Lecture Notes 6: Dynamic Equations
Part D: Differential Equations

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Lecture Outline

First-Order Differential Equations in One Variable

Introduction
First-Order Affine Equation with Constant Coefficients
General First-Order Affine Equation
Constant and Undetermined Coefficients
Stability in the Autonomous Case

Second-Order Differential Equations in One Variable

Introduction
The Inhomogeneous Equation
Constant and Undetermined Coefficients
Stability

First-Order Multivariable Differential Equations

Introduction
Stability with Many Variables
Autonomous Nonlinear Equations in Many Variables
First-Order Differential Equations

The typical first-order differential equation in one variable $x$ is

$$\dot{x} = \frac{dx}{dt} = f(x, t)$$

The equation is autonomous just in case $f$ is independent of $t$, so it can be written as $\dot{x} = f(x)$.

Typically one imposes an initial condition requiring $x(s) = \bar{x}_s$ at time $s$ (not necessarily the earliest time).

Then any solution is a fixed function $t \mapsto x(t)$ that satisfies the corresponding integral equation $x(t) = \bar{x}_s + \int_s^t f(x(u), u) \, du$.

Picard’s method of successive approximations starts with an arbitrary function $t \mapsto x^{(0)}(t)$ satisfying $x^{(0)}(s) = \bar{x}_s$.

Then it computes $x^{(n)}(t) = \bar{x}_s + \int_s^t f(x^{(n-1)}(u), u) \, du$ for $n \in \mathbb{N}$.

If convergence occurs, the limit as $n \to \infty$ will be a solution.
Right-Hand Side Independent of $x$

A special case occurs when the right-hand side $f(x, t)$ is independent of $x$.

Then the differential equation can be written as

$$\frac{dx}{dt} = g(t)$$

Its solution can be written as the indefinite integral

$$x(t) = \int g(t)dt$$

Introducing an initial condition $x(s) = \bar{x}_s$ at a particular start time $s$ allows the solution to be written as the definite integral

$$x(t) = \bar{x}_s + \int_s^t g(\tau)d\tau$$

CHECK that this alleged solution satisfies $x(s) = \bar{x}_s$ and $\dot{x}(t) = g(t)$ for all $t \geq s$. 
Leibnitz Rule for Differentiating an Integral

Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F(a, b, u) := \int_a^b f(t, u) dt$$

Its three first-order partial derivatives are:

(i) $F'_a = -f(a, u)$; (ii) $F'_b = f(b, u)$; (iii) $F'_u = \int_a^b \frac{\partial}{\partial u} f(t, u) dt$

Applying the chain rule, the total derivative of the integral function $y \mapsto I(y) := \int_{a(y)}^{b(y)} f(t, y) dt$ satisfies

$$I'(y) = \frac{d}{dy} F(a(y), b(y), y) = a'(y)F'_a + b'(y)F'_b + F'_u$$

$$= b'(y)f(b(y), y) - a'(y)f(a(y), y) + \int_{a(y)}^{b(y)} \frac{\partial}{\partial y} f(t, y) dt$$
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Picard’s Method of Successive Approximations

The simplest first-order equation with constant coefficients takes the form

\[ \dot{x}(t) = ax(t), \text{ with } x(0) \text{ given} \]

It corresponds to the integral equation

\[ x(t) - x(0) = \int_0^t ax(u) du \text{ for all } t \geq 0 \]

Starting with an approximation such as \( x^{(0)}(t) \equiv x(0) \) for all \( t \geq 0 \), we can calculate a sequence \( t \mapsto x^{(n)}(t) \) \((n \in \mathbb{N})\) of successive approximations to a solution \([0, \infty) \ni t \mapsto x(t) \in \mathbb{R}\) using the iterative rule

\[ x^{(n)}(t) - x(0) = \int_0^t ax^{(n-1)}(u) du \text{ for all } t \geq 0 \]
First Iterations

Iterating once starting from \( x^{(0)}(t) \equiv x(0) \) gives

\[
x^{(1)}(t) - x(0) = \int_{0}^{t} a x^{(0)}(u) \, du = a x(0) t
\]

Iterating a second time gives

\[
x^{(2)}(t) - x(0) = \int_{0}^{t} a x(0)(1 + au) \, du = a x(0) t + \frac{1}{2} a^2 x(0) t^2
\]

Iterating a third time gives

\[
x^{(3)}(t) - x(0) = \int_{0}^{t} [a x(0) + a^2 x(0) u + \frac{1}{2} a^3 x(0) u^2] \, du
\]
\[= a x(0) t + \frac{1}{2} a^2 x(0) t^2 + \frac{1}{6} a^3 x(0) u^3
\]
Terms of the Sum

Each time we are adding a term to a sum, so define the new incremental variable $y^{(n)}(t) := x^{(n)}(t) - x^{(n-1)}(t)$ with $y^{(0)}(t) \equiv x(0)$.

This implies that $x^{(n)}(t) = x(0) + \sum_{k=1}^{n} y^{(k)}(t)$.

Subtract $x^{(n)}(t) - x(0) = \int_{0}^{t} ax^{(n-1)}(u) \, du$ from $x^{(n+1)}(t) - x(0) = \int_{0}^{t} ax^{(n)}(u) \, du$ to obtain $y^{(n+1)}(t) = \int_{0}^{t} ay^{(n)}(u) \, du$.

Now we obtain successively

\[
\begin{align*}
y^{(1)}(t) &= \int_{0}^{t} a \, x(0) \, du = a \, x(0) \, t \\
y^{(2)}(t) &= \int_{0}^{t} a^2 \, x(0) \, u \, du = \frac{1}{2} a^2 \, x(0) \, t^2 \\
y^{(3)}(t) &= \int_{0}^{t} \frac{1}{2} a^3 \, x(0) \, u^2 \, du = \frac{1}{6} a^3 \, x(0) \, t^3
\end{align*}
\]

This suggests the induction hypothesis $y^{(n)}(t) = \frac{1}{n!} a^n x(0) \, t^n$. 

Constructing the Sum

The induction hypothesis $y^{(n)}(t) = \frac{1}{n!} a^n \times(0) \, t^n$
and the relation $y^{(n+1)}(t) = \int_0^t a y^{(n)}(u) \, du$ together imply that

$$
y^{(n+1)}(t) = \int_0^t a \frac{1}{n!} a^n \times(0) \, u^n \, du = \frac{1}{n!} a^{n+1} \times(0) \int_0^t u^n \, du
$$

$$
= \frac{1}{n!} a^{n+1} \times(0) \frac{1}{n+1} t^{n+1} = \frac{1}{(n+1)!} a^{n+1} \times(0) \, t^{n+1}
$$

This confirms the induction hypothesis with $n$ replaced by $n + 1$.

It follows that $y^{(n)}(t) = \frac{1}{n!} a^n \times(0) \, t^n$ for all $n \in \mathbb{N}$
and then that $x^{(n)}(t) = x(0) + \sum_{k=1}^{n} \frac{1}{k!} a^k \times(0) \, t^k$. 

The Exponential Solution

Recall that

$$\exp x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} x^n = e^x$$

where

$$e = 1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3!} + \ldots = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.718281828$$

My late co-author Knut Sydsæter who, as a cultured Norwegian, recognized that 1828 is the year when their great playwright Henrik Ibsen was born, remembers this 10 digit approximation as “2.7 Ibsen Ibsen”.

As $n \to \infty$, the solution $x^{(n)}(t)$ we found converges to the infinite series

$$x(0) + \sum_{k=1}^{\infty} \frac{1}{k!} a^k x(0) t^k = x(0) \exp(at) = x(0)e^{at}$$
Full Decimal Expansions of Some Important Numbers

\[ \frac{1}{3} = 0.3333333333333333333 \ldots \]

\[ \sqrt{2} = 1.4142135623730950488016887242097 \ldots \]

\[ e = 2.7182818284590452353602874713526 \ldots \]

\[ \pi = 3.1415926535897932384626433832795 \ldots \]
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General First-Order Affine Equation

The general first-order affine equation takes the form

\[ \dot{x}(t) = a(t)x(t) + b(t) \]

for arbitrary integrable functions \( t \mapsto a(t) \) and \( t \mapsto b(t) \).

In the homogeneous case one has \( b(t) \equiv 0 \), and the equation takes the linear form \( \dot{x}(t) = a(t)x(t) \).

Assuming that \( x > 0 \) for all \( t \), we can take logs and write the equation as

\[ \frac{d}{dt} \ln x = \frac{\dot{x}}{x} = a(t) \]

After introducing the new variable \( y(t) := \ln x(t) \), the equation becomes \( \dot{y} = a(t) \) whose solution is obviously

\[ y(t) = y(s) + \int_s^t a(\tau)d\tau \]
Because \( x(t) = \exp y(t) \), the solution for \( x \) is

\[
x(t) = \exp[y(t)] = \exp[y(s)] \exp \left[ \int_s^t a(\tau) d\tau \right] = x(s) \alpha_s(t)
\]

where \( \alpha_s(t) \) denotes the integrating factor \( \exp \left[ \int_s^t a(\tau) d\tau \right] \).

In the special case of an autonomous equation where \( a(\tau) = a \) constant, one has \( \int_s^t a(\tau) d\tau = a(t - s) \) and so \( \alpha_s(t) = e^{a(t-s)} \).
The Non-Homogenous Case

The solution \(x(t) = x(s)\alpha_s(t)\)
to the homogeneous equation \(\dot{x}(t) - a(t)x(t) = 0\)
can be used to help solve the corresponding
non-homogeneous equation \(\dot{x}(t) - a(t)x(t) = f(t)\).

Indeed, consider the result of dividing
each side of this non-homogeneous equation
by the integrating factor \(\alpha_s(t) := \exp\left[\int_s^t a(\tau)d\tau\right]\)
whose reciprocal is \(1/\alpha_s(t) := \exp\left[-\int_s^t a(\tau)d\tau\right]\).

Note that \(\frac{d}{dt}\left[-\int_s^t a(\tau)d\tau\right] = -a(t)\),
implying that \(\frac{d}{dt}[1/\alpha_s(t)] = -a(t)/\alpha_s(t)\) so, by the product rule

\[ \frac{d}{dt}[x(t)/\alpha_s(t)] = \frac{1}{\alpha_s(t)}\dot{x}(t) - \frac{a(t)}{\alpha_s(t)}x(t) = f(t)/\alpha_s(t) \]

for any solution of the equation \(\dot{x}(t) - a(t)x(t) = f(t)\).
Solving the Non-Homogeneous Equation

Integrating each side of the equation \( \frac{d}{dt}[x(t)/\alpha_s(t)] = f(t)/\alpha_s(t) \) over the interval from \( s \) to \( t \) gives us

\[
\left| t \frac{x(u)}{\alpha_s(u)} \right|_s^t = \frac{x(t)}{\alpha_s(t)} - \frac{x(s)}{\alpha_s(s)} = \int_s^t \frac{f(u)}{\alpha_s(u)} \, du
\]

The definition \( \alpha_s(t) = \exp \left[ \int_s^t a(\tau)d\tau \right] \)
implies that \( \alpha_s(s) = 1 \) and also \( \alpha_s(t)/\alpha_s(u) = \alpha_u(t) \).

Hence, multiplying each side by \( \alpha_s(t) \) gives the solution

\[
x(t) = \alpha_s(t) \left[ x(s) + \int_s^t \frac{1}{\alpha_s(u)} \, f(u) \, du \right]
= \alpha_s(t)x(s) + \int_s^t \alpha_u(t) \, f(u) \, du
= \exp \left[ \int_s^t a(\tau)d\tau \right] x(s) + \int_s^t \exp \left[ \int_u^t a(\tau)d\tau \right] f(u) \, du
\]
Linearity in the Forcing Term

Theorem

Suppose that \( x^P(t) \) and \( y^P(t) \) are particular solutions of the two respective differential equations

\[
\dot{x}(t) - a(t)x(t) = d(t) \quad \text{and} \quad \dot{y}(t) - a(t)y(t) = e(t)
\]

Then, for any scalars \( \alpha \) and \( \beta \), the equation \( \dot{z}(t) - a(t)z(t) = f(t) = \alpha d(t) + \beta e(t) \) has as a particular solution

the corresponding linear combination \( z^P(t) := \alpha x^P(t) + \beta y^P(t) \).

Consider any equation of the form \( \dot{x}(t) - a(t)x(t) = f(t) \)
where \( f(t) \) is a linear combination \( \sum_{k=1}^{n} \alpha_k f^k(t) \)
of \( n \) forcing terms.

The theorem implies that a particular solution is the corresponding linear combination \( \sum_{k=1}^{n} \alpha_k x^{P^k}(t) \)
of particular solutions to the \( n \) equations \( \dot{x}(t) - a(t)x(t) = f^k(t) \).
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First-Order Linear Equation with a Constant Coefficient

Next, consider the equation \( \dot{x}(t) - ax(t) = f(t) \) where the coefficient \( a \) of \( x(t) \) has become the constant \( a \neq 0 \).

The solution we found for the general case was

\[
x(t) = \exp \left[ \int_s^t a(\tau) d\tau \right] x(s) + \int_s^t \exp \left[ \int_u^t a(\tau) d\tau \right] f(u) du
\]

When \( a(t) = a \), independent of \( t \), this reduces to

\[
x(t) = e^{a(t-s)} x(s) + \int_s^t e^{a(t-u)} f(u) du
\]

We simplify further by choosing the initial time \( s = 0 \).

Then

\[
x(t) = e^{at} x(0) + \int_0^t e^{a(t-u)} f(u) du
\]
First Special Case

An interesting special case occurs when the forcing term \( f(t) \) is the exponential function \( t \mapsto e^{\mu t} \).

Then the solution is

\[
x(t) = e^{at}x(0) + \int_0^t e^{a(t-u)+\mu u} du = e^{at} \left[ x(0) + \int_0^t e^{(\mu-a)u} du \right]
\]

In the degenerate case when \( \mu = a \), one has \( \int_0^t e^{(\mu-a)u} du = \int_0^t 1 du = t \), so the solution collapses to

\[
x(t) = e^{at} [x(0) + t]
\]

This solution can be written as \( x(t) = x^H(t) + x^P(t) \) where:

1. \( x^H(t) = \xi^H e^{at} \) with \( \xi^H := x(0) \) is a complementary solution of the homogeneous equation \( \dot{x}(t) - ax(t) = 0 \);

2. \( x^P(t) = \xi^P e^{at} t \) with \( \xi^P := 1 \) is a particular solution of the inhomogeneous equation \( \dot{x}(t) - ax(t) = e^{at} \).
Non-Degenerate Case When $\mu \neq a$

In the non-degenerate case when $\mu \neq a$, one has

$$(\mu - a) \int_0^t e^{(\mu-a)u} \, du = \left|_0^t e^{(\mu-a)u} = e^{(\mu-a)t} - 1 \right.$$

So the solution is

$$x(t) = e^{at} \left[ x(0) + \frac{e^{(\mu-a)t} - 1}{\mu - a} \right] = e^{at}x(0) + \frac{e^{\mu t} - e^{at}}{\mu - a}$$

Again, this solution can be written as $x(t) = x^H(t) + x^P(t)$ where:

1. $x^H(t) = \xi^H e^{at}$ with $\xi^H := x(0) - 1/(\mu - a)$ is a solution of the homogeneous equation $\dot{x}(t) - ax(t) = 0$;
2. $x^P(t) = \xi^P e^{\mu t}$ with $\xi^P := 1/(\mu - a)$ is a particular solution of the inhomogeneous equation $\dot{x}(t) - ax(t) = e^{\mu t}$.
Second Special Case

Another interesting special case occurs when \( f(t) = t^r e^{\mu t} \) for some \( r \in \mathbb{N} \).

Then the solution \( x(t) = e^{at} x(0) + \int_0^t e^{a(t-u)} f(u) du \) becomes

\[
\begin{align*}
x(t) &= e^{at} x(0) + \int_0^t e^{a(t-u)} u^r e^{\mu u} du \\
&= e^{at} \left[ x(0) + \int_0^t u^r e^{(\mu-a)u} du \right]
\end{align*}
\]

In the degenerate case when \( \mu = a \), the solution collapses to

\[
x(t) = e^{at} \left[ x(0) + \int_0^t (r+1)^{-1} u^{r+1} \right] = e^{at} \left[ x(0) + (r+1)^{-1} t^{r+1} \right]
\]

This solution can be written as \( x(t) = x^H(t) + x^P(t) \) where:

1. \( x^H(t) = \xi^H e^{at} \) with \( \xi^H := x(0) \) is a solution of the homogeneous equation \( \dot{x}(t) - ax(t) = 0 \);
2. \( x^P(t) = \xi^P e^{at} t^{r+1} \) with \( \xi^P := (r+1)^{-1} \)
   is a particular solution of the inhomogeneous equation \( \dot{x}(t) - ax(t) = t^r e^{at} \).
Non-Degenerate Case When $\mu \neq a$

In the non-degenerate case when $\mu \neq a$, the solution is

$$x(t) = e^{at} \left[ x(0) + \int_0^t u^r e^{(\mu-a)u} \, du \right] = e^{at} \left[ x(0) + l_r(t) \right]$$

where $l_r(t) := \int_0^t u^r e^{(\mu-a)u} \, du$.

In particular, $l_0(t) = \int_0^t e^{(\mu-a)u} \, du = (\mu - a)^{-1} [e^{(\mu-a)t} - 1]$.

Integrating by parts gives the first-order linear difference equation

$$l_r(t) = \int_0^t u^r e^{(\mu-a)u} \, du$$
$$= (\mu - a)^{-1} \left[ t u^r e^{(\mu-a)u} - r e^{(\mu-a)u} \right]$$
$$= (\mu - a)^{-1} \left[ r l_{r-1}(t) - t^r e^{(\mu-a)t} \right]$$
Solving the First-Order Linear Difference Equation

Let us divide each side of the difference equation

\[ I_r(t) = (a - \mu)^{-1} \left[ r I_{r-1}(t) - t^r e^{(\mu-a)t} \right] \]

by the “summing factor” \( \prod_{k=1}^{r} k(a - \mu)^{-1} = r!(a - \mu)^{-r} \) to get

\[
J_r(t) := \frac{1}{r!} (a - \mu)^r I_r(t)
\]

\[
= \frac{1}{r!} \left[ r(a - \mu)^{r-1} I_{r-1}(t) - (a - \mu)^{-1} t^r e^{(\mu-a)t} \right]
\]

\[
= \frac{1}{(r-1)!} (a - \mu)^{r-1} I_{r-1}(t) - \frac{1}{r!} (a - \mu)^{r-1} t^r e^{(\mu-a)t}
\]

\[
= J_{r-1}(t) - \frac{1}{r!} (a - \mu)^{r-1} t^r e^{(\mu-a)t}
\]

This obviously implies that

\[
J_r(t) = J_0(t) - \sum_{k=1}^{r} \frac{1}{k!} (a - \mu)^{k-1} t^k e^{(\mu-a)t}
\]
Solving the Differential Equation

Because \( J_0(t) = I_0(t) = (\mu - a)^{-1}[e^{(\mu-a)t} - 1] \), this implies that

\[
J_r(t) = (\mu - a)^{-1}[e^{(\mu-a)t} - 1] - \sum_{k=1}^{r} \frac{1}{k!} (a - \mu)^{k-1} t^k e^{(\mu-a)t}
\]

But \( J_r(t) = \frac{1}{r!} (a - \mu)^r I_r(t) \), so

\[
I_r(t) := r!(a - \mu)^{-r} J_r(t) = -r!(a - \mu)^{-r-1}[e^{(\mu-a)t} - 1] - \sum_{k=1}^{r} \frac{r!}{k!} (a - \mu)^{k-r-1} t^k e^{(\mu-a)t}
\]

Then

\[
x(t) = e^{at} \left[ x(0) + I_r(t) \right] = e^{at} \left[ x(0) + r!(a - \mu)^{-r-1} \right] - r!(a - \mu)^{-r-1} e^{\mu t} \left[ 1 + \sum_{k=1}^{r} \frac{1}{k!} (a - \mu)^k t^k \right]
\]
Particular and General Solution

For the equation \( \dot{x}(t) - ax(t) = t^r e^{\mu t} \) with \( \mu \neq a \), the solution

\[
x(t) = e^{at} \left[ x(0) + r!(a - \mu)^{-r-1} \right] - r!(a - \mu)^{-r-1} e^{\mu t} \left[ 1 + \sum_{k=1}^{r} \frac{1}{k!} (a - \mu)^k t^k \right]
\]

can be written as \( x(t) = x^H(t) + x^P(t) \) where:

1. \( x^H(t) = \xi^H e^{at} \) with \( \xi^H := x(0) + r!(a - \mu)^{-r-1} \)

is a solution of the **homogeneous** equation \( \dot{x}(t) - ax(t) = 0 \);

2. \( x^P(t) = \xi^P(t)e^{\mu t} \), where the polynomial

\[
t \mapsto \xi^P(t) := -r!(a - \mu)^{-r-1} \left[ 1 + \sum_{k=1}^{r} \frac{1}{k!} (a - \mu)^k t^k \right]
\]

of degree \( r \) in \( t \) is a **particular** solution of the **inhomogeneous** equation \( \dot{x}(t) - ax(t) = t^r e^{\mu t} \).
Method of Undetermined Coefficients

A practical issue is finding what polynomial

\[ t \mapsto \xi^P(t) = \sum_{k=0}^r \xi_k t^k \]

of degree \( r \) (the power of \( t \) on the right-hand side) makes \( \xi^P(t)e^{\mu t} \) a particular solution of the inhomogeneous differential equation \( \dot{x}(t) - ax(t) = t^r e^{\mu t} \).

The coefficients \((\xi_0, \xi_1, \ldots, \xi_r)\) of the polynomial are undetermined till we choose the associated polynomial \( t \mapsto \xi^P(t) \) to make \( \xi^P(t)e^{\mu t} \) satisfy the differential equation.
Determining the Undetermined Coefficients

For \( x^P(t) = e^{\mu t} \sum_{k=0}^{r} \xi_k t^k \) to solve \( \dot{x}(t) - ax(t) = t^r e^{\mu t} \), we need

\[
t^r e^{\mu t} = \mu e^{\mu t} \sum_{k=0}^{r} \xi_k t^k + e^{\mu t} \sum_{k=1}^{r} \xi_k k t^{k-1} - ae^{\mu t} \sum_{k=0}^{r} \xi_k t^k
\]

\[
= (\mu - a) e^{\mu t} \xi_r t^r + e^{\mu t} \sum_{k=0}^{r-1} [ (\mu - a) \xi_k + \xi_{k+1} (k + 1) ] t^k
\]

First consider the non-degenerate case \( \mu \neq a \).

For \( k = r \), this implies that \( (\mu - a) \xi_r = 1 \), so \( \xi_r = (\mu - a)^{-1} \).

For \( k = 0, 1, \ldots, r - 1 \), it implies that \( (\mu - a) \xi_k + \xi_{k+1} (k + 1) = 0 \) or that \( \xi_k = (a - \mu)^{-1} (k + 1) \xi_{k+1} \), and so

\[
\xi_k = \left[ \prod_{j=k}^{r-1} (a - \mu)^{-1} (j + 1) \right] \xi_r
\]

\[
= \frac{r!}{k!} (a - \mu)^{k-r} \xi_r = -\frac{r!}{k!} (a - \mu)^{k-r+1}
\]

This matches our previous answer.
Degenerate Case

But in the degenerate case $\mu = a$, the method does not work.

Instead, to solve $\dot{x}(t) - ax(t) = t^r e^{at}$, we introduce the new variable $y(t) = e^{-at}x(t)$.

Then $\dot{y}(t) = e^{-at} [\dot{x}(t) - ax(t)] = e^{-at} t^r e^{at} = t^r$.

The solution to this differential equation is $y(t) = y(0) + \int_0^t u^r du = y(0) + (r + 1)^{-1}t^{r+1}$.

The solution to the original differential equation is therefore $x(t) = e^{at}y(t) = e^{at} [x(0) + (r + 1)^{-1}t^{r+1}]$.

The polynomial in $t$ that occurs in this solution is now of degree $r + 1$ rather than $r$. 
Theorem

Consider the inhomogeneous first-order linear differential equation

\[ \dot{x}(t) - ax(t) = t^r e^{\mu t}, \text{ where } a \neq 0 \text{ and } r \in \mathbb{Z}_+. \]

There exists a particular solution of the form \( x^P(t) = Q(t) e^{\mu t} \) where the function \( t \mapsto Q(t) \) is a polynomial in \( t \) of degree:

- \( r \) in the regular case when \( \mu \neq a \);
- \( r + 1 \) in the degenerate case when \( \mu = a \).

The general solution takes the form \( x(t) = x^P(t) + x^C(t) \) where:

- \( x^P(t) \) is any particular solution;
- \( x^C(t) \) is any member of the one-dimensional linear space of complementary solutions to the corresponding homogeneous equation \( \dot{x}(t) - ax(t) = 0 \).
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- General First-Order Affine Equation
- Constant and Undetermined Coefficients
- Stability in the Autonomous Case

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First-Order Multivariable Differential Equations
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The Autonomous Case

The **autonomous case** occurs
when the first-order affine equation takes the form

\[ \dot{x} = ax + b \]

with the right-hand side independent of \( t \).

The **steady state** at which \( \dot{x}(t) = 0 \) occurs when \( ax + b = 0 \),
and so at \( x^* := -\frac{b}{a} \).

Then the **deviation** \( y(t) := x(t) - x^* \) of \( x(t) \)
from the steady state \( x^* \) satisfies the homogeneous equation

\[ \dot{y}(t) = \dot{x}(t) = ax(t) + b = a[y(t) + x^*] + b = ay(t) \]

Hence \( y(t) = e^{at}y(0) \), implying that \( x(t) = x^* + e^{at}[x(0) - x^*] \).
The steady state $x^* := -b/a$ is stable just in case, for all $x(0)$, the solution $x(t) = x^* + e^{at}[x(0) - x^*]$ satisfies $x(t) \to x^*$ as $t \to \infty$.

A necessary and sufficient condition for stability is obviously that $a < 0$. 
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Second-Order Equations with Constant Coefficients

A general second-order differential equation takes the form

\[ \ddot{x}(t) = F(\dot{x}(t), x(t), t) \]

To obtain a unique solution (if any solution exists), one typically needs two initial conditions such as \( x(s) = x_s \) and \( \dot{x}(s) = \dot{x}_s \) at an initial time \( s \).

The equation is autonomous just in case it takes the form \( \ddot{x}(t) = F(\dot{x}(t), x(t)) \), with \( F \) independent of \( t \).

The equation is linear just in case it takes the form \( \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0 \), with \( F \) linear in \( (\dot{x}(t), x(t)) \).

The equation is linear with constant coefficients just in case it takes the form \( \ddot{x}(t) + a\dot{x}(t) + bx(t) = 0 \).
We know that the first-order equation \( \dot{x}(t) + ax(t) = 0 \) has a solution of the form \( x(t) = x(0)e^{\lambda t} \) where \( \lambda \) solves the characteristic equation \( \lambda + a = 0 \).

So we look for solutions of the form \( x(t) = \xi e^{\lambda t} \) to the second-order equation \( \ddot{x}(t) + a\dot{x}(t) + bx(t) = 0 \).

Note that when \( x(t) = \xi e^{\lambda t} \), then \( \dot{x}(t) = \lambda \xi e^{\lambda t} \) and \( \ddot{x}(t) = \lambda^2 \xi e^{\lambda t} \).

So \( x(t) = \xi e^{\lambda t} \) is a non-trivial solution (with \( \xi \neq 0 \)) if and only if

\[
0 = \lambda^2 \xi e^{\lambda t} + a\lambda \xi e^{\lambda t} + b\xi e^{\lambda t} = (\lambda^2 + a\lambda + b)\xi e^{\lambda t}
\]

and so, given that \( \xi e^{\lambda t} \neq 0 \), if and only if \( \lambda \) is a root of the characteristic equation \( \lambda^2 + a\lambda + b = 0 \).
Characteristic Equation for an Equation of Order $n$

**Definition**

A homogeneous differential equation of order $n$ with constant coefficients takes the form

$$
\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} x(t) = 0
$$

where the coefficient of the $n$ derivative satisfies $a_n \neq 0$, and so can be normalized to take the value $a_n = 1$.

**Remark**

A similar technique based on roots of the characteristic equation applies to this $n$th order equation.

It implies that $x(t) = \xi e^{\lambda t}$ is a non-trivial solution if and only if $\lambda$ is a root of the characteristic equation

$$
\sum_{k=0}^{n} a_k \lambda^k = 0
$$
Characteristic Roots

One can factorize the quadratic function $q(\lambda) := \lambda^2 + a\lambda + b$ as $q(\lambda) := (\lambda - \lambda_1)(\lambda - \lambda_2)$ where $\lambda_1$ and $\lambda_2$ are the two roots of the equation $q(\lambda) = 0$.

As with the corresponding discussion of second-order difference equations, there are three cases:

1. in case $a^2 > 4b$, there are two distinct real roots $\lambda_1$ and $\lambda_2$ given by $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.

2. in case $a^2 < 4b$, there are two complex conjugate roots given by $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}i\sqrt{4b - a^2}$.

3. in case $a^2 = 4b$, there are two coincident real roots given by $\lambda = -\frac{1}{2}a = \sqrt{b}$. 
Case 1: Two Distinct Real Roots

In this case $a^2 > 4b$, when the two characteristic roots are $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.

Because $\lambda_1 \neq \lambda_2$, one has

$$\begin{vmatrix}
e^{\lambda_1 0} & e^{\lambda_2 0} \\
e^{\lambda_1 1} & e^{\lambda_2 1}
\end{vmatrix} = \begin{vmatrix}1 & 1 \\
e^{\lambda_1} & e^{\lambda_2}\end{vmatrix} = e^{\lambda_2} - e^{\lambda_1} \neq 0$$

and so $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two linearly independent solutions.

The general solution of the homogeneous equation is

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$
Case 2: Two Complex Conjugate Roots, I

In this case $a^2 < 4b$, when the two characteristic roots are the complex conjugates $\lambda_{1,2} = -\frac{1}{2}a \pm i\theta$, with $\theta := \frac{1}{2}\sqrt{4b - a^2}$.

Then $x(t) = e^{\lambda_1 t} = e^{-\frac{1}{2}at}e^{i\theta t} = e^{-\frac{1}{2}at}(\cos \theta t + i \sin \theta t)$

and $x(t) = e^{\lambda_2 t} = e^{-\frac{1}{2}at}e^{-i\theta t} = e^{-\frac{1}{2}at}(\cos \theta t - i \sin \theta t)$

are two different solutions, where $\theta \neq 0$.

For any $t$ such that $\sin \theta t \neq 0$, one has

$$\begin{vmatrix} e^{\lambda_1 0} & e^{\lambda_2 0} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{vmatrix} = e^{\lambda_2 t} - e^{\lambda_1 t} = -2e^{-\frac{1}{2}at}i \sin \theta t \neq 0$$

It follows that $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two linearly independent solutions in the complex plane $\mathbb{C}$.
Case 2: Two Complex Conjugate Roots, II

Focusing on solutions in the real line $\mathbb{R}$, we can consider $e^{-\frac{1}{2}a t} \cos \theta t$ and $e^{-\frac{1}{2}a t} \sin \theta t$.

Again, for any $t$ such that $\sin \theta t \neq 0$, one has

$$
\begin{vmatrix}
   e^{-\frac{1}{2}a t} \cos \theta t & e^{-\frac{1}{2}a t} \sin \theta t \\
   e^{-\frac{1}{2}a t} \cos \theta 0 & e^{-\frac{1}{2}a t} \sin \theta 0
\end{vmatrix} =
\begin{vmatrix}
   1 & 0 \\
   -\frac{1}{2}a t \cos \theta t & e^{-\frac{1}{2}a t} \sin \theta t
\end{vmatrix} =
\begin{vmatrix}
   e^{-\frac{1}{2}a t} \sin \theta t \\
   e^{-\frac{1}{2}a t} \sin \theta t \neq 0
\end{vmatrix}
$$

It follows that $e^{-\frac{1}{2}a t} \cos \theta t$ and $e^{-\frac{1}{2}a t} \sin \theta t$ are two linearly independent real-valued solutions in the complex plane $\mathbb{C}$.

The general solution of the homogeneous equation is $x = e^{-\frac{1}{2}a t}(A \cos \theta t + B \sin \theta t)$. 

Case 3: Two Coincident Real Roots

In this case $a^2 = 4b$, and so

$$q(\lambda) = \lambda^2 + a\lambda + b = (\lambda + \frac{1}{2}a)^2 = (\lambda - \sqrt{b})^2$$

The homogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$ has one solution given by $x = e^{\lambda t}$ where $\lambda = -\frac{1}{2}a = \sqrt{b}$.

To find a second linearly independent solution, introduce the new variable $y(t) := e^{-\lambda t}x(t)$.

Then $\dot{y}(t) = e^{-\lambda t}\dot{x}(t) - \lambda e^{-\lambda t}x(t)$ and so, when $x = e^{\lambda t}$, one has

$$\ddot{y}(t) = e^{-\lambda t}\ddot{x}(t) - 2\lambda e^{-\lambda t}\dot{x}(t) + \lambda^2 e^{-\lambda t}x(t)$$
$$= e^{-\lambda t}[\ddot{x}(t) - 2\lambda \dot{x}(t) + \lambda^2 x(t)]$$
$$= e^{-\lambda t}[\lambda^2 e^{\lambda t} - 2\lambda \cdot \lambda e^{\lambda t} + \lambda^2 e^{\lambda t}] = 0$$

The obvious general solution to $\ddot{y}(t) = 0$ satisfies $\dot{y}(t) = \text{constant}$ and so $y(t) = A + Bt = e^{-\lambda t}x(t)$.

Hence $x(t) = (A + Bt)e^{\lambda t}$ is the general solution.
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The Inhomogeneous Equation

Consider next the inhomogeneous equation

\[ \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t) \]

with a non-zero forcing term on the right-hand side.

Suppose that \( y(t) \) and \( z(t) \) are both solutions, implying that

\[ \ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) = f(t) \]
and \[ \ddot{z}(t) + a(t)\dot{z}(t) + b(t)z(t) = f(t) \]

Subtracting the second equation from the first tells us that the function \( x_H(t) := y(t) - z(t) \) is a solution of the corresponding homogeneous equation \( \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0 \).

So the general solution of \( \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t) \)
is the sum \( x_G(t) = x_P(t) + x_H(t) \) of:

- any particular solution \( x_P(t) \) of the inhomogeneous equation;
- any function \( x_H(t) \) in the two dimensional linear space of solutions to the homogeneous equation.
Theorem

Suppose that \( x^P(t) \) and \( y^P(t) \) are particular solutions of the two respective differential equations

\[
\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = d(t)
\]
and
\[
\ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) = e(t)
\]

Then, for any scalars \( \alpha \) and \( \beta \), a particular solution of the equation

\[
\ddot{z}(t) + a(t)\dot{z}(t) + b(t)z(t) = f(t) = \alpha d(t) + \beta e(t)
\]

is the linear combination \( z^P(t) := \alpha x^P(t) + \beta y^P(t) \).
Consider the equation $\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t)$ whose forcing term $f(t)$ is a linear combination $\sum_{k=1}^{n} \alpha_k f^k(t)$ of $n$ forcing terms.

The theorem implies that a particular solution is the corresponding linear combination $\sum_{k=1}^{n} \alpha_k x^{P_k}(t)$ of particular solutions to the $n$ equations

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f^k(t) \quad (k = 1, 2, \ldots, n)$$
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A Newtonian Example, I

Newton’s law: force = mass \times \text{acceleration}.

A force of 1 Newton, by definition, accelerates a mass of 1 kilogram at the rate of 1 metre per second per second.

So we consider the equation $\ddot{x}(t) = f(t)$ whose solution $t \mapsto x(t)$ is the position (in one dimension) of a 1 kilogram weight that has been subjected to a force function $t \mapsto f(t)$.

Integrating once gives us the equation $\dot{x}(t) = \dot{x}(0) + \int_0^t f(u)du$.

Integrating a second time gives us the solution

$$x(t) = x(0) + \int_0^t \dot{x}(\nu)d\nu = x(0) + \int_0^t \left[ \dot{x}(0) + \int_0^\nu f(u)du \right] d\nu = x(0) + \dot{x}(0)t + \int_0^t \left[ \int_0^\nu f(u)du \right] d\nu$$

Note that $x(0) + \dot{x}(0)t$ solves the homogeneous equation $\ddot{x}(t) = 0$, whereas the iterated double integral $\int_0^t \left[ \int_0^\nu f(u)du \right] d\nu$ is a particular solution.
An Important Theorem on Iterated Double Integrals, I

Theorem

For any integrable function \((x, y) \mapsto \phi(x, y) \in \mathbb{R}\) defined on the square domain \([a, b] \times [a, b] \subset \mathbb{R}^2\), one has

\[
\int_a^b \left[ \int_a^y \phi(x, y) \, dx \right] \, dy = \int_a^b \left[ \int_x^b \phi(x, y) \, dy \right] \, dx
\]

Proof.

Define the indicator function \(1_{x \leq y}(x, y) := \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y \end{cases}\). Then

\[
\int_a^b \left[ \int_a^y \phi(x, y) \, dx \right] \, dy = \int_a^b \left[ \int_a^b 1_{x \leq y}(x, y) \phi(x, y) \, dx \right] \, dy
\]

\[
\int_a^b \left[ \int_x^b \phi(x, y) \, dy \right] \, dx = \int_a^b \left[ \int_a^b 1_{x \leq y}(x, y) \phi(x, y) \, dy \right] \, dx
\]

But both right-hand sides equal \(\int_a^b \int_a^b 1_{x \leq y}(x, y) \phi(x, y) \, dx \, dy\). \qed
An Important Theorem on Iterated Double Integrals, II

An alternative simple proof involves noticing that the two integrals

\[
\int_a^b \left[ \int_a^y \phi(x, y) \, dx \right] \, dy \quad \text{and} \quad \int_a^b \left[ \int_x^b \phi(x, y) \, dy \right] \, dx
\]

are simply two different ways of writing the integral \( \int_T \phi(x, y) \, dx \, dy \) of the function \( \phi \) of two variables over the isosceles right-angled triangle

\[
T := \{(x, y) \in [a, b] \times [a, b] \subset \mathbb{R}^2 \mid x \leq y\}
\]

Note that \( T \) consists of points above and to the left of the diagonal that joins the two corner points \((a, a)\) and \((b, b)\) of the square \([a, b] \times [a, b]\).

The set \( T \) is also the convex hull of the three points \((a, a)\), \((a, b)\) and \((b, b)\).
Reversing the order of integration allows the particular solution in the form of the iterated double integral $\int_0^t \left[ \int_0^t f(u) \, du \right] \, dv$ to be rewritten as

$$\int_0^t \left[ \int_u^t f(u) \, dv \right] \, du = \int_0^t \left[ \int_u^t 1 \, dv \right] f(u) \, du = \int_0^t (t - u)f(u) \, du$$

Ultimately, then, one has

$$x(t) = x(0) + \dot{x}(0)t + \int_0^t (t - u)f(u) \, du$$
Next, consider the equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = f(t)$
where the coefficients $a$ of $\dot{x}(t)$ and $b$ of $\dot{x}(t)$
have both become constants, with $b \neq 0$.

Consider the quadratic function $q(\lambda) := \lambda^2 + a\lambda + b$
that appears in the characteristic equation $\lambda^2 + a\lambda + b = 0$.

One can factorize it as

$$q(\lambda) = \lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

where $\lambda_1$ and $\lambda_2$ are the two roots of the equation $q(\lambda) = 0$.

Recall that $\lambda_1 + \lambda_2 = -a$ and $\lambda_1\lambda_2 = b$.

Define the new variable $y(t) := \dot{x}(t) - \lambda_1 x(t)$.

Note that, if we could find the function $t \mapsto y(t)$,
then we would have

$$x(t) = e^{\lambda_1 t}x(0) + \int_0^t e^{\lambda_1(t-u)}y(u)du$$
We are considering the equation \( \ddot{x}(t) + a\dot{x}(t) + b x(t) = f(t) \), with \( b \neq 0 \).

We have introduced the new variable \( y(t) := \dot{x}(t) - \lambda_1 x(t) \), implying that \( x(t) = e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} y(u) du \).

But the characteristic roots satisfy \( \lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2) \), implying that \( \lambda_1 + \lambda_2 = -a \) and \( \lambda_1 \lambda_2 = b \), and so

\[
\dot{y}(t) - \lambda_2 y(t) = \ddot{x}(t) - \lambda_1 \dot{x}(t) - \lambda_2 \dot{x}(t) + \lambda_1 \lambda_2 x(t) = \dot{x}(t) + a\dot{x}(t) + b x(t)
\]

Hence \( y(t) \) satisfies the first-order equation \( \dot{y}(t) - \lambda_2 y(t) = f(t) \) whose solution is

\[
y(t) = e^{\lambda_2 t} y(0) + \int_0^t e^{\lambda_2 (t-v)} f(v) dv
\]
Linear Equation with Constant Coefficients, III

Substituting \( y(t) = e^{\lambda_2 t} y(0) + \int_0^t e^{\lambda_2 (t-v)} f(v)dv \)
in the expression \( x(t) = e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} y(u)du \) gives

\[
x(t) = e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} y(u)du \\
= e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} \left[ e^{\lambda_2 u} y(0) + \int_0^u e^{\lambda_2 (u-v)} f(v)dv \right] du
\]

We split this form of the solution into two parts:

1. the complementary solution

\[
t \mapsto x^C(t) := e^{\lambda_1 t} x(0) + y(0) \int_0^t e^{\lambda_1 (t-u)} e^{\lambda_2 u}du \\
= e^{\lambda_1 t} \left[ x(0) + y(0) \int_0^t e^{(\lambda_2-\lambda_1)u}du \right]
\]

to the homogeneous equation \( \ddot{x}(t) + a \dot{x}(t) + bx(t) = 0; \)

2. a particular solution in the form of the iterated double integral

\[
t \mapsto x^P(t) := \int_0^t e^{\lambda_1 (t-u)} \left[ \int_0^u e^{\lambda_2 (u-v)} f(v)dv \right] du
\]

to the inhomogeneous equation \( \ddot{x}(t) + a \dot{x}(t) + bx(t) = f(t). \)
Degenerate Case

In the degenerate case when $\lambda_1 = \lambda_2 = \lambda$,

1. the **complementary** solution takes the form:

$$x^C(t) = e^{\lambda t}x(0) + y(0) \int_0^t e^{\lambda u} du$$

$$= e^{\lambda t} [x(0) + y(0)t]$$

2. the **particular** solution takes the form:

$$x^P(t) = \int_0^t e^{\lambda(t-u)} \left[ \int_0^u e^{\lambda(u-v)} f(v) dv \right] du$$

$$= e^{\lambda t} \int_0^t \left[ \int_0^u e^{-\lambda v} f(v) dv \right] du$$

$$= e^{\lambda t} \int_0^t \left[ \int_v^t 1 du \right] e^{-\lambda v} f(v) dv$$

$$= \int_0^t (t-v)e^{\lambda(t-v)} f(v) dv$$

The overall solution is therefore

$$x(t) = e^{\lambda t} \left[ x(0) + y(0)t + \int_0^t (t-v)e^{-\lambda v} f(v) dv \right]$$
Non-Degenerate Case: Complementary Solution

In the non-degenerate case when $\lambda_1 \neq \lambda_2$, the complementary solution takes the form

$$x^C(t) = e^{\lambda_1 t} \left[ x(0) + y(0) \int_0^t e^{(\lambda_2 - \lambda_1)u} du \right]$$

$$= e^{\lambda_1 t} x(0) + \frac{1}{\lambda_2 - \lambda_1} e^{\lambda_1 t} y(0) \left[ e^{(\lambda_2 - \lambda_1)t} - 1 \right]$$

$$= x(0)e^{\lambda_1 t} + y(0)\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}$$

After substituting $\dot{x}(0) - \lambda_1 x(0)$ for $y(0)$, the right-hand side becomes

$$\frac{1}{\lambda_2 - \lambda_1} \left\{(\lambda_2 - \lambda_1)x(0)e^{\lambda_1 t} + [\dot{x}(0) - \lambda_1 x(0)](e^{\lambda_2 t} - e^{\lambda_1 t})\right\}$$

and so

$$x^C(t) = \frac{1}{\lambda_2 - \lambda_1} \left[ x(0)(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}) + \dot{x}(0)(e^{\lambda_2 t} - e^{\lambda_1 t}) \right]$$
Non-Degenerate Case: Particular Solution

Using our rule for reversing the order of recursive integration, the particular solution takes the form

\[ x^P(t) = \int_0^t e^{\lambda_1(t-u)} \left[ \int_0^u e^{\lambda_2(u-v)} f(v)dv \right] du \]

\[ = \int_0^t \left[ \int_0^t e^{\lambda_1(t-u)} e^{\lambda_2(u-v)} du \right] f(v)dv \]

\[ = \int_0^t e^{\lambda_1 t - \lambda_2 v} \left[ \int_0^t e^{(\lambda_2 - \lambda_1)u} du \right] f(v)dv \]

\[ = \frac{1}{\lambda_2 - \lambda_1} \int_0^t e^{\lambda_1 t - \lambda_2 v} \left[ e^{(\lambda_2 - \lambda_1)t} - e^{(\lambda_2 - \lambda_1)v} \right] f(v)dv \]

\[ = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[ e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)} \right] f(v)dv \]
First Special Case

An interesting first special case of the particular solution

\[ x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[ e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)} \right] f(v) dv \]

occurs when \( f(t) \) is the exponential function \( e^{\mu t} \), and so

\[ x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[ e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)} \right] e^{\mu v} dv \]

In the degenerate case when \( \lambda_2 = \mu \neq \lambda_1 \), this reduces to

\[
\begin{align*}
  x^P(t) &= \frac{1}{\lambda_2 - \lambda_1} \left[ e^{\lambda_2 t} t - e^{\lambda_1 t} \int_0^t e^{(\mu - \lambda_1)v} dv \right] \\
  &= \frac{1}{\lambda_2 - \lambda_1} \left[ e^{\lambda_2 t} t - e^{\lambda_1 t} \frac{e^{(\lambda_2 - \lambda_1)t} - 1}{\lambda_2 - \lambda_1} \right] \\
  &= \frac{e^{\lambda_2 t} t}{\lambda_2 - \lambda_1} - \frac{e^{\lambda_1 t} (e^{(\lambda_2 - \lambda_1)t} - 1)}{(\lambda_2 - \lambda_1)^2} \\
  &= \frac{e^{\lambda_2 t} t}{\lambda_2 - \lambda_1} - \frac{e^{\lambda_1 t} (e^{(\lambda_2 - \lambda_1)t} - 1)}{(\lambda_2 - \lambda_1)^2}
\end{align*}
\]
Non-Degenerate Case

In the non-degenerate case when $\lambda_1, \lambda_2$ and $\mu$ are all different, one has the particular solution

$$x^P(t) = \frac{e^{\lambda_2 t}}{\lambda_2 - \lambda_1} \int_0^t e^{(\mu - \lambda_2)v} \, dv - \frac{e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \int_0^t e^{(\mu - \lambda_1)v} \, dv$$

$$= \frac{e^{\lambda_2 t} \left[ e^{(\mu - \lambda_2)t} - 1 \right]}{(\lambda_2 - \lambda_1)(\mu - \lambda_2)} - \frac{e^{\lambda_1 t} \left[ e^{(\mu - \lambda_1)t} - 1 \right]}{(\lambda_2 - \lambda_1)(\mu - \lambda_1)}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \left( \frac{e^{\mu t} - e^{\lambda_2 t}}{\mu - \lambda_2} - \frac{e^{\mu t} - e^{\lambda_1 t}}{\mu - \lambda_1} \right)$$

But the multiples of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ can be incorporated in the complementary solution to the homogeneous equation, so this particular solution can be reduced to

$$\tilde{x}^P(t) = \frac{e^{\mu t}}{\lambda_2 - \lambda_1} \left( \frac{1}{\mu - \lambda_2} - \frac{1}{\mu - \lambda_1} \right) = \frac{e^{\mu t}}{(\mu - \lambda_1)(\mu - \lambda_2)}$$
Second Special Case

An interesting second special case of the particular solution

\[ x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[ e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)} \right] f(v)dv \]

occurs when \( f(t) \) is the exponential function \( t^r e^{\mu t} \), and so

\[ x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[ e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)} \right] v^r e^{\mu v}dv \]

In the non-degenerate case when \( \lambda_1, \lambda_2 \) and \( \mu \) are all different, this becomes

\[
\begin{align*}
    x^P(t) &= \frac{1}{\lambda_2 - \lambda_1} \left[ e^{\lambda_2 t} \int_0^t v^r e^{(\mu - \lambda_2)v}dv - e^{\lambda_1 t} \int_0^t v^r e^{(\mu - \lambda_1)v}dv \right] \\
    &= P_2(t)e^{\lambda_2 t} - P_1(t)e^{\lambda_1 t}
\end{align*}
\]

for polynomials \( t \mapsto P_1(t) \) and \( t \mapsto P_2(t) \) of degree \( r \) whose coefficients are functions of the parameter triple \((\lambda_1, \lambda_2, \mu)\).
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First-Order Multivariable Differential Equations
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- Autonomous Nonlinear Equations in Many Variables
Now consider the autonomous equation

\[ \ddot{x}(t) + a\dot{x}(t) + bx(t) = c \]

with a constant right-hand side.

There is a constant solution \( x(t) = \bar{x} \)
where \( \bar{x} = c/b \) is the unique steady state.

The new variable \( y(t) := x(t) - \bar{x} \) satisfies
the homogeneous equation \( \ddot{x}(t) + a\dot{x}(t) + bx(t) = 0 \).

The associated characteristic equation is

\[ \lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \]
A Stability Condition

1. In case there are two real characteristic roots

   \[ \lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b} \]

   the general solution \( Ae^{\lambda_1 t} + Be^{\lambda_2 t} \rightarrow 0 \text{ as } t \rightarrow \infty \)

   if and only if both \( \lambda_1 \) and \( \lambda_2 \) are negative.

2. In case there are two complex conjugate characteristic roots

   \[ \lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}i\sqrt{4b - a^2} \]

   one has \( e^{\lambda t} = e^{-\frac{1}{2}at} e^{\pm \frac{1}{2}it\sqrt{4b - a^2}} \).

   The general solution \( Ae^{\lambda_1 t} + Be^{\lambda_2 t} \rightarrow 0 \text{ as } t \rightarrow \infty \)

   iff \( a > 0 \), or iff both \( \lambda_1 \) and \( \lambda_2 \) have negative real parts.

3. In case there are two coincident real characteristic roots,

   the general solution \( (A + Bt)e^{\lambda t} \rightarrow 0 \text{ as } t \rightarrow \infty \)

   iff \( \lambda < 0 \).

All these conditions can be subsumed into one: stability holds iff each characteristic root has a negative real part.
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A linear differential equation in $n$ variables specifies the time derivative $\dot{x}(t)$ of the $n$-vector $x(t)$ as an affine function $A(t)x(t) + b(t)$ of $x(t)$.

That is

$$\dot{x}(t) = A(t)x(t) + b(t)$$

where

- $t \mapsto A(t) \in \mathbb{R}^{n \times n}$ is a matrix-valued function of time;
- $t \mapsto b(t) \in \mathbb{R}^n$ is a vector-valued function of time.
Matrix Differentiation

Consider the $m \times n$ matrix function $t \mapsto A(t)$ whose elements $(a_{ij}(t))_{m \times n}$ are differentiable functions of $t$.

For all $h \neq 0$, the Newton quotient matrix $\frac{1}{h}[A(t+h) - A(t)]$ has elements equal to the Newton quotients $\frac{1}{h}(a_{ij}(t+h) - a_{ij}(t))_{m \times n}$ of the matrix $(a_{ij}(t))_{m \times n}$.

As $h \to 0$, these converge to the derivatives $(\frac{d}{dt} a_{ij}(t))_{m \times n}$.

For this reason, the matrix $A(t)$ is said to be differentiable with derivative $\dot{A}(t) = \frac{d}{dt} A(t)$ whose elements are $(\frac{d}{dt} a_{ij}(t))_{m \times n}$.
Differentiating the Product of Matrices

Suppose that $t \mapsto A(t)$ and $t \mapsto B(t)$ are differentiable, where each $A(t)$ is $\ell \times m$ and each $B(t)$ is $m \times n$.

Then $t \mapsto C(t) = A(t)B(t)$ is well defined as a matrix product with elements given by $c_{ik}(t) = \sum_{j=1}^{m} a_{ij}(t)b_{jk}(t)$ whose time derivatives are

$$
\dot{c}_{ik}(t) = \sum_{j=1}^{m} [\dot{a}_{ij}(t)b_{jk}(t) + a_{ij}(t)\dot{b}_{jk}(t)]
$$

Hence $t \mapsto C(t)$ is differentiable,

with $\dot{C}(t) = \dot{A}(t)B(t) + A(t)\dot{B}(t)$. 
Differentiating the Square of a Square Matrix

Suppose that $A(t)$ is an $n \times n$ matrix for all $t$, and that each element is a differentiable function of $t$.

Then the square matrix $A^2(t)$ is well defined and differentiable, with derivative $\frac{d}{dt} A^2(t) = \dot{A}(t)A(t) + A(t)\dot{A}(t)$.

Unless the matrices $\dot{A}(t)$ and $A(t)$ happen to commute, in the sense that $\dot{A}(t)A(t) = A(t)\dot{A}(t)$, this will not be equal to $2\dot{A}(t)A(t)$ or to $2A(t)\dot{A}(t)$. 

Example

Note that, even if each $\mathbf{A}(t)$ is square, it may not commute with $\dot{\mathbf{A}}(t)$.

For example, when $\mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix}$, then $\dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, implying that $\mathbf{A}(t) \dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \neq \dot{\mathbf{A}}(t) \mathbf{A}(t) = \begin{pmatrix} 0 & 0 \\ 1 & t \end{pmatrix}$.

Note that in this example $\mathbf{A}$ is symmetric; so therefore is $\dot{\mathbf{A}}$.

Hence $\mathbf{A}(t) \dot{\mathbf{A}}(t) = \mathbf{A}(t)^\top \dot{\mathbf{A}}^\top(t) = [\dot{\mathbf{A}}(t) \mathbf{A}(t)]^\top$.

Also $\mathbf{A}^2(t) = \begin{pmatrix} 1 & t \\ t & 1 + t^2 \end{pmatrix}$ whose derivative satisfies

$$
\frac{d}{dt} \mathbf{A}^2(t) = \dot{\mathbf{A}}(t) \mathbf{A}(t) + \mathbf{A}(t) \dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 2t \end{pmatrix}
$$

This differs from both $2 \mathbf{A}(t) \dot{\mathbf{A}}(t)$ and $2 \dot{\mathbf{A}}(t) \mathbf{A}(t)$. 
The Exponential of a Square Matrix

Recall that the exponential function of a scalar is defined so that the solution of the differential equation \( \dot{x} = ax \) is \( x(t) = e^{at} x(0) \).

Similarly, we define the exponential function of a square matrix so that the solution of the differential equation system \( \dot{x} = Ax \) is \( x(t) = \exp(At) x(0) \).

The function \( t \mapsto \exp(At) \) is often called the resolvent.

Recall that, for a scalar, there is the convergent power series

\[
e^{at} = 1 + \frac{1}{1!} at + \frac{1}{2!} (at)^2 + \frac{1}{3!} (at)^3 \ldots = \sum_{r=0}^{\infty} \frac{1}{r!} (at)^r
\]

with the convention that \( 0! = 1 \).

Similarly, for a square matrix, with the convention that \( (At)^0 = I \) one can use a convergent power series to give,

\[
\exp(At) = I + \frac{1}{1!} At + \frac{1}{2!} (At)^2 + \frac{1}{3!} (At)^3 \ldots = \sum_{r=0}^{\infty} \frac{1}{r!} (At)^r
\]
The Exponential of a Diagonal Matrix

Dropping the time argument, it follows that we define

\[ \exp(C) := I + \frac{1}{1!}C + \frac{1}{2!}(C)^2 + \frac{1}{3!}(C)^3 \ldots = \sum_{r=0}^{\infty} \frac{1}{r!}(C)^r \]

Suppose that \( C \) is the diagonal matrix \( \text{diag}(c_1, c_2, \ldots, c_n) = \text{diag} c \) where \( c \) is the vector \((c_1, c_2, \ldots, c_n)\).

Now, each matrix power \( (\text{diag} c)^r = \text{diag}(c_1^r, c_2^r, \ldots, c_n^r) \) as is readily proved by induction on \( r \).

So, with this notation for the exponential of a matrix, we have

\[ \exp(C) = \sum_{r=0}^{\infty} \frac{1}{r!}C^r = \sum_{r=0}^{\infty} \frac{1}{r!} \text{diag}(c_1^r, c_2^r, \ldots, c_n^r) = \text{diag} (e^{c_1}, e^{c_2}, \ldots, e^{c_n}) \]

Also, suppose matrix \( C \) has \( C = V\Lambda V^{-1} \) as a diagonalization.

Then each matrix power \( C^r = V\Lambda^r V^{-1} \) implying that \( \exp(C) = V \exp(\Lambda)V^{-1} \).
Integrating and Differentiating an Exponential Matrix

From the definition \( \exp(As) = \sum_{r=0}^{\infty} \frac{1}{r!} (As)^r \)
either post- or premultiplying by \( A \) and then integrating gives
\[
\int_{0}^{t} \exp(As) \, A \, ds = \int_{0}^{t} A \exp(As) \, ds = \int_{0}^{t} \sum_{r=0}^{\infty} \frac{1}{r!} A^{r+1} s^r \, ds
\]
Next, integrating term by term, the last expression becomes
\[
\sum_{r=0}^{\infty} \frac{1}{r!} A^{r+1} \int_{0}^{t} s^r \, ds = \sum_{r=0}^{\infty} \frac{1}{r!} A^{r+1} \left. t^{r+1} \right|_{0}^{t}
\]
Simplifying converts this to
\[
\sum_{r=0}^{\infty} \frac{1}{(r + 1)!} A^{r+1} t^{r+1} = \sum_{r=1}^{\infty} \frac{1}{r!} A^{r} t^{r} = \exp(At) - I
\]
So \( \int_{0}^{t} \exp(As) \, A \, ds = \int_{0}^{t} A \exp(As) \, ds = \exp(At) - I \), implying that
\[
\frac{d}{dt} \exp(At) = A \exp(At) = \exp(At) \, A
\]
Affine Equation in $n$ Variables

Consider what happens when we multiply each side of the non-homogeneous equation $\dot{x}(t) - Ax(t) = b(t)$ by the matrix integrating factor $\exp(-At)$.

Because the product rule of differentiation applies to matrices,

$$
\frac{d}{dt} \left[ \exp(-At) \, x(t) \right] = \exp(-At) \, \dot{x}(t) + \frac{d}{dt} \left[ \exp(-At) \right] \, x(t)
$$

$$
= \exp(-At) \, \dot{x}(t) - \exp(-At) \, A \, x(t)
$$

$$
= \exp(-At) \, b(t)
$$

if and only if $x(t)$ solves the equation $\dot{x}(t) - Ax(t) = b(t)$.

Hence $\exp(-At) \, x(t) - \exp(-As) \, x(s) = \int_s^t \exp(-A\tau) \, b(\tau) \, d\tau$.

Multiplying each side by $\exp(A \, t)$ gives the unique solution

$$
x(t) = \exp[A(t - s)] \, x(s) + \int_s^t \exp[A(t - \tau)] \, b(\tau) \, d\tau
$$
The **diagonal case** occurs when $A = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

Then the system $\dot{x}(t) - Ax(t) = b(t)$ of $n$ coupled equations reduces to the system of $n$ uncoupled equations

$$
\dot{x}_i(t) = a_{ii}x_i(t) + b_i(t) = \lambda_i x_i(t) + b_i(t) \ (i = 1, \ldots, n)
$$

one in each variable $x_i$, with respective solutions

$$
x_i(t) = e^{\lambda_i t} x_i(s) + \int_s^t e^{\lambda_i (t-\tau)} b_i(\tau) \, d\tau
$$
The Diagonalizable Case

Suppose that $A$ has $n$ distinct eigenvalues — or if not, then $n$ linearly independent eigenvectors that make up the columns of the matrix $V$.

Then $A = V\Lambda V^{-1}$ and $At = V(\Lambda t)V^{-1}$ implying that $\exp(At) = V \exp(\Lambda t)V^{-1}$.

Hence the solution

$$x(t) = \exp[A(t - s)] x(s) + \int_s^t \exp[A(t - \tau)] b(\tau) d\tau$$

simplifies to

$$x(t) = V \exp[\Lambda(t - s)] V^{-1} x(s) + \int_s^t V \exp[\Lambda(t - \tau)] V^{-1} b(\tau) d\tau$$

Of course, the transformation $y(t) := V^{-1} x(t)$ takes us back to the diagonal case, with $\dot{y}(t) := \Lambda y(t) + V^{-1} b(t)$. 
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A Stability Condition

When $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, one has $\exp(\Lambda) = \text{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})$.

Furthermore $\exp(\Lambda t) = \text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})$.

This converges to the zero matrix as $t \to \infty$ if and only if each $e^{\lambda_i t} \to 0$, which is true iff each eigenvalue $\lambda_i$ has a negative real part.

Similarly, if $A$ is diagonalizable, with $A = V \Lambda V^{-1}$, then consider the new variables $y = V^{-1}x$.

The differential equation $\dot{x}(t) = Ax(t)$ becomes transformed to

$$\dot{y}(t) = V^{-1} \dot{x}(t) = V^{-1}Ax(t) = \Lambda V^{-1}x(t) = \Lambda y(t)$$

Because $V$ is invertible, one has $x(t) \to 0 \iff y(t) \to 0$.

Once again, stability holds iff each eigenvalue of the matrix $A$ has a negative real part.

This is true even when $A$ is not diagonalizable.
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Autonomous Nonlinear Equations in Many Variables
Autonomous First-Order Equations

Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a general function that may be non-linear.

Consider the autonomous differential equation \( \dot{x} = F(x) \).

A solution satisfying the initial condition \( x(s) = \bar{x} \) is a differentiable function \( [s, t) \ni t \mapsto x(t) \) that satisfies \( \dot{x}(t) = F(x(t)) \) for almost all \( t \geq s \).

Equivalently, for almost all \( t \geq s \), one must have

\[
x(t) = x(s) + \int_s^t F(x(\tau)) \, d\tau
\]
Stationary States and Rest Points

A stationary state is a point \( x^* \in \mathbb{R}^n \) with the property that if \( x(s) = x^* \) at any time \( s \), then \( x(t) = x^* \) at all times \( t \geq s \).

A rest point is a state \( \bar{x} \in \mathbb{R}^n \) with the property that \( F(\bar{x}) = 0 \).

Theorem
Any rest point is a stationary state, and conversely.

Proof.
If \( F(\bar{x}) = 0 \), then the solution of \( x(t) = x(s) + \int_s^t F(x(\tau)) \, d\tau \)
with \( x(s) = \bar{x} \) satisfies \( x(t) = x(s) = \bar{x} \) for all \( t \geq s \).

Conversely, if that solution satisfies \( x(t) = x(s) = x^* \) for all \( t \geq s \),
then \( \dot{x}(t) = F(x(t)) = F(x^*) = 0 \) for all \( t \geq s \).
Let $F'(x)$ denote the $n \times n$ Jacobian matrix whose elements are the partial derivatives $F'_{ij}(x) = \frac{\partial}{\partial x_j} F_i(x)$ of the different components $(F_i(x))_{i=1}^n$.

Any particular steady state $x^*$ is locally asymptotically stable if and only if all the eigenvalues of $F'(x^*)$ have negative real parts.
A System with Two Variables

Consider the coupled pair \( \dot{x} = f(x, y), \quad \dot{y} = g(x, y) \) of differential equations.

Let \((a, b)\) be any stationary point satisfying both \( f(a, b) = 0 \) and \( g(a, b) = 0 \).

The Jacobian matrix at the stationary point takes the form

\[
J(a, b) = \begin{pmatrix}
\frac{\partial f}{\partial x}(a, b) & \frac{\partial f}{\partial y}(a, b) \\
\frac{\partial g}{\partial x}(a, b) & \frac{\partial g}{\partial y}(a, b)
\end{pmatrix}
\]
Local Saddle Point with Two Variables

The product of the two eigenvalues $\lambda_1, \lambda_2$ of $J(a, b)$ equals its determinant $|J(a, b)|$.

The two eigenvalues are real and have opposite signs if and only if $|J(a, b)| < 0$.

This is a sufficient condition for the steady state to be unstable.

But if $(x(0) - a, y(0) - b)^\top$ is an eigenvector corresponding to the negative eigenvalue, then in the case when the equations are linear and so $J$ is constant, the solution will converge to the steady state.

This is saddle point stability.
The Lotka–Volterra Predator–Prey Model

Foxes are predators; their prey includes rabbits.

Let $x$ denote the expected population of rabbits, and $y$ denote expected population of foxes.

Assume these populations are linked by the differential equations

\[ \frac{dx}{dt} = x(k - ay) \]
\[ \frac{dy}{dt} = y(bx - h) \]

where $a, b, h, k$ are all positive parameters.

Thus:

1. the rabbit population growth rate $\frac{d}{dt} \ln x = \frac{\dot{x}}{x}$ is a decreasing affine function of the fox population;
2. whereas the fox population growth rate $\frac{d}{dt} \ln y = \frac{\dot{y}}{y}$ is an increasing affine function of the rabbit population.
Lotka–Volterra: Phase Plane Analysis

Given the system \( \dot{x} = x(k - ay) \) and \( \dot{y} = y(bx - h) \), the two nullclines where \( \dot{x} = 0 \) and \( \dot{y} = 0 \) are given by \( y = k/a \) and \( x = h/b \) respectively.

So the steady state is at \((x, y) = (h/b, k/a)\).

The Jacobian matrix is

\[
J(x, y) = \begin{pmatrix}
\frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\
\frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y}
\end{pmatrix} = \begin{pmatrix}
k - ay & -ax \\
yb & bx - h
\end{pmatrix} = \begin{pmatrix}
0 & -ah/b \\
bk/a & 0
\end{pmatrix}
\]

at the steady state.

The characteristic equation is \( \lambda^2 + hk = 0 \), whose roots are \( \pm i \sqrt{hk} \).

As the following diagram suggests, there can be limit cycles with \( x(t) = \xi \cos \sqrt{hk} t \) and \( y(t) = \eta \sin \sqrt{hk} t \).
Lotka–Volterra: Phase Plane Diagram
Saddle Point Example

Consider a macro model where: (i) \( K \) denotes capital; (ii) \( Y \) denotes output; and (iii) \( C \) denotes consumption.

Suppose that net investment \( \dot{K} = Y - C \), that \( Y = aK - bK^2 \), and \( \dot{C} = w(a - 2bk)C \), where \( a, b, k, w \) are positive constants.

This gives the coupled system with

\[
\dot{K} = aK - bK^2 - C \quad \text{and} \quad \dot{C} = w(a - 2bK)C
\]

The two nullclines are \( C = aK - bK^2 \) and \( K = a/2b \).

These intersect at the stationary point \( (K^*, C^*) = (a/2b, a^2/4b) \).

The Jacobian matrix is

\[
J(K, C) = \begin{pmatrix}
\frac{\partial \dot{K}}{\partial K} & \frac{\partial \dot{K}}{\partial C} \\
\frac{\partial \dot{C}}{\partial K} & \frac{\partial \dot{C}}{\partial C}
\end{pmatrix} = \begin{pmatrix}
a - 2bK & -1 \\
-2wbC & w(a - 2bK)
\end{pmatrix}
\]

which reduces to \( \begin{pmatrix} 0 & -1 \\ \frac{1}{2}a^2w & 0 \end{pmatrix} \) at the steady state.
In Fig. 5, the direction of motion on a path at a point in each of the four sectors is indicated by arrows. In accordance with common practice, a separate arrow is drawn for each of the $x$ and $y$ directions. We usually make all the arrows have the same length. (If they were drawn with their correct lengths, they would correspond to the vectors $(\dot{x}, 0)$ and $(0, \dot{y})$. It follows that the actual direction of the path through the point would correspond to the sum of these two vectors.)

**EXAMPLE 3**

In a model of economic growth, capital $K = K(t)$ and consumption $C = C(t)$ satisfy the pair of differential equations

$$\dot{K} = aK - bK^2 - C \quad \dot{C} = w(a - 2bK)C$$

Here $a$, $b$, and $w$ are positive constants. Construct a phase diagram for this system, assuming that $K \geq 0$ and $C \geq 0$.

**Solution:**

The nullcline $\dot{K} = 0$ is the parabola $C = aK - bK^2$, and the nullcline $\dot{C} = 0$ consists of the two lines $C = 0$ and $K = a/2b$. In Fig. 6 the two nullclines are drawn.

There are three equilibrium points, $(0, 0)$, $(a/b, 0)$, and $(a/2b, a^2/4b)$.

In sector (I), $C > a - bK^2$ and $K > a/2b$, so $\dot{K} < 0$ and $\dot{C} < 0$. In sectors (II), (III), and (IV), we have $\dot{K} < 0$, $\dot{C} > 0$, then $\dot{K} > 0$, $\dot{C} > 0$, and $\dot{K} > 0$, $\dot{C} < 0$, respectively.

The appropriate arrows are drawn in Fig. 7, which indicates some paths consistent with the arrows.

These examples show how useful information about the solution paths can be obtained by partitioning the phase plane into regions to indicate whether each of the two variables is increasing or decreasing. In particular, the partition will often suggest whether or not a certain equilibrium point is stable, in the sense that paths starting near the equilibrium point tend to that point as $t \to \infty$. However, to determine whether an equilibrium point really is stable or not, a phase diagram analysis should be supplemented with tests based on analytical methods like those set out in the subsequent sections.
In Fig. 5, the direction of motion on a path at a point in each of the four sectors is indicated by arrows. In accordance with common practice, a separate arrow is drawn for each of the \( x \) and \( y \) directions. We usually make all the arrows have the same length. (If they were drawn with their correct lengths, they would correspond to the vectors \((\dot{x}, 0)\) and \((0, \dot{y})\). It follows that the actual direction of the path through the point would correspond to the sum of these two vectors.)

**Example 3**

In a model of economic growth, capital \( K = K(t) \) and consumption \( C = C(t) \) satisfy the pair of differential equations:

\[
\dot{K} = aK - bK^2 - C \\
\dot{C} = w(a - 2bK)C
\]

Here \( a, b, \) and \( w \) are positive constants. Construct a phase diagram for this system, assuming that \( K \geq 0 \) and \( C \geq 0 \).

**Solution:**

The nullcline \( \dot{K} = 0 \) is the parabola \( C = aK - bK^2 \), and the nullcline \( \dot{C} = 0 \) consists of the two lines \( C = 0 \) and \( K = a/2b \). In Fig. 6 the two nullclines are drawn.

There are three equilibrium points, \((0, 0)\), \((a/b, 0)\), and \((a/2b, a^2/4b)\).

In sector (I), \( C > aK - bK^2 \) and \( K > a/2b \), so \( \dot{K} < 0 \) and \( \dot{C} < 0 \). In sectors (II), (III), and (IV), we have \( \dot{K} < 0 \), \( \dot{C} > 0 \), then \( \dot{K} > 0 \), \( \dot{C} > 0 \), and \( \dot{K} > 0 \), \( \dot{C} < 0 \), respectively.

The appropriate arrows are drawn in Fig. 7, which indicates some paths consistent with the arrows.

These examples show how useful information about the solution paths can be obtained by partitioning the phase plane into regions to indicate whether each of the two variables is increasing or decreasing. In particular, the partition will often suggest whether or not a certain equilibrium point is stable, in the sense that paths starting near the equilibrium point tend to that point as \( t \to \infty \). However, to determine whether an equilibrium point really is stable or not, a phase diagram analysis should be supplemented with tests based on analytical methods like those set out in the subsequent sections.
Stability Analysis

The Jacobian matrix \( \begin{pmatrix} 0 & -1 \\ \frac{1}{2}a^2w & 0 \end{pmatrix} \) at the steady state has trace 0 and negative determinant \(-\frac{1}{2}a^2w\).

So the two eigenvalues are \( \pm \lambda \) where \( \lambda^2 = \frac{1}{2}a^2w \) and so \( \lambda = a\sqrt{w/2} \).

The general solution near the steady state takes the form \( x = Ae^{\lambda t} + Be^{-\lambda t} \) for arbitrary constants \( A, B \).

This converges to the steady state at \( (K^*, C^*) = (a/2b, a^2/4b) \) if and only if \( A = 0 \).