Lecture Notes 6: Dynamic Equations
Part D: Differential Equations

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Lecture Outline

First-Order Differential Equations in One Variable

- Introduction
- First-Order Affine Equation with Constant Coefficients
- General First-Order Affine Equation
- Constant and Undetermined Coefficients
- Stability in the Autonomous Case

Second-Order Differential Equations in One Variable

- Introduction
- The Inhomogeneous Equation
- Constant and Undetermined Coefficients
- Stability

First-Order Multivariable Differential Equations

- Introduction
- Stability with Many Variables
- Autonomous Nonlinear Equations in Many Variables

First-Order Differential Equations

The typical first-order differential equation in one variable x is

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t)$$

The equation is autonomous just in case f is independent of t, so it can be written as $\dot{x} = f(x)$.

Typically one imposes an initial condition requiring $x(s) = \bar{x}_s$ at time s (not necessarily the earliest time).

Then any solution is a fixed function $t \mapsto x(t)$ that satisfies the corresponding integral equation $x(t) = \bar{x}_s + \int_s^t f(x(u), u)\,\mathrm{d}u$.

Picard's method of successive approximations starts with an arbitrary function $t \mapsto x^{(0)}(t)$ satisfying $x^{(0)}(s) = \bar{x}_s$.

Then it computes $x^{(n)}(t) = \bar{x}_s + \int_s^t f(x^{(n-1)}(u), u)\,\mathrm{d}u$ for $n \in \mathbb{N}$.

If convergence occurs, the limit as $n \to \infty$ will be a solution.

Right-Hand Side Independent of x

A special case occurs when the right-hand side $f(x, t)$ is independent of x.

Then the differential equation can be written as

$$\frac{dx}{dt} = g(t)$$

Its solution can be written as the indefinite integral

$$x(t) = \int g(t) dt$$

Introducing an initial condition $x(s) = \bar{x}_s$
at a particular start time s
allows the solution to be written as the definite integral

$$x(t) = \bar{x}_s + \int_s^t g(\tau) d\tau$$

CHECK that this alleged solution
satisfies $x(s) = \bar{x}_s$ and $\dot{x}(t) = g(t)$ for all $t \geq s$.

Leibnitz Rule for Differentiating an Integral

Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F(a, b, u) := \int_a^b f(t, u) \mathrm{d}t$$

Its three first-order partial derivatives are:

$$\text{(i) } F'_a = -f(a, u); \text{ (ii) } F'_b = f(b, u); \text{ (iii) } F'_u = \int_a^b \frac{\partial}{\partial u} f(t, u) \mathrm{d}t$$

Applying the chain rule, the total derivative of the integral function $y \mapsto I(y) := \int_{a(y)}^{b(y)} f(t, y) \mathrm{d}t$ satisfies

$$\begin{aligned} I'(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F(a(y), b(y), y) = a'(y) F'_a + b'(y) F'_b + F'_u \\ &= b'(y) f(b(y), y) - a'(y) f(a(y), y) + \int_{a(y)}^{b(y)} \frac{\partial}{\partial y} f(t, y) \mathrm{d}t \end{aligned}$$

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Picard's Method of Successive Approximations

The simplest first-order equation with constant coefficients takes the form

$$\dot{x}(t) = ax(t), \text{ with } x(0) \text{ given}$$

It corresponds to the integral equation

$$x(t) - x(0) = \int_0^t ax(u)\mathrm{d}u \text{ for all } t \geq 0$$

Starting with a approximation such as $x^{(0)}(t) \equiv x(0)$ for all $t \geq 0$, we can calculate a sequence $t \mapsto x^{(n)}(t)$ $(n \in N)$ of successive approximations to a solution $[0, \infty) \ni t \mapsto x(t) \in \mathbb{R}$ using the iterative rule

$$x^{(n)}(t) - x(0) = \int_0^t ax^{(n-1)}(u)\,\mathrm{d}u \text{ for all } t \geq 0$$

First Iterations

Iterating once starting from $x^{(0)}(t) \equiv x(0)$ gives

$$x^{(1)}(t) - x(0) = \int_0^t a\, x^{(0)}(u)\, \mathrm{d}u = a\, x(0)\, t$$

Iterating a second time gives

$$x^{(2)}(t) - x(0) = \int_0^t a\, x(0)(1 + au)\, \mathrm{d}u = a\, x(0)\, t + \tfrac{1}{2} a^2\, x(0)\, t^2$$

Iterating a third time gives

$$\begin{aligned} x^{(3)}(t) - x(0) &= \textstyle\int_0^t [a\, x(0) + a^2\, x(0)\, u + \frac{1}{2} a^3\, x(0)\, u^2]\, \mathrm{d}u \\ &= a\, x(0)\, t + \tfrac{1}{2} a^2\, x(0)\, t^2 + \tfrac{1}{6} a^3\, x(0)\, u^3 \end{aligned}$$

Terms of the Sum

Each time we are adding a term to a sum, so define the new incremental variable $y^{(n)}(t) := x^{(n)}(t) - x^{(n-1)}(t)$ with $y^{(0)}(t) \equiv x(0)$.

This implies that $x^{(n)}(t) = x(0) + \sum_{k=1}^{n} y^{(k)}(t)$.

Subtract $x^{(n)}(t) - x(0) = \int_0^t a x^{(n-1)}(u)\,\mathrm{d}u$
from $x^{(n+1)}(t) - x(0) = \int_0^t a x^{(n)}(u)\,\mathrm{d}u$
to obtain $y^{(n+1)}(t) = \int_0^t a y^{(n)}(u)\,\mathrm{d}u$.

Now we obtain successively

$$
\begin{aligned}
y^{(1)}(t) &= \int_0^t a\,x(0)\,\mathrm{d}u &&= a\,x(0)\,t \\
y^{(2)}(t) &= \int_0^t a^2\,x(0)\,u\,\mathrm{d}u &&= \tfrac{1}{2}a^2\,x(0)\,t^2 \\
y^{(3)}(t) &= \int_0^t \tfrac{1}{2}a^3\,x(0)\,u^2\,\mathrm{d}u &&= \tfrac{1}{6}a^3\,x(0)\,t^3
\end{aligned}
$$

This suggests the induction hypothesis $y^{(n)}(t) = \dfrac{1}{n!}a^n\,x(0)\,t^n$.

Constructing the Sum

The induction hypothesis $y^{(n)}(t) = \dfrac{1}{n!} a^n \, x(0) \, t^n$
and the relation $y^{(n+1)}(t) = \int_0^t a y^{(n)}(u) \, \mathrm{d}u$ together imply that

$$\begin{aligned} y^{(n+1)}(t) &= \int_0^t a \frac{1}{n!} a^n \, x(0) \, u^n \, \mathrm{d}u \;=\; \frac{1}{n!} a^{n+1} \, x(0) \int_0^t u^n \, \mathrm{d}u \\ &= \frac{1}{n!} a^{n+1} \, x(0) \, \frac{1}{n+1} t^{n+1} \;=\; \frac{1}{(n+1)!} a^{n+1} \, x(0) \, t^{n+1} \end{aligned}$$

This confirms the induction hypothesis with n replaced by $n+1$.

It follows that $y^{(n)}(t) = \dfrac{1}{n!} a^n \, x(0) \, t^n$ for all $n \in \mathbb{N}$
and then that $x^{(n)}(t) = x(0) + \sum_{k=1}^n \dfrac{1}{k!} a^k \, x(0) \, t^k$.

The Exponential Solution

Recall that

$$\exp x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = 1 + \sum\nolimits_{n=1}^{\infty} \frac{1}{n!} x^n = e^x$$

where

$$e = 1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \ldots = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.718281828$$

My late co-author Knut Sydsæter who, as a cultured Norwegian, recognized that 1828 is the year
when their great playright Henrik Ibsen was born,
remembers this 10 digit approximation as "2.7 Ibsen Ibsen".

As $n \to \infty$, the solution $x^{(n)}(t)$ we found
converges to the infinite series

$$x(0) + \sum\nolimits_{k=1}^{\infty} \frac{1}{k!} a^k \, x(0) \, t^k = x(0) \exp(at) = x(0) e^{at}$$

Full (??) Decimal Expansions of Some Important Numbers

$$1/3 = 0.3333333333333333333333333333\ldots$$

$$\sqrt{2} = 1.414213562373095048801688724209698\ldots$$

$$e = 2.71828182845904523536028747135266249\ldots$$

$$\pi = 3.14159265358979323846264338327950288\ldots$$

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General First-Order Affine Equation

The general first-order affine equation takes the form

$$\dot{x}(t) = a(t)x(t) + b(t)$$

for arbitrary integrable functions $t \mapsto a(t)$ and $t \mapsto b(t)$.

In the homogeneous case one has $b(t) \equiv 0$,
and the equation takes the linear form $\dot{x}(t) = a(t)x(t)$.

Assuming that $x > 0$ for all t,
we can take logs and write the equation as

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln x = \frac{\dot{x}}{x} = a(t)$$

After introducing the new variable $y(t) := \ln x(t)$,
the equation becomes $\dot{y} = a(t)$ whose solution is obviously

$$y(t) = y(s) + \int_s^t a(\tau)\mathrm{d}\tau$$

Solution in the Homogenenous Case

Because $x(t) = \exp y(t)$, the solution for x is

$$x(t) = \exp[y(t)] = \exp[y(s)] \exp\left[\int_s^t a(\tau) \mathrm{d}\tau\right] = x(s)\, \alpha_s(t)$$

where $\alpha_s(t)$ denotes the integrating factor $\exp\left[\int_s^t a(\tau) \mathrm{d}\tau\right]$.

In the special case of an autonomous equation
where $a(\tau) = a$ constant, one has $\int_s^t a(\tau) \mathrm{d}\tau = a(t - s)$
and so $\alpha_s(t) = e^{a(t-s)}$.

The Non-Homogenous Case

The solution $x(t) = x(s)\alpha_s(t)$
to the homogeneous equation $\dot{x}(t) - a(t)x(t) = 0$
can be used to help solve the corresponding
non-homogeneous equation $\dot{x}(t) - a(t)x(t) = f(t)$.

Indeed, consider the result of dividing
each side of this non-homogeneous equation
by the integrating factor $\alpha_s(t) := \exp\left[\int_s^t a(\tau)\mathrm{d}\tau\right]$
whose reciprocal is $1/\alpha_s(t) := \exp\left[-\int_s^t a(\tau)\mathrm{d}\tau\right]$.

Note that $\frac{\mathrm{d}}{\mathrm{d}t}\left[-\int_s^t a(\tau)\mathrm{d}\tau\right] = -a(t)$,
implying that $\frac{\mathrm{d}}{\mathrm{d}t}[1/\alpha_s(t)] = -a(t)/\alpha_s(t)$ so, by the product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)/\alpha_s(t)] = [1/\alpha_s(t)]\dot{x}(t) - [a(t)/\alpha_s(t)]x(t) = f(t)/\alpha_s(t)$$

for any solution of the equation $\dot{x}(t) - a(t)x(t) = f(t)$.

Solving the Non-Homogenenous Equation

Integrating each side of the equation $\frac{\mathrm{d}}{\mathrm{d}t}[x(t)/\alpha_s(t)] = f(t)/\alpha_s(t)$ over the interval from s to t gives us

$$\Big|_s^t [x(u)/\alpha_s(u)] = \frac{x(t)}{\alpha_s(t)} - \frac{x(s)}{\alpha_s(s)} = \int_s^t \frac{f(u)}{\alpha_s(u)}\,\mathrm{d}u$$

The definition $\alpha_s(t) = \exp\left[\int_s^t a(\tau)\mathrm{d}\tau\right]$
implies that $\alpha_s(s) = 1$ and also $\alpha_s(t)/\alpha_s(u) = \alpha_u(t)$.

Hence, multiplying each side by $\alpha_s(t)$ gives the solution

$$\begin{aligned} x(t) &= \alpha_s(t)\left[x(s) + \int_s^t [1/\alpha_s(u)]\,f(u)\,\mathrm{d}u\right] \\ &= \alpha_s(t)x(s) + \int_s^t \alpha_u(t)\,f(u)\,\mathrm{d}u \\ &= \exp\left[\int_s^t a(\tau)\mathrm{d}\tau\right]x(s) + \int_s^t \exp\left[\int_u^t a(\tau)\mathrm{d}\tau\right]f(u)\mathrm{d}u \end{aligned}$$

Linearity in the Forcing Term

Theorem

Suppose that $x^P(t)$ and $y^P(t)$ are particular solutions of the two respective differential equations

$$\dot{x}(t) - a(t)x(t) = d(t) \quad \textit{and} \quad \dot{y}(t) - a(t)y(t) = e(t)$$

Then, for any scalars α and β,
the equation $\dot{z}(t) - a(t)z(t) = f(t) = \alpha d(t) + \beta e(t)$
has as a particular solution
the corresponding linear combination $z^P(t) := \alpha x^P(t) + \beta y^P(t)$.

Consider any equation of the form $\dot{x}(t) - a(t)x(t) = f(t)$
where $f(t)$ is a linear combination $\sum_{k=1}^n \alpha_k f^k(t)$
of n forcing terms.

The theorem implies that a particular solution
is the corresponding linear combination $\sum_{k=1}^n \alpha_k x^{Pk}(t)$
of particular solutions to the n equations $\dot{x}(t) - a(t)x(t) = f^k(t)$.

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First-Order Linear Equation with a Constant Coefficient

Next, consider the equation $\dot{x}(t) - ax(t) = f(t)$
where the coefficient a of $x(t)$ has become the constant $a \neq 0$.

The solution we found for the general case was

$$x(t) = \exp\left[\int_s^t a(\tau)\mathrm{d}\tau\right] x(s) + \int_s^t \exp\left[\int_u^t a(\tau)\mathrm{d}\tau\right] f(u)\mathrm{d}u$$

When $a(t) = a$, independent of t, this reduces to

$$x(t) = e^{a(t-s)}x(s) + \int_s^t e^{a(t-u)}f(u)\mathrm{d}u$$

We simplify further by choosing the initial time $s = 0$.

Then

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-u)}f(u)\mathrm{d}u$$

First Special Case

An interesting special case occurs
when the forcing term $f(t)$ is the exponential function $t \mapsto e^{\mu t}$.

Then the solution is

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-u)+\mu u} \mathrm{d}u = e^{at} \left[x(0) + \int_0^t e^{(\mu - a)u} \mathrm{d}u \right]$$

In the degenerate case when $\mu = a$,
one has $\int_0^t e^{(\mu-a)u} \mathrm{d}u = \int_0^t 1 \mathrm{d}u = t$, so the solution collapses to

$$x(t) = e^{at} [x(0) + t]$$

This solution can be written as $x(t) = x^H(t) + x^P(t)$ where:

1. $x^H(t) = \xi^H e^{at}$ with $\xi^H := x(0)$ is a complementary solution of the homogeneous equation $\dot{x}(t) - ax(t) = 0$;
2. $x^P(t) = \xi^P e^{at} t$ with $\xi^P := 1$ is a particular solution of the inhomogeneous equation $\dot{x}(t) - ax(t) = e^{at}$.

Non-Degenerate Case When $\mu \neq a$

In the non-degenerate case when $\mu \neq a$, one has

$$(\mu - a) \int_0^t e^{(\mu-a)u} \mathrm{d}u = |_0^t e^{(\mu-a)u} = e^{(\mu-a)t} - 1$$

So the solution is

$$x(t) = e^{at} \left[x(0) + \frac{e^{(\mu-a)t} - 1}{\mu - a} \right] = e^{at} x(0) + \frac{e^{\mu t} - e^{at}}{\mu - a}$$

Again, this solution can be written as $x(t) = x^H(t) + x^P(t)$ where:

1. $x^H(t) = \xi^H e^{at}$ with $\xi^H := x(0) - 1/(\mu - a)$ is a solution of the homogeneous equation $\dot{x}(t) - ax(t) = 0$;
2. $x^P(t) = \xi^P e^{\mu t}$ with $\xi^P := 1/(\mu - a)$ is a particular solution of the inhomogeneous equation $\dot{x}(t) - ax(t) = e^{\mu t}$.

Second Special Case

Another interesting special case occurs when $f(t) = t^r e^{\mu t}$ for some $r \in \mathbb{N}$.

Then the solution $x(t) = e^{at}x(0) + \int_0^t e^{a(t-u)} f(u) \mathrm{d}u$ becomes

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-u)} u^r e^{\mu u} \mathrm{d}u = e^{at} \left[x(0) + \int_0^t u^r e^{(\mu - a)u} \mathrm{d}u \right]$$

In the degenerate case when $\mu = a$, the solution collapses to

$$x(t) = e^{at} \left[x(0) + |_0^t (r+1)^{-1} u^{r+1} \right] = e^{at} \left[x(0) + (r+1)^{-1} t^{r+1} \right]$$

This solution can be written as $x(t) = x^H(t) + x^P(t)$ where:

1. $x^H(t) = \xi^H e^{at}$ with $\xi^H := x(0)$ is a solution of the homogeneous equation $\dot{x}(t) - ax(t) = 0$;
2. $x^P(t) = \xi^P e^{at} t^{r+1}$ with $\xi^P := (r+1)^{-1}$ is a particular solution of the inhomogeneous equation $\dot{x}(t) - ax(t) = t^r e^{at}$.

Non-Degenerate Case When $\mu \neq a$

In the non-degenerate case when $\mu \neq a$, the solution is

$$x(t) = e^{at} \left[x(0) + \int_0^t u^r e^{(\mu - a)u} \mathrm{d}u \right] = e^{at} \left[x(0) + I_r(t) \right]$$

where $I_r(t) := \int_0^t u^r e^{(\mu-a)u} \mathrm{d}u$.

In particular, $I_0(t) = \int_0^t e^{(\mu-a)u} \mathrm{d}u = (\mu - a)^{-1}[e^{(\mu-a)t} - 1]$.

Integrating by parts gives the first-order linear difference equation

$$\begin{aligned} I_r(t) &= \int_0^t u^r e^{(\mu-a)u} \mathrm{d}u \\ &= (\mu - a)^{-1} |_0^t u^r e^{(\mu-a)u} - r(\mu - a)^{-1} \int_0^t u^{r-1} e^{(\mu-a)u} \mathrm{d}u \\ &= (a - \mu)^{-1} \left[r\, I_{r-1}(t) - t^r e^{(\mu-a)t} \right] \end{aligned}$$

Solving the First-Order Linear Difference Equation

Let us divide each side of the difference equation

$$I_r(t) = (a-\mu)^{-1}\left[r\, I_{r-1}(t) - t^r e^{(\mu-a)t}\right]$$

by the "summing factor" $\prod_{k=1}^r k(a-\mu)^{-1} = r!(a-\mu)^{-r}$ to get

$$\begin{aligned}
J_r(t) &:= \frac{1}{r!}(a-\mu)^r I_r(t) \\
&= \frac{1}{r!}\left[r(a-\mu)^{r-1} I_{r-1}(t) - (a-\mu)^{-1} t^r e^{(\mu-a)t}\right] \\
&= \frac{1}{(r-1)!}(a-\mu)^{r-1} I_{r-1}(t) - \frac{1}{r!}(a-\mu)^{r-1} t^r e^{(\mu-a)t} \\
&= J_{r-1}(t) - \frac{1}{r!}(a-\mu)^{r-1} t^r e^{(\mu-a)t}
\end{aligned}$$

This obviously implies that

$$J_r(t) = J_0(t) - \sum_{k=1}^r \frac{1}{k!}(a-\mu)^{k-1} t^k e^{(\mu-a)t}$$

Solving the Differential Equation

Because $J_0(t) = I_0(t) = (\mu - a)^{-1}[e^{(\mu-a)t} - 1]$, this implies that

$$J_r(t) = (\mu - a)^{-1}[e^{(\mu-a)t} - 1] - \sum_{k=1}^{r} \frac{1}{k!}(a-\mu)^{k-1} t^k e^{(\mu-a)t}$$

But $J_r(t) = \frac{1}{r!}(a-\mu)^r I_r(t)$, so

$$\begin{aligned} I_r(t) &:= r!(a-\mu)^{-r} J_r(t) \\ &= -r!(a-\mu)^{-r-1}[e^{(\mu-a)t} - 1] \\ &\qquad - \sum_{k=1}^{r} \frac{r!}{k!}(a-\mu)^{k-r-1} t^k e^{(\mu-a)t} \end{aligned}$$

Then

$$\begin{aligned} x(t) &= e^{at}[x(0) + I_r(t)] \\ &= e^{at}\left[x(0) + r!(a-\mu)^{-r-1}\right] \\ &\qquad - r!(a-\mu)^{-r-1} e^{\mu t}\left[1 + \sum_{k=1}^{r} \frac{1}{k!}(a-\mu)^k t^k\right] \end{aligned}$$

Particular and General Solution

For the equation $\dot{x}(t) - ax(t) = t^r e^{\mu t}$ with $\mu \neq a$, the solution

$$x(t) = e^{at}\left[x(0) + r!(a-\mu)^{-r-1}\right] - r!(a-\mu)^{-r-1}e^{\mu t}\left[1 + \sum_{k=1}^{r} \frac{1}{k!}(a-\mu)^k t^k\right]$$

can be written as $x(t) = x^H(t) + x^P(t)$ where:

1. $x^H(t) = \xi^H e^{at}$ with $\xi^H := x(0) + r!(a-\mu)^{-r-1}$
   is a solution of the homogeneous equation $\dot{x}(t) - ax(t) = 0$;
2. $x^P(t) = \xi^P(t)e^{\mu t}$, where the polynomial

$$t \mapsto \xi^P(t) := -r!(a-\mu)^{-r-1}\left[1 + \sum_{k=1}^{r} \frac{1}{k!}(a-\mu)^k t^k\right]$$

   of degree r in t is a particular solution
   of the inhomogeneous equation $\dot{x}(t) - ax(t) = t^r e^{\mu t}$.

Method of Undetermined Coefficients

A practical issue is finding what polynomial

$$t \mapsto \xi^P(t) = \sum\nolimits_{k=0}^{r} \xi_k t^k$$

of degree r (the power of t on the right-hand side)
makes $\xi^P(t)e^{\mu t}$ a particular solution
of the inhomogeneous differential equation $\dot{x}(t) - ax(t) = t^r e^{\mu t}$.

The coefficients $(\xi_0, \xi_1, \ldots, \xi_r)$ of the polynomial
are undetermined
till we choose the associated polynomial $t \mapsto \xi^P(t)$
to make $\xi^P(t)e^{\mu t}$ satisfy the differential equation.

Determining the Undetermined Coefficients

For $x^P(t) = e^{\mu t} \sum_{k=0}^{r} \xi_k t^k$ to solve $\dot{x}(t) - ax(t) = t^r e^{\mu t}$, we need

$$\begin{aligned} t^r e^{\mu t} &= \mu e^{\mu t} \sum_{k=0}^{r} \xi_k t^k + e^{\mu t} \sum_{k=1}^{r} \xi_k k t^{k-1} - a e^{\mu t} \sum_{k=0}^{r} \xi_k t^k \\ &= (\mu - a) e^{\mu t} \xi_r t^r + e^{\mu t} \sum_{k=0}^{r-1} \left[(\mu - a)\xi_k + \xi_{k+1}(k+1) \right] t^k \end{aligned}$$

First consider the non-degenerate case $\mu \neq a$.

For $k = r$, this implies that $(\mu - a)\xi_r = 1$, so $\xi_r = (\mu - a)^{-1}$.

For $k = 0, 1, \ldots, r-1$, it implies that $(\mu - a)\xi_k + \xi_{k+1}(k+1) = 0$ or that $\xi_k = (a - \mu)^{-1}(k+1)\xi_{k+1}$, and so

$$\begin{aligned} \xi_k &= \left[\prod_{j=k}^{r-1} (a - \mu)^{-1}(j+1) \right] \xi_r \\ &= \frac{r!}{k!}(a - \mu)^{k-r} \xi_r \;=\; -\frac{r!}{k!}(a - \mu)^{k-r+1} \end{aligned}$$

This matches our previous answer.

Degenerate Case

But in the degenerate case $\mu = a$, the method does not work.

Instead, to solve $\dot{x}(t) - ax(t) = t^r e^{at}$,
we introduce the new variable $y(t) = e^{-at}x(t)$.

Then $\dot{y}(t) = e^{-at}[\dot{x}(t) - ax(t)] = e^{-at}t^r e^{at} = t^r$.

The solution to this differential equation
is $y(t) = y(0) + \int_0^t u^r \mathrm{d}u = y(0) + (r+1)^{-1}t^{r+1}$.

The solution to the original differential equation
is therefore $x(t) = e^{at}y(t) = e^{at}\left[x(0) + (r+1)^{-1}t^{r+1}\right]$.

The polynomial in t that occurs in this solution
is now of degree $r + 1$ rather than r.

Main Theorem

Theorem

Consider the inhomogeneous first-order linear differential equation

$$\dot{x}(t) - ax(t) = t^r e^{\mu t}, \text{ where } a \neq 0 \text{ and } r \in \mathbb{Z}_+.$$

There exists a particular solution of the form $x^P(t) = Q(t)\, e^{\mu t}$

where the function $t \mapsto Q(t)$ *is a polynomial in* t *of degree:*

- r *in the regular case when* $\mu \neq a$;
- $r + 1$ *in the degenerate case when* $\mu = a$.

The general solution takes the form $x(t) = x^P(t) + x^C(t)$ *where:*

- $x^P(t)$ *is any particular solution;*
- $x^C(t)$ *is any member of the one-dimensional linear space of complementary solutions to the corresponding homogeneous equation* $\dot{x}(t) - ax(t) = 0$.

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The Autonomous Case

The autonomous case occurs
when the first-order affine equation takes the form

$$\dot{x} = ax + b$$

with the right-hand side independent of t.

The steady state at which $\dot{x}(t) = 0$ occurs when $ax + b = 0$,
and so at $x^* := -b/a$.

Then the deviation $y(t) := x(t) - x^*$ of $x(t)$
from the steady state x^* satisfies the homogeneous equation

$$\dot{y}(t) = \dot{x}(t) = ax(t) + b = a[y(t) + x^*] + b = ay(t)$$

Hence $y(t) = e^{at}y(0)$, implying that $x(t) = x^* + e^{at}[x(0) - x^*]$.

Stability

The steady state $x^* := -b/a$ is stable just in case, for all $x(0)$, the solution $x(t) = x^* + e^{at}[x(0) - x^*]$ satisfies $x(t) \to x^*$ as $t \to \infty$.

A necessary and sufficient condition for stability is obviously that $a < 0$.

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Second-Order Equations with Constant Coefficients

A general second-order differential equation takes the form

$$\ddot{x}(t) = F(\dot{x}(t), x(t), t)$$

To obtain a unique solution (if any solution exists),
one typically needs two initial conditions
such as $x(s) = x_s$ and $\dot{x}(s) = \dot{x}_s$ at an initial time s.

The equation is autonomous just in case
it takes the form $\ddot{x}(t) = F(\dot{x}(t), x(t))$, with F independent of t.

The equation is linear just in case
it takes the form $\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0$,
with F linear in $(\dot{x}(t), x(t))$.

The equation is linear with constant coefficients just in case
it takes the form $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$.

Characteristic Equation

We know that the first-order equation $\dot{x}(t) + ax(t) = 0$
has a solution of the form $x(t) = x(0)e^{\lambda t}$
where λ solves the characteristic equation $\lambda + a = 0$.

So we look for solutions of the form $x(t) = \xi e^{\lambda t}$
to the second-order equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$.

Note that when $x(t) = \xi e^{\lambda t}$,
then $\dot{x}(t) = \lambda \xi e^{\lambda t}$ and $\ddot{x}(t) = \lambda^2 \xi e^{\lambda t}$.

So $x(t) = \xi e^{\lambda t}$ is a non-trivial solution (with $\xi \neq 0$) if and only if

$$0 = \lambda^2 \xi e^{\lambda t} + a\lambda \xi e^{\lambda t} + b\xi e^{\lambda t} = (\lambda^2 + a\lambda + b)\xi e^{\lambda t}$$

and so, given that $\xi e^{\lambda t} \neq 0$, if and only if λ is a root
of the characteristic equation $\lambda^2 + a\lambda + b = 0$.

Characteristic Equation for an Equation of Order n

Definition

A homogeneous differential equation of order n
with constant coefficients takes the form

$$\sum_{k=0}^{n} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t) = 0$$

where the coefficient of the n derivative satisfies $a_n \neq 0$,
and so can be normalized to take the value $a_n = 1$.

Remark

A similar technique based on roots of the characteristic equation applies to this nth order equation.

It implies that $x(t) = \xi e^{\lambda t}$ is a non-trivial solution if and only if λ is a root of the characteristic equation

$$\sum_{k=0}^{n} a_k \lambda^k = 0$$

Characteristic Roots

One can factorize the quadratic function $q(\lambda) := \lambda^2 + a\lambda + b$
as $q(\lambda) := (\lambda - \lambda_1)(\lambda - \lambda_2)$
where λ_1 and λ_2 are the two roots of the equation $q(\lambda) = 0$.

As with the corresponding discussion
of second-order difference equations, there are three cases:

1. in case $a^2 > 4b$, there are two distinct real roots λ_1 and λ_2 given by $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.
2. in case $a^2 < 4b$, there are two complex conjugate roots given by $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}i\sqrt{4b - a^2}$.
3. in case $a^2 = 4b$, there are two coincident real roots given by $\lambda = -\frac{1}{2}a = \sqrt{b}$.

Case 1: Two Distinct Real Roots

In this case $a^2 > 4b$, when the two characteristic roots are $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.

Because $\lambda_1 \neq \lambda_2$, one has

$$\begin{vmatrix} e^{\lambda_1 0} & e^{\lambda_2 0} \\ e^{\lambda_1 1} & e^{\lambda_2 1} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} \end{vmatrix} = e^{\lambda_2} - e^{\lambda_1} \neq 0$$

and so $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two linearly independent solutions.

The general solution of the homogeneous equation is

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

Case 2: Two Complex Conjugate Roots, I

In this case $a^2 < 4b$, when the two characteristic roots are the complex conjugates $\lambda_{1,2} = -\frac{1}{2}a \pm i\theta$, with $\theta := \frac{1}{2}\sqrt{4b - a^2}$.

Then $x(t) = e^{\lambda_1 t} = e^{-\frac{1}{2}at} e^{i\theta t} = e^{-\frac{1}{2}at}(\cos\theta t + i \sin\theta t)$ and $x(t) = e^{\lambda_2 t} = e^{-\frac{1}{2}at} e^{-i\theta t} = e^{-\frac{1}{2}at}(\cos\theta t - i \sin\theta t)$ are two different solutions, where $\theta \neq 0$.

For any t such that $\sin\theta t \neq 0$, one has

$$\begin{vmatrix} e^{\lambda_1 0} & e^{\lambda_2 0} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{vmatrix} = e^{\lambda_2 t} - e^{\lambda_1 t} = -2e^{-\frac{1}{2}at} i \sin\theta t \neq 0$$

It follows that $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two linearly independent solutions in the complex plane $\mathbb{C}$.

Case 2: Two Complex Conjugate Roots, II

Focusing on solutions in the real line $\mathbb{R}$,
we can consider $e^{-\frac{1}{2}at}\cos\theta t$ and $e^{-\frac{1}{2}at}\sin\theta t$.

Again, for any t such that $\sin\theta t \neq 0$, one has

$$\begin{aligned}
\begin{vmatrix} e^{-\frac{1}{2}a0}\cos\theta 0 & e^{-\frac{1}{2}a0}\sin\theta 0 \\ e^{-\frac{1}{2}at}\cos\theta t & e^{-\frac{1}{2}at}\sin\theta t \end{vmatrix} &= \begin{vmatrix} 1 & 0 \\ -\frac{1}{2}at\cos\theta t & e^{-\frac{1}{2}at}\sin\theta t \end{vmatrix} \\
&= e^{-\frac{1}{2}at}\sin\theta t \neq 0
\end{aligned}$$

It follows that $e^{-\frac{1}{2}at}\cos\theta t$ and $e^{-\frac{1}{2}at}\sin\theta t$ are two linearly independent real-valued solutions in the complex plane $\mathbb{C}$.

The general solution of the homogeneous equation
is $x = e^{-\frac{1}{2}at}(A\cos\theta t + B\sin\theta t)$.

Case 3: Two Coincident Real Roots

In this case $a^2 = 4b$, and so

$$q(\lambda) = \lambda^2 + a\lambda + b = (\lambda + \tfrac{1}{2}a)^2 = (\lambda - \sqrt{b})^2$$

The homogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$
has one solution given by $x = e^{\lambda t}$ where $\lambda = -\frac{1}{2}a = \sqrt{b}$.

To find a second linearly independent solution,
introduce the new variable $y(t) := e^{-\lambda t}x(t)$.

Then $\dot{y}(t) = e^{-\lambda t}\dot{x}(t) - \lambda e^{-\lambda t}x(t)$ and so, when $x = e^{\lambda t}$, one has

$$\begin{aligned}
\ddot{y}(t) &= e^{-\lambda t}\ddot{x}(t) - 2\lambda e^{-\lambda t}\dot{x}(t) + \lambda^2 e^{-\lambda t}x(t) \\
&= e^{-\lambda t}[\ddot{x}(t) - 2\lambda\dot{x}(t) + \lambda^2 x(t)] \\
&= e^{-\lambda t}[\lambda^2 e^{\lambda t} - 2\lambda \cdot \lambda e^{\lambda t} + \lambda^2 e^{\lambda t}] = 0
\end{aligned}$$

The obvious general solution to $\ddot{y}(t) = 0$
satisfies $\dot{y}(t) =$ constant and so $y(t) = A + Bt = e^{-\lambda t}x(t)$.

Hence $x(t) = (A + Bt)e^{\lambda t}$ is the general solution.

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The Inhomogeneous Equation

Consider next the inhomogeneous equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t)$$

with a non-zero forcing term on the right-hand side.

Suppose that $y(t)$ and $z(t)$ are both solutions, implying that

$$\begin{array}{rrcl} & \ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) & = & f(t) \\ \text{and} & \ddot{z}(t) + a(t)\dot{z}(t) + b(t)z(t) & = & f(t) \end{array}$$

Subtracting the second equation from the first tells us that the function $x_H(t) := y(t) - z(t)$ is a solution of the corresponding homogenenous equation $\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0$.

So the general solution of $\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t)$ is the sum $x_G(t) = x_P(t) + x_H(t)$ of:

- any particular solution $x_P(t)$ of the inhomogeneous equation;
- any function $x_H(t)$ in the two dimensional linear space of solutions to the homogeneous equation.

Linearity in the Forcing Term, I

Theorem

Suppose that $x^P(t)$ and $y^P(t)$ are particular solutions of the two respective differential equations

$$\begin{aligned} \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) &= d(t) \\ \text{and} \quad \ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) &= e(t) \end{aligned}$$

Then, for any scalars α and β, a particular solution of the equation

$$\ddot{z}(t) + a(t)\dot{z}(t) + b(t)z(t) = f(t) = \alpha d(t) + \beta e(t)$$

is the linear combination $z^P(t) := \alpha x^P(t) + \beta y^P(t)$.

Linearity in the Forcing Term, II

Consider the equation $\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t)$
whose forcing term $f(t)$ is a linear combination $\sum_{k=1}^{n} \alpha_k f^k(t)$
of n forcing terms.

The theorem implies that a particular solution
is the corresponding linear combination $\sum_{k=1}^{n} \alpha_k x^{Pk}(t)$
of particular solutions to the n equations

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f^k(t) \quad (k = 1, 2, \ldots, n)$$

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A Newtonian Example, I

Newton's law: force $=$ mass $\times$ acceleration.

A force of 1 Newton, by definition, accelerates a mass of 1 kilogram at the rate of 1 metre per second per second.

So we consider the equation $\ddot{x}(t) = f(t)$ whose solution $t \mapsto x(t)$ is the position (in one dimension) of a 1 kilogram weight that has been subjected to a force function $t \mapsto f(t)$.

Integrating once gives us the equation $\dot{x}(t) = \dot{x}(0) + \int_0^t f(u)\mathrm{d}u$.

Integrating a second time gives us the solution

$$\begin{aligned} x(t) \ = \ x(0) + \int_0^t \dot{x}(v)\mathrm{d}v \quad &= \quad x(0) + \int_0^t \left[\dot{x}(0) + \int_0^v f(u)\mathrm{d}u\right]\mathrm{d}v \\ &= \quad x(0) + \dot{x}(0)t + \int_0^t \left[\int_0^v f(u)\mathrm{d}u\right]\mathrm{d}v \end{aligned}$$

Note that $x(0) + \dot{x}(0)t$ solves the homogeneous equation $\ddot{x}(t) = 0$, whereas the iterated double integral $\int_0^t \left[\int_0^v f(u)\mathrm{d}u\right]\mathrm{d}v$ is a particular solution.

An Important Theorem on Iterated Double Integrals, I

Theorem

For any integrable function $(x, y) \mapsto \phi(x, y) \in \mathbb{R}$
defined on the square domain $[a, b] \times [a, b] \subset \mathbb{R}^2$*, one has*

$$\int_a^b \left[\int_a^y \phi(x, y) \mathrm{d}x \right] \mathrm{d}y = \int_a^b \left[\int_x^b \phi(x, y) \mathrm{d}y \right] \mathrm{d}x$$

Proof.

Define the indicator function $1_{x \le y}(x, y) := \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{if } x > y \end{cases}$. Then

$$\begin{aligned} \int_a^b \left[\int_a^y \phi(x, y) \mathrm{d}x \right] \mathrm{d}y &= \int_a^b \left[\int_a^b 1_{x \le y}(x, y) \phi(x, y) \mathrm{d}x \right] \mathrm{d}y \\ \int_a^b \left[\int_x^b \phi(x, y) \mathrm{d}y \right] \mathrm{d}x &= \int_a^b \left[\int_a^b 1_{x \le y}(x, y) \phi(x, y) \mathrm{d}y \right] \mathrm{d}x \end{aligned}$$

But both right-hand sides equal $\int_a^b \int_a^b 1_{x \le y}(x, y) \phi(x, y) \mathrm{d}x \mathrm{d}y$. □

An Important Theorem on Iterated Double Integrals, II

An alternative simple proof involves noticing that the two integrals

$$\int_a^b \left[\int_a^y \phi(x, y) \mathrm{d}x \right] \mathrm{d}y \quad \text{and} \quad \int_a^b \left[\int_x^b \phi(x, y) \mathrm{d}y \right] \mathrm{d}x$$

are simply two different ways of writing
the integral $\iint_T \phi(x, y) \mathrm{d}x \mathrm{d}y$ of the function ϕ of two variables
over the isosceles right-angled triangle

$$T := \{(x, y) \in [a, b] \times [a, b] \subset \mathbb{R}^2 \mid x \leq y\}$$

Note that T consists of points above and to the left
of the diagonal that joins the two corner points (a, a) and (b, b)
of the square $[a, b] \times [a, b]$.

The set T is also the convex hull
of the three points (a, a), (a, b) and (b, b).

A Newtonian Example, II

Reversing the order of integration allows the particular solution in the form of the iterated double integral $\int_0^t \left[\int_0^v f(u)\mathrm{d}u\right]\mathrm{d}v$ to be rewritten as

$$\int_0^t \left[\int_u^t f(u)\mathrm{d}v\right]\mathrm{d}u = \int_0^t \left[\int_u^t 1\mathrm{d}v\right] f(u)\mathrm{d}u = \int_0^t (t-u)f(u)\mathrm{d}u$$

Ultimately, then, one has

$$x(t) = x(0) + \dot{x}(0)t + \int_0^t (t-u)f(u)\mathrm{d}u$$

Linear Equation with Constant Coefficients, I

Next, consider the equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = f(t)$
where the coefficients a of $\dot{x}(t)$ and b of $\dot{x}(t)$
have both become constants, with $b \neq 0$.

Consider the quadratic function $q(\lambda) := \lambda^2 + a\lambda + b$
that appears in the characteristic equation $\lambda^2 + a\lambda + b = 0$.

One can factorize it as

$$q(\lambda) = \lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

where λ_1 and λ_2 are the two roots of the equation $q(\lambda) = 0$.

Recall that $\lambda_1 + \lambda_2 = -a$ and $\lambda_1 \lambda_2 = b$.

Define the new variable $y(t) := \dot{x}(t) - \lambda_1 x(t)$.

Note that, if we could find the function $t \mapsto y(t)$,
then we would have

$$x(t) = e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} y(u) \mathrm{d}u$$

Linear Equation with Constant Coefficients, II

We are considering the equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = f(t)$, with $b \neq 0$.

We have introduced the new variable $y(t) := \dot{x}(t) - \lambda_1 x(t)$, implying that $x(t) = e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} y(u) \mathrm{d}u$.

But the characteristic roots satisfy $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$, implying that $\lambda_1 + \lambda_2 = -a$ and $\lambda_1 \lambda_2 = b$, and so

$$\begin{aligned}
\dot{y}(t) - \lambda_2 y(t) &= \ddot{x}(t) - \lambda_1 \dot{x}(t) - \lambda_2 \dot{x}(t) + \lambda_1 \lambda_2 x(t) \\
&= \ddot{x}(t) + a\dot{x}(t) + bx(t)
\end{aligned}$$

Hence $y(t)$ satisfies the first-order equation $\dot{y}(t) - \lambda_2 y(t) = f(t)$ whose solution is

$$y(t) = e^{\lambda_2 t} y(0) + \int_0^t e^{\lambda_2 (t-v)} f(v) \mathrm{d}v$$

Linear Equation with Constant Coefficients, III

Substituting $y(t) = e^{\lambda_2 t} y(0) + \int_0^t e^{\lambda_2(t-v)} f(v) \mathrm{d}v$

in the expression $x(t) = e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1(t-u)} y(u) \mathrm{d}u$ gives

$$\begin{aligned} x(t) &= e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1(t-u)} y(u) \mathrm{d}u \\ &= e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1(t-u)} \left[e^{\lambda_2 u} y(0) + \int_0^u e^{\lambda_2(u-v)} f(v) \mathrm{d}v \right] \mathrm{d}u \end{aligned}$$

We split this form of the solution into two parts:

1. the complementary solution

$$\begin{aligned} t \mapsto x^C(t) &:= e^{\lambda_1 t} x(0) + y(0) \int_0^t e^{\lambda_1(t-u)} e^{\lambda_2 u} \mathrm{d}u \\ &= e^{\lambda_1 t} \left[x(0) + y(0) \int_0^t e^{(\lambda_2 - \lambda_1) u} \mathrm{d}u \right] \end{aligned}$$

   to the homogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$;

2. a particular solution in the form of the iterated double integral

$$t \mapsto x^P(t) := \int_0^t e^{\lambda_1(t-u)} \left[\int_0^u e^{\lambda_2(u-v)} f(v) \mathrm{d}v \right] \mathrm{d}u$$

   to the inhomogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = f(t)$.

Degenerate Case

In the degenerate case when $\lambda_1 = \lambda_2 = \lambda$,

1. the complementary solution takes the form:

$$\begin{aligned} x^C(t) &= e^{\lambda t} x(0) + y(0) \int_0^t e^{\lambda t} \mathrm{d}u \\ &= e^{\lambda t} \left[x(0) + y(0) t \right] \end{aligned}$$

2. the particular solution takes the form:

$$\begin{aligned} x^P(t) &= \int_0^t e^{\lambda(t-u)} \left[\int_0^u e^{\lambda(u-v)} f(v) \mathrm{d}v \right] \mathrm{d}u \\ &= e^{\lambda t} \int_0^t \left[\int_0^u e^{-\lambda v} f(v) \mathrm{d}v \right] \mathrm{d}u \\ &= e^{\lambda t} \int_0^t \left[\int_v^t 1 \mathrm{d}u \right] e^{-\lambda v} f(v) \mathrm{d}v \\ &= \int_0^t (t - v) e^{\lambda(t-v)} f(v) \mathrm{d}v \end{aligned}$$

The overall solution is therefore

$$x(t) = e^{\lambda t} \left[x(0) + y(0) t + \int_0^t (t - v) e^{-\lambda v} f(v) \mathrm{d}v \right]$$

Non-Degenerate Case: Complementary Solution

In the non-degenerate case when $\lambda_1 \neq \lambda_2$,
the complementary solution takes the form

$$\begin{aligned} x^C(t) &= e^{\lambda_1 t}\left[x(0) + y(0)\int_0^t e^{(\lambda_2-\lambda_1)u}\mathrm{d}u\right] \\ &= e^{\lambda_1 t}x(0) + \frac{1}{\lambda_2-\lambda_1}e^{\lambda_1 t}y(0)\left[e^{(\lambda_2-\lambda_1)t}-1\right] \\ &= x(0)e^{\lambda_1 t} + y(0)\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1} \end{aligned}$$

After substituting $\dot{x}(0) - \lambda_1 x(0)$ for $y(0)$,
the right-hand side becomes

$$\frac{1}{\lambda_2-\lambda_1}\left\{(\lambda_2-\lambda_1)x(0)e^{\lambda_1 t} + [\dot{x}(0)-\lambda_1 x(0)](e^{\lambda_2 t}-e^{\lambda_1 t})\right\}$$

and so

$$x^C(t) = \frac{1}{\lambda_2-\lambda_1}\left[x(0)(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}) + \dot{x}(0)(e^{\lambda_2 t}-e^{\lambda_1 t})\right]$$

Non-Degenerate Case: Particular Solution

Using our rule for reversing the order of recursive integration, the particular solution takes the form

$$\begin{aligned}
x^P(t) &= \int_0^t e^{\lambda_1(t-u)} \left[\int_0^u e^{\lambda_2(u-v)} f(v) \mathrm{d}v\right] \mathrm{d}u \\
&= \int_0^t \left[\int_v^t e^{\lambda_1(t-u)} e^{\lambda_2(u-v)} \mathrm{d}u\right] f(v) \mathrm{d}v \\
&= \int_0^t e^{\lambda_1 t - \lambda_2 v} \left[\int_v^t e^{(\lambda_2-\lambda_1)u} \mathrm{d}u\right] f(v) \mathrm{d}v \\
&= \frac{1}{\lambda_2 - \lambda_1} \int_0^t e^{\lambda_1 t - \lambda_2 v} \left[e^{(\lambda_2-\lambda_1)t} - e^{(\lambda_2-\lambda_1)v}\right] f(v) \mathrm{d}v \\
&= \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)}\right] f(v) \mathrm{d}v
\end{aligned}$$

First Special Case

An interesting first special case of the particular solution

$$x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[e^{\lambda_2 (t-v)} - e^{\lambda_1 (t-v)} \right] f(v) \mathrm{d}v$$

occurs when $f(t)$ is the exponential function $e^{\mu t}$, and so

$$x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[e^{\lambda_2 (t-v)} - e^{\lambda_1 (t-v)} \right] e^{\mu v} \mathrm{d}v$$

In the degenerate case when $\lambda_2 = \mu \neq \lambda_1$, this reduces to

$$\begin{aligned} x^P(t) &= \frac{1}{\lambda_2 - \lambda_1} \left[e^{\lambda_2 t} t - e^{\lambda_1 t} \int_0^t e^{(\mu - \lambda_1) v} \mathrm{d}v \right] \\ &= \frac{e^{\lambda_2 t} t}{\lambda_2 - \lambda_1} - \frac{e^{\lambda_1 t} (e^{(\lambda_2 - \lambda_1) t} - 1)}{(\lambda_2 - \lambda_1)^2} \\ &= \frac{e^{\lambda_2 t} t}{\lambda_2 - \lambda_1} - \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{(\lambda_2 - \lambda_1)^2} \end{aligned}$$

Non-Degenerate Case

In the non-degenerate case when λ_1, λ_2 and μ are all different, one has the particular solution

$$\begin{aligned} x^P(t) &= \frac{e^{\lambda_2 t}}{\lambda_2 - \lambda_1} \int_0^t e^{(\mu-\lambda_2)v} \mathrm{d}v - \frac{e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \int_0^t e^{(\mu-\lambda_1)v} \mathrm{d}v \\ &= \frac{e^{\lambda_2 t}\left[e^{(\mu-\lambda_2)t} - 1\right]}{(\lambda_2 - \lambda_1)(\mu - \lambda_2)} - \frac{e^{\lambda_1 t}\left[e^{(\mu-\lambda_1)t} - 1\right]}{(\lambda_2 - \lambda_1)(\mu - \lambda_1)} \\ &= \frac{1}{\lambda_2 - \lambda_1}\left(\frac{e^{\mu t} - e^{\lambda_2 t}}{\mu - \lambda_2} - \frac{e^{\mu t} - e^{\lambda_1 t}}{\mu - \lambda_1}\right) \end{aligned}$$

But the multiples of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ can be incorporated in the complementary solution to the homogeneous equation, so this particular solution can be reduced to

$$\tilde{x}^P(t) = \frac{e^{\mu t}}{\lambda_2 - \lambda_1}\left(\frac{1}{\mu - \lambda_2} - \frac{1}{\mu - \lambda_1}\right) = \frac{e^{\mu t}}{(\mu - \lambda_1)(\mu - \lambda_2)}$$

Second Special Case

An interesting second special case of the particular solution

$$x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)} \right] f(v) \mathrm{d}v$$

occurs when $f(t)$ is the exponential function $t^r e^{\mu t}$, and so

$$x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)} \right] v^r e^{\mu v} \mathrm{d}v$$

In the non-degenerate case when λ_1, λ_2 and μ are all different, this becomes

$$\begin{aligned} x^P(t) &= \frac{1}{\lambda_2 - \lambda_1} \left[e^{\lambda_2 t} \int_0^t v^r e^{(\mu - \lambda_2)v} \mathrm{d}v - e^{\lambda_1 t} \int_0^t v^r e^{(\mu - \lambda_1)v} \mathrm{d}v \right] \\ &= P_2(t) e^{\lambda_2 t} - P_1(t) e^{\lambda_1 t} \end{aligned}$$

for polynomials $t \mapsto P_1(t)$ and $t \mapsto P_2(t)$ of degree r whose coefficients are functions of the parameter triple $(\lambda_1, \lambda_2, \mu)$.

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The Autonomous Equation

Now consider the autonomous equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = c$$

with a constant right-hand side.

There is a constant solution $x(t) = \bar{x}$
where $\bar{x} = c/b$ is the unique steady state.

The new variable $y(t) := x(t) - \bar{x}$ satisfies
the homogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$.

The associated characteristic equation is

$$\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

A Stability Condition

1. In case there there are two real characteristic roots
$$\lambda_{1,2} = -\tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{a^2 - 4b}$$
the general solution $Ae^{\lambda_1 t} + Be^{\lambda_2 t} \to 0$ as $t \to \infty$
if and only if both λ_1 and λ_2 are negative.
2. In case there are two complex conjugate characteristic roots
$$\lambda_{1,2} = -\tfrac{1}{2}a \pm \tfrac{1}{2}i\sqrt{4b - a^2}$$
one has $e^{\lambda t} = e^{-\frac{1}{2}at} e^{\pm\frac{1}{2}it\sqrt{4b-a^2}}$.
The general solution $Ae^{\lambda_1 t} + Be^{\lambda_2 t} \to 0$ as $t \to \infty$
iff $a > 0$, or iff both λ_1 and λ_2 have negative real parts.
3. In case there are two coincident real characteristic roots,
the general solution $(A + Bt)e^{\lambda t} \to 0$ as $t \to \infty$ iff $\lambda < 0$.

All these conditions can be subsumed into one:
stability holds iff each characteristic root has a negative real part.

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Linear Differential Equation in n Variables

A linear differential equation in n variables specifies the time derivative $\dot{\mathbf{x}}(t)$ of the n-vector $\mathbf{x}(t)$ as an affine function $\mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)$ of $\mathbf{x}(t)$.

That is

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)$$

where

- $t \mapsto \mathbf{A}(t) \in \mathbb{R}^{n \times n}$ is a matrix-valued function of time;
- $t \mapsto \mathbf{b}(t) \in \mathbb{R}^n$ is a vector-valued function of time.

Matrix Differentiation

Consider the $m \times n$ matrix function $t \mapsto \mathbf{A}(t)$ whose elements $(a_{ij}(t))_{m \times n}$ are differentiable functions of t.

For all $h \neq 0$, the Newton quotient matrix $\frac{1}{h}[\mathbf{A}(t+h) - \mathbf{A}(t)]$ has elements equal to the Newton quotients $\frac{1}{h}(a_{ij}(t+h) - a_{ij}(t))_{m \times n}$ of the matrix $(a_{ij}(t))_{m \times n}$.

As $h \to 0$, these converge to the derivatives $\left(\frac{d}{dt} a_{ij}(t)\right)_{m \times n}$.

For this reason, the matrix $\mathbf{A}(t)$ is said to be differentiable with derivative $\dot{\mathbf{A}}(t) = \frac{d}{dt}\mathbf{A}(t)$ whose elements are $(\frac{d}{dt} a_{ij}(t))_{m \times n}$.

Differentiating the Product of Matrices

Suppose that $t \mapsto \mathbf{A}(t)$ and $t \mapsto \mathbf{B}(t)$ are differentiable, where each $\mathbf{A}(t)$ is $\ell \times m$ and each $\mathbf{B}(t)$ is $m \times n$.

Then $t \mapsto \mathbf{C}(t) = \mathbf{A}(t)\,\mathbf{B}(t)$ is well defined as a matrix product with elements given by $c_{ik}(t) = \sum_{j=1}^m a_{ij}(t) b_{jk}(t)$ whose time derivatives are

$$\dot{c}_{ik}(t) = \sum\nolimits_{j=1}^m [\dot{a}_{ij}(t) b_{jk}(t) + a_{ij}(t) \dot{b}_{jk}(t)]$$

Hence $t \mapsto \mathbf{C}(t)$ is differentiable,

with $\dot{\mathbf{C}}(t) = \dot{\mathbf{A}}(t)\mathbf{B}(t) + \mathbf{A}(t)\dot{\mathbf{B}}(t)$.

Differentiating the Square of a Square Matrix

Suppose that $\mathbf{A}(t)$ is an $n \times n$ matrix for all t,
and that each element is a differentiable function of t.

Then the square matrix $\mathbf{A}^2(t)$ is well defined and differentiable,
with derivative $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^2(t) = \dot{\mathbf{A}}(t)\mathbf{A}(t) + \mathbf{A}(t)\dot{\mathbf{A}}(t)$.

Unless the matrices $\dot{\mathbf{A}}(t)$ and $\mathbf{A}(t)$ happen to commute,
in the sense that $\dot{\mathbf{A}}(t)\mathbf{A}(t) = \mathbf{A}(t)\dot{\mathbf{A}}(t)$,
this will not be equal to $2\dot{\mathbf{A}}(t)\mathbf{A}(t)$ or to $2\mathbf{A}(t)\dot{\mathbf{A}}(t)$.

Example

Note that, even if each $\mathbf{A}(t)$ is square,
it may not commute with $\dot{\mathbf{A}}(t)$.

For example, when $\mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix}$, then $\dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,
implying that $\mathbf{A}(t)\,\dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \neq \dot{\mathbf{A}}(t)\,\mathbf{A}(t) = \begin{pmatrix} 0 & 0 \\ 1 & t \end{pmatrix}$.

Note that in this example $\mathbf{A}$ is symmetric; so therefore is $\dot{\mathbf{A}}$.

Hence $\mathbf{A}(t)\,\dot{\mathbf{A}}(t) = \mathbf{A}(t)^\top\,\dot{\mathbf{A}}^\top(t) = [\dot{\mathbf{A}}(t)\mathbf{A}(t)]^\top$.

Also $\mathbf{A}^2(t) = \begin{pmatrix} 1 & t \\ t & 1 + t^2 \end{pmatrix}$ whose derivative satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^2(t) = \dot{\mathbf{A}}(t)\mathbf{A}(t) + \mathbf{A}(t)\,\dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 2t \end{pmatrix}$$

This differs from both $2\,\mathbf{A}(t)\,\dot{\mathbf{A}}(t)$ and $2\,\dot{\mathbf{A}}(t)\,\mathbf{A}(t)$.

The Exponential of a Square Matrix

Recall that the exponential function of a scalar is defined so that the solution of the differential equation $\dot{x} = ax$ is $x(t) = e^{at} x(0)$.

Similarly, we define the exponential function of a square matrix so that the solution of the differential equation system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is $\mathbf{x}(t) = \exp(\mathbf{A}t)\,\mathbf{x}(0)$.

The function $t \mapsto \exp(\mathbf{A}t)$ is often called the resolvent.

Recall that, for a scalar, there is the convergent power series

$$e^{at} = 1 + \frac{1}{1!}at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 \ldots = \sum_{r=0}^{\infty} \frac{1}{r!}(at)^r$$

with the convention that $0! = 1$.

Similarly, for a square matrix, with the convention that $(\mathbf{A}t)^0 = \mathbf{I}$ one can use a convergent power series to give,

$$\exp(\mathbf{A}t) = \mathbf{I} + \frac{1}{1!}\mathbf{A}t + \frac{1}{2!}(\mathbf{A}t)^2 + \frac{1}{3!}(\mathbf{A}t)^3 \ldots = \sum_{r=0}^{\infty} \frac{1}{r!}(\mathbf{A}t)^r$$

The Exponential of a Diagonal Matrix

Dropping the time argument, it follows that we define

$$\exp(\mathbf{C}) := \mathbf{I} + \frac{1}{1!}\mathbf{C} + \frac{1}{2!}(\mathbf{C})^2 + \frac{1}{3!}(\mathbf{C})^3 \ldots = \sum_{r=0}^{\infty} \frac{1}{r!}(\mathbf{C})^r$$

Suppose that $\mathbf{C}$ is the diagonal matrix $\mathbf{diag}(c_1, c_2, \ldots, c_n) = \mathbf{diag}\,\mathbf{c}$ where $\mathbf{c}$ is the vector $(c_1, c_2, \ldots, c_n)$.

Now, each matrix power $(\mathbf{diag}\,\mathbf{c})^r = \mathbf{diag}(c_1^r, c_2^r, \ldots, c_n^r)$ as is readily proved by induction on r.

So, with this notation for the exponential of a matrix, we have

$$\begin{aligned} \exp(\mathbf{C}) &= \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{C}^r = \sum_{r=0}^{\infty} \frac{1}{r!}\,\mathbf{diag}(c_1^r, c_2^r, \ldots, c_n^r) \\ &= \mathbf{diag}\left(e^{c_1}, e^{c_2}, \ldots, e^{c_n}\right) \end{aligned}$$

Also, suppose matrix $\mathbf{C}$ has $\mathbf{C} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$ as a diagonalization.

Then each matrix power $\mathbf{C}^r = \mathbf{V}\boldsymbol{\Lambda}^r\mathbf{V}^{-1}$ implying that $\exp(\mathbf{C}) = \mathbf{V}\exp(\boldsymbol{\Lambda})\mathbf{V}^{-1}$.

Integrating and Differentiating an Exponential Matrix

From the definition $\exp(\mathbf{A}s) = \sum_{r=0}^{\infty} \frac{1}{r!}(\mathbf{A}s)^r$,
either post- or premultiplying by $\mathbf{A}$ and then integrating gives

$$\int_0^t \exp(\mathbf{A}s)\ \mathbf{A}\, \mathrm{d}s = \int_0^t \mathbf{A} \exp(\mathbf{A}s)\, \mathrm{d}s = \int_0^t \sum_{r=0}^{\infty} \frac{1}{r!} \mathbf{A}^{r+1}\, s^r\, \mathrm{d}s$$

Next, integrating term by term, the last expression becomes

$$\sum_{r=0}^{\infty} \frac{1}{r!} \mathbf{A}^{r+1} \int_0^t s^r \mathrm{d}s = \sum_{r=0}^{\infty} \frac{1}{r!} \mathbf{A}^{r+1} \ \cdot \ \Big|_0^t \tfrac{1}{r+1} s^{r+1}$$

Simplifying converts this to

$$\sum_{r=0}^{\infty} \frac{1}{(r+1)!} \mathbf{A}^{r+1} t^{r+1} = \sum_{r=1}^{\infty} \frac{1}{r!} \mathbf{A}^r t^r = \exp(\mathbf{A}t) - \mathbf{I}$$

So $\int_0^t \exp(\mathbf{A}s)\, \mathbf{A}\, \mathrm{d}s = \int_0^t \mathbf{A}\ \exp(\mathbf{A}s)\, \mathrm{d}s = \exp(\mathbf{A}t) - \mathbf{I}$,
implying that

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(\mathbf{A}t) = \mathbf{A} \exp(\mathbf{A}t) = \exp(\mathbf{A}t)\, \mathbf{A}$$

Affine Equation in n Variables

Consider what happens when we multiply each side of the non-homogeneous equation $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}(t)$ by the matrix integrating factor $\exp(-\mathbf{A}t)$.

Because the product rule of differentiation applies to matrices,

$$\begin{aligned}
\frac{d}{dt}[\exp(-\mathbf{A}t)\,\mathbf{x}(t)] &= \exp(-\mathbf{A}t)\,\dot{\mathbf{x}}(t) + \frac{d}{dt}[\exp(-\mathbf{A}t)]\;\mathbf{x}(t) \\
&= \exp(-\mathbf{A}t)\,\dot{\mathbf{x}}(t) - \exp(-\mathbf{A}t)\,\mathbf{A}\,\mathbf{x}(t) \\
&= \exp(-\mathbf{A}t)\,\mathbf{b}(t)
\end{aligned}$$

if and only if $\mathbf{x}(t)$ solves the equation $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}(t)$.

Hence $\exp(-\mathbf{A}t)\,\mathbf{x}(t) - \exp(-\mathbf{A}s)\,\mathbf{x}(s) = \int_s^t \exp(-\mathbf{A}\tau)\,\mathbf{b}(\tau)\,d\tau$.

Multiplying each side by $\exp(\mathbf{A}t)$ gives the unique solution

$$\mathbf{x}(t) = \exp[\mathbf{A}(t-s)]\,\mathbf{x}(s) + \int_s^t \exp[\mathbf{A}(t-\tau)]\,\mathbf{b}(\tau)\,d\tau$$

The Diagonal Case

The diagonal case occurs when $\mathbf{A} = \mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$.

Then the system $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}(t)$ of n coupled equations reduces to the system of n uncoupled equations

$$\dot{x}_i(t) = a_{ii} x_i(t) + b_i(t) = \lambda_i x_i(t) + b_i(t) \ (i = 1, \ldots, n)$$

one in each variable x_i, with respective solutions

$$x_i(t) = e^{\lambda_i t} x_i(s) + \int_s^t e^{\lambda_i (t-\tau)} b_i(\tau) \, \mathrm{d}\tau$$

The Diagonalizable Case

Suppose that $\mathbf{A}$ has n distinct eigenvalues
— or if not, then n linearly independent eigenvectors
that make up the columns of the matrix $\mathbf{V}$.

Then $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ and $\mathbf{A}t = \mathbf{V}(\mathbf{\Lambda}t)\mathbf{V}^{-1}$
implying that $\exp(\mathbf{A}t) = \mathbf{V}\exp(\mathbf{\Lambda}t)\mathbf{V}^{-1}$.

Hence the solution

$$\mathbf{x}(t) = \exp[\mathbf{A}(t-s)]\,\mathbf{x}(s) + \int_s^t \exp[\mathbf{A}(t-\tau)]\,\mathbf{b}(\tau)\,\mathrm{d}\tau$$

simplifies to

$$\mathbf{x}(t) = \mathbf{V}\exp[\mathbf{\Lambda}(t-s)]\mathbf{V}^{-1}\,\mathbf{x}(s) + \int_s^t \mathbf{V}\exp[\mathbf{\Lambda}(t-\tau)]\mathbf{V}^{-1}\,\mathbf{b}(\tau)\,\mathrm{d}\tau$$

Of course, the transformation $\mathbf{y}(t) := \mathbf{V}^{-1}\,\mathbf{x}(t)$ takes us back
to the diagonal case, with $\dot{\mathbf{y}}(t) := \mathbf{\Lambda}\,\mathbf{y}(t) + \mathbf{V}^{-1}\,\mathbf{b}(t)$.

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A Stability Condition

When $\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$,
one has $\exp(\mathbf{\Lambda}) = \mathbf{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})$.

Furthermore $\exp(\mathbf{\Lambda}t) = \mathbf{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})$.

This converges to the zero matrix as $t \to \infty$
if and only if each $e^{\lambda_i t} \to 0$,
which is true iff each eigenvalue λ_i has a negative real part.

Similarly, if $\mathbf{A}$ is diagonalizable, with $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$,
then consider the new variables $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$.

The differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ becomes transformed to

$$\dot{\mathbf{y}}(t) = \mathbf{V}^{-1}\dot{\mathbf{x}}(t) = \mathbf{V}^{-1}\mathbf{A}\mathbf{x}(t) = \mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{x}(t) = \mathbf{\Lambda}\mathbf{y}(t)$$

Because $\mathbf{V}$ is invertible, one has $\mathbf{x}(t) \to \mathbf{0} \iff \mathbf{y}(t) \to \mathbf{0}$.

Once again, stability holds
iff each eigenvalue of the matrix $\mathbf{A}$ has a negative real part.

This is true even when $\mathbf{A}$ is not diagonalizable.

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Autonomous First-Order Equations

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a general function that may be non-linear.

Consider the autonomous differential equation $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$.

A solution satisfying the initial condition $\mathbf{x}(s) = \bar{x}$
is a differentiable function $[s, t) \ni t \mapsto \mathbf{x}(t)$
that satisfies $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t))$ for almost all $t \geq s$.

Equivalently, for almost all $t \geq s$, one must have

$$\mathbf{x}(t) = \mathbf{x}(s) + \int_s^t \mathbf{F}(\mathbf{x}(\tau))\, \mathrm{d}\tau$$

Stationary States and Rest Points

A stationary state is a point $\mathbf{x}^* \in \mathbb{R}^n$ with the property that if $\mathbf{x}(s) = \mathbf{x}^*$ at any time s, then $\mathbf{x}(t) = \mathbf{x}^*$ at all times $t \geq s$.

A rest point is a state $\bar{\mathbf{x}} \in \mathbb{R}^n$ with the property that $\mathbf{F}(\bar{\mathbf{x}}) = \mathbf{0}$.

Theorem

Any rest point is a stationary state, and conversely.

Proof.

If $\mathbf{F}(\bar{\mathbf{x}}) = \mathbf{0}$, then the solution of $\mathbf{x}(t) = \mathbf{x}(s) + \int_s^t \mathbf{F}(\mathbf{x}(\tau))\, d\tau$ with $\mathbf{x}(s) = \bar{\mathbf{x}}$ satisfies $\mathbf{x}(t) = \mathbf{x}(s) = \bar{\mathbf{x}}$ for all $t \geq s$.

Conversely, if that solution satisfies $\mathbf{x}(t) = \mathbf{x}(s) = \mathbf{x}^*$ for all $t \geq s$, then $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) = \mathbf{F}(\mathbf{x}^*) = \mathbf{0}$ for all $t \geq s$. □

Local Stability of a Stationary State

Let $\mathbf{F}'(\mathbf{x})$ denote the $n \times n$ Jacobian matrix
whose elements are the partial derivatives $\mathbf{F}'_{ij}(\mathbf{x}) = \frac{\partial}{\partial x_j} F_i(\mathbf{x})$
of the different components $(F_i(\mathbf{x}))_{i=1}^n$.

Any particular steady state $\mathbf{x}^*$ is locally asymptotically stable
if and only if all the eigenvalues of $\mathbf{F}'(\mathbf{x}^*)$ have negative real parts.

A System with Two Variables

Consider the coupled pair $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ of differential equations.

Let (a, b) be any stationary point satisfying both $f(a, b) = 0$ and $g(a, b) = 0$.

The Jacobian matrix at the stationary point takes the form

$$\mathbf{J}(a, b) = \begin{pmatrix} \frac{\partial f}{\partial x}(a, b) & \frac{\partial f}{\partial y}(a, b) \\ \frac{\partial g}{\partial x}(a, b) & \frac{\partial g}{\partial y}(a, b) \end{pmatrix}$$

Local Saddle Point with Two Variables

The product of the two eigenvalues λ_1, λ_2 of $\mathbf{J}(a, b)$
equals its determinant $|\mathbf{J}(a, b)|$.

The two eigenvalues are real and have opposite signs
if and only if $|\mathbf{J}(a, b)| < 0$.

This is a sufficient condition for the steady state to be unstable.

But if $(x(0) - a, y(0) - b)^\top$ is an eigenvector
corresponding to the negative eigenvalue,
then in the case when the equations are linear and so $\mathbf{J}$ is constant,
the solution will converge to the steady state.

This is saddle point stability.

The Lotka–Volterra Predator–Prey Model

Foxes are predators; their prey includes rabbits.

Let x denote the expected population of rabbits,
and y denote expected population of foxes.

Assume these populations are linked by the differential equations

$$\begin{aligned} \dot{x} &= x(k - ay) \\ \dot{y} &= y(bx - h) \end{aligned}$$

where a, b, h, k are all positive parameters.

Thus:

1. the rabbit population growth rate $\frac{d}{dt} \ln x = \dot{x}/x$ is a decreasing affine function of the fox population;
2. whereas the fox population growth rate $\frac{d}{dt} \ln y = \dot{y}/y$ is an increasing affine function of the rabbit population.

Lotka–Volterra: Phase Plane Analysis

Given the system $\dot{x} = x(k - ay)$ and $\dot{y} = y(bx - h)$,
the two nullclines where $\dot{x} = 0$ and $\dot{y} = 0$
are given by $y = k/a$ and $x = h/b$ respectively.

So the steady state is at $(x, y) = (h/b, k/a)$.

The Jacobian matrix is

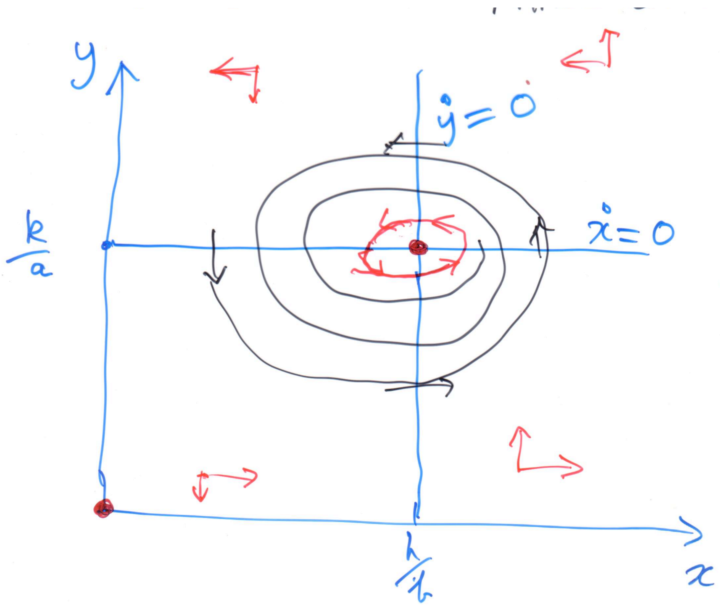
$$\mathbf{J}(x, y) = \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial y} \\ \\ \dfrac{\partial \dot{y}}{\partial x} & \dfrac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} k - ay & -ax \\ yb & bx - h \end{pmatrix} = \begin{pmatrix} 0 & -ah/b \\ bk/a & 0 \end{pmatrix}$$

at the steady state.

The characteristic equation is $\lambda^2 + hk = 0$,
whose roots are $\pm i\sqrt{hk}$.

As the following diagram suggests, there can be limit cycles
with $x(t) = \xi \cos \sqrt{hk}\, t$ and $y(t) = \eta \sin \sqrt{hk}\, t$.

Lotka–Volterra: Phase Plane Diagram



Saddle Point Example

Consider a macro model where: (i) K denotes capital; (ii) Y denotes output; and (iii) C denotes consumption.

Suppose that net investment $\dot{K} = Y - C$, that $Y = aK - bK^2$, and $\dot{C} = w(a - 2bk)C$, where a, b, k, w are positive constants.

This gives the coupled system with

$$\dot{K} = aK - bK^2 - C \text{ and } \dot{C} = w(a - 2bK)C$$

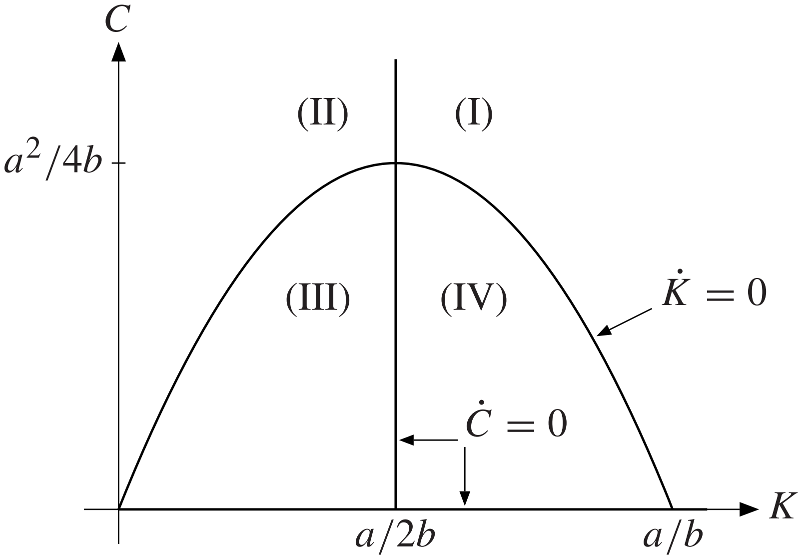
The two nullclines are $C = aK - bK^2$ and $K = a/2b$.

These intersect at the stationary point $(K^*, C^*) = (a/2b, a^2/4b)$. The Jacobian matrix is

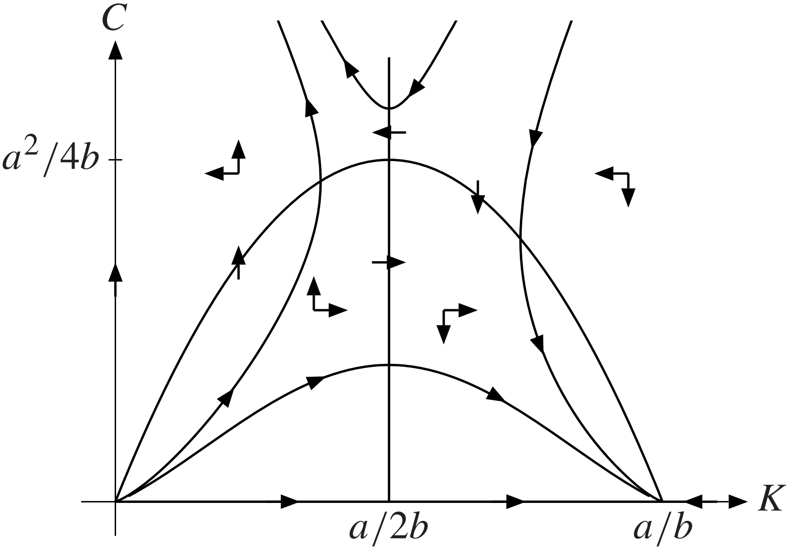
$$\mathbf{J}(K, C) = \begin{pmatrix} \frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial C} \\ \frac{\partial \dot{C}}{\partial K} & \frac{\partial \dot{C}}{\partial C} \end{pmatrix} = \begin{pmatrix} a - 2bK & -1 \\ -2wbC & w(a - 2bK) \end{pmatrix}$$

which reduces to $\begin{pmatrix} 0 & -1 \\ \frac{1}{2}a^2 w & 0 \end{pmatrix}$ at the steady state.

Phase Diagram I



Phase Diagram II



Stability Analysis

The Jacobian matrix $\begin{pmatrix} 0 & -1 \\ \frac{1}{2}a^2 w & 0 \end{pmatrix}$ at the steady state has trace 0 and negative determinant $-\frac{1}{2}a^2 w$.

So the two eigenvalues are $\pm\lambda$ where $\lambda^2 = \frac{1}{2}a^2 w$ and so $\lambda = a\sqrt{w/2}$.

The general solution near the steady state takes the form $x = Ae^{\lambda t} + Be^{-\lambda t}$ for arbitrary constants A, B.

This converges to the steady state at $(K^*, C^*) = (a/2b, a^2/4b)$ if and only if $A = 0$.