Lecture Notes 10: Dynamic Programming
Part A: Stochastic Difference Equations

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Lecture Outline

Basic Equation

A simple stochastic linear difference equation of the first order in one variable takes the form

$$x_t = a x_{t-1} + \epsilon_t \quad (t \in \mathbb{N})$$

Here a is a real parameter,
and each ϵ_t is a real random disturbance.

Assume that:

1. there is a given or pre-determined initial state x_0;
2. the random variables ϵ_t
   are independent and identically distributed (IID)
   with mean $\mathbb{E}\epsilon_t = 0$ and variance $\mathbb{E}\epsilon_t^2 = \sigma^2$.

A special case is when the disturbances are all normally distributed — i.e., $\epsilon_t \sim N(0, \sigma^2)$.

Explicit Solution and Conditional Mean

For each fixed outcome $\epsilon^{\mathbb{N}} = (\epsilon_t)_{t \in \mathbb{N}}$ of the random sequence, there is a unique solution which can be written as

$$x_t = a^t x_0 + \sum\nolimits_{s=1}^{t} a^{t-s} \epsilon_s$$

The main stable case occurs when $|a| < 1$.

Then each term of the sum $a^t x_0 + \sum_{s=1}^{t} a^{t-s} \epsilon_s$ converges to 0 as $t \to \infty$.

This is what econometricians or statisticians call a first-order autoregressive (or AR(1)) process.

In fact, given x_0 at time 0, our assumption that $\mathbb{E}\epsilon_s = 0$ for all $s = 1, 2, \ldots, t$ implies that the conditional mean of x_t is

$$m_t := \mathbb{E}[x_t | x_0] = \mathbb{E}\left[a^t x_0 + \sum\nolimits_{s=1}^{t} a^{t-s} \epsilon_s | x_0\right] = a^t x_0$$

Conditional Variance

The conditional variance, however, is given by

$$v_t := \mathbb{E}\left[(x_t - m_t)^2 | x_0\right] = \mathbb{E}[(x_t - a^t x_0)^2 | x_0] = \mathbb{E}\left[\sum_{s=1}^{t} a^{t-s}\epsilon_s\right]^2$$

In the case we are considering
with independently distributed disturbances ϵ_s,
the variance of a sum is the sum of the variances.

Hence

$$v_t = \sum_{s=1}^{t} \mathbb{E}\left[a^{t-s}\epsilon_s\right]^2 = \sum_{s=1}^{t} a^{2(t-s)}\mathbb{E}\epsilon_s^2 = \sigma^2 \sum_{s=1}^{t} a^{2(t-s)}$$

Using the rule for summing the geometric series $\sum_{s=1}^{t} a^{2(t-s)}$,
we finally obtain

$$v_t = \frac{1 - a^{2t}}{1 - a^2}\sigma^2$$

Sums of Normally Distributed Random Variables, I

Recall that if $X \sim N(\mu, \sigma^2)$, then the characteristic function defined by $\phi_X(t) = \mathbb{E}[e^{iXt}]$ takes the form

$$\phi_X(t) = \mathbb{E}[e^{iXt}] = \int_{-\infty}^{+\infty} e^{ixt} \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 dx$$

This reduces to $\phi_X(t) = \exp\left(it\mu - \frac{1}{2}\sigma^2 t^2 \right)$.

Hence, if $Z = X + Y$ where $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ are independent random, variables, then

$$\phi_Z(t) = \mathbb{E}[e^{iZt}] = \mathbb{E}[e^{i(X+Y)t}] = \mathbb{E}[e^{iXt} e^{iYt}] = \mathbb{E}[e^{iXt}]\mathbb{E}[e^{iYt}]$$

Sums of Normally Distributed Random Variables, II

So

$$\begin{aligned}\phi_Z(t) &= \exp\left(it\mu_X - \tfrac{1}{2}\sigma_X^2 t^2\right)\exp\left(it\mu_Y - \tfrac{1}{2}\sigma_Y^2 t^2\right) \\ &= \exp\left(it(\mu_X + \mu_Y) - \tfrac{1}{2}(\sigma_X^2 + \sigma_Y^2)t^2\right) \\ &= \exp\left(it\mu_Z - \tfrac{1}{2}\sigma_Z^2 t^2\right)\end{aligned}$$

where $\mu_Z = \mu_X + \mu_Y = \mathbb{E}(X + Y)$ is the mean of $X + Y$,

and $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$ is the variance of $X + Y$.

It follows that $t \mapsto \phi_Z(t)$
is the characteristic function of a random variable $Z \sim N(\mu_Z, \sigma_Z^2)$
where $\mu_Z = \mu_X + \mu_Y = \mathbb{E}(X + Y)$ and $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$.

That is, the sum $Z = X + Y$
of two independent normally distributed random variables X and Y
is also normally distributed, with:

1. mean equal to the sum of the means;
2. variance equal to the sum of the variances.

The Gaussian Case and the Asymptotic Distribution

In the particular case when each ϵ_t is normally distributed
as well as IID,
then x_t is also normally distributed with mean m_t and variance v_t.

As $t \to \infty$, the conditional mean $m_t = a^t x_0 \to 0$
and the conditional variance

$$v_t = \frac{1 - a^{2t}}{1 - a^2} \sigma^2 \to v := \frac{\sigma^2}{1 - a^2}$$

In the case when each ϵ_t is normally distributed,
this implies that the asymptotic distribution of x_t is also normal,
with mean 0 and variance $v = \sigma^2/(1 - a^2)$.

Stationarity

Now suppose that x_0 itself has this asymptotic normal distribution — suppose that $x_0 \sim N(0, \sigma^2/(1-a^2))$.

This is what the distribution of x_0 would be
if the process had started at $t = -\infty$ instead of at $t = 0$.

Then the unconditional mean of each x_t is $\mathbb{E}x_t = a^t \mathbb{E}x_0 = 0$.

On the other hand, because $x_{t+k} = a^k x_t + \sum_{s=1}^k a^{k-s} \epsilon_{t+s}$,
the unconditional covariance of x_t and x_{t+k} is

$$\mathbb{E}(x_{t+k} x_t) = \mathbb{E}[a^k x_t^2] = a^k v = \frac{a^k}{1-a^2}\sigma^2 \quad (k = 0, 1, 2 \ldots)$$

In fact, given any t, the joint distribution
of the r random variables $x_t, x_{t+1}, \ldots, x_{t+r-1}$
is multivariate normal with variance–covariance matrix
having elements $\mathbb{E}(x_{t+k} x_t) = a^k \sigma^2/(1-a^2)$, independent of t.

Because of this independence, the process is said to be stationary.

Lecture Outline

Intertemporal Utility

Consider a household which at time s is planning its intertemporal consumption stream $\mathbf{c}_s^T := (c_s, c_{s+1}, \ldots, c_T)$ over periods t in the set $\{s, s+1, \ldots, T\}$.

Its intertemporal utility function $\mathbb{R}^{T-s+1} \ni \mathbf{c}_s^T \mapsto U_s^T(\mathbf{c}_s^T) \in \mathbb{R}$ is assumed to take the additively separable form

$$U_s^T(\mathbf{c}_s^T) := \sum_{t=s}^{T} u_t(c_t)$$

where the one period felicity functions $c \mapsto u_t(c)$ are differentiably increasing and strictly concave (DISC) — i.e., $u_t'(c) > 0$, and $u_t''(c) < 0$ for all t and all $c > 0$.

As before, the household faces:

1. fixed initial wealth w_s;
2. a terminal wealth constraint $w_{T+1} \geq 0$.

Risky Wealth Accumulation

Also as before, we assume
a wealth accumulation equation $w_{t+1} = \tilde{r}_t(w_t - c_t)$,
where $\tilde{r}_t$ is the household's gross rate of return
on its wealth in period t.

It is assumed that:

1. the return $\tilde{r}_t$ in each period t is a random variable with positive values;
2. the return distributions for different times t are stochastically independent;
3. starting with predetermined wealth w_s at time s, the household seeks to maximize the expectation $\mathbb{E}_s[U_s^T(\mathbf{c}_s^T)]$ of its intertemporal utility.

Lecture Outline

Two Period Case

We work backwards from the last period, when $s = T$.

In this last period the household will obviously choose $c_T = w_T$, yielding a maximized utility equal to $V_T(w_T) = u_T(w_T)$.

Next, consider the penultimate period, when $s = T - 1$.
The consumer will want to choose c_{T-1} in order to maximize

$$\underbrace{u_{T-1}(c_{T-1})}_{\text{period } T-1} \quad + \quad \underbrace{\mathbb{E}_{T-1} V_T(w_T)}_{\text{result of an optimal policy in period } T}$$

subject to the wealth constraint

$$w_T = \underbrace{\tilde{r}_{T-1}}_{\text{random gross return}} \underbrace{(w_{T-1} - c_{T-1})}_{\text{saving}}$$

First-Order Condition

Substituting both the function $V_T(w_T) = u_T(w_T)$
and the wealth constraint into the objective reduces the problem to

$$\max_{c_{T-1}} \{ u_{T-1}(c_{T-1}) + \mathbb{E}_{T-1} [u_T(\tilde{r}_{T-1}(w_{T-1} - c_{T-1}))] \}$$

subject to $0 \leq c_{T-1} \leq w_{T-1}$ and $\tilde{c}_T := \tilde{r}_{T-1}(w_{T-1} - c_{T-1})$.

Assume we can differentiate under the integral sign,
and that there is an interior solution with $0 < c_{T-1} < w_{T-1}$.

Then the first-order condition (FOC) is

$$0 = u'_{T-1}(c_{T-1}) + \mathbb{E}_{T-1}[(-\tilde{r}_{T-1}) u'_T(\tilde{r}_{T-1}(w_{T-1} - c_{T-1}))]$$

The Stochastic Euler Equation

Rearranging the first-order condition while recognizing that $\tilde{c}_T := \tilde{r}_{T-1}(w_{T-1} - c_{T-1})$, one obtains

$$u'_{T-1}(c_{T-1}) = \mathbb{E}_{T-1}[\tilde{r}_{T-1} u'_T(\tilde{r}_{T-1}(w_{T-1} - c_{T-1}))]$$

Dividing by $u'_{T-1}(c_{T-1})$ gives the stochastic Euler equation

$$1 = \mathbb{E}_{T-1}\left[\tilde{r}_{T-1}\frac{u'_T(\tilde{c}_T)}{u'_{T-1}(c_{T-1})}\right] = \mathbb{E}_{T-1}\left[\tilde{r}_{T-1}\mathrm{MRS}^T_{T-1}(c_{T-1};\tilde{c}_T)\right]$$

involving the marginal rate of substitution function

$$\mathrm{MRS}^T_{T-1}(c_{T-1};\tilde{c}_T) := \frac{u'_T(\tilde{c}_T)}{u'_{T-1}(c_{T-1})}$$

The CES Case

For the marginal utility function $c \mapsto u'(c)$,
its elasticity of substitution is defined
for all $c > 0$ by $\eta(c) := d \ln u'(c) / d \ln c$.

Then $\eta(c)$ is both the degree of relative risk aversion,
and the degree of relative fluctuation aversion.

A constant elasticity of substitution (or CES) utility function
satisfies $d \ln u'(c) / d \ln c = -\epsilon < 0$ for all $c > 0$.

The marginal rate of substitution
satisfies $u'(c)/u'(\bar{c}) = (c/\bar{c})^{-\epsilon}$ for all $c, \bar{c} > 0$.

Normalized Utility

Normalize by putting $u'(1) = 1$, implying that $u'(c) \equiv c^{-\epsilon}$.

Then integrating gives

$$\begin{aligned} u(c;\epsilon) &= u(1) + \int_1^c x^{-\epsilon} dx \\ &= \begin{cases} u(1) + \dfrac{c^{1-\epsilon} - 1}{1-\epsilon} & \text{if } \epsilon \neq 1 \\ u(1) + \ln c & \text{if } \epsilon = 1 \end{cases} \end{aligned}$$

Introduce the final normalization

$$u(1) = \begin{cases} \dfrac{1}{1-\epsilon} & \text{if } \epsilon \neq 1 \\ 0 & \text{if } \epsilon = 1 \end{cases}$$

The utility function is reduced to

$$u(c;\epsilon) = \begin{cases} \dfrac{c^{1-\epsilon} - 1}{1-\epsilon} & \text{if } \epsilon \neq 1 \\ \ln c & \text{if } \epsilon = 1 \end{cases}$$

The Stochastic Euler Equation in the CES Case

Consider the CES case when $u_t'(c) \equiv \delta_t c^{-\epsilon}$,
where each δ_t is the discount factor for period t.

Definition

The one-period discount factor in period t
is defined as $\beta_t := \delta_{t+1}/\delta_t$.

Then the stochastic Euler equation takes the form

$$1 = \mathbb{E}_{T-1}\left[\tilde{r}_{T-1}\beta_{T-1}\left(\frac{\tilde{c}_T}{c_{T-1}}\right)^{-\epsilon}\right]$$

Because c_{T-1} is being chosen at time $T-1$, this implies that

$$(c_{T-1})^{-\epsilon} = \mathbb{E}_{T-1}\left[\tilde{r}_{T-1}\beta_{T-1}(\tilde{c}_T)^{-\epsilon}\right]$$

The Two Period Problem in the CES Case

In the two-period case, we know that

$$\tilde{c}_T = \tilde{w}_T = \tilde{r}_{T-1}(w_{T-1} - c_{T-1})$$

in the last period, so the Euler equation becomes

$$\begin{aligned}(c_{T-1})^{-\epsilon} &= \mathbb{E}_{T-1}\left[\tilde{r}_{T-1}\beta_{T-1}(\tilde{c}_T)^{-\epsilon}\right] \\ &= \beta_{T-1}(w_{T-1} - c_{T-1})^{-\epsilon}\mathbb{E}_{T-1}\left[(\tilde{r}_{T-1})^{1-\epsilon}\right]\end{aligned}$$

Take the $(-1/\epsilon)$ th power of each side and define

$$\rho_{T-1} := \left(\beta_{T-1}\mathbb{E}_{T-1}\left[(\tilde{r}_{T-1})^{1-\epsilon}\right]\right)^{-1/\epsilon}$$

to reduce the Euler equation to $c_{T-1} = \rho_{T-1}(w_{T-1} - c_{T-1})$ whose solution is evidently $c_{T-1} = \gamma_{T-1} w_{T-1}$ where

$$\gamma_{T-1} := \rho_{T-1}/(1 + \rho_{T-1}) \quad \text{and} \quad 1 - \gamma_{T-1} = 1/(1 + \rho_{T-1})$$

are respectively the optimal consumption and savings ratios.
It follows that $\rho_{T-1} = \gamma_{T-1}/(1 - \gamma_{T-1})$
is the consumption/savings ratio.

Optimal Discounted Expected Utility

The optimal policy in periods T and $T-1$
is $c_t = \gamma_t w_t$ where $\gamma_T = 1$ and γ_{T-1} has just been defined.

In this CES case, the discounted utility of consumption in period T
is $V_T(w_T) := \delta_T u(w_T; \epsilon)$.

The discounted expected utility at time $T-1$
of consumption in periods T and $T-1$ together is

$$V_{T-1}(w_{T-1}) = \delta_{T-1} u(\gamma_{T-1} w_{T-1}; \epsilon) + \delta_T \mathbb{E}_{T-1}[u(\tilde{w}_T; \epsilon)]$$

where $\tilde{w}_T = \tilde{r}_{T-1}(1 - \gamma_{T-1}) w_{T-1}$.

Discounted Expected Utility in the Logarithmic Case

In the logarithmic case when $\epsilon = 1$, one has

$$V_{T-1}(w_{T-1}) = \delta_{T-1} \ln(\gamma_{T-1} w_{T-1}) \\ + \delta_T \mathbb{E}_{T-1}[\ln(\tilde{r}_{T-1}(1-\gamma_{T-1})w_{T-1})]$$

It follows that

$$V_{T-1}(w_{T-1}) = \alpha_{T-1} + (\delta_{T-1} + \delta_T) u(w_{T-1}; \epsilon)$$

where

$$\alpha_{T-1} := \delta_{T-1} \ln \gamma_{T-1} + \delta_T \{\ln(1-\gamma_{T-1}) + \mathbb{E}_{T-1}[\ln \tilde{r}_{T-1}]\}$$

Discounted Expected Utility in the CES Case

In the CES case when $\epsilon \neq 1$, one has

$$\begin{aligned}(1-\epsilon)V_{T-1}(w_{T-1}) &= \delta_{T-1}(\gamma_{T-1}w_{T-1})^{1-\epsilon} \\ &\quad + \delta_T[(1-\gamma_{T-1})w_{T-1}]^{1-\epsilon}\,\mathbb{E}_{T-1}[(\tilde{r}_{T-1})^{1-\epsilon}]\end{aligned}$$

so $V_{T-1}(w_{T-1}) = v_{T-1}u(w_{T-1};\epsilon)$ where

$$v_{T-1} := \delta_{T-1}(\gamma_{T-1})^{1-\epsilon} + \delta_T(1-\gamma_{T-1})^{1-\epsilon}\,\mathbb{E}_{T-1}[(\tilde{r}_{T-1})^{1-\epsilon}]$$

In both cases,
one can write $V_{T-1}(w_{T-1}) = \alpha_{T-1} + v_{T-1}u(w_{T-1};\epsilon)$
for a suitable additive constant α_{T-1} (which is 0 in the CES case)
and a suitable multiplicative constant v_{T-1}.

Lecture Outline

Linear Equations in One Variable

Optimal Saving

The Two Period Problem

The T Period Problem

A General Problem

Infinite Time Horizon

Main Theorem

Policy Improvement

Unboundedness

The Time Line

In each period t, suppose:

- the consumer starts with known wealth w_t;
- then the consumer chooses consumption c_t,
  along with savings or residual wealth $w_t - c_t$;
- there is a cumulative distribution function $F_t(r)$ on $\mathbb{R}$
  that determines the gross return $\tilde{r}_t$
  as a positive-valued random variable.

After these three steps have been completed,
the problem starts again in period $t + 1$,
with the consumer's wealth known to be $w_{t+1} = \tilde{r}_t(w_t - c_t)$.

Expected Conditionally Expected Utility

Starting at any t, suppose the consumer's choices, together with the random returns, jointly determine a cdf F_t^T over the space of intertemporal consumption streams $\mathbf{c}_t^T$.

The associated expected utility is $\mathbb{E}_t \left[U_t^T(\mathbf{c}_t^T)\right]$, using the shorthand $\mathbb{E}_t$ to denote integration w.r.t. the cdf F_t^T.

Then, given that the consumer has chosen c_t at time t, let $\mathbb{E}_{t+1}[\cdot|c_t]$ denote the conditional expected utility.

This is found by integrating w.r.t. the conditional cdf $F_{t+1}^T(\mathbf{c}_{t+1}^T|c_t)$.

The law of iterated expectations allows us to write the unconditional expectation $\mathbb{E}_t \left[U_t^T(\mathbf{c}_t^T)\right]$ as the expectation $\mathbb{E}_t[\mathbb{E}_{t+1}[U_t^T(\mathbf{c}_t^T)|c_t]]$ of the conditional expectation.

The Expectation of Additively Separable Utility

Our hypothesis is that the intertemporal von Neumann–Morgenstern utility function takes the additively separable form

$$U_t^T(\mathbf{c}_t^T) = \sum_{\tau=t}^{T} u_\tau(c_\tau)$$

The conditional expectation given c_t must then be

$$\mathbb{E}_{t+1}[U_t^T(\mathbf{c}_t^T)|c_t] = u_t(c_t) + \mathbb{E}_{t+1}\left[\sum_{\tau=t+1}^{T} u_\tau(c_\tau)|c_t\right]$$

whose expectation is

$$\mathbb{E}_t\left[\sum_{\tau=t}^{T} u_\tau(c_\tau)\right] = u_t(c_t) + \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\sum_{\tau=t+1}^{T} u_\tau(c_\tau)\right]|c_t\right]$$

The Continuation Value

Let $V_{t+1}(w_{t+1})$ be the state valuation function
expressing the maximum of the continuation value

$$\mathbb{E}_{t+1}\left[U_{t+1}^T(\mathbf{c}_{t+1}^T)|w_{t+1}\right] = \mathbb{E}_{t+1}\left[\sum\nolimits_{\tau=t+1}^T u_\tau(c_\tau)|w_{t+1}\right]$$

as a function of the wealth level or state w_{t+1}.

Assume this maximum value is achieved
by following an optimal policy from period $t+1$ on.

Then total expected utility at time t will then reduce to

$$\begin{aligned}
\mathbb{E}_t\left[U_t^T(\tilde{\mathbf{c}}_\mathbf{t}^\mathbf{T})|c_t\right] &= u_t(c_t) + \mathbb{E}_t\left[\mathbb{E}_{t+1}\left[\sum\nolimits_{\tau=t+1}^T u_\tau(c_\tau)|w_{t+1}\right]|c_t\right] \\
&= u_t(c_t) + \mathbb{E}_t[V_{t+1}(\tilde{w}_{t+1})|c_t] \\
&= u_t(c_t) + \mathbb{E}_t[V_{t+1}(\tilde{r}_t(w_t - c_t))]
\end{aligned}$$

The Principle of Optimality

Maximizing $\mathbb{E}_s\left[U_s^T(\mathbf{c}_s^T)\right]$ w.r.t. c_s, taking as fixed the optimal consumption plans $c_t(w_t)$ at times $t = s+1, \ldots, T$, therefore requires choosing c_s to maximize

$$u_s(c_s) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))]$$

Let $c_s^*(w_s)$ denote a solution to this maximization problem.

Then the value of an optimal plan $(c_t^*(w_t))_{t=s}^T$ that starts with wealth w_s at time s is

$$V_s(w_s) := u_s(c_s^*(w_s)) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s^*(w_s)))]$$

Together, these two properties can be expressed as

$$\left.\begin{array}{rcl} V_s(w_s) & = & \\ c_s^*(w_s) & = & \arg \end{array}\right\} \max_{0 \le c_s \le w_s} \{u_s(c_s) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))]\}$$

which can be described as the the principle of optimality.

An Induction Hypothesis

Consider once again the case when $u_t(c) \equiv \delta_t u(c; \epsilon)$
for the CES (or logarithmic) utility function
that satisfies $u'(c; \epsilon) \equiv c^{-\epsilon}$ and, specifically

$$u(c; \epsilon) = \begin{cases} c^{1-\epsilon}/(1-\epsilon) & \text{if } \epsilon \neq 1; \\ \ln c & \text{if } \epsilon = 1. \end{cases}$$

Inspired by the solution we have already found
for the final period T and penultimate period $T - 1$,
we adopt the induction hypothesis that there are
constants α_t, γ_t, v_t $(t = T, T-1, \ldots, s+1, s)$ for which

$$c_t^*(w_t) = \gamma_t w_t \quad \text{and} \quad V_t(w_t) = \alpha_t + v_t u(w_t; \epsilon)$$

In particular, the consumption ratio γ_t and savings ratio $1 - \gamma_t$
are both independent of the wealth level w_t.

Applying Backward Induction

Under the induction hypotheses that

$$c_t^*(w_t) = \gamma_t w_t \quad \text{and} \quad V_t(w_t) = \alpha_t + v_t u(w_t; \epsilon)$$

the maximand

$$u_s(c_s) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))]$$

takes the form

$$\delta_s u(c_s; \epsilon) + \mathbb{E}_s[\alpha_{s+1} + v_{s+1} u(\tilde{r}_s(w_s - c_s); \epsilon)]$$

The first-order condition for this to be maximized w.r.t. c_s is

$$0 = \delta_s u'(c_s; \epsilon) - v_{s+1} \mathbb{E}_s[\tilde{r}_s u'(\tilde{r}_s(w_s - c_s); \epsilon)]$$

or, equivalently, that

$$\delta_s (c_s)^{-\epsilon} = v_{s+1} \mathbb{E}_s[\tilde{r}_s (\tilde{r}_s(w_s - c_s))^{-\epsilon})] = v_{s+1}(w_s - c_s)^{-\epsilon} \mathbb{E}_s[(\tilde{r}_s)^{1-\epsilon}]$$

Solving the Logarithmic Case

When $\epsilon = 1$ and so $u(c; \epsilon) = \ln c$,
the first-order condition reduces to $\delta_s (c_s)^{-1} = v_{s+1}(w_s - c_s)^{-1}$.

Its solution is indeed $c_s = \gamma_s w_s$ where $\delta_s (\gamma_s)^{-1} = v_{s+1}(1 - \gamma_s)^{-1}$,
implying that $\gamma_s = \delta_s / (\delta_s + v_{s+1})$.

The state valuation function then becomes

$$\begin{aligned} V_s(w_s) &= \delta_s u(\gamma_s w_s; \epsilon) + \alpha_{s+1} + v_{s+1} \mathbb{E}_s [u(\tilde{r}_s (1 - \gamma_s) w_s; \epsilon)] \\ &= \delta_s \ln(\gamma_s w_s) + \alpha_{s+1} + v_{s+1} \mathbb{E}_s [\ln(\tilde{r}_s (1 - \gamma_s) w_s)] \\ &= \delta_s \ln(\gamma_s w_s) + \alpha_{s+1} + v_{s+1} \{\ln(1 - \gamma_s) w_s + \ln R_s\} \end{aligned}$$

where we define the geometric mean certainty equivalent return R_s
so that $\ln R_s := \mathbb{E}_s [\ln(\tilde{r}_s)]$.

The State Valuation Function

The formula

$$V_s(w_s) = \delta_s \ln(\gamma_s w_s) + \alpha_{s+1} + v_{s+1}\{\ln(1-\gamma_s)w_s + \ln R_s\}$$

reduces to the desired form $V_s(w_s) = \alpha_s + v_s \ln w_s$
provided we take $v_s := \delta_s + v_{s+1}$, which implies that $\gamma_s = \delta_s / v_s$, and also

$$\begin{aligned} \alpha_s &:= \delta_s \ln \gamma_s + \alpha_{s+1} + v_{s+1}\left\{\ln(1-\gamma_s) + \ln R_s\right\} \\ &= \delta_s \ln(\delta_s / v_s) + \alpha_{s+1} + v_{s+1}\{\ln(v_{s+1}/v_s) + \ln R_s\} \\ &= \delta_s \ln \delta_s + \alpha_{s+1} - v_s \ln v_s + v_{s+1}\{\ln v_{s+1} + \ln R_s\} \end{aligned}$$

This confirms the induction hypothesis for the logarithmic case.

The relevant constants v_s are found by summing backwards, starting with $v_T = \delta_T$, implying that $v_s = \sum_{\tau=s}^{T} \delta_s$.

The Stationary Logarithmic Case

In the stationary logarithmic case:

- the felicity function in each period t is $\beta^t \ln c_t$,
  so the one period discount factor is the constant β;
- the certainty equivalent return R_t is also a constant R.

Then $v_s = \sum_{\tau=s}^{T} \delta_s = \sum_{\tau=s}^{T} \beta^\tau = (\beta^s - \beta^{T+1})/(1-\beta)$,
implying that $\gamma_s = \beta^s / v_s = \beta^s(1-\beta)/(\beta^s - \beta^{T+1})$.

It follows that

$$c_s = \gamma_s w_s = \frac{(1-\beta)w_s}{1-\beta^{T-s+1}} = \frac{(1-\beta)w_s}{1-\beta^{H+1}}$$

when there are $H := T - s$ periods left before the horizon T.

As $H \to \infty$, this solution converges to $c_s = (1-\beta)w_s$,
so the savings ratio equals the constant discount factor β.

Remarkably, this is also independent of the gross return to saving.

First-Order Condition in the CES Case

Recall that the first-order condition in the CES Case is

$$\delta_s (c_s)^{-\epsilon} = v_{s+1}(w_s - c_s)^{-\epsilon} \mathbb{E}_s[(\tilde{r}_s)^{1-\epsilon}] = v_{s+1}(w_s - c_s)^{-\epsilon} R_s^{1-\epsilon}$$

where we have defined the certainty equivalent return R_s as the solution to $R_s^{1-\epsilon} := \mathbb{E}_s[(\tilde{r}_s)^{1-\epsilon}]$.

The first-order condition indeed implies that $c_s^*(w_s) = \gamma_s w_s$, where $\delta_s(\gamma_s)^{-\epsilon} = v_{s+1}(1-\gamma_s)^{-\epsilon} R_s^{1-\epsilon}$.

This implies that

$$\frac{\gamma_s}{1-\gamma_s} = \left(v_{s+1} R_s^{1-\epsilon} / \delta_s \right)^{-1/\epsilon}$$

or

$$\gamma_s = \frac{\left(v_{s+1} R_s^{1-\epsilon} / \delta_s \right)^{-1/\epsilon}}{1 + \left(v_{s+1} R_s^{1-\epsilon} / \delta_s \right)^{-1/\epsilon}} = \frac{\left(v_{s+1} R_s^{1-\epsilon} \right)^{-1/\epsilon}}{(\delta_s)^{-1/\epsilon} + \left(v_{s+1} R_s^{1-\epsilon} \right)^{-1/\epsilon}}$$

Completing the Solution in the CES Case

Under the induction hypothesis that $V_{s+1}(w) = v_{s+1} w^{1-\epsilon}/(1-\epsilon)$, one also has

$$(1-\epsilon)V_s(w_s) = \delta_s(\gamma_s w_s)^{1-\epsilon} + v_{s+1}\mathbb{E}_s[(\tilde{r}_s(1-\gamma_s)w_s)^{1-\epsilon}]$$

This reduces to the desired form $(1-\epsilon)V_s(w_s) = v_s(w_s)^{1-\epsilon}$, where

$$\begin{aligned}
v_s &:= \delta_s(\gamma_s)^{1-\epsilon} + v_{s+1}\mathbb{E}_s[(\tilde{r}_s)^{1-\epsilon}](1-\gamma_s)^{1-\epsilon} \\
&= \frac{\delta_s(v_{s+1}R_s^{1-\epsilon})^{1-1/\epsilon} + v_{s+1}R_s^{1-\epsilon}(\delta_s)^{1-1/\epsilon}}{[(\delta_s)^{-1/\epsilon} + \left(v_{s+1}R_s^{1-\epsilon}\right)^{-1/\epsilon}]^{1-\epsilon}} \\
&= \delta_s v_{s+1} R_s^{1-\epsilon} \frac{(v_{s+1}R_s^{1-\epsilon})^{-1/\epsilon} + (\delta_s)^{-1/\epsilon}}{[(\delta_s)^{-1/\epsilon} + \left(v_{s+1}R_s^{1-\epsilon}\right)^{-1/\epsilon}]^{1-\epsilon}} \\
&= \delta_s v_{s+1} R_s^{1-\epsilon}[(\delta_s)^{-1/\epsilon} + \left(v_{s+1}R_s^{1-\epsilon}\right)^{-1/\epsilon}]^{\epsilon}
\end{aligned}$$

This confirms the induction hypothesis for the CES case.

Again, the relevant constants are found by working backwards.

Histories and Strategies

For each time $t = s, s + 1, \ldots, T$
between the start s and the horizon T,
let h^t denote a known history $(w_\tau, c_\tau, \tilde{r}_\tau)_{\tau=s}^t$
of the triples $(w_\tau, c_\tau, \tilde{r}_\tau)$
at successive times $\tau = s, s + 1, \ldots, t$ up to time t.

A general policy the consumer can choose
involves a measurable function $h^t \mapsto \psi_t(h^t)$
mapping each known history up to time t,
which determines the consumer's information set,
into a consumption level at that time.

The collection of successive functions $\psi_s^T = \langle \psi_t \rangle_{t=s}^T$
is what a game theorist would call the consumer's strategy
in the extensive form game "against nature".

Markov Strategies

We found an optimal solution
for the two-period problem when $t = T - 1$.

It took the form of a Markov strategy $\psi_t(h^t) := c_t^*(w_t)$,
which depends only on w_t as the particular state variable.

The following analysis will demonstrate in particular
that at each time $t = s, s+1, \ldots, T$,
under the induction hypothesis that the consumer will follow
a Markov strategy in periods $\tau = t+1, t+2, \ldots, T$,
there exists a Markov strategy that is optimal in period t.

It will follow by backward induction
that there exists an optimal strategy $h^t \mapsto \psi_t(h^t)$
for every period $t = s, s+1, \ldots, T$
that takes the Markov form $h^t \mapsto w_t \mapsto c_t^*(w_t)$.

This treats history as irrelevant, except insofar as it determines
current wealth w_t at the time when c_t has to be chosen.

A Stochastic Difference Equation

Accordingly, suppose that the consumer pursues a Markov strategy taking the form $w_t \mapsto c_t^*(w_t)$.

Then the Markov state variable w_t will evolve over time according to the stochastic difference equation

$$w_{t+1} = \phi_t(w_t, \tilde{r}_t) := \tilde{r}_t(w_t - c_t^*(w_t)).$$

Starting at any time t, conditional on initial wealth w_t, this equation will have a random solution $\tilde{\mathbf{w}}_{t+1}^T = (\tilde{w}_\tau)_{\tau=t+1}^T$ described by a unique joint conditional cdf $F_{t+1}^T(\mathbf{w}_{t+1}^T | w_t)$ on $\mathbb{R}^{T-s}$.

Combined with the Markov strategy $w_t \mapsto c_t^*(w_t)$, this generates a random consumption stream $\tilde{\mathbf{c}}_{t+1}^T = (\tilde{c}_\tau)_{\tau=t+1}^T$ described by a unique joint conditional cdf $G_{t+1}^T(\mathbf{c}_{t+1}^T | w_t)$ on $\mathbb{R}^{T-s}$.

General Finite Horizon Problem

Consider the objective of choosing y_s in order to maximize

$$\mathbb{E}_s \left[\sum_{t=s}^{T-1} u_s(x_s, y_s) + \phi_T(x_T) \right]$$

subject to the law of motion $x_{t+1} = \xi_t(x_t, y_t, \epsilon_t)$,
where the random shocks ϵ_t
at different times $t = s, s+1, s+2, \ldots, T-1$
are conditionally independent given x_t, y_t.

Here $x_T \mapsto \phi_T(x_T)$ is the terminal state valuation function.

The stochastic law of motion can also be expressed
through successive conditional probabilities $\mathbb{P}_{t+1}(x_{t+1} | x_t, y_t)$.

The choices of y_t at successive times determine
a controlled Markov process governing the stochastic transition
from each state x_t to its immediate successor x_{t+1}.

Backward Recurrence Relation

The optimal solution can be derived
by solving the backward recurrence relation

$$\left.\begin{array}{rl} V_s(x_s) & = \\ y_s^*(x_s) & = \arg \end{array}\right\} \max_{y_s \in F_s(x_s)} \{u_s(x_s, y_s) + \mathbb{E}_s [V_{s+1}(x_{s+1}) | x_s, y_s]\}$$

where

1. x_s denotes the "inherited state" at time s;
2. $V_s(x_s)$ is the current value in state x_s of the state value function $X \ni x \mapsto V_s(x) \in \mathbb{R}$;
3. $X \ni x \mapsto\!\!\!\rightarrow F_s(x) \subset Y$ is the feasible set correspondence;
4. $(x, y) \mapsto u_s(x, y)$ denotes the immediate return function in period s;
5. $X \ni x \mapsto y_s^*(x) \in F_s(x_s)$ is the optimal "strategy" or policy function;
6. The relevant terminal condition is that $V_T(x_T)$ is given by the exogenously specified function $\phi_T(x_T)$.

Lecture Outline

An Infinite Horizon Savings Problem

Game theorists speak of the "one-shot" deviation principle.

This states that if any deviation
from a particular policy or strategy improves a player's payoff,
then there exists a one-shot deviation that improves the payoff.

We consider the infinite horizon extension
of the consumption/investment problem already considered.

Given the initial time s and initial wealth w_s,
This takes the form of choosing a consumption policy $c_t(w_t)$
at times $t = s, s+1, s+2, \ldots$ in order to maximize
the discounted sum of total utility, given by

$$\sum_{t=s}^{\infty} \beta^{t-s} u(c_t)$$

subject to the accumulation equation $w_{t+1} = \tilde{r}_t(w_t - c_t)$
as well as the inequality constraint $w_t \geq 0$ for $t = s+1, s+2, \ldots$.

Some Assumptions

The parameter $\beta \in (0, 1)$ is the constant discount factor.

Note that utility function $\mathbb{R} \ni c \mapsto u(c)$ is independent of t;
its first two derivatives are assumed to satisfy
the inequalities $u'(c) > 0$ and $u''(c) < 0$ for all $c \in \mathbb{R}_+$.

The investment returns $\tilde{r}_t$ in successive periods
are assumed to be i.i.d. random variables.

It is assumed that w_t in each period t is known at time t,
but not before.

Terminal Constraint

There has to be an additional constraint that imposes a lower bound on wealth at some time t.

Otherwise there would be no optimal policy — the consumer can always gain by increasing debt (negative wealth), no matter how large existing debt may be.

In the finite horizon, there was a constraint $w_T \geq 0$ on terminal wealth.

But here T is effectively infinite.

One might try an alternative like

$$\liminf_{t \to \infty} \beta^t w_t \geq 0$$

But this places no limit on wealth at any finite time.

We use the alternative constraint requiring that $w_t \geq 0$ for all time.

The Stationary Problem

Our modified problem can be written
in the following form that is independent of s:

$$\max_{c_0, c_1, \ldots, c_t, \ldots} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the constraints $0 \leq c_t \leq w_t$ and $w_{t+1} = \tilde{r}_t(w_t - c_t)$ for all $t = 0, 1, 2, \ldots$, with $w_0 = w$, where w is given.

Because the starting time s is irrelevant,
this is a stationary problem.

Define the state valuation function $w \mapsto V(w)$
as the maximum value of the objective,
as a function of initial wealth w.

Bellman's Equation

For the finite horizon problem, the principle of optimality was

$$\left.\begin{array}{rcl} V_s(w_s) & = & \\ c_s^*(w_s) & = & \arg \end{array}\right\} \max_{0 \le c_s \le w_s} \{u_s(c_s) + \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))]\}$$

For the stationary infinite horizon problem, however,
the time starting time s is irrelevant.

So the principle of optimality can be expressed as

$$\left.\begin{array}{rcl} V(w) & = & \\ c^*(w) & = & \arg \end{array}\right\} \max_{0 \le c \le w} \{u(c) + \beta\mathbb{E}[V(\tilde{r}(w - c))]\}$$

The state valuation function $w \mapsto V(w)$ appears
on both left and right hand sides of this equation.

Solving it therefore involves finding a fixed point, or function,
in an appropriate function space.

Isoelastic Case

We consider yet again the isoelastic case
with a CES (or logarithmic) utility function
that satisfies $u'(c;\epsilon) \equiv c^{-\epsilon}$ and, specifically

$$u(c;\epsilon) = \begin{cases} c^{1-\epsilon}/(1-\epsilon) & \text{if } \epsilon \neq 1; \\ \ln c & \text{if } \epsilon = 1. \end{cases}$$

Recall the corresponding finite horizon case,
where we found that the solution to the corresponding equations

$$\left.\begin{array}{rcl} V_s(w_s) & = & \\ c_s^*(w_s) & = & \arg \end{array}\right\} \max_{0 \leq c_s \leq w_s} \{u_s(c_s) + \beta \mathbb{E}_s[V_{s+1}(\tilde{r}_s(w_s - c_s))]\}$$

takes the form $V_s(w) = \alpha_s + v_s u(w;\epsilon)$
for suitable real constants α_s and $v_s > 0$, where $\alpha_s = 0$ if $\epsilon \neq 1$.

First-Order Condition

Accordingly, we look for a solution to the stationary problem

$$\left.\begin{array}{rcl} V(w) & = & \\ c^*(w) & = & \arg \end{array}\right\} \max_{0 \leq c \leq w} \{u(c;\epsilon) + \beta \mathbb{E}[V(\tilde{r}(w-c))]\}$$

taking the isoelastic form $V(w) = \alpha + vu(w;\epsilon)$
for suitable real constants α and $v > 0$, where $\alpha = 0$ if $\epsilon \neq 1$.

The first-order condition for solving
this concave maximization problem is

$$c^{-\epsilon} = \beta \mathbb{E}[\tilde{r}(\tilde{r}(w-c))^{-\epsilon}] = \zeta^\epsilon (w-c)^{-\epsilon}$$

where $\zeta^\epsilon := \beta R^{1-\epsilon}$ with R as the certainty equivalent return
defined by $R^{1-\epsilon} := \mathbb{E}[\tilde{r}^{1-\epsilon}]$.
Hence $c = \gamma w$ where $\gamma^{-\epsilon} = \zeta^\epsilon (1-\gamma)^{-\epsilon}$,
implying that $\gamma = 1/(1+\zeta)$.

Solution in the Logarithmic Case

When $\epsilon = 1$ and so $u(c;\epsilon) = \ln c$, one has

$$\begin{aligned} V(w) &= u(\gamma w;\epsilon) + \beta\{\alpha + v\mathbb{E}[u(\tilde{r}(1-\gamma)w;\epsilon)]\} \\ &= \ln(\gamma w) + \beta\{\alpha + v\mathbb{E}[\ln(\tilde{r}(1-\gamma)w)]\} \\ &= \ln\gamma + (1+\beta v)\ln w + \beta\{\alpha + v\ln(1-\gamma) + \mathbb{E}[\ln\tilde{r}]\} \end{aligned}$$

This is consistent with $V(w) = \alpha + v\ln w$ in case:

1. $v = 1 + \beta v$, implying that $v = (1-\beta)^{-1}$;
2. and also $\alpha = \ln\gamma + \beta\{\alpha + v\ln(1-\gamma) + \mathbb{E}[\ln\tilde{r}]\}$, which implies that

$$\alpha = (1-\beta)^{-1}\left[\ln\gamma + \beta\left\{(1-\beta)^{-1}\ln(1-\gamma) + \mathbb{E}[\ln\tilde{r}]\right\}\right]$$

This confirms the solution for the logarithmic case.

Solution in the CES Case

When $\epsilon \neq 1$ and so $u(c;\epsilon) = c^{1-\epsilon}/(1-\epsilon)$, the equation

$$V(w) = u(\gamma w;\epsilon) + \beta v \mathbb{E}[u(\tilde{r}(1-\gamma)w;\epsilon)]$$

implies that

$$(1-\epsilon)V(w) = (\gamma w)^{1-\epsilon} + \beta v \mathbb{E}[(\tilde{r}(1-\gamma)w)^{1-\epsilon}] = v w^{1-\epsilon}$$

where $v = \gamma^{1-\epsilon} + \beta v (1-\gamma)^{1-\epsilon} R^{1-\epsilon}$ and so

$$v = \frac{\gamma^{1-\epsilon}}{1 - \beta(1-\gamma)^{1-\epsilon} R^{1-\epsilon}} = \frac{\gamma^{1-\epsilon}}{1 - (1-\gamma)^{1-\epsilon}\zeta^{\epsilon}}$$

But optimality requires $\gamma = 1/(1+\zeta)$, implying finally that

$$v = \frac{(1+\zeta)^{\epsilon-1}}{1 - \zeta(1+\zeta)^{\epsilon-1}} = \frac{1}{(1+\zeta)^{1-\epsilon} - \zeta}$$

This confirms the solution for the CES case.

Lecture Outline

Uniformly Bounded Returns

Suppose that the stochastic transition from each state x
to the immediately succeeding state $\tilde{x}$
is specified by a conditional probability measure $B \mapsto \mathbb{P}(\tilde{x} \in B | x, u)$
on a σ-algebra of the state space.

Consider the stationary problem of choosing a policy $x \mapsto u^*(x)$
in order to maximize the infinite discounted sum of utility

$$\mathbb{E} \sum_{t=1}^{\infty} \beta^{t-1} f(x_t, u_t)$$

where $0 < \beta < 1$, with x_1 given
and subject to $u_t \in U(x_t)$ for $t = 1, 2, \ldots$.

The return function $(x, u) \mapsto f(x, u) \in \mathbb{R}$ is uniformly bounded
provided there exist a uniform lower bound M_*
and a uniform upper bound M^* such that

$$M_* \leq f(x, u) \leq M^* \quad \text{for all } (x, u)$$

The Function Space

The boundedness assumption $M_* \le f(x,u) \le M^*$ for all (x,u) ensures that, because $0 < \beta < 1$ and so $\sum_{t=1}^\infty \beta^{t-1} = \dfrac{1}{1-\beta}$, the infinite discounted sum of utility

$$W := \mathbb{E} \sum_{t=1}^\infty \beta^{t-1} f(x_t, u_t)$$

satisfies $(1-\beta)\, W \in [M_*, M^*]$.

This makes it natural to consider the linear space $\mathcal{V}$ of all bounded functions $X \ni x \mapsto V(x) \in \mathbb{R}$ equipped with its sup norm defined by $\|V\| := \sup_{x \in X} |V(x)|$.

We will pay special attention to the subset

$$\mathcal{V}_M := \{ V \in \mathcal{V} \mid x \in X \Longrightarrow (1-\beta) V(x) \in [M_*, M^*] \}$$

of state valuation functions whose values $V(x)$ lie within the range of the possible values of W.

Existence and Uniqueness

Theorem

Consider the Bellman equation system

$$\left.\begin{array}{rcl} V(x) & = & \\ u^*(x) & \in & \arg \end{array}\right\} \max_{u \in F(x)} \{f(x,u) + \beta \mathbb{E}[V(\tilde{x})|x,u]\}$$

Under the assumption of uniformly bounded returns satisfying $M_* \leq f(x,u) \leq M^*$ for all (x,u):

1. *among the set $\mathcal{V}_M$ of state valuation functions that satisfy the inequalities $M_* \leq (1-\beta)V(x) \leq M^*$ for all x, there is a unique state valuation function $x \mapsto V(x)$ that satisfies the Bellman equation system.*
2. *any associated policy solution $x \mapsto u^*(x)$ determines an optimal policy that is stationary — i.e., independent of time.*

Two Mappings

Given any measurable policy function $X \ni x \mapsto u(x)$ denoted by $\mathbf{u}$, define the mapping $T^{\mathbf{u}} : \mathcal{V}_M \to \mathcal{V}$ by

$$[T^{\mathbf{u}} V](x) := f(x, u(x)) + \beta \mathbb{E}\left[V(\tilde{x}) | x, u(x)\right]$$

When the state is x, this gives the value $[T^{\mathbf{u}} V](x)$
of choosing the policy $u(x)$ for one period,
and then experiencing a future discounted return $V(\tilde{x})$
after reaching each possible subsequent state $\tilde{x} \in X$.

Define also the mapping $T^* : \mathcal{V}_M \to \mathcal{V}$ by

$$[T^* V](x) := \max_{u \in F(x)} \{f(x, u) + \beta \mathbb{E}\left[V(\tilde{x}) | x, u\right]\}$$

These definitions allow the Bellman equation system
to be rewritten as

$$\begin{aligned} V(x) &= [T^* V](x) \\ u^*(x) &\in \arg\max_{u \in F(x)} [T^{\mathbf{u}} V](x) \end{aligned}$$

Two Mappings of $\mathcal{V}_M$ into Itself

For all $V \in \mathcal{V}_M$, policies $\mathbf{u}$, and $x \in X$, we have defined

$$\begin{array}{rrcl} & [T^{\mathbf{u}} V](x) & := & f(x, u(x)) + \beta \, \mathbb{E}\,[V(\tilde{x}) | x, u(x)] \\ \text{and} & [T^* V](x) & := & \max_{u \in F(x)} \{ f(x, u) + \beta \, \mathbb{E}\,[V(\tilde{x}) | x, u] \} \end{array}$$

Recall the uniform boundedness condition $M_* \leq f(x, u) \leq M^*$, together with the assumption that V belongs to the domain $\mathcal{V}_M$ of functions satisfying $M_* \leq (1 - \beta) V(\tilde{x}) \leq M^*$ for all $\tilde{x}$.

So these two definitions jointly imply that

$$\begin{array}{rrcl} & [T^{\mathbf{u}} V](x) & \geq & M_* + \beta \, (1 - \beta)^{-1} M_* \; = \; (1 - \beta)^{-1} M_* \\ \text{and} & [T^{\mathbf{u}} V](x) & \leq & M^* + \beta \, (1 - \beta)^{-1} M^* \; = \; (1 - \beta)^{-1} M^* \end{array}$$

Similarly, given any $V \in \mathcal{V}_M$,
one has $M_* \leq (1 - \beta) \, [T^* V](x) \leq M^*$ for all $x \in X$.

Therefore both $V \mapsto T^{\mathbf{u}} V$ and $V \mapsto T^* V$ map $\mathcal{V}_M$ into itself.

A First Contraction Mapping

The definition $[T^{\mathbf{u}} V](x) := f(x, u(x)) + \beta \, \mathbb{E} \, [V(\tilde{x}) | x, u(x)]$ implies that for any two functions $V_1, V_2 \in \mathcal{V}_M$, one has

$$[T^{\mathbf{u}} V_1](x) - [T^{\mathbf{u}} V_2](x) = \beta \, \mathbb{E} \, [V_1(\tilde{x}) - V_2(\tilde{x}) | x, u(x)]$$

The definition of the sup norm therefore implies that

$$\begin{aligned} \| T^{\mathbf{u}} V_1 - T^{\mathbf{u}} V_2 \| &= \sup_{x \in X} |[T^{\mathbf{u}} V_1](x) - [T^{\mathbf{u}} V_2](x)| \\ &= \sup_{x \in X} |\beta \, \mathbb{E} \, [V_1(\tilde{x}) - V_2(\tilde{x}) | x, u(x)] \, | \\ &= \beta \, \sup_{x \in X} |\mathbb{E} \, [V_1(\tilde{x}) - V_2(\tilde{x}) | x, u(x)] \, | \\ &\leq \beta \, \sup_{\tilde{x} \in X} |V_1(\tilde{x}) - V_2(\tilde{x})| \\ &= \beta \, \| V_1 - V_2 \| \end{aligned}$$

Hence $V \mapsto T^{\mathbf{u}} V$ is a contraction mapping with factor $\beta < 1$ that maps the bounded subset $\mathcal{V}_M$ of the normed linear space $\mathcal{V}$ into itself.

Applying the Contraction Mapping Theorem, I

For each fixed policy $\mathbf{u}$, the contraction mapping $V \mapsto T^{\mathbf{u}} V$ mapping the space $\mathcal{V}_M$ into itself has a unique fixed point in the form of a function $V^{\mathbf{u}} \in \mathcal{V}_M$.

Furthermore, given any initial function $V \in \mathcal{V}_M$, consider the infinite sequence of mappings $[T^{\mathbf{u}}]^k V$ $(k \in \mathbb{N})$ that result from applying the operator $T^{\mathbf{u}}$ iteratively k times.

The contraction mapping property of $T^{\mathbf{u}}$ implies that $\|[T^{\mathbf{u}}]^k V - V^{\mathbf{u}}\| \to 0$ as $k \to \infty$.

Characterizing the Fixed Point, I

Starting from $V_0 = 0$ and given any initial state $x \in X$, note that

$$\begin{aligned} [T^{\mathbf{u}}]^k V_0(x) &= [T^{\mathbf{u}}]\left([T^{\mathbf{u}}]^{k-1} V_0\right)(x) \\ &= f(x, u(x)) + \beta\, \mathbb{E}\left[\left([T^{\mathbf{u}}]^{k-1} V_0\right)(\tilde{x}) | x, u(x)\right] \end{aligned}$$

It follows by induction on k that $[T^{\mathbf{u}}]^k V_0(\bar{x})$ equals
the expected discounted total payoff $\mathbb{E} \sum_{t=1}^{k} \beta^{t-1} f(x_t, u_t)$
of starting from $x_0 = \bar{x}$
and then following the policy $x \mapsto u(x)$ for k subsequent periods.

Taking the limit as $k \to \infty$, it follows that for any state $\bar{x} \in X$,
the value $V^{\mathbf{u}}(\bar{x})$ of the fixed point in $\mathcal{V}_M$
is the expected discounted total payoff

$$\mathbb{E} \sum_{t=1}^{\infty} \beta^{t-1} f(x_t, u_t)$$

of starting from $x_0 = \bar{x}$
and then following the policy $x \mapsto u(x)$ for ever thereafter.

A Second Contraction Mapping

Recall the definition

$$[T^* V](x) := \max_{u \in F(x)} \{f(x, u) + \beta \mathbb{E}[V(\tilde{x})|x, u]\}$$

Given any state $x \in X$ and any two functions $V_1, V_2 \in \mathcal{V}_M$, define $u_1, u_2 \in F(x)$ so that for $k = 1, 2$ one has

$$[T^* V_k](x) = f(x, u_k) + \beta \mathbb{E}[V_k(\tilde{x})|x, u_k]\}$$

Note that $[T^* V_2](x) \geq f(x, u_1) + \beta \mathbb{E}[V_2(\tilde{x})|x, u_1]\}$ implying that

$$\begin{aligned} [T^* V_1](x) - [T^* V_2](x) &\leq \beta \mathbb{E}[V_1(\tilde{x}) - V_2(\tilde{x})|x, u_1]\} \\ &\leq \beta \|V_1 - V_2\| \end{aligned}$$

Similarly, interchanging 1 and 2 in the above argument gives $[T^* V_2](x) - [T^* V_1](x) \leq \beta \|V_1 - V_2\|$.

Hence $\|T^* V_1 - T^* V_2\| \leq \beta \|V_1 - V_2\|$, so T^* is also a contraction.

Applying the Contraction Mapping Theorem, II

Similarly the contraction mapping $V \mapsto T^* V$
has a unique fixed point in the form of a function $V^* \in \mathcal{V}_M$
such that $V^*(\bar{x})$ is the maximized expected discounted total payoff
of starting in state $x_0 = \bar{x}$
and following an optimal policy for ever thereafter.

Moreover, $V^* = T^* V^* = T^{\mathbf{u}^*} V$.

This implies that V^* is also the value
of following the policy $x \mapsto u^*(x)$ throughout,
which must therefore be an optimal policy.

Characterizing the Fixed Point, II

Starting from $V_0 = 0$ and given any initial state $x \in X$, note that

$$\begin{aligned} [T^*]^k V_0(x) &= [T^*]\left([T^*]^{k-1} V_0\right)(x) \\ &= \max_{u \in F(x)}\{f(x,u) + \beta\,\mathbb{E}\left[\left([T^*]^{k-1} V_0\right)(\tilde{x})|x,u\right]\} \end{aligned}$$

It follows by induction on k
that $[T^*]^k V_0(\bar{x})$ equals the maximum possible
expected discounted total payoff $\mathbb{E}\sum_{t=1}^{k} \beta^{t-1} f(x_t, u_t)$
of starting from $x_1 = \bar{x}$
and then following the "backward" sequence
of optimal policies $(u_k^*, u_{k-1}^*, u_{k-2}^*, \ldots, u_2^*, u_1^*)$,
where for each k the policy $x \mapsto u_k^*(x)$
is optimal when k periods remain.

Method of Successive Approximation

The method of successive approximation starts with an arbitrary function $V_0 \in \mathcal{V}_M$.

For $k = 1, 2, \ldots,$, it then repeatedly solves the pair of equations $V_k = T^* V_{k-1} = T^{u_k^*} V_{k-1}$ to construct sequences of:

1. state valuation functions $X \ni x \mapsto V_k(x) \in \mathbb{R}$;
2. policies $X \ni x \mapsto u_k^*(x) \in F(x)$ that are optimal given that one applies the preceding state valuation function $X \ni \tilde{x} \mapsto V_{k-1}(\tilde{x}) \in \mathbb{R}$ to each immediately succeeding state $\tilde{x}$.

Because the operator $V \mapsto T^* V$ on $\mathcal{V}_M$ is a contraction mapping, the method produces a convergent sequence $(V_k)_{k=1}^\infty$ of state valuation functions whose limit satisfies $V^* = T^* V^* = T^{u^*} V^*$ for a suitable policy $X \ni x \mapsto u^*(x) \in F(x)$.

Lecture Outline

Monotonicity

For all functions $V \in \mathcal{V}_M$, policies $\mathbf{u}$, and states $x \in X$, we have defined

$$\begin{array}{rrcl} & [T^{\mathbf{u}} V](x) & := & f(x, u(x)) + \beta \, \mathbb{E}\,[V(\tilde{x})|x, u(x)] \\ \text{and} & [T^* V](x) & := & \max_{u \in F(x)} \{f(x, u) + \beta \, \mathbb{E}\,[V(\tilde{x})|x, u]\} \end{array}$$

Notation

Given any pair $V_1, V_2 \in \mathcal{V}_M$, we write $V_1 \geqq V_2$ to indicate that the inequality $V_1(x) \geq V_2(x)$ holds for all $x \in X$.

Definition

An operator $\mathcal{V}_M \ni V \mapsto TV \in \mathcal{V}_M$ is monotone just in case whenever $V_1, V_2 \in \mathcal{V}_M$ satisfy $V_1 \geqq V_2$, one has $TV_1 \geqq TV_2$.

Theorem

The following operators on $\mathcal{V}_M$ are monotone:

1. $V \mapsto T^{\mathbf{u}} V$ *for all policies* $\mathbf{u}$*;*
2. $V \mapsto T^* V$ *for the optimal policy.*

Proof that $T^{\mathbf{u}}$ is Monotone

Given any state $x \in X$ and any two functions $V_1, V_2 \in \mathcal{V}_M$,
the definition of $T^{\mathbf{u}}$ implies that

$$\begin{array}{rrcl} & [T^{\mathbf{u}} V_1](x) & := & f(x, u(x)) + \beta \, \mathbb{E}\,[V_1(\tilde{x}) | x, u(x)] \\ \text{and} & [T^{\mathbf{u}} V_2](x) & := & f(x, u(x)) + \beta \, \mathbb{E}\,[V_2(\tilde{x}) | x, u(x)] \end{array}$$

Subtracting the second equation from the first implies that

$$[T^{\mathbf{u}} V_1](x) - [T^{\mathbf{u}} V_2](x) = \beta \, \mathbb{E}\,[V_1(\tilde{x}) - V_2(\tilde{x}) | x, u(x)]$$

If $V_1 \geqq V_2$ and so the inequality $V_1(\tilde{x}) \geq V_2(\tilde{x})$ holds for all $\tilde{x} \in X$,
it follows that $[T^{\mathbf{u}} V_1](x) \geq [T^{\mathbf{u}} V_2](x)$.

Since this holds for all $x \in X$,
we have proved that $T^{\mathbf{u}} V_1 \geqq T^{\mathbf{u}} V_2$. □

Proof that T^* is Monotone

Given any state $x \in X$ and any two functions $V_1, V_2 \in \mathcal{V}_M$, define $u_1, u_2 \in F(x)$ so that for $k = 1, 2$ one has

$$
\begin{aligned}
[T^* V_k](x) &= \max_{u \in F(x)} \{ f(x,u) + \beta\, \mathbb{E}[V_k(\tilde{x}) | x, u] \} \\
= \quad [T^{\mathbf{u_k}} V_k](x) &= f(x, u_k) + \beta\, \mathbb{E}[V_k(\tilde{x}) | x, u_k]
\end{aligned}
$$

It follows that

$$
\begin{aligned}
[T^* V_1](x) &\geq f(x, u_2) + \beta\, \mathbb{E}[V_1(\tilde{x}) | x, u_2] \\
\text{and} \quad [T^* V_2](x) &= f(x, u_2) + \beta\, \mathbb{E}[V_2(\tilde{x}) | x, u_2]
\end{aligned}
$$

Subtracting the second equation from the first inequality gives

$$
[T^* V_1](x) - [T^* V_2](x) \geq \beta\, \mathbb{E}[V_1(\tilde{x}) - V_2(\tilde{x}) | x, u_2]
$$

If $V_1 \geqq V_2$ and so the inequality $V_1(\tilde{x}) \geq V_2(\tilde{x})$ holds for all $\tilde{x} \in X$, it follows that $[T^* V_1](x) \geq [T^* V_2](x)$.

Since this holds for all $x \in X$,
we have proved that $T^* V_1 \geqq T^* V_2$. □

Starting Policy Improvement

The method of policy improvement starts
with any fixed policy $\mathbf{u}_0$ or $X \ni x \mapsto u_0(x) \in F(x)$,
along with the value $V^{\mathbf{u}_0} \in \mathcal{V}_M$ of following that policy for ever.

The value $V^{\mathbf{u}_0}$ is the unique fixed point satisfying $V^{\mathbf{u}_0} = T^{\mathbf{u}_0} V^{\mathbf{u}_0}$
which belongs to the domain $\mathcal{V}_M$ of suitably bounded functions.

At each step $k = 1, 2, \ldots$, given the previous policy $\mathbf{u}_{k-1}$
and associated value $V^{\mathbf{u}_{k-1}}$ satisfying $V^{\mathbf{u}_{k-1}} = T^{\mathbf{u}_{k-1}} V^{\mathbf{u}_{k-1}}$:

1. the policy $\mathbf{u}_k$ is chosen so that $T^* V^{\mathbf{u}_{k-1}} = T^{\mathbf{u}_k} V^{\mathbf{u}_{k-1}}$;
2. the state valuation function $x \mapsto V_k(x)$ is chosen
   as the unique fixed point in $\mathcal{V}_M$ of the operator $T^{\mathbf{u}_k}$.

Policy Improvement Theorem

Theorem

The infinite sequence $(\mathbf{u}_k, V^{\mathbf{u}_k})_{k \in \mathbb{N}}$
consisting of pairs of policies $\mathbf{u}_k$
with their associated valuation functions $V^{\mathbf{u}_k} \in \mathcal{V}_M$ *satisfies*

1. $V^{\mathbf{u}_k} \geqq V^{\mathbf{u}_{k-1}}$ *for all* $k \in \mathbb{N}$ *(policy improvement);*
2. $\|V^{\mathbf{u}_k} - V^*\| \to 0$ *as* $k \to \infty$,
   where V^* *is the infinite-horizon optimal state valuation function in* $\mathcal{V}_M$ *that satisfies* $T^* V^* = V^*$.

Proof of Policy Improvement

By definition of the optimality operator T^*,
one has $T^* V \geqq T^{\mathbf{u}} V$ for all functions $V \in \mathcal{V}_M$ and all policies $\mathbf{u}$.

So at each step k of the policy improvement routine, one has

$$T^{\mathbf{u}_k} V^{\mathbf{u}_{k-1}} = T^* V^{\mathbf{u}_{k-1}} \geqq T^{\mathbf{u}_{k-1}} V^{\mathbf{u}_{k-1}} = V^{\mathbf{u}_{k-1}}$$

In particular, $T^{\mathbf{u}_k} V^{\mathbf{u}_{k-1}} \geqq V^{\mathbf{u}_{k-1}}$.

Now, applying successive iterations of the monotonic operator $T^{\mathbf{u}_k}$ implies that

$$V^{\mathbf{u}_{k-1}} \leqq T^{\mathbf{u}_k} V^{\mathbf{u}_{k-1}} \leqq [T^{\mathbf{u}_k}]^2 V^{\mathbf{u}_{k-1}} \leqq \dots$$
$$\dots \leqq [T^{\mathbf{u}_k}]^r V^{\mathbf{u}_{k-1}} \leqq [T^{\mathbf{u}_k}]^{r+1} V^{\mathbf{u}_{k-1}} \leqq \dots$$

But the definition of $V^{\mathbf{u}_k}$ implies that for all $V \in \mathcal{V}_M$,
including $V = V^{\mathbf{u}_{k-1}}$, one has $\|[T^{\mathbf{u}_k}]^r V - V^{\mathbf{u}_k}\| \to 0$ as $r \to \infty$.

Hence $V^{\mathbf{u}_k} = \sup_r [T^{\mathbf{u}_k}]^r V^{\mathbf{u}_{k-1}} \geqq V^{\mathbf{u}_{k-1}}$,
thus confirming that the policy $\mathbf{u}_k$ does improve $\mathbf{u}_{k-1}$. □

Proof of Convergence

Recall that at each step k of the policy improvement routine, one has $T^{\mathbf{u}_k} V^{\mathbf{u}_{k-1}} = T^* V^{\mathbf{u}_{k-1}}$ and also $T^{\mathbf{u}_k} V^{\mathbf{u}_k} = V^{\mathbf{u}_k}$.

Now, for each state $x \in X$, define $\hat{V}(x) := \sup_{k \in \mathbb{N}} V^{\mathbf{u}_k}(x)$.

Because $V^{\mathbf{u}_k} \geqq V^{\mathbf{u}_{k-1}}$ and $T^{\mathbf{u}_k}$ is monotonic, one has $V^{\mathbf{u}_k} = T^{\mathbf{u}_k} V^{\mathbf{u}_k} \geqq T^{\mathbf{u}_k} V^{\mathbf{u}_{k-1}} = T^* V^{\mathbf{u}_{k-1}}$.

Next, because T^* is monotonic, it follows that

$$\hat{V} = \sup_k V^{\mathbf{u}_k} \geqq \sup_k T^* V^{\mathbf{u}_{k-1}} = T^*(\sup_k V^{\mathbf{u}_{k-1}}) = T^* \hat{V}$$

Similarly, monotonicity of and the definition of T^* imply that

$$\hat{V} = \sup_k V^{\mathbf{u}_k} = \sup_k T^{\mathbf{u}_k} V^{\mathbf{u}_k} \leqq \sup_k T^* V^{\mathbf{u}_k} = T^*(\sup_k V^{\mathbf{u}_k}) = T^* \hat{V}$$

Hence $\hat{V} = T^* \hat{V} = V^*$, because T^* has a unique fixed point.

Therefore $V^* = \sup_k V^{\mathbf{u}_k}$ and so, because the sequence $V^{\mathbf{u}_k}(x)$ is non-decreasing, one has $V^{\mathbf{u}_k}(x) \to V^*(x)$ for each $x \in X$. $\square$

Lecture Outline

Unbounded Utility

In economics the boundedness condition $M_* \leq f(x, u) \leq M^*$ is rarely satisfied!

Consider for example the isoelastic utility function

$$u(c; \epsilon) = \begin{cases} \dfrac{c^{1-\epsilon}}{1-\epsilon} & \text{if } \epsilon > 0 \text{ and } \epsilon \neq 1 \\ \ln c & \text{if } \epsilon = 1 \end{cases}$$

This function is obviously:

1. bounded below but unbounded above in case $0 < \epsilon < 1$;
2. unbounded both above and below in case $\epsilon = 1$;
3. bounded above but unbounded below in case $\epsilon > 1$.

Also commonly used is the negative exponential utility function defined by $u(c) = -e^{-\alpha c}$
where α is the constant absolute rate of risk aversion (CARA).

This function is bounded above and, provided that $c \geq 0$, also below.

Warning Example: Statement of Problem

The following example shows that there can be irrelevant unbounded solutions to the Bellman equation.

Example

Consider the problem of maximizing $\sum_{t=0}^{\infty} \beta^t (1 - u_t)$

where $u_t \in [0, 1]$, $0 < \beta < 1$, and $x_{t+1} = \frac{1}{\beta}(x_t + u_t)$, with $x_0 > 0$.

Notice that $x_{t+1} \geq \frac{1}{\beta} x_t$ implying that $x_t \geq \beta^{-t} x_0 \to \infty$ as $t \to \infty$.

Of course the return function $[0, 1] \ni u \mapsto f(x, u) = 1 - u \in [0, 1]$ is uniformly bounded.

Warning Example: Unbounded Spurious Solution

The Bellman equation is

$$\left.\begin{array}{rcl} J(x) & = & \\ u^*(x) & = & \arg \end{array}\right\} \max_{u\in[0,1]} \left\{ 1 - u + \beta J\left(\frac{1}{\beta}(x+u)\right) \right\}$$

Even though the return function is uniformly bounded, this Bellman equation has an unbounded spurious solution.

Indeed, we find a spurious solution with $J(x) \equiv \gamma + x$ for a suitable constant γ.

The condition for this to solve the Bellman equation is that

$$\begin{array}{rcl} \gamma + x & = & \max_{u\in[0,1]} \left\{ 1 - u + \beta \left[\gamma + \frac{1}{\beta}(x+u) \right] \right\} \\ & = & \max_{u\in[0,1]} \{1 + \beta\gamma + x\} \; = \; 1 + \beta\gamma + x \end{array}$$

which is true iff $\gamma = 1 + \beta\gamma$ and so $\gamma = (1-\beta)^{-1}$.

Warning Example: True Solution

The problem is to maximize $\sum_{t=0}^{\infty} \beta^t (1 - u_t)$

where $u_t \in [0, 1]$, $0 < \beta < 1$, and $x_{t+1} = \dfrac{1}{\beta}(x_t + u_t)$, with $x_0 > 0$.

The obvious optimal policy is to choose $u_t = 0$ for all t,
giving the maximized value $J(x) = \sum_{t=0}^{\infty} \beta^t = (1 - \beta)^{-1}$.

Indeed the bounded function $J(x) = (1 - \beta)^{-1}$,
together with $u^* = 0$, both independent of x,
do indeed solve the Bellman equation

$$
\begin{aligned}
J(x) &= \max_{u \in [0,1]} \left\{ 1 - u + \beta J\left(\frac{1}{\beta}(x + u)\right)\right\} \\
&= \max_{u \in [0,1]} \left\{ 1 - u + \beta (1 - \beta)^{-1} \right\} \\
&= 1 + \frac{\beta}{1 - \beta} = \frac{1}{1 - \beta}
\end{aligned}
$$