Lecture Notes 8: Dynamic Optimization
Part 1: Calculus of Variations

Peter J. Hammond

Revised 2018 September 25th
typeset from `calcVar18.tex`

Outline

Vickrey–Mirrlees Model

Problem: how much to pay workers of different skills.

Goal: achieve fairness while preserving incentives.

References: William S. Vickrey (1945)
"Measuring Marginal Utility by Reactions to Risk"
Econometrica 13: 319–333.

James A. Mirrlees (1971)
"An Exploration in the Theory of Optimal Income Taxation"
Review of Economic Studies 38: 175–208.

Let $n \in \mathbb{R}_+$ denote a person's skill level, defined to mean that there is a constant rate of marginal substitution of n_1/n_2 between hours of work supplied by workers of the two skill levels n_1 and n_2.

Thus, a worker's productivity is proportional to n, personal skill.

Assume that the distribution of workers' skills can be described by a continuous density function $\mathbb{R}_+ \ni n \mapsto f(n) \in \mathbb{R}_+$ which, like a probability density function, satisfies $\int_0^\infty f(n) dn = 1$.

Objective and Constraints

"Macro" model with a "representative consumer/worker"
whose preferences for consumption/labour supply pairs $(c, \ell) \in \mathbb{R}^2_+$
are represented by the utility function $u(c) - v(\ell)$,
where $u' > 0$, $v' > 0$, $u'' < 0$ $v'' > 0$.

The **social objective** is to maximize
the utility integral $\int_0^\infty [u(c(n)) - v(\ell(n))] f(n) dn$
w.r.t. the functions $\mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}^2_+$.

The **resource balance constraint** takes the form $C \leq F(L)$ where

- $C := \int_0^\infty c(n) f(n) dn$ is mean consumption;
- $L := \int_0^\infty n\ell(n) f(n) dn$ is mean effective labour supply.

The aggregate production function $\mathbb{R}_+ \ni L \mapsto F(L) \in \mathbb{R}_+$
is assumed to satisfy $F'(L) > 0$ and $F''(L) \leq 0$ for all $L \geq 0$.

Pseudo First-Order Conditions

Consider the Lagrangian

$$\begin{aligned}\mathcal{L}(c(\cdot),\ell(\cdot)) := &\int_0^\infty [u(c(n)) - v(\ell(n))] f(n)\,dn \\ &- \lambda \left[\int_0^\infty c(n) f(n)\,dn - F\left(\int_0^\infty n\ell(n) f(n)\,dn \right) \right]\end{aligned}$$

as a functional (rather than a mere function) of the functions $\mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}^2_+$.

We derive "pseudo" first-order conditions by pretending that the derivatives $\dfrac{\partial \mathcal{L}}{\partial c(n)}$ and $\dfrac{\partial \mathcal{L}}{\partial \ell(n)}$ both exist, for all $n \geq 0$.

This gives the pseudo first-order conditions

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial c(n)} = [u'(c(n)) - \lambda] f(n) \\ 0 &= \frac{\partial \mathcal{L}}{\partial \ell(n)} = -v'(\ell(n)) f(n) + \lambda F'(L) n f(n) \end{aligned}$$

Marxist First Best

For any skill level n such that $f(n) > 0$, these two equations

$$0 = [u'(c(n)) - \lambda]f(n) = -v'(\ell(n))f(n) + \lambda F'(L)nf(n)$$

imply that:

- $u'(c(n)) = \lambda$ and so $c(n) = c^*$,
  where the constant c^* uniquely solves $u'(c^*) = \lambda$
  ("to each according to their need");
- $v'(\ell(n)) = \lambda F'(L)n$, implying that $v''(\ell(n)) \cdot \dfrac{d\ell}{dn} = \lambda F' > 0$,
  so $\dfrac{d\ell}{dn} > 0$ ("from each according to their ability")

Exercise

Use concavity arguments to prove
that this is the (essentially unique) solution.

What makes this solution practically infeasible?

Sufficiency Theorem: Statement

Theorem

Suppose that there exists $\lambda > 0$
such that c^* *and the function* $\mathbb{R}_+ \ni n \mapsto \ell^*(n)$
jointly satisfy the first-order conditions:

$$u'(c^*) = \lambda \quad \textit{and} \quad v'(\ell^*(n)) = \lambda F'(L^*) n \quad \textit{for all } n \in \mathbb{R}_+$$

where $c^* = F(L^*)$ *and* $L^* = \int_0^\infty n \ell^*(n) f(n) \, dn$.

Let $\mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}_+^2$
be any other policy satisfying $C = F(L)$
where $C = \int_0^\infty c(n) f(n) \, dn$ *and* $L = \int_0^\infty n \ell(n) f(n) \, dn$.

Then

$$\int_0^\infty [u(c(n)) - v(\ell(n))] f(n) dn \leq u(c^*) - \int_0^\infty v(\ell^*(n)) f(n) dn$$

with strict inequality unless $c(n) = c^*$ *wherever* $f(n) > 0$.

Sufficiency Theorem: Proof, I

Because $u'' < 0$ and so u is strictly concave,
the supergradient property of concave functions implies that

$$u(c(n)) - u(c^*) \leq u'(c^*)[c(n) - c^*] = \lambda[c(n) - c^*]$$

for all n, with strict inequality unless $c(n) = c^*$.

Integrating gives $\int_0^\infty [u(c(n)) - u(c^*)]f(n)\,dn \leq \lambda(C - c^*)$,
with strict inequality unless $c(n) = c^*$ wherever $f(n) > 0$.

Similarly, because $v'' \geq 0$ and so v is convex, for all n
the subgradient property of convex functions implies that

$$v(\ell(n)) - v(\ell^*(n)) \geq v'(\ell^*(n))[\ell(n) - \ell^*(n)] = \lambda F'(L^*)[\ell(n) - \ell^*(n)]$$

Integrating gives

$$\int_0^\infty [v(\ell(n)) - v(\ell^*(n))]f(n)\,dn \geq \lambda F'(L^*)(L - L^*)$$

Sufficiency Theorem: Proof, II

Subtracting the second inequality from the first, then rearranging, one has

$$\begin{aligned} & \textstyle\int_0^\infty \{[u(c(n)) - v(\ell(n))] - [u(c^*) - v(\ell^*(n))]\} f(n)\, dn \\ \leq \quad & \lambda[(C - c^*) - F'(L^*)(L - L^*)] \end{aligned}$$

Next, because $F'' \leq 0$ and so F is concave, one has

$$C - c^* = F(L) - F(L^*) \leq F'(L^*)(L - L^*)$$

Finally, because $\lambda > 0$, it follows that

$$\int_0^\infty [u(c(n)) - v(\ell(n))] f(n)\, dn \leq \int_0^\infty [u(c^*) - v(\ell^*(n))] f(n)\, dn$$

as required for $\mathbb{R}_+ \ni n \mapsto (c^*, \ell^*(n)) \in \mathbb{R}^2_+$ to be optimal. $\square$

Outline

Problem Formulation

The calculus of variations is used to optimize a functional that maps functions into real numbers.

A typical problem is to choose a function $[t_0, t_1] \ni t \mapsto x(t) \in \mathbb{R}$, often denoted simply by $\mathbf{x}$, in order to maximize the integral objective functional

$$J(\mathbf{x}) = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t)) dt$$

subject to the fixed end point conditions $x(t_0) = x_0$, $x(t_1) = x_1$.

A variation involves moving away from the first path $\mathbf{x}$ to the variant path $\mathbf{x} + \epsilon\mathbf{u}$, where $\mathbf{u}$ denotes the differentiable function $[t_0, t_1] \ni t \mapsto u(t) \in \mathbb{R}$, and $\epsilon \in \mathbb{R}$ is a scalar.

To ensure that the end point conditions $x(t_0) + \epsilon u(t_0) = x_0$ and $x(t_1) + \epsilon u(t_1) = x_1$ remain satisfied by $\mathbf{x} + \epsilon\mathbf{u}$, one imposes the conditions $u(t_0) = u(t_1) = 0$.

Toward a Necessary First-Order Condition

A maximum is a path $\mathbf{x}^*$ satisfying the end point conditions such that $J(\mathbf{x}^*) \geq J(\mathbf{x})$ for any alternative path $\mathbf{x}$ that also the satisfies the end point conditions.

A necessary condition for $\mathbf{x}^*$ to maximize $J(\mathbf{x})$ w.r.t. $\mathbf{x}$ is that $J(\mathbf{x}^*) \geq J(\mathbf{x}^* + \epsilon\mathbf{u})$ for all small ϵ.

Alternatively, the function

$$\mathbb{R} \ni \epsilon \mapsto f_{\mathbf{x}^*,\mathbf{u}}(\epsilon) := J(\mathbf{x}^* + \epsilon\mathbf{u})$$

must satisfy, for all small ϵ, the inequality

$$f_{\mathbf{x}^*,\mathbf{u}}(0) = J(\mathbf{x}^*) \geq J(\mathbf{x}^* + \epsilon\mathbf{u}) = f_{\mathbf{x}^*,\mathbf{u}}(\epsilon)$$

In case the function $\epsilon \mapsto f_{\mathbf{x}^*,\mathbf{u}}(\epsilon)$ is differentiable at $\epsilon = 0$, a necessary first-order condition is therefore $f'_{\mathbf{x}^*,\mathbf{u}}(0) = 0$.

Evaluating the Derivative

Our definitions of the functions J and $f_{\mathbf{x}^*,\mathbf{u}}$ imply that

$$f_{\mathbf{x}^*,\mathbf{u}}(\epsilon) = J(\mathbf{x}^* + \epsilon\mathbf{u}) = \int_{t_0}^{t_1} F(t, x^*(t) + \epsilon u(t), \dot{x}^*(t) + \epsilon \dot{u}(t))dt$$

Differentiating the integrand w.r.t. ϵ at $\epsilon = 0$ implies that

$$f'_{\mathbf{x}^*,\mathbf{u}}(0) = \int_{t_0}^{t_1} [F'_x(t)u(t) + F'_{\dot{x}}(t)\dot{u}(t)]dt$$

where for each $t \in [t_0, t_1]$, the partial derivatives $F'_x(t)$ and $F'_{\dot{x}}(t)$ of $F(t, x, \dot{x})$ are evaluated at the triple $(t, x^*(t), \dot{x}^*(t))$.

Integrating by Parts

The product rule for differentiation implies that

$$\frac{d}{dt}[F'_{\dot{x}}(t)u(t)] = \left[\frac{d}{dt}F'_{\dot{x}}(t)\right] u(t) + F'_{\dot{x}}(t)\dot{u}(t)$$

and so, integrating by parts, one has

$$\int_{t_0}^{t_1} F'_{\dot{x}}(t)\dot{u}(t)dt = |_{t_0}^{t_1} F'_{\dot{x}}(t)u(t) - \int_{t_0}^{t_1} \left[\frac{d}{dt}F'_{\dot{x}}(t)\right] u(t)dt$$

But the end point conditions imply that $u(t_0) = u(t_1) = 0$, so the first term on the right-hand side vanishes.

The Euler Equation

Substituting $-\int_{t_0}^{t_1} \left[\frac{d}{dt} F'_{\dot{x}}(t)\right] u(t) dt$ for the term $\int_{t_0}^{t_1} F'_{\dot{x}}(t)\dot{u}(t)dt$ in the equation $f'_{\mathbf{x}^*,\mathbf{u}}(0) = \int_{t_0}^{t_1} [F'_x(t)u(t) + F'_{\dot{x}}(t)\dot{u}(t)]dt$, then recognizing the common factor $u(t)$, we finally obtain

$$f'_{\mathbf{x}^*,\mathbf{u}}(0) = \int_{t_0}^{t_1} \left[F'_x(t) - \frac{d}{dt} F'_{\dot{x}}(t)\right] u(t) dt$$

The first-order condition is $f'_{\mathbf{x}^*,\mathbf{u}}(0) = 0$
for every differentiable function $t \mapsto u(t)$
satisfying the two end point conditions $u(t_0) = u(t_1) = 0$.

This condition holds
iff the integrand is zero for (almost) all $t \in [t_0, t_1]$,
which is true iff the Euler equation $\frac{d}{dt} F'_{\dot{x}}(t) = F'_x(t)$
holds for (almost) all $t \in [t_0, t_1]$.

Outline

Are We Saving Too Little?

Kenneth Arrow, Gretchen Daily, Partha Dasgupta, Paul Ehrlich, Lawrence Goulder, Geoffrey Heal, Simon Levin, Karl-Göran Mäler, Stephen Schneider, David Starrett and Brian Walker (2004) "Are We Consuming Too Much?" *Journal of Economic Perspectives* 18: 147–172.

Macroeconomic variation of the Solow–Swan growth model.

Given a capital stock K, output Y is given by
the production function $Y = f(K)$, where $f' > 0$, and $f'' \leq 0$.

Net investment = gross investment, without depreciation.

So given capital K and consumption C, investment I is given by

$$I = \dot{K} = f(K) - C$$

The Ramsey Problem and Beyond

The economy's intertemporal objective is taken to be

$$\int_0^T e^{-rt} U(C(t)) dt = \int_0^T e^{-rt} U(f(K) - \dot{K}) dt$$

Frank Ramsey (EJ, 1928) assumed $T = \infty$ (infinite horizon) and $r = 0$ (no discounting).

Nicholas Stern (of the *Stern Report on Climate Change*) and others advocate:

- $T = \infty$;
- r as the hazard rate in a Poisson process that determines when extinction occurs; this implies that e^{-rt} is the probability that the human race has not become extinct by time t.

Chichilnisky, Hammond, and Stern (2018) TWERPS 1174

Applying the Calculus of Variations

We apply the calculus of variations
to the objective $\int_0^T e^{-rt} U(f(K) - \dot{K}) dt$
with the end conditions $K(0) = \bar{K}$, which is exogenous,
and $K(T) = 0$ at the finite time horizon T.

Euler's equation takes the form $\frac{d}{dt} F'_{\dot{K}}(t) = F'_K(t)$
where $F(t, K, \dot{K}) = e^{-rt} U(f(K) - \dot{K}) = e^{-rt} U(C)$.

So Euler's equation becomes $\frac{d}{dt}[-e^{-rt} U'(C)] = e^{-rt} U'(C) f'(K)$.

Equivalently, after evaluating the time derivative,

$$-U''(C)\dot{C}e^{-rt} + rU'(C)e^{-rt} = e^{-rt}U'(C)f'(K)$$

Cancelling the common factor e^{-rt} and dividing by $U'(C) > 0$,
then rearranging, one obtains

$$-\frac{U''(C)}{U'(C)}\dot{C} = f'(K) - r$$

Further Interpretation

Define the (negative) elasticity of marginal utility as

$$\eta(C) := -\frac{d \ln U'(C)}{d \ln C} = -\frac{U''(C)C}{U'(C)}$$

This is related to the curvature of the utility function,
and to how quickly marginal utility $U'(C)$ decreases as C increases.

Rearranging the equation $-U''(C)\dot{C}/U'(C) = f'(K) - r$ yet again,
one obtains the equation

$$\eta(C)\frac{\dot{C}}{C} = f'(K) - r$$

whose left hand side is the proportional rate of consumption growth
multiplied by: (i) the elasticity of marginal utility;
or (ii) the elasticity of an intertemporal MRS;
or (iii) the degree of relative fluctuation aversion.

Final Recommendation

Morton I. Kamien and Nancy L. Schwartz (2012)
Dynamic Optimization, Second Edition:
The Calculus of Variations and Optimal Control
in Economics and Management (Dover Publications)