

Lecture Notes 6: Dynamic Equations Part B: Second and Higher-Order Linear Difference Equations in One Variable

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Lecture Outline

Solving Second-Order Equations

Inhomogeneous Equations

Particular Solutions in Two Special Cases

Method of Undetermined Coefficients

Higher-Order Linear Equations with Constant Coefficients

Stationary States and Stability for Second-Order Equations

Second-Order Equations

A general **second-order difference equation** specifies the state x_t at each time t as a function $x_t = F_t(x_{t-1}, x_{t-2})$ of the state at **two** previous times.

Suppose we define a new variable defined by $y_t := x_{t-1}$. Then the equation $x_t = F_t(x_{t-1}, x_{t-2})$ can be converted into the coupled pair

$$\begin{aligned}x_t &= F_t(x_{t-1}, y_{t-1}) \\ y_t &= x_{t-1}\end{aligned}$$

of **first**-order equations that express the vector $(x_t, y_t)^\top \in \mathbb{R}^2$ as a function of the vector $(x_{t-1}, y_{t-1})^\top \in \mathbb{R}^2$.

The Linear Case

We focus on linear equations in one variable with constant coefficients, which take the form

$$x_{t+1} + ax_t + bx_{t-1} = f_t$$

Here a, b are scalars, and f_t is the forcing term.

We assume that $b \neq 0$ because otherwise we have the first-order equation $x_{t+1} + ax_t = f_t$.

If we define $y_t = x_{t-1}$, the equation becomes the coupled pair

$$x_{t+1} = -ax_t - by_t + f_t; \quad y_{t+1} = x_t$$

In matrix form, these can be written as

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} f_t \\ 0 \end{pmatrix}$$

Such **vector difference equations** are the subject of part C.

The Homogeneous Case

Nevertheless, consider the homogeneous case when the vector equation is

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} - \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solution in matrix form is evidently

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

for an arbitrary initial state $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

Inspired by our earlier discussion of matrix powers, consider the case when $(\lambda, (x_0, y_0)^\top)$ is an eigenpair, that is

$$\begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving the Homogeneous Case

In case $\begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, the solution takes the form

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

For this to work, the initial vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ must solve

the matrix equation $\begin{pmatrix} -a - \lambda & -b \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

For a non-trivial solution to exist,

the matrix $\begin{pmatrix} -a - \lambda & -b \\ 1 & -\lambda \end{pmatrix}$ must be singular, implying that

$$\begin{vmatrix} -a - \lambda & -b \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + a\lambda + b = 0$$

The Auxiliary Equation

Instead of treating the second-order equation as a coupled pair, consider directly the homogeneous second-order equation

$$x_{t+1} + ax_t + bx_{t-1} = 0$$

Inspired by our previous analysis using eigenvalues of a suitable matrix, we look for a solution of the form $x_t = \lambda^t x_0$, for suitable constants λ and x_0 .

It is a solution provided that $\lambda^{t+1}x_0 + a\lambda^t x_0 + b\lambda^{t-1}x_0 = 0$.

Ignoring the trivial solutions when $x_0 = 0$ or $\lambda = 0$, cancel $\lambda^{t-1}x_0$ to obtain the **auxiliary** or **characteristic equation**

$$\lambda^2 + a\lambda + b = 0$$

This, of course, is the condition for λ to be an eigenvalue.

The Auxiliary Equation and Its Roots

The auxiliary equation $\lambda^2 + a\lambda + b = 0$ is quadratic.

It therefore has two roots λ_1, λ_2
satisfying $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$.

In particular $\lambda_1 + \lambda_2 = -a$ and $\lambda_1\lambda_2 = b$.

The assumption that $b \neq 0$ implies
that the two roots λ_1, λ_2 are both non-zero.

This leaves three cases:

1. two distinct real roots $\lambda_1, \lambda_2 \in \mathbb{R}$,
which is true iff $a^2 > 4b$;
2. two complex conjugate roots $\lambda_1, \lambda_2 = re^{\pm i\theta} \in \mathbb{C}$,
which is true iff $a^2 < 4b$;
3. two coincident real roots $\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$,
which is true iff $a^2 = 4b$.

Case 1: Two Distinct Real Roots

In this case $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$,

where $\lambda_1, \lambda_2 = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.

Note that $a = \lambda_1 + \lambda_2$ and $b = \lambda_1\lambda_2$

with $a^2 - 4b = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2 > 0$.

There are two degrees of freedom in the difference equation, so we look for two **linearly independent** solutions $x_t^{H(1)}$ and $x_t^{H(2)}$ of the homogeneous difference equation $x_{t+1} + ax_t + bx_{t-1} = 0$.

— that is two solutions for which $Ax_t^{H(1)} + Bx_t^{H(2)} \equiv 0$ implies that the two scalars A and B satisfy $A = B = 0$.

Two Linearly Independent Solutions

Note that $A\lambda_1^t + B\lambda_2^t = 0$ for both $t = 0$ and $t = 1$ if and only if

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This has a non-trivial solution in the two constants A and B

iff $0 = \begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix}$, or if and only if $0 = \lambda_2 - \lambda_1$.

So when $\lambda_1 \neq \lambda_2$, the only solution is trivial, with $A = B = 0$.

Hence, the two functions $x_t^{(1)} = x_0\lambda_1^t$ and $x_t^{(2)} = x_0\lambda_2^t$ with $x_0 \neq 0$ are linearly independent solutions of $x_{t+1} + ax_t + bx_{t-1} = 0$.

There are two degrees of freedom in the difference equation.

Therefore, its general solution with these two degrees of freedom is $x_t = A\lambda_1^t + B\lambda_2^t$ for arbitrary real constants A and B .

Example: The Fibonacci Sequence

The **Fibonacci sequence** is

$$(x_t)_{t=0}^{\infty} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \dots)$$

It is the unique solution with $x_0 = 0$ and $x_1 = 1$ of the **Fibonacci difference equation** $x_{t+1} - x_t - x_{t-1} = 0$.

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$, with characteristic roots $\lambda_{1,2} = -\frac{1}{2}(-1 \pm \sqrt{5})$.

Its two roots are:

(i) the **golden ratio** $\varphi := \lambda_1 = \frac{1}{2}(1 + \sqrt{5}) \approx 1.61803398875$;
and (ii) $\lambda_2 = 1 - \lambda_1 = \frac{1}{2}(1 - \sqrt{5}) \approx -0.61803398875$.

The general solution of the Fibonacci difference equation is $x_t = A\lambda_1^t + B\lambda_2^t$ for arbitrary constants A and B .

To obtain the Fibonacci sequence with $x_0 = 0$ and $x_1 = 1$ requires $B = -A$ and $1 = A(\lambda_1 - \lambda_2) = A\sqrt{5}$, so $B = -A = -\frac{1}{5}\sqrt{5}$.

Hence $x_t = \frac{1}{5}\sqrt{5} \cdot 2^{-t} [(1 + \sqrt{5})^t - (1 - \sqrt{5})^t]$, so $x_t \in \mathbb{N}$.

Case 2: Two Complex Conjugate Roots

Consider next the case where the equation $\lambda^2 + a\lambda + b = 0$ has two complex conjugate roots that we write as

$$\lambda = re^{\pm i\theta} = r(\cos \theta \pm i \sin \theta) \quad \text{where } \sin \theta \neq 0$$

In this case $\lambda^2 + a\lambda + b = (\lambda - re^{i\theta})(\lambda - re^{-i\theta})$ where

$$a = re^{i\theta} + re^{-i\theta} = r(\cos \theta + i \sin \theta) + r(\cos \theta - i \sin \theta) = 2r \cos \theta$$

and $b = (re^{i\theta})(re^{-i\theta}) = r^2$ with $\sin \theta \neq 0$.

It follows that $a^2 - 4b = 4r^2 \cos^2 \theta - 4r^2 = -4r^2 \sin^2 \theta < 0$.

Note that $r = \sqrt{|b|}$ and $\theta = \arccos\left(\frac{a}{2r}\right) = \arccos\left(\frac{1}{2}a|b|^{-\frac{1}{2}}\right)$.

Case 2: Oscillating Solutions

In the complex plane \mathbb{C} , two possible solutions of the difference equation $x_{t+1} + ax_t + bx_{t-1} = 0$ with $x_0 \neq 0$ are

$$x_t^{(1)} = x_0(re^{i\theta})^t = x_0r^te^{i\theta t} = x_0r^t(\cos \theta t + i \sin \theta t)$$

$$\text{and } x_t^{(2)} = x_0(re^{-i\theta})^t = x_0r^te^{-i\theta t} = x_0r^t(\cos \theta t - i \sin \theta t)$$

In the real line \mathbb{R} , two possible solutions are

$$x_t^{(1)} = r^t \cos \theta t \quad \text{and} \quad x_t^{(2)} = r^t \sin \theta t$$

These are linearly independent because

$$\begin{vmatrix} x_0^{(1)} & x_0^{(2)} \\ x_1^{(1)} & x_1^{(2)} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ r \cos \theta & r \sin \theta \end{vmatrix} = r \sin \theta \neq 0$$

The general solution is therefore $x_t = r^t(A \cos \theta t + B \sin \theta t)$ for arbitrary real constants A and B , where $A = x_0$.

Case 3: Two Coincident Roots

In this case $\lambda^2 + a\lambda + b = (\lambda - \bar{\lambda})^2$,
where $a = -2\bar{\lambda}$ and $b = \bar{\lambda}^2$.

Consider the perturbed equation $x_{t+1} + ax_t + \tilde{b}x_{t-1} = 0$
where $\tilde{a} = -2\bar{\lambda}$ and $\tilde{b} = \bar{\lambda}^2 - \epsilon^2$ with ϵ a small positive number.

We consider the behaviour of its general solution as $\epsilon \rightarrow 0$.

The auxiliary equation $\lambda^2 + a\lambda + \tilde{b} = 0$
can be written as $\lambda^2 + 2\bar{\lambda}\lambda + \bar{\lambda}^2 - \epsilon^2 = 0$.

Note that $\lambda^2 + 2\bar{\lambda}\lambda + \bar{\lambda}^2 - \epsilon^2 = (\lambda + \bar{\lambda} + \epsilon)(\lambda + \bar{\lambda} - \epsilon)$.

So the perturbed auxiliary equation
has the two real roots $\lambda = \bar{\lambda} \pm \epsilon$.

The Solution with Fixed Initial Conditions

Fix \bar{x}_0 and \bar{x}_1 .

The general solution satisfying $x_0 = \bar{x}_0$ and $x_1 = \bar{x}_1$ is $x_t = A(\bar{\lambda} + \epsilon)^t + B(\bar{\lambda} - \epsilon)^t$ where $\bar{x}_0 = A + B$ and $\bar{x}_1 = A(\bar{\lambda} + \epsilon) + B(\bar{\lambda} - \epsilon) = (A + B)\bar{\lambda} + (A - B)\epsilon$.

Hence $A + B = \bar{x}_0$ and $A - B = (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})$, implying that $A = \frac{1}{2} [\bar{x}_0 + (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})]$ and $B = \frac{1}{2} [\bar{x}_0 - (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})]$.

The solution for fixed ϵ is therefore

$$x_t^\epsilon = \frac{1}{2} [\bar{x}_0 + (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})] (\bar{\lambda} + \epsilon)^t + \frac{1}{2} [\bar{x}_0 - (1/\epsilon)(\bar{x}_1 - \bar{x}_0\bar{\lambda})] (\bar{\lambda} - \epsilon)^t$$

which can be rewritten as

$$x_t^\epsilon = \frac{1}{2}\bar{x}_0 [(\bar{\lambda} + \epsilon)^t + (\bar{\lambda} - \epsilon)^t] + \frac{1}{2}(\bar{x}_1 - \bar{x}_0\bar{\lambda})(1/\epsilon) [(\bar{\lambda} + \epsilon)^t - (\bar{\lambda} - \epsilon)^t]$$

The Limiting Solution as $\epsilon \rightarrow 0$

The limit of x_t^ϵ as $\epsilon \rightarrow 0$ takes the form

$$\bar{x}_0 \bar{\lambda}^t + \frac{1}{2}(\bar{x}_1 - \bar{x}_0 \bar{\lambda}) \lim_{\epsilon \rightarrow 0} (1/\epsilon) [(\bar{\lambda} + \epsilon)^t - (\bar{\lambda} - \epsilon)^t]$$

To evaluate the last limit, apply l'Hôpital's rule to obtain

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} [(\bar{\lambda} + \epsilon)^t - (\bar{\lambda} - \epsilon)^t] / \epsilon \\ &= \lim_{\epsilon \rightarrow 0} [t(\bar{\lambda} + \epsilon)^{t-1} + t(\bar{\lambda} - \epsilon)^{t-1}] / 1 = 2t\bar{\lambda}^t \end{aligned}$$

Two linearly independent possible solutions

of the difference equation $x_{t+1} + ax_t + bx_{t-1} = 0$

with $x_0 \neq 0$ are $x_t^{(1)} = x_0 \lambda^t$ and $x_t^{(2)} = x_0 t \lambda^t$.

There are two degrees of freedom in the difference equation.

Its general solution is $x_t = (C + Dt)\lambda^t$

for arbitrary real constants C and D .

A Simpler Approach, I

We are trying to solve the homogeneous second-order difference equation with a repeated root λ , taking the form

$$x_{t+1} - 2\lambda x_t + \lambda^2 x_{t-1} = 0$$

We know that one solution is $x_t = x_0 \lambda^t$ for arbitrary x_0 .

To find a second linearly independent solution that we know must exist, try putting $x_t = \lambda^t y_t$.

Substituting into the original equation gives

$$\lambda^{t+1} y_{t+1} - 2\lambda^{t+1} y_t + \lambda^{t+1} y_{t-1} = 0$$

Disregarding the trivial case when $\lambda = 0$, one has $y_{t+1} - 2y_t + y_{t-1} = 0$.

A Simpler Approach, II

To solve $y_{t+1} - 2y_t + y_{t-1} = 0$,

try introducing yet another new variable $z_t = y_{t+1} - y_t$.

This leads to the new difference equation $z_t - z_{t-1} = 0$ whose solution is obviously $z_t = z_0$ for all $t = 1, 2, \dots$

Then $y_{t+1} - y_t = z_0$ for all t , implying that $y_t = y_0 + z_0 t$.

It follows that $x_t = \lambda^t y_t = (y_0 + z_0 t)\lambda^t$.

To conclude, two solutions are $x_t^{(1)} = \lambda^t$ and $x_t^{(2)} = t\lambda^t$.

These are linearly independent because

$$\begin{vmatrix} x_0^{(1)} & x_0^{(2)} \\ x_1^{(1)} & x_1^{(2)} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \lambda & \lambda \end{vmatrix} = \lambda \neq 0$$

The general solution is therefore $x_t = (A + Bt)\lambda^t$ for arbitrary real constants A and B , where $A = x_0$.

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From Particular to General Solutions

The **homogeneous equation** with constant coefficients takes the form

$$x_{t+1} + ax_t + bx_{t-1} = 0$$

The associated **inhomogeneous equation** takes the form

$$x_{t+1} + ax_t + bx_{t-1} = f_t$$

for a general **forcing term** f_t on the RHS.

Let x_t^P denote a **particular solution**,
and x_t^G any alternative **general solution**,
of the inhomogeneous equation.

Characterizing the General Solution

Our assumptions imply that, for each $t = 1, 2, \dots$, one has

$$\begin{aligned}x_{t+1}^P + ax_t^P + bx_{t-1}^P &= f_t \\x_{t+1}^G + ax_t^G + bx_{t-1}^G &= f_t\end{aligned}$$

Subtracting the first equation from the second implies that

$$x_{t+1}^G - x_{t+1}^P + a(x_t^G - x_t^P) + b(x_{t-1}^G - x_{t-1}^P) = 0$$

This shows that $x_t^H := x_t^G - x_t^P$ solves the homogeneous equation $x_{t+1} + ax_t + bx_{t-1} = 0$.

So the general solution x_t^G of the inhomogeneous equation $x_{t+1} + ax_t + bx_{t-1} = f_t$ with forcing term f_t is the sum $x_t^P + x_t^H$ of

- ▶ any particular solution x_t^P of the inhomogeneous equation;
- ▶ the general solution x_t^H of the homogeneous equation.

Linearity in the Forcing Term

Theorem

Suppose that x_t^P and y_t^P are particular solutions of the two respective difference equations

$$x_{t+1} + ax_t + bx_{t-1} = d_t \quad \text{and} \quad y_{t+1} + ay_t + by_{t-1} = e_t$$

Then, for any scalars α and β , the linear combination $z_t^P := \alpha x_t^P + \beta y_t^P$ is a particular solution of the equation $z_{t+1} + az_t + bz_{t-1} = \alpha d_t + \beta e_t$.

Proof.

Routine algebra. □

Consider any equation of the form $x_{t+1} + ax_t + bx_{t-1} = f_t$ where f_t is a linear combination $\sum_{k=1}^n \alpha_k f_t^k$ of n forcing terms.

The theorem implies that a particular solution is the corresponding linear combination $\sum_{k=1}^n \alpha_k x_t^{Pk}$ of particular solutions to the equations $x_{t+1} + ax_t + bx_{t-1} = f_t^k$.

Deriving an Explicit Particular Solution, I

In part A we were able to derive an explicit solution to the general first-order linear equation $x_t - a_t x_{t-1} = f_t$.

Here, for the special case of **constant coefficients**, we derive an explicit particular solution satisfying $x_0 = x_1 = 0$ to the general second-order linear equation $x_{t+1} + ax_t + bx_{t-1} = f_t$.

Indeed, suppose that $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$ because λ_1 and λ_2 are the roots (possibly coincident, or possibly complex conjugates) of the auxiliary equation $\lambda^2 + a\lambda + b = 0$.

Introduce the new variable $y_t = x_t - \lambda_1 x_{t-1}$, implying that

$$\begin{aligned}y_{t+1} - \lambda_2 y_t &= x_{t+1} - \lambda_1 x_t - \lambda_2 x_t + \lambda_1 \lambda_2 x_{t-1} \\ &= x_{t+1} - (\lambda_1 + \lambda_2)x_t + \lambda_1 \lambda_2 x_{t-1} \\ &= x_{t+1} + ax_t + bx_{t-1} = f_t\end{aligned}$$

Deriving an Explicit Particular Solution, II

Instead of the second-order equation $x_{t+1} + ax_t + bx_{t-1} = f_t$, we have the recursive pair of first-order equations

$$x_t - \lambda_1 x_{t-1} = y_t \quad \text{and} \quad y_{t+1} - \lambda_2 y_t = f_t \quad (\text{for } t = 1, 2, \dots)$$

where λ_1 and λ_2 are the roots of $\lambda^2 + a\lambda + b = 0$.

Given the initial conditions $x_0 = x_1 = 0$ and so $y_1 = 0$, the explicit solutions like those derived in Part A are the sums

$$y_t = \sum_{k=1}^{t-1} \lambda_2^{t-k-1} f_k \quad \text{and} \quad x_t = \sum_{s=2}^t \lambda_1^{t-s} y_s \quad \text{for } t = 1, 2, \dots$$

Substituting the first equation in the second yields the double sum

$$x_t = \sum_{s=2}^t \lambda_1^{t-s} \sum_{k=1}^{s-1} \lambda_2^{s-k-1} f_k$$

which we would like to reduce to $x_t = \sum_{k=1}^{t-1} \xi_{t-k-1} f_k$
— i.e., a linear combination of the forcing terms $(f_1, f_2, \dots, f_{t-1})$.

Deriving an Explicit Particular Solution, III

We begin by introducing

the mapping $\mathbb{N} \times \mathbb{N} \ni (k, s) \mapsto 1_{ks}\{k < s\} \in \{0, 1\}$ defined by

$$1_{ks}\{k < s\} := \begin{cases} 1 & \text{if } k < s \\ 0 & \text{if } k \geq s \end{cases}$$

Then we can rewrite $x_t = \sum_{s=2}^t \lambda_1^{t-s} \sum_{k=1}^{s-1} \lambda_2^{s-k-1} f_k$

as the double sum $x_t = \sum_{s=2}^t \sum_{k=1}^{t-1} 1_{ks}\{k < s\} \lambda_1^{t-s} \lambda_2^{s-k-1} f_k$.

Interchanging the order of summation gives

$$\begin{aligned} x_t &= \sum_{k=1}^{t-1} \sum_{s=2}^t 1_{ks}\{k < s\} \lambda_1^{t-s} \lambda_2^{s-k-1} f_k \\ &= \sum_{k=1}^{t-1} \left(\sum_{s=k+1}^t \lambda_1^{t-s} \lambda_2^{s-k-1} \right) f_k \\ &= \sum_{k=1}^{t-1} \left(\lambda_1^{t-k-1} + \lambda_1^{t-k-2} \lambda_2 + \dots + \lambda_1 \lambda_2^{t-k-2} + \lambda_2^{t-k-1} \right) f_k \end{aligned}$$

This reduces to $x_t = \sum_{k=1}^{t-1} \xi_{t-k-1} f_k$ where $\xi_m := \sum_{j=0}^m \lambda_1^{m-j} \lambda_2^j$.

Deriving an Explicit Particular Solution: IV

The value of the sum $\xi_m = \sum_{j=0}^m \lambda_1^{m-j} \lambda_2^j$ depends on whether:

- ▶ we are in the **general case** when $\lambda_1 \neq \lambda_2$;
- ▶ we are in the **degenerate case** when $\lambda_1 = \lambda_2 = \lambda$.

In the general case one has

$$(\lambda_1 - \lambda_2)\xi_m = \sum_{j=0}^m \left(\lambda_1^{m+1-j} \lambda_2^j - \lambda_1^{m-j} \lambda_2^{j+1} \right) = \lambda_1^{m+1} - \lambda_2^{m+1}$$

implying the particular solution

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \sum_{k=1}^{t-1} \left(\lambda_1^{t-k} - \lambda_2^{t-k} \right) f_k$$

In the degenerate case one has $\xi_m = (m+1)\lambda^m$,

implying the particular solution

$$x_t^P = \sum_{k=1}^{t-1} (t-k)\lambda^{t-k} f_k$$

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First Special Case with Distinct Real Roots, I

Consider the equation $x_{t+1} + ax_t + bx_{t-1} = f_t$
in the first special case when $f_t = \mu^t$ with $\mu \neq 0$.

In the general case when the two roots λ_1 and λ_2
of the auxiliary equation $\lambda^2 + a\lambda + b = 0$ are distinct,
the particular solution with $x_0^P = x_1^P = 0$ is

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \sum_{k=1}^{t-1} (\lambda_1^{t-k} - \lambda_2^{t-k}) \mu^k$$

But $(\lambda - \mu) \sum_{k=1}^{t-1} \lambda^{t-k} \mu^k = \sum_{k=1}^{t-1} (\lambda^{t-k+1} \mu^k - \lambda^{t-k} \mu^{k+1})$, so

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \left(\frac{\lambda_1^t \mu - \lambda_1 \mu^t}{\lambda_1 - \mu} - \frac{\lambda_2^t \mu - \lambda_2 \mu^t}{\lambda_2 - \mu} \right)$$

in case $\mu \notin \{\lambda_1, \lambda_2\}$.

Disregarding the terms in λ_1^t and λ_2^t
that solve the corresponding homogeneous equation,
the solution reduces to $x_t^P = \alpha \mu^t$ for a suitable constant α .

First Special Case with Distinct Real Roots, II

The degenerate case when $\mu \in \{\lambda_1, \lambda_2\}$ is more complicated.

In case $\lambda_1 \neq \lambda_2 = \mu$, the particular solution with $x_0^P = x_1^P = 0$ is still

$$x_t^P = \frac{1}{\lambda_1 - \lambda_2} \sum_{k=1}^{t-1} \left(\lambda_1^{t-k} - \lambda_2^{t-k} \right) \mu^k$$

Because $\lambda_2 = \mu$, this reduces to

$$\begin{aligned} x_t^P &= \frac{1}{\lambda_1 - \mu} \sum_{k=1}^{t-1} \left(\lambda_1^{t-k} \mu^k - \mu^t \right) \\ &= \frac{1}{\lambda_1 - \mu} \left[\frac{\lambda_1^t \mu - \lambda_1 \mu^t}{\lambda_1 - \mu} - (t-1) \mu^t \right] \end{aligned}$$

Disregarding the terms in λ_1^t and in $\lambda_2^t = \mu^t$ that solve the corresponding homogeneous equation, the solution reduces to $x_t^P = \alpha t \mu^t$ for a suitable constant α .

First Special Case with Coincident Real Roots

Consider now the degenerate case
with coincident real roots $\lambda_1 = \lambda_2 = \lambda$.

So the inhomogeneous equation is $x_{t+1} - 2\lambda x_t + \lambda^2 x_{t-1} = \mu^t$.

As before, put $y_t = x_t - \lambda x_{t-1}$ so that

$$y_{t+1} - \lambda y_t = x_{t+1} - \lambda x_t - \lambda x_t + \lambda^2 x_{t-1} = \mu^t$$

We consider again the particular solution
with $x_0 = x_1 = 0$ and so $y_1 = 0$.

First Special Case with Coincident Real Roots: $\lambda \neq \mu$

Provided that $\lambda \neq \mu$, for $t = 2, 3, \dots$ one has

$$y_t^P = \sum_{k=2}^t \lambda^{t-k} \mu^{k-1} = \frac{\mu(\lambda^{t-1} - \mu^{t-1})}{\lambda - \mu}$$

$$\begin{aligned} \text{and then } x_t^P &= \sum_{k=2}^t \lambda^{t-k} y_k^P = \sum_{k=2}^t \lambda^{t-k} \mu \frac{\lambda^{k-1} - \mu^{k-1}}{\lambda - \mu} \\ &= \sum_{k=2}^t \frac{\mu \lambda^{t-1} - \lambda^{t-k} \mu^k}{\lambda - \mu} \\ &= \frac{\mu(t-1)\lambda^{t-1}}{\lambda - \mu} - \frac{\lambda^{t-1}\mu^2 - \mu^{t+1}}{(\lambda - \mu)^2} \end{aligned}$$

Hence $x_t^P = (\alpha + \beta t)\lambda^t + \gamma\mu^t$ for suitable constants α , β and γ that depend on λ and μ , but not on t .

Because $(\alpha + \beta t)\lambda^t$ is a complementary solution of the homogeneous equation,

the particular solution can be reduced to $x_t^P = \gamma\mu^t$.

First Special Case with Coincident Real Roots: $\lambda = \mu$

In case $\lambda = \mu$, however, for $t = 2, 3, \dots$ one has

$$y_t^P = \sum_{k=2}^t \lambda^{t-k} \mu^{k-1} = (t-1)\lambda^{t-1}$$

and then

$$\begin{aligned} x_t^P &= \sum_{k=2}^t \lambda^{t-k} y_k^P = \sum_{k=2}^t \lambda^{t-k} (k-1)\lambda^{k-1} \\ &= \sum_{k=2}^t (k-1)\lambda^{t-1} = \frac{1}{2}t(t-1)\lambda^{t-1} \end{aligned}$$

Hence $x_t^P = (\alpha t + \beta t^2)\lambda^t$ for suitable constants α and β that depend on $\lambda = \mu$, but not on t .

Because $\alpha t \lambda^t$ is a complementary solution of the homogeneous equation, the particular solution can be reduced to $x_t^P = \beta t^2 \mu^t$.

Second Special Case: General Theorem

Consider next the equation $x_{t+1} + ax_t + bx_{t-1} = f_t$
in the second special case when $f_t = t^r \mu^t$ with $\mu \neq 0$ and $r \in \mathbb{N}$.

As before, let λ_1 and λ_2 denote the roots
of the auxiliary equation $\lambda^2 + a\lambda + b = 0$.

Theorem

*The difference equation $x_{t+1} + ax_t + bx_{t-1} = t^r \mu^t$
has a particular solution of the form $x_t^P = \xi^P(t) \mu^t$
where $\xi^P(t) = \sum_{j=0}^d \xi_{rj} t^j$ is a polynomial in t which has degree:*

- ▶ $d = r$ in case $\mu \notin \{\lambda_1, \lambda_2\}$;
- ▶ $d = r + 2$ in case $\mu = \lambda_1 = \lambda_2$;
- ▶ $d = r + 1$ otherwise.

We begin the proof by introducing, as before,
the new variable $y_t := x_t - \lambda_1 x_{t-1}$, implying that

$$\begin{aligned} y_{t+1} - \lambda_2 y_t &= x_{t+1} - \lambda_1 x_t - \lambda_2 x_t + \lambda_1 \lambda_2 x_{t-1} \\ &= x_{t+1} + ax_t + bx_{t-1} = t^r \mu^t \end{aligned}$$

Continuing the Proof of the General Theorem

By the result in part A, the first-order equation $y_{t+1} - \lambda_2 y_t = t^r \mu^t$ has a particular solution of the form $y_t = Q(t)\mu^t$, where $Q(t) = \sum_{j=0}^d q_{rj} t^j$ is a polynomial in t which has degree:

- (i) $d = r$ in case $\mu \neq \lambda_2$; (ii) $d = r + 1$ in case $\mu = \lambda_2$.

By the linearity property of particular solutions, the equation

$$x_t - \lambda_1 x_{t-1} = y_t = Q(t)\mu^t = \sum_{j=0}^d q_{rj} t^j \mu^t$$

has a particular solution $x_t^P = \xi^P(t)\mu^t$ where

$$x_t^P = \xi^P(t)\mu^t = \sum_{j=0}^d q_{rj} P_j(t)\mu^t$$

is the appropriate linear combination

of the particular solutions $x_t = P_j(t)\mu^t$ ($j = 0, 1, 2, \dots, d$)

of the $d + 1$ first-order equations $x_t - \lambda_1 x_{t-1} = t^j \mu^t$.

Ending the Proof of the General Theorem

Again, using the result in part A,

for each $j = 0, 1, 2, \dots, r$, the solution $x_t = P_j(t)\mu^t$ of the first-order difference equation $x_t - \lambda_1 x_{t-1} = t^j \mu^t$ involves a polynomial $P_j(t)$ in t which has degree:

- (i) j in case $\mu \neq \lambda_1$; (ii) $j + 1$ in case $\mu = \lambda_1$.

So the degree of the highest order polynomial $P_d(t)$ is

- (i) d in case $\mu \neq \lambda_1$; (ii) $d + 1$ in case $\mu = \lambda_1$.

Combined with our previous result on whether $d = r$ or $d = r + 1$, the degree d of $\xi^P(t)$ is now easily seen to be

- ▶ $d = r$ in case $\mu \notin \{\lambda_1, \lambda_2\}$;
- ▶ $d = r + 2$ in case $\mu = \lambda_1 = \lambda_2$;
- ▶ $d = r + 1$ otherwise.



Using the notation $\#S$ for the number of elements in a set S , these three cases can be summarized as $d = r + 3 - \#\{\lambda_1, \lambda_2, \mu\}$.

Lecture Outline

Solving Second-Order Equations

Inhomogeneous Equations

Particular Solutions in Two Special Cases

Method of Undetermined Coefficients

Higher-Order Linear Equations with Constant Coefficients

Stationary States and Stability for Second-Order Equations

First Special Case: A Simpler Approach

We have proved that the second-order difference equation

$$x_{t+1} + ax_t + bx_{t-1} = \mu^t$$

has a particular solution of the form $x_t^P = \alpha\mu^t$.

But there is a much easier way to find x_t^P , treating the parameter α as an **undetermined coefficient**.

Indeed, for $x_t = \alpha\mu^t$ to be a solution, one needs $\alpha\mu^{t+1} + a\alpha\mu^t + b\alpha\mu^{t-1} = \mu^t$.

Dividing each side by μ^{t-1} yields the equation $\alpha(\mu^2 + a\mu + b) = \mu$.

In the **non-degenerate** case when $\mu^2 + a\mu + b \neq 0$

because μ is not a root

of the characteristic equation $\lambda^2 + a\lambda + b = 0$,

one has $\alpha = \mu(\mu^2 + a\mu + b)^{-1}$.

Hence, a particular solution is $x_t^P = (\mu^2 + a\mu + b)^{-1}\mu^{t+1}$.

Degenerate Case When μ is a Characteristic Root

The **simple degenerate case** occurs when $\mu^2 + a\mu + b = 0$ because μ equals one of the two **distinct** roots λ_1 and λ_2 of the characteristic equation $\lambda^2 + a\lambda + b = 0$.

Then we have proved that the second-order difference equation

$$x_{t+1} + ax_t + bx_{t-1} = \mu^t$$

has a particular solution of the form $x_t^P = \alpha t \mu^t$.

To determine the undetermined coefficient α , we must solve

$$\alpha(t+1)\mu^{t+1} + a\alpha t\mu^t + b\alpha(t-1)\mu^{t-1} = \mu^t$$

Dividing each side by μ^{t-1} and gathering terms yields the equation $\alpha t(\mu^2 + a\mu + b) + \alpha(\mu^2 - b) = \mu$.

Provided that $\mu^2 \neq b$, this reduces to $\alpha = (\mu^2 - b)^{-1}\mu$.

Doubly Degenerate Case

When $\mu^2 = b$, however, the degenerate case is more complicated.

Indeed, the equation $\mu^2 + a\mu + b = 0$ implies that $a\mu + 2b = 0$.

Hence $\mu = -2b/a$, so $\mu^2 = b = 4b^2/a^2$ implying that $a^2 = 4b$.

Then the characteristic equation $\lambda^2 + a\lambda + b = 0$ reduces to $(\lambda - \mu)^2 = 0$, with μ as its repeated root.

Inspired by the earlier theorem,

we look for a particular solution of the form $x_t^P = \alpha t^2 \mu^t$.

To determine the undetermined coefficient α , we must solve

$$\alpha(t+1)^2 \mu^{t+1} + a\alpha t^2 \mu^t + b\alpha(t-1)^2 \mu^{t-1} = \mu^t$$

Dividing each side by μ^{t-1} and gathering terms yields

$$\alpha t^2 (\mu^2 + a\mu + b) + \alpha(2t+1)\mu^2 + \alpha b(-2t+1) = \mu$$

Because $\mu^2 + a\mu + b = 0$ and $0 \neq b = \mu^2$,

this equation reduces to $2\alpha\mu^2 = \mu$, implying that $\alpha = 1/2\mu$.

Second Special Case

Again, inspired by earlier theorems, we can apply the method of undetermined coefficients to the equation

$$x_{t+1} + ax_t + bx_{t-1} = \sum_{k=1}^m \sum_{j=1}^{r_k} \alpha_{kj} t^j \mu_k^t$$

where we naturally assume that the constants μ_k ($k = 1, 2, \dots, m$) are all different.

A particular solution takes the form

$$x_t^P = \sum_{k=1}^m \sum_{j=1}^{d_k} \beta_{kj} t^j \mu_k^t$$

where the degree d_k of each polynomial $\sum_{j=1}^{d_k} \beta_{kj} t^j$ with undetermined coefficients $\langle \langle \beta_{kj} \rangle_{j=1}^{d_k} \rangle_{k=1}^m$ is

- ▶ r_k in case $\mu_k \notin \{\lambda_1, \lambda_2\}$;
- ▶ $r_k + 2$ in case $\mu_k = \lambda_1 = \lambda_2$;
- ▶ $r_k + 1$ otherwise.

Determining the Coefficients

The coefficients $\langle\langle\beta_{kj}\rangle_{j=1}^{d_k}\rangle_{k=1}^m$ of the particular solution

$$x_t^P = \sum_{k=1}^m \sum_{j=1}^{d_k} \beta_{kj} t^j \mu_k^t$$

can be found (in principle!) by solving, for $k = 1, 2, \dots, m$, the m independent systems of linear equations that result from equating coefficients of powers of t in the expansions

$$\sum_{j=1}^{d_k} \beta_{kj} [(t+1)^j \mu_k^2 + at^j \mu_k^t + b(t-1)^j] = \sum_{j=1}^{r_k} \alpha_{kj} t^j \mu_k$$

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Higher-Order Linear Equations with Constant Coefficients

An n th order linear equation with constant coefficients takes the form

$$x_t + \sum_{r=1}^n a_r x_{t-r} = f_t$$

in the inhomogeneous case, and

$$x_t + \sum_{r=1}^n a_r x_{t-r} = 0$$

in the homogeneous case.

The corresponding auxiliary equation is $\lambda^n + \sum_{r=1}^n a_r \lambda^{n-r} = 0$.

Roots of the Auxiliary Equation

The auxiliary equation can be written as $p_n(\lambda) = 0$ whose LHS is the polynomial $\lambda^n + \sum_{r=1}^n a_r \lambda^{n-r}$ of degree n .

By the **fundamental theorem of algebra**, this equation has at least one root λ_1 , which may be complex.

Then $p_n(\lambda)$ can be factored as $p_n(\lambda) \equiv (\lambda - \lambda_1)p_{n-1}(\lambda)$.

But now the equation $p_{n-1}(\lambda) = 0$ also has at least one root λ_2 , which may also be complex.

Repeating this argument n times, the auxiliary equation $p_n(\lambda) = 0$ has n roots $\lambda_1, \lambda_2, \dots, \lambda_n$, some of which may be repeated.

In particular, $p_n(\lambda) \equiv \prod_{i=1}^n (\lambda - \lambda_i)$.

Solving the Homogeneous Equation

Theorem

Consider the homogeneous equation $x_t + \sum_{r=1}^n a_r x_{t-r} = 0$, and suppose that the auxiliary equation can be written as

$$0 = \lambda^n + \sum_{r=1}^n a_r \lambda^{t-r} = \prod_{j=1}^k (\lambda - \rho_j)^{m_j}$$

with k distinct roots ρ_j ($j = 1, 2, \dots, k$)

whose respective **multiplicities** m_j satisfy $\sum_{j=1}^k m_j = n$.

Then the general solution of the homogeneous equation takes the form

$$x_t = \sum_{j=1}^k \sum_{h=1}^{m_j} \alpha_{jh} t^{h-1} \rho_j^t$$

for n arbitrary constants α_{jh} ($h = 1, 2, \dots, m_j$ and $j = 1, 2, \dots, k$).

That is, the general solution is an arbitrary linear combination of the functions $t^{h-1} \rho_j^t$ ($h = 1, 2, \dots, m_j$ and $j = 1, 2, \dots, k$).

Solving the Inhomogeneous Equation

Theorem

The general solution of the inhomogeneous equation

$$x_t + \sum_{r=1}^n a_r x_{t-r} = \sum_{h=1}^i \sum_{j=1}^{q_h} \alpha_{hj} t^j \mu_h^t$$

is the sum of: (i) the general **complementary** solution to the corresponding homogeneous equation $x_t + \sum_{r=1}^n a_r x_{t-r} = 0$; and (ii) any particular solution.

One particular solution takes the form $x_t^P = \sum_{h=1}^i \sum_{j=1}^{d_h} \beta_{hj} t^j \mu_h^t$ where the degree d_h of each polynomial $\sum_{j=1}^{d_h} \beta_{hj} t^j$ with undetermined coefficients $\langle \langle \beta_{hj} \rangle_{j=1}^{d_h} \rangle_{h=1}^i$ is

- ▶ q_h in case $\mu_h \notin \{\rho_1, \rho_2, \dots, \rho_k\}$;
- ▶ $q_h + m_j$ in case $\mu_h = \rho_j$, a root of multiplicity m_j .

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Stationary States of a Linear Equation

Consider the second-order equation $x_{t+1} + ax_t + bx_{t-1} = f$
for a constant **forcing term** $f \in \mathbb{R}$.

Here a stationary state $x^* \in \mathbb{R}$ has the defining property
that $x_{t-1} = x_t = x^* \implies x_{t+1} = x^*$.

This is satisfied if and only if $x^* + ax^* + bx^* = f$,
or equivalently, if and only if $(1 + a + b)x^* = f$.

In case $a + b = -1$, there is:

- ▶ no stationary state unless $f = 0$;
- ▶ a whole real line \mathbb{R} of stationary states if $f = 0$.

Otherwise, if $a + b \neq -1$,
the only stationary state is $x^* = (1 + a + b)^{-1}f$.

Stability of a Linear Equation

If $a + b \neq -1$, let $y_t := x_t - x^*$ denote the **deviation** of state x_t from the stationary state $x^* = (1 + a + b)^{-1}f$. Then

$$\begin{aligned}y_{t+1} &= x_{t+1} - x^* = -ax_t - bx_{t-1}f - x^* \\ &= -a(y_t + x^*) - b(y_{t-1} + x^*) + f - x^* = -ay_t - by_{t-1}\end{aligned}$$

Thus y_t solves the homogenous equation $x_{t+1} + ax_t + bx_{t-1} = 0$.

As already seen, the solution to this homogeneous equation depends on the two roots $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$ of the quadratic characteristic equation

$$f(\lambda) \equiv \lambda^2 + a\lambda + b \equiv (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

There are three cases to consider:

1. two distinct real roots because $a^2 - 4b > 0$;
2. two complex conjugate roots because $a^2 - 4b < 0$;
3. two coincident real roots because $a^2 - 4b = 0$.

Stability Condition

With two distinct roots λ_1 and λ_2 , real or complex, the general solution of the homogeneous equation is $x_t = A\lambda_1^t + B\lambda_2^t$.

Stability is satisfied if and only if for all $A, B \in \mathbb{R}$ one has $x_t \rightarrow 0$ as $t \rightarrow \infty$.

This is true if and only if the absolute values of both roots satisfy $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

With two coincident roots $\lambda_1 = \lambda_2 = -\frac{1}{2}a = \sqrt{b}$, the general solution of the homogeneous equation is $x_t = (A + Bt)\lambda^t$.

Again, stability is satisfied if and only if for all $A, B \in \mathbb{R}$ one has $x_t \rightarrow 0$ as $t \rightarrow \infty$.

This is true if and only if the absolute value of the double root satisfies $|\lambda| < 1$.

Two Distinct Real Roots

Here $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$

where λ_1 and λ_2 are both real.

Note that the quadratic function $f(\lambda) \equiv \lambda^2 + a\lambda + b$ is convex and satisfies $f(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow \pm\infty$.

So the real roots of $f(\lambda) = 0$ satisfy $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff

$$f(-1) > 0 \text{ and } f(1) > 0 \text{ with } f'(-1) < 0 \text{ and } f'(1) > 0$$

These conditions are equivalent to

$$1 - a + b > 0 \text{ and } 1 + a + b > 0 \text{ with } -2 + a < 0 \text{ and } 2 + a > 0$$

or to $|a| < 2$ and $|a| < 1 + b$.

Together with the condition $a^2 > 4b$

for the equation $f(\lambda) = 0$ to have two distinct real roots, these inequalities are equivalent to $|a| - 1 < b < 1$.

Two Complex Conjugate Roots

The characteristic equation $\lambda^2 + a\lambda + b = 0$ has two complex conjugate roots when $a^2 - 4b < 0$.

In this case, these characteristic roots are

$$\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}i\sqrt{4b - a^2} = r e^{\pm i\theta} = r(\cos \theta \pm i \sin \theta)$$

where $r = \sqrt{b}$ and $\theta = \arccos(a/2\sqrt{b})$

Then the general solution of the homogeneous equation can be written as $x_t = r^t(A \cos \theta t + B \sin \theta t)$.

Stability is satisfied if and only if for all $A, B \in \mathbb{R}$ one has $x_t \rightarrow 0$ as $t \rightarrow \infty$.

This is true if and only if $b < 1$, as well as $a^2 - 4b < 0$ which implies that $b > 0$.

A Repeated Real Root

The characteristic equation $\lambda^2 + a\lambda + b = 0$ has two coincident real roots when $a^2 = 4b$.

In this case, $\lambda^2 + a\lambda + b = (\lambda^2 + \frac{1}{2}a)^2$.

The coincident real roots both equal $-\frac{1}{2}a$.

Then the general solution of the homogeneous equation is $x_t = (A + Bt)(-\frac{1}{2}a)^t$.

Stability is satisfied if and only if for all $A, B \in \mathbb{R}$ one has $x_t \rightarrow 0$ as $t \rightarrow \infty$.

This is true if and only if the modulus of the repeated root $\lambda = -\frac{1}{2}a$ satisfies $|\lambda| < 1$, and so if and only if $|a| < 2$.

A Simpler Test

Theorem

The linear autonomous equation $x_{t+1} + ax_t + bx_{t-1} = f$ is stable, both locally and globally, if and only if $|a| < 1 + b < 2$.

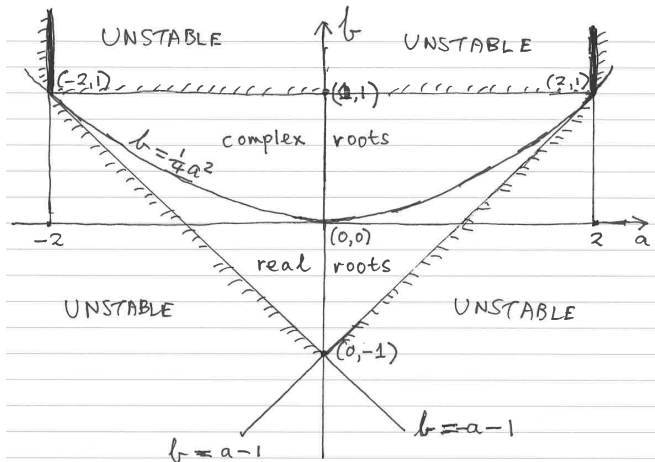
Proof.

Stability requires one of the following three to hold:

1. distinct real roots because $a^2 > 4b$, with $|a| - 1 < b < 1$;
2. complex conjugate roots because $a^2 < 4b$, with $b < 1$;
3. a repeated real root because $a^2 = 4b$, with $|a| < 2$.

A diagram in the (a, b) -plane shows that one of these three holds if and only if $|a| < 1 + b < 2$. □

Diagram of Stable Region



The stable region occurs where $|a| - 1 < b < 1$, in the interior of an isosceles right-angled triangle with corners at $(a, b) = (0, -1)$ and $(a, b) = (\pm 2, 1)$.

Stability with a Variable Forcing Term

Consider now the second-order equation $x_{t+1} + ax_t + bx_{t-1} = f_t$ for a **variable** forcing term f_t .

The general solution takes the form $x_t^G = x_t^H + x_t^P$ where:

- ▶ x_t^P is one particular solution of $x_{t+1} + ax_t + bx_{t-1} = f_t$;
- ▶ x_t^H is any one of a two-dimension continuum of solutions to the homogeneous equation $x_{t+1} + ax_t + bx_{t-1} = 0$.

The stability condition $|a| < 1 + b < 2$ is necessary and sufficient for any solution of the homogeneous equation to satisfy $x_t^H \rightarrow 0$ as $t \rightarrow \infty$.

It is therefore also necessary and sufficient for the difference between any two solutions $x_t^{(1)}$ and $x_t^{(2)}$ of the inhomogeneous equation $x_{t+1} + ax_t + bx_{t-1} = f_t$ to satisfy $x_t^{(1)} - x_t^{(2)} \rightarrow 0$ as $t \rightarrow \infty$.

In the long run, this means that there is an **asymptotically unique** solution to $x_{t+1} + ax_t + bx_{t-1} = f_t$.