

Lecture Notes 8: Dynamic Optimization

Part 1: Calculus of Variations

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Outline

Introduction

Vickrey–Mirrlees Model of Optimal Income Taxation

Typical Calculus of Variations Problem

Economic Application

Vickrey–Mirrlees Model

Problem: how much to pay workers of different skills.

Goal: achieve fairness while preserving incentives.

References: William S. Vickrey (1945)

“Measuring Marginal Utility by Reactions to Risk”

Econometrica 13: 319–333.

James A. Mirrlees (1971)

“An Exploration in the Theory of Optimal Income Taxation”

Review of Economic Studies 38: 175–208.

Let $n \in \mathbb{R}_+$ denote a person’s skill level, defined to mean that there is a constant rate of marginal substitution of n_1/n_2 between hours of work supplied by workers of the two skill levels n_1 and n_2 .

Thus, a worker’s productivity is proportional to n , personal skill.

Assume that the distribution of workers’ skills can be described by a continuous density function $\mathbb{R}_+ \ni n \mapsto f(n) \in \mathbb{R}_+$ which, like a probability density function, satisfies $\int_0^\infty f(n)dn = 1$.

Objective and Constraints

We consider a “macro” model with a “representative consumer/worker”, whose preferences for consumption/labour supply pairs $(c, \ell) \in \mathbb{R}_+^2$ are represented by the utility function $u(c) - v(\ell)$, where $u' > 0$, $v' > 0$, $u'' < 0$, and $v'' > 0$.

The problem is to maximize a **social objective** by choosing the pair of functions $\mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}_+^2$.

The maximand is “mean utility” which is specified by the **utility integral** $\int_0^\infty [u(c(n)) - v(\ell(n))]f(n) \, dn$.

The **resource balance constraint** takes the form $C \leq F(L)$ where

- ▶ $C := \int_0^\infty c(n)f(n)dn$ is **mean consumption**;
- ▶ $L := \int_0^\infty n \ell(n)f(n)dn$ is **mean effective labour supply**.

The **aggregate production function** $\mathbb{R}_+ \ni L \mapsto F(L) \in \mathbb{R}_+$ is assumed to satisfy $F'(L) > 0$ and $F''(L) \leq 0$ for all $L \geq 0$.

Pseudo First-Order Conditions

Consider the Lagrangian

$$\mathcal{L}(c(\cdot), \ell(\cdot)) := \int_0^{\infty} [u(c(n)) - v(\ell(n))] f(n) dn - \lambda \left[\int_0^{\infty} c(n) f(n) dn - F \left(\int_0^{\infty} n \ell(n) f(n) dn \right) \right]$$

as a **functional** (rather than a mere function) of the functions $\mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}_+^2$.

We derive “pseudo” first-order conditions by pretending that, for all $n \geq 0$, the derivatives $\frac{\partial \mathcal{L}}{\partial c(n)}$ and $\frac{\partial \mathcal{L}}{\partial \ell(n)}$ both exist.

For each $n \geq 0$, this gives the two pseudo first-order conditions

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial c(n)} = [u'(c(n)) - \lambda] f(n) \\ 0 &= \frac{\partial \mathcal{L}}{\partial \ell(n)} = -v'(\ell(n)) f(n) + \lambda F'(L) n f(n) \end{aligned}$$

The Marxist First Best

Karl Marx (1875) *Critique of the Gotha Programme*

For any skill level n such that $f(n) > 0$, we have the two equations

$$0 = [u'(c(n)) - \lambda]f(n) = -v'(\ell(n))f(n) + \lambda F'(L)nf(n)$$

These imply that, for any skill level n with $f(n) > 0$, we want:

- ▶ $u'(c(n)) = \lambda$ and so $c(n) = c^*$,
where the constant c^* uniquely solves $u'(c^*) = \lambda$
("to each according to their need");
- ▶ $v'(\ell(n)) = \lambda F'(L)n$, implying that $v''(\ell(n)) \cdot \frac{d\ell}{dn} = \lambda F' > 0$,
so $\frac{d\ell}{dn} > 0$ ("from each according to their ability")

Exercise

Use concavity and convexity arguments to prove that this is the (essentially unique) solution.

What makes this solution practically infeasible?

Sufficiency Theorem: Statement

Theorem

Suppose that there exists $\lambda > 0$
such that c^* and the function $\mathbb{R}_+ \ni n \mapsto \ell^*(n)$
jointly satisfy the first-order conditions:

$$u'(c^*) = \lambda \quad \text{and} \quad v'(\ell^*(n)) = \lambda F'(L^*)n \quad \text{for all } n \in \mathbb{R}_+$$

where $c^* = F(L^*)$ and $L^* = \int_0^\infty n \ell^*(n) f(n) dn$.

Let $\mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}_+^2$
be any other policy satisfying $C = F(L)$
where $C = \int_0^\infty c(n) f(n) dn$ and $L = \int_0^\infty n \ell(n) f(n) dn$.

Then

$$\int_0^\infty [u(c(n)) - v(\ell(n))] f(n) dn \leq u(c^*) - \int_0^\infty v(\ell^*(n)) f(n) dn$$

with strict inequality unless $c(n) = c^*$ wherever $f(n) > 0$.

Sufficiency Theorem: Proof, I

Because $u'' < 0$ and so u is strictly concave, the supergradient property of concave functions implies that

$$u(c(n)) - u(c^*) \leq u'(c^*)[c(n) - c^*] = \lambda[c(n) - c^*]$$

for all n , with strict inequality unless $c(n) = c^*$.

Integrating this inequality gives the **first integral inequality**

$$\int_0^\infty [u(c(n)) - u(c^*)] f(n) \, dn \leq \lambda(C - c^*)$$

with strict inequality unless $c(n) = c^*$ wherever $f(n) > 0$.

Similarly, because $v'' \geq 0$ and so v is convex, for all n the subgradient property of convex functions implies that

$$v(\ell(n)) - v(\ell^*(n)) \geq v'(\ell^*(n))[\ell(n) - \ell^*(n)] = \lambda F'(L^*)[\ell(n) - \ell^*(n)]$$

Integrating this inequality gives the **second integral inequality**

$$\int_0^\infty [v(\ell(n)) - v(\ell^*(n))] f(n) \, dn \geq \lambda F'(L^*)(L - L^*)$$

Sufficiency Theorem: Proof, II

We have derived the following two integral inequalities:

$$\int_0^{\infty} [u(c(n)) - u(c^*)] f(n) \, dn \leq \lambda(C - c^*)$$
$$\int_0^{\infty} [v(\ell(n)) - v(\ell^*(n))] f(n) \, dn \geq \lambda F'(L^*)(L - L^*)$$

Subtracting the second integral inequality from the first, then rearranging, one has

$$D := \int_0^{\infty} \{[u(c(n)) - v(\ell(n))] - [u(c^*) - v(\ell^*(n))]\} f(n) \, dn$$
$$\leq \lambda[(C - c^*) - F'(L^*)(L - L^*)]$$

Note that: (i) $C \leq F(L)$, by feasibility; (ii) $c^* = F(L^*)$;
(iii) because $F'' \leq 0$ and so F is concave, one has $F(L) - F(L^*) \leq F'(L^*)(L - L^*)$.

It follows that $C - c^* \leq F(L) - F(L^*) \leq F'(L^*)(L - L^*)$ and so

$$C - c^* - F'(L^*)(L - L^*) \leq 0$$

Sufficiency Theorem: Proof, III

Because $\lambda > 0$, the above definition of D implies that

$$\begin{aligned} D &= \int_0^\infty \{[u(c(n)) - v(\ell(n))] - [u(c^*) - v(\ell^*(n))]\} f(n) \, d n \\ &\leq \lambda[(C - c^*) - F'(L^*)(L - L^*)] \leq 0 \end{aligned}$$

This proves that no feasible policy $\mathbb{R}_+ \ni n \mapsto (c(n), \ell(n)) \in \mathbb{R}_+^2$ can yield more mean utility $\int_0^\infty \{[u(c(n)) - v(\ell(n))]\} f(n) \, d n$ than the mean utility $\int_0^\infty \{[u(c^*) - v(\ell^*(n))]\} f(n) \, d n$ which the policy $\mathbb{R}_+ \ni n \mapsto (c^*, \ell^*(n)) \in \mathbb{R}_+^2$ yields.

The latter policy is therefore optimal. □

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Problem Formulation

The calculus of variations is used to optimize a **functional** that maps functions into real numbers.

A typical problem is to choose a **path** \mathbf{x} , in the form of a function $[t_0, t_1] \ni t \mapsto \mathbf{x}(t) \in \mathbb{R}$, in order to maximize the integral **objective functional**

$$J(\mathbf{x}) = \int_{t_0}^{t_1} F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) dt$$

subject to the **fixed end point conditions** $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{x}(t_1) = \mathbf{x}_1$.

A **variation** involves moving away from the first path \mathbf{x} to the **variant path** $\mathbf{x} + \epsilon \mathbf{u}$, where \mathbf{u} denotes the differentiable function $[t_0, t_1] \ni t \mapsto \mathbf{u}(t) \in \mathbb{R}$, and $\epsilon \in \mathbb{R}$ is a (small) scalar.

To ensure that the end point conditions $\mathbf{x}(t_0) + \epsilon \mathbf{u}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) + \epsilon \mathbf{u}(t_1) = \mathbf{x}_1$ remain satisfied by $\mathbf{x} + \epsilon \mathbf{u}$, one imposes the conditions $\mathbf{u}(t_0) = \mathbf{u}(t_1) = 0$.

Toward a Necessary First-Order Condition

A **maximum** of the integral objective $J(\mathbf{x}) = \int_{t_0}^{t_1} F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) dt$ is a path \mathbf{x}^* or function $[t_0, t_1] \ni t \mapsto \mathbf{x}^*(t) \in \mathbb{R}^n$:

(i) that satisfies the end point conditions $\mathbf{x}^*(t_0) = \mathbf{x}_0$, $\mathbf{x}^*(t_1) = \mathbf{x}_1$;

(ii) with the property that $J(\mathbf{x}^*) \geq J(\mathbf{x})$

for any alternative path $\mathbf{x} = (\mathbf{x}(t))_{t \in [t_0, t_1]}$

that also satisfies the end point conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{x}(t_1) = \mathbf{x}_1$.

A necessary condition for \mathbf{x}^* to maximize $J(\mathbf{x})$ w.r.t. \mathbf{x}

is that $J(\mathbf{x}^*) \geq J(\mathbf{x}^* + \epsilon \mathbf{u})$ for all small ϵ .

Alternatively, the function

$$\mathbb{R} \ni \epsilon \mapsto f_{\mathbf{x}^*, \mathbf{u}}(\epsilon) := J(\mathbf{x}^* + \epsilon \mathbf{u}) \in \mathbb{R}$$

must satisfy, for all small ϵ , the inequality

$$f_{\mathbf{x}^*, \mathbf{u}}(0) = J(\mathbf{x}^*) \geq J(\mathbf{x}^* + \epsilon \mathbf{u}) = f_{\mathbf{x}^*, \mathbf{u}}(\epsilon)$$

In case the function $\epsilon \mapsto f_{\mathbf{x}^*, \mathbf{u}}(\epsilon)$ is differentiable at $\epsilon = 0$, a necessary first-order condition is therefore $f'_{\mathbf{x}^*, \mathbf{u}}(0) = 0$.

Evaluating the Derivative

Our definitions of the functions J and $f_{\mathbf{x}^*, \mathbf{u}}$ imply that

$$f_{\mathbf{x}^*, \mathbf{u}}(\epsilon) = J(\mathbf{x}^* + \epsilon \mathbf{u}) = \int_{t_0}^{t_1} F(t, \mathbf{x}^*(t) + \epsilon \mathbf{u}(t), \dot{\mathbf{x}}^*(t) + \epsilon \dot{\mathbf{u}}(t)) dt$$

By Leibnitz's formula, the derivative $f'_{\mathbf{x}^*, \mathbf{u}}(0)$ w.r.t. ϵ at $\epsilon = 0$ of the function $\mathbb{R} \ni \epsilon \mapsto f_{\mathbf{x}^*, \mathbf{u}}(\epsilon) = J(\mathbf{x}^* + \epsilon \mathbf{u}) \mapsto \mathbb{R}$ equals the integral of the derivative of the integrand.

It follows that $f'_{\mathbf{x}^*, \mathbf{u}}(0) = \int_{t_0}^{t_1} [F'_x(t)u(t) + F'_{\dot{x}}(t)\dot{u}(t)] dt$ where, for each $t \in [t_0, t_1]$, the partial derivatives $F'_x(t)$ and $F'_{\dot{x}}(t)$ of $F(t, x, \dot{x})$ are evaluated at the triple $(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t))$.

Integrating by Parts

The product rule for differentiation implies that

$$\frac{d}{dt}[F'_{\dot{x}}(t)u(t)] = \left[\frac{d}{dt} F'_{\dot{x}}(t) \right] u(t) + F'_{\dot{x}}(t)\dot{u}(t)$$

and so, integrating by parts, one has

$$\int_{t_0}^{t_1} F'_{\dot{x}}(t)\dot{u}(t)dt = \left. F'_{\dot{x}}(t)u(t) \right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left[\frac{d}{dt} F'_{\dot{x}}(t) \right] u(t) dt$$

But the end point conditions imply that $u(t_0) = u(t_1) = 0$, so the first term on the right-hand side vanishes.

The Euler Equation

Substituting $-\int_{t_0}^{t_1} \left[\frac{d}{dt} F'_x(t) \right] u(t) dt$ for the term $\int_{t_0}^{t_1} F'_x(t) \dot{u}(t) dt$ in the equation $f'_{x^*,u}(0) = \int_{t_0}^{t_1} [F'_x(t)u(t) + F'_x(t)\dot{u}(t)] dt$, then recognizing the common factor $u(t)$, we finally obtain

$$f'_{x^*,u}(0) = \int_{t_0}^{t_1} \left[F'_x(t) - \frac{d}{dt} F'_x(t) \right] u(t) dt$$

The **first-order condition** is $f'_{x^*,u}(0) = 0$ for **every** differentiable function $t \mapsto u(t)$ satisfying the two end point conditions $u(t_0) = u(t_1) = 0$.

This condition holds if and only if, for (almost) all $t \in [t_0, t_1]$, the integrand is zero, or equivalently, if and only if the **Euler equation** $\frac{d}{dt} F'_x(t) = F'_x(t)$ holds.

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Are We Saving Too Little?

Kenneth Arrow, Gretchen Daily, Partha Dasgupta, Paul Ehrlich, Lawrence Goulder, Geoffrey Heal, Simon Levin, Karl-Göran Mäler, Stephen Schneider, David Starrett and Brian Walker (2004)

“Are We Consuming Too Much?”

Journal of Economic Perspectives 18: 147–172.

Macroeconomic variation in the Solow–Swan growth model.

Given a capital stock K , output Y is given by the production function $Y = f(K)$, where $f' > 0$, and $f'' \leq 0$.

Net investment = gross investment, without depreciation.

So given capital K and consumption C , investment I is given by

$$I = \dot{K} = f(K) - C$$

The Ramsey Problem and Beyond

The economy's **intertemporal objective** is taken to be

$$\int_0^T e^{-rt} U(C(t)) dt = \int_0^T e^{-rt} U(f(K) - \dot{K}) dt$$

Frank Ramsey (*Economic Journal*, 1928)

assumed $T = \infty$ (infinite horizon) and $r = 0$ (no discounting).

Nicholas Stern (of the *Stern Review* on Climate Change)

and others advocate:

- ▶ $T = \infty$;
- ▶ r as the hazard rate in an **exogenous** Poisson process that determines the latest date at which extinction occurs; this implies that e^{-rt} is the exogenous maximum probability that the human race has not become extinct by time t .

Chichilnisky, Hammond, and Stern in a special (2020) issue of *Social Choice and Welfare* honouring Kenneth Arrow.

Applying the Calculus of Variations

We apply the calculus of variations to the objective $\int_0^T e^{-rt} U(f(K) - \dot{K}) dt$ with the end conditions $K(0) = \bar{K}$, which is exogenous, and $K(T) = 0$ at the **finite time horizon** T .

Euler's equation takes the form $\frac{d}{dt} F'_K(t) = F'_K(t)$ where $F(t, K, \dot{K}) = e^{-rt} U(f(K) - \dot{K}) = e^{-rt} U(C)$.

So Euler's equation becomes $\frac{d}{dt} [-e^{-rt} U'(C)] = e^{-rt} U'(C) f'(K)$.

Equivalently, after evaluating the time derivative,

$$-U''(C)\dot{C}e^{-rt} + rU'(C)e^{-rt} = e^{-rt}U'(C)f'(K)$$

Cancelling the common factor e^{-rt} and dividing by $U'(C) > 0$, then rearranging, one obtains

$$-\frac{U''(C)}{U'(C)}\dot{C} = f'(K) - r$$

Further Interpretation

Define the (negative) **elasticity of marginal utility** as

$$\eta(C) := -\frac{d \ln U'(C)}{d \ln C} = -\frac{U''(C)C}{U'(C)}$$

This is related to the curvature of the utility function, and to how quickly marginal utility $U'(C)$ decreases as C increases.

Rearranging the equation $-U''(C)\dot{C}/U'(C) = f'(K) - r$ yet again, one obtains the equation

$$\eta(C)\frac{\dot{C}}{C} = f'(K) - r$$

whose left hand side is the proportional rate of consumption growth multiplied by: (i) the elasticity of marginal utility; or (ii) the elasticity of what macroeconomists call “an intertemporal marginal rate of substitution”; or (iii) by analogy with the theories of risk and inequality aversion, the degree of relative fluctuation aversion.

Final Recommendation

Morton I. Kamien and Nancy L. Schwartz (2012)
*Dynamic Optimization, Second Edition:
The Calculus of Variations and Optimal Control
in Economics and Management* (Dover Publications)