Lecture 1: Basic Models of Growth

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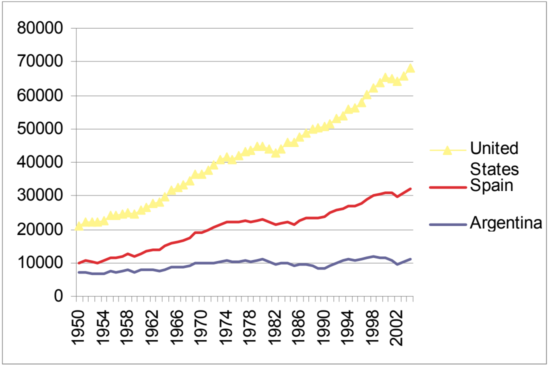
2. Physical Capital per worker grows over time
3. The growth rate of output per worker differs substantially across countries

Different paths of Growth

- 

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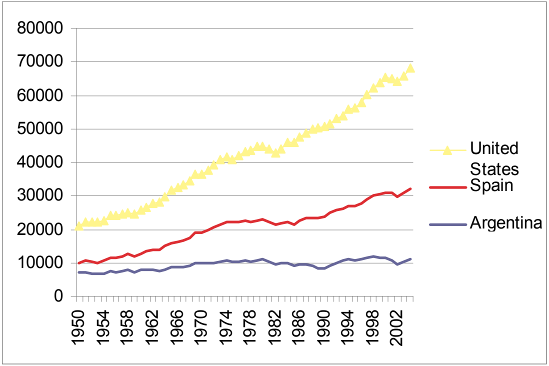
- 

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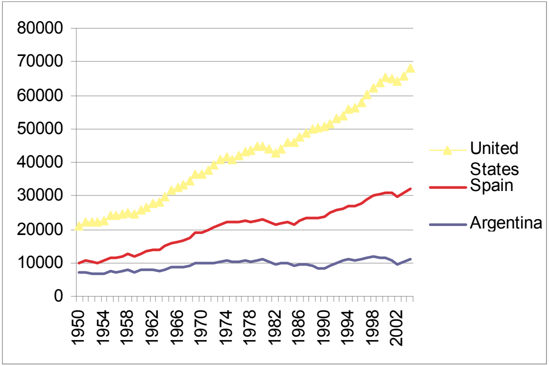

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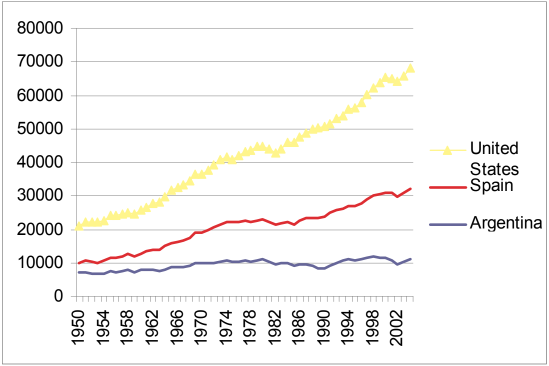

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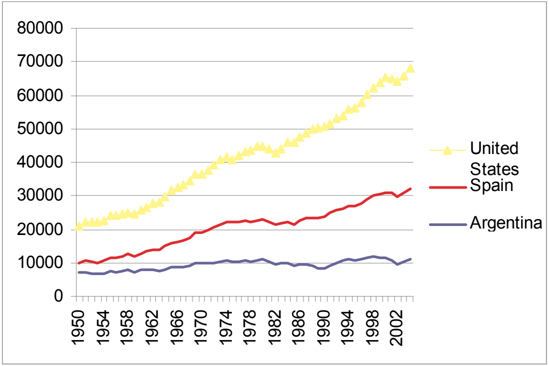
- 

- Kaldor facts do not apply to stagnating countries
- Macroeconomic growth consider growing countries
- Development and growth proceed ed separately
- Generating a unique model for growth and development is still far away

Ramsey-Samuelson model

- Household behavior

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- Firms' Behavior
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  - increasing and concave in K and $\hat{L}$,
  - Constant Return to scale:

$$F(\lambda K, \lambda \hat{L}) = \lambda F(K, \hat{L})$$

Static Equilibria

- Household Optimal choice:

$$\max_c U(c)$$

$$w + ra - c - na \geq 0$$

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  - Euler Condition

$$r = \rho + [\frac{-u''(c)c}{u'(c)}] * \dot{\hat{c}}/\hat{c}$$

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  - with $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$ (CIES):

$$\dot{\hat{c}}/\hat{c} = (1/\theta)(r - \rho)$$

Static Equilibrium (cont'd)

- Firms' optimal Choice

$$\max F(K, \hat{L}) - (r+\delta)K - wL =$$
$$\max f(\hat{k}) - (r+\delta)\hat{k} - \frac{w}{T(t)}$$

Static Equilibrium (cont'd)

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- FOCs

$$f'(\hat{k}) = r + \delta$$
$$[f(\hat{k}) - \hat{k}f'(\hat{k})]e^{xt} = w$$

Steady state

- Dynamics ($k = a$)

$$\dot{\hat{k}} = f(\hat{k}) - \hat{c} - (x + n + \delta)\hat{k}$$
$$\dot{\hat{c}}/\hat{c} = (1/\theta)(f'(k) - \delta - \rho - \theta x)$$
$$\lim\{a(t)\exp[-\int_0^t [r(v) - n]dv\} = 0$$

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- Equilibrium

$$\begin{aligned} \dot{\hat{c}} &= 0 \rightarrow f'(k^*) = \delta + \rho + \theta x \\ \dot{\hat{k}} &= 0 \rightarrow \hat{c}^* = f(\hat{k}^*) - (x + n + \delta)\hat{k}^* \end{aligned}$$

with

$$\frac{\dot{\hat{y}}}{\hat{y}} = \alpha \frac{\dot{\hat{k}}}{\hat{k}} = 0$$

Steady state analysis

- Growth in steady state

$$
\begin{aligned}
\frac{\dot{\hat{c}}}{\hat{c}} &= \frac{\partial\left(\frac{C(t)}{e^{(n+x)t}}\right)}{\partial t} \Big/ \frac{C(t)}{e^{(n+x)t}} = \dot{c} - x = 0 \\
\frac{\dot{\hat{k}}}{\hat{k}} &= \frac{\partial\left(\frac{K(t)}{e^{(n+x)t}}\right)}{\partial t} \Big/ \frac{K(t)}{e^{(n+x)t}} = \dot{k} - x = 0 \\
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\end{aligned}
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Steady state analysis

- Growth in steady state

$$\frac{\dot{\hat{c}}}{\hat{c}} = \frac{\partial\left(\frac{C(t)}{e^{(n+x)t}}\right)}{\partial t} / \frac{C(t)}{e^{(n+x)t}} = \dot{c} - x = 0$$

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$$\frac{\dot{\hat{y}}}{\hat{y}} = \dot{y} - x = \alpha\frac{\dot{\hat{k}}}{\hat{k}} = 0$$

- y (per capita income) in steady state grows with $T(t) = e^{xt}$
  GROWTH IS EXOGENOUS

AK model

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  - $r = A - \delta$

- Dynamics ($k = a$)

$$\dot{k} = (A - \delta - n) - c/k$$
$$\dot{c}/c = (1/\theta)(A - \delta - \rho)$$
$$\lim\{k(t)e^{-(A-\delta-\rho)t}\} = 0$$

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- Equilibrium

$$\begin{aligned} \dot{c}/c &= cons \\ \dot{k}/k &= \dot{c}/c \\ \dot{y}/y &= \dot{k}/k = cons \end{aligned}$$

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- Define $A = f(H/K)$ and we obtain a AK model

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$$F_1(k, K) = f(K/k) - f'(K/k)\frac{K}{k_i^2}k_i = f(L) - f'(L)L$$

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  private marginal product of capital is non decreasing in k..
- $\dot{c}/c = (1/\theta)[f(L) - Lf'(L) - \delta - \rho]$ constant
- Generates long run growth