figure 21. Principal component analyses of four-bar mechanism (the 2nd eigenvalues of P(P_x, P_y) for different sample size *n*).

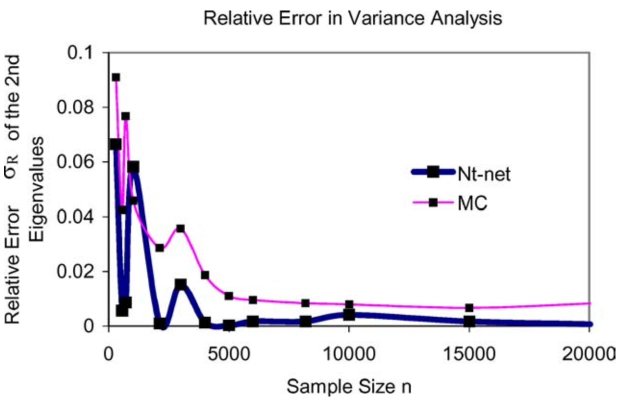

Figure 22. Relative error σ_R of the 2nd eigenvalues of the position P(P_x, P_y) for a four-bar mechanism.

5. Conclusions

The number-theory (NT-net) based tolerance analyses method is proposed and demonstrated in this article with the potential application towards statistical simulation of engineering design tolerances. The proposed method is especially applicable for tolerance simulations of complex products and multistage manufacturing systems. The proposed method can be used alternatively with the Monte Carlo (MC) based method in design tolerance simulations; however, the NT-net tolerance approach allows for achieving the same accuracy of simulations with significantly smaller sample size when compared with the MC-based approaches.

The article presents the NT-net methodology and then provides a comparative analyses of both the MC and NT-net methods using five different design response functions used in design of mechanical and electrical systems. The simulation results for all the case studies are summarized in Table 5. The comparative analyses shows that the sample size needed by the NT-net approach is about 5–10% of that necessary by the MC-based method to achieve the same level of accuracy (Table 5). In addition, the MC-based methodology shows larger random fluctuations of results, while the results of NT-net are much more consistent and stable. It seems that the NT-net simulations will show similar results when compared with the MC-based methods for all design response functions, which are (i) polynomials, (ii) trigonometric, or (iii) rational. Additionally, similar to the MC method, there are no limitations regarding variable distributions and model complexity for the NT net method, so that the NT-net is as versatile as the MC-based method.

In addition, the proposed method can also be used with high computational efficiency for (i) yield prediction and circuit simulation and (ii) solving numerically multidimensional integrals, which are frequently encountered in tolerance synthesis for yields calculation. As such tolerance synthesis usually involves large-scale direct optimization with implicit functions and the NT-net shows a promising alternative in this aspect as well.

Appendix: Background on the NT-net approach

A.1. Discrepancy

Definition. Let $P = \{x_k, k = 1, \ldots, n\}$ represent a set of points on C^s. For any $\gamma \in C^s$, let $N(\gamma, P)$ be the number of the points satisfying $x_k \leq \gamma$. Then $D(n, P) = \sup_{\gamma \in C^s} |\frac{N(\gamma,P)}{n} - v([0, \gamma])|$ is called the discrepancy of *P*, where $v([0, \gamma]) = \gamma_1, \ldots, \gamma_s$ denotes the volume of the super-rectangle $[0, \gamma]$.

Intuitively the $D(n, P)$ in the definition represents how the point set P scatters in C^s. It is a measure of the uniformity of a point set. The difference of the percentage of points fall in $v([0, \gamma])$, and the percentage of the volume ratio of $v([0, \gamma])$ to $C^s (= 1)$ represents the uniformity of points distribution. For example, the fact that 30% of points from the point set P falls in the super-rectangle precisely corresponds to the volume $v([0, \gamma]) = 0.3$, thus, we get zero discrepancy or the point set P scatters evenly.

A.2. The lattice point, good-lattice points (glp set) and NT-net

Definition. The set $P_n = \{x_k = \{x_{k1}, \ldots, x_{ks}\}, k = 1, \ldots, n\}$ is called the *lattice point* set of the generating vector $(n, h_1, \ldots, h_s)$, if P_n is generated from the vector $(n; h_1, \ldots, h_s)$ with integral components: $1 \leq h_i < n, h_i \neq h_j (i \neq j), s < n$ and the greatest common divisors $(n, h_i) = 1, i = 1, \ldots, s$ by

$$\begin{cases} q_{ki} \equiv kh_i(\text{mod} n) \\ x_{ki} = (2q_{ki} - 1)/2n, \end{cases} \quad k = 1, \ldots, n, \quad i = 1, \ldots, s,$$

where q_{ki} is confined by $1 \leq q_{ki} \leq n$ through the usual multiplicative operation modulo n. Then, a set of points is defined as the glp set when it has the smallest discrepancy among the sets of points generated by all possible generating vectors.

Theorem 1. *For any given prime number* n, *there exists an integral vector* $h_p = (h_1, \ldots, h_s)$ *such that the lattice point set of* $(n; h_1, \ldots, h_s)$ *has discrepancy* $D(n, P) < c(s)n^{-1}(\log n)^s$, *where* $c(s)$ *is a constant, which depends on* s. *The net on* C^s *constructed from* P *is called NT-net.*

The existence of glp set is given by this theorem. It ensures that the NT-net has the smallest discrepancy or scatters more uniformly among all other potential point sets (MC point set, for example) with the same sample size. The generating vectors of $s \leq 18$ have been provided in the literature giving us a way to create NT-net to replace MC point set in simulation.

A.3. NT-net on a simplex A_s

The set of points in tolerance analyses/synthesis usually falls into a simplex (a super-rectangle domain) instead of an s-dimensional unit tolerance cube C^s. For example, if the dimensions x_i's tolerance range is defined by $[x_{li}, x_{ui}], i = 1, 2, \ldots s, x_{li} \neq x_{lj}, x_{ui} \neq x_{uj}$, for $i \neq j$, then the "tolerance cubic" forms a simplex. Hence, an inverse transform method is used to map the set of points between the domain of C^s and the simplex A_s which can produce uniformly scattered points on A_s. The inverse transform method is given by the following definition and Theorem 2.

Definition. Let $\mathbf{x} \in R^s$ be a random vector with a c.d.f. $F(x)$ and has a stochastic representation $x = h(z)$, where $z \sim$ uniform distribution on $C^t, t \leq s$. Let $\{c_k, k = 1, \ldots, n\}$ be a NT-net on C^t with discrepancy d. Then, we understand the set $P_f = \{h(c_k), k = 1, \ldots, n\}$ to have quasi F-discrepancy d with respect to $F(x)$.

It has been proven in number theory that the quasi F-discrepancy d is the measure for uniformity of $\mathbf{x}$ on $D \in R^s$ and the quasi F-discrepancy d is indeed the discrepancy d on $D \in R^s$, then we can conveniently get the NT-net on $D \in R^s$ through a mapping by using the following theorem.

Theorem 2. *Let D be a closed and bounded domain in R^s and $\mathbf{x} \sim$ uniform distribution on D. Suppose $\mathbf{x}$ has a stochastic representation $\mathbf{x} = h(\phi)$, where ϕ is a t-dimensional random vector with independent marginal pdf $p_i(\phi_i)$ and cdf $F_i(\phi_i)$. Let $P = \{c_k\}$ be a set on C^t with discrepancy d. Then the set $P_f = \{h(c_k), k = 1, \ldots, n\}$ has a quasi F-discrepancy d which equals to $GD(P_D, D) = \sup_{r \in C^t} |\frac{N(P_f, G_r)}{n} - F(r)|$, where $N(P_f, G_r)$ denotes the number of points $x = h(\phi)$ falling in G_r, and G_r is a subdomain: $G_r = \{x : x = h(\phi), \phi \leq r\}$, and $F(r) = \prod_{i=1}^{t} F_i(r_i)$.*

Obviously, the $GD(P_D, D)$ is the discrepancy of the point set $P_f = \{h(c_k), k = 1, \ldots, n\}$. By applying Theorem 2 to some specific domain D, uniformly scattered NT-nets on D can be obtained.

Acknowledgments

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