# Lecture Notes 7: Dynamic Equations Part A: First-Order Difference Equations in One Variable

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# Lecture Outline

# Walking as a Simple Difference Equation

What is the difference
between difference and differential equations?

It is relatively common to indicate by:

a subscript a discrete time function like $m \mapsto x_m$;

parentheses a continuous time function like $t \mapsto x(t)$.

Walking on two feet can be modelled as a discrete time process,
with time domain $T = \{0, 1, 2, \ldots\} = \mathbb{Z}_+ = \{0\} \cup \mathbb{N}$
that counts the number of completed steps.

After $m$ steps, the respective positions $\ell, r \in \mathbb{R}^2$
of the left and right feet on the ground
can be described by the two functions $T \ni m \mapsto (\ell_m, r_m)$.

# Walking as a More Complicated Difference Equation

Athletics rules limit a walking step to be no longer than a stride.

So a walking process that starts with the left foot
might be described by the two coupled equations

$$\ell_m = \begin{cases} \lambda(r_{m-1}) & \text{if } m \text{ is odd} \\ \ell_{m-1} & \text{if } m \text{ is even} \end{cases} \quad \text{and } r_m = \begin{cases} \rho(\ell_{m-1}) & \text{if } m \text{ is even} \\ r_{m-1} & \text{if } m \text{ is odd} \end{cases}$$

for $m = 0, 1, 2, \ldots$.

Or, if the length and direction of each pace are affected
by the length and direction of its predecessor, by

$$\begin{aligned} \ell_m &= \begin{cases} \lambda(r_{m-1}, \ell_{m-2}) & \text{if } m \text{ is odd} \\ \ell_{m-1} & \text{if } m \text{ is even} \end{cases} \\ \text{and } r_m &= \begin{cases} \rho(\ell_{m-1}, r_{m-2}) & \text{if } m \text{ is even} \\ r_{m-1} & \text{if } m \text{ is odd} \end{cases} \end{aligned}$$

for $m = 0, 1, 2, \ldots$.

# Walking as a Differential Equation

Newtonian physics implies that a walker's centre of mass must be a continuous function of time, described by a 3-vector valued mapping $\mathbb{R}_+ \ni t \mapsto (x(t), y(t), z(t)) \in \mathbb{R}^3$.

The time domain is therefore $T := \mathbb{R}_+$.

The same will be true for the position of, for instance, the extreme end of the walker's left big toe.

Newtonian physics requires that the acceleration 3-vector described by the second derivative $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x(t), y(t), z(t)) \in \mathbb{R}^3$ should be well defined for all $t$.

The biology of survival requires it to be bounded.

Actually, the motion becomes seriously uncomfortable unless the acceleration (or deceleration) is continuous — as my driving instructor taught me more than 50 years ago!

# Lecture Outline

# Basic Definition

Let $T = \mathbb{Z}_+ \ni t \mapsto x_t \in X$ describe a discrete time process, with $X = \mathbb{R}$ (or $X = \mathbb{R}^m$) as the state space.

Its difference at time $t$ is defined as

$$\Delta x_t := x_{t+1} - x_t$$

A standard first-order difference equation takes the form

$$x_{t+1} - x_t = \Delta x_t = d_t(x_t)$$

where each $d_t : X \to X$, or equivalently,

$$T \times X \ni (t, x) \mapsto d_t(x)$$

# Equivalent Recurrence Relations

Obviously, the difference equation $x_{t+1} - x_t = \Delta x_t = d_t(x_t)$
is equivalent to the recurrence relation $x_{t+1} = r_t(x_t)$
where $T \times X \ni (t, x) \mapsto r_t(x) = x + d_t(x)$,
or equivalently, $d_t(x) = r_t(x) - x$.

Thus difference equations and recurrence relations
are entirely equivalent.

We follow standard mathematical practice
in using the notation for recurrence relations,
even when discussing difference equations.

We may write "difference equation"
even when considering a recurrence relation.

# Existence of Solutions

## Example

Consider the difference equation $x_t = \sqrt{x_{t-1} - 1}$ with $x_0 = 5$.

Evidently $x_1 = \sqrt{5-1} = 2$, then $x_2 = \sqrt{2-1} = 1$,
and next $x_3 = \sqrt{1-1} = 0$,
leaving $x_4 = \sqrt{0-1}$ undefined as a real number.

The domain of $(t, x) \mapsto \sqrt{x-1}$ is limited to
$D := \mathbb{Z}_+ \times [1, \infty)$. $\square$

Generally, consider a mapping $D \ni (t, x) \mapsto r_t(x)$
whose domain is restricted to a subset $D \subset \mathbb{Z}_+ \times X$.

For the difference equation $x_{t+1} = r_t(x_t)$ to have a solution
one must ensure that

$$(t, x) \in D \Longrightarrow (t+1, r_t(x_t)) \in D \quad \text{for all } t \in \mathbb{Z}_+$$

# Lecture Outline

# Application: Wealth Accumulation in Discrete Time

Consider a consumer who, in discrete time $t = 0, 1, 2, \ldots$:

- starts each period $t$
  with an amount $w_t$ of accumulated wealth;
- receives income $y_t$;
- spends an amount $e_t$;
- earns interest on the residual wealth $w_t + y_t - e_t$ at the rate $r_t$.

The process of wealth accumulation is then described
by any of the equivalent equations

$$w_{t+1} = (1 + r_t)(w_t + y_t - e_t) = \rho_t(w_t - x_t) = \rho_t(w_t + s_t)$$

where, at each time $t$,

- $\rho_t := 1 + r_t$ is the interest factor;
- $x_t = e_t - y_t$ denotes net expenditure;
- $s_t = y_t - e_t = -x_t$ denotes net saving.

# Compound Interest

Define the compound interest factor

$$R_t := \prod_{k=0}^{t-1}(1 + r_k) = \prod_{k=0}^{t-1}\rho_k$$

with the convention that the product of zero terms equals 1 — just as the sum of zero terms equals 0.

This compound interest factor is the unique solution to the recurrence relation $R_{t+1} = (1 + r_t)R_t$ that satisfies the initial condition $R_0 = 1$.

In the special case when $r_t = r$ (all $t$), it reduces to $R_t = (1 + r)^t = \rho^t$.

# Present Discounted Value (PDV)

We transform the difference equation $w_{t+1} = \rho_t(w_t - x_t)$
by using the compound interest factor $R_t = \prod_{k=0}^{t-1} \rho_k$
in order to discount both future wealth and expenditure.

To do so, define new variables $\omega_t, \xi_t$
for the present discounted values (PDVs) of, respectively:

1. wealth $w_t$ at time $t$ as $\omega_t := (1/R_t)w_t$;
2. net expenditure $x_t$ at time $t$ as $\xi_t := (1/R_t)x_t$.

With these new variables,
the wealth equation $w_{t+1} = \rho_t(w_t - x_t)$ becomes

$$R_{t+1}\omega_{t+1} = \rho_t R_t(\omega_t - \xi_t)$$

But $R_{t+1} = \rho_t R_t$, so eliminating this common factor
reduces the equation to $\omega_{t+1} = \omega_t - \xi_t$,
with the evident solution $\omega_t = \omega_0 - \sum_{k=0}^{t-1} \xi_k$ for $k = 1, 2, \ldots$.

# Lecture Outline

# General First-Order Linear Equation

The general first-order linear difference equation can be written in the form

$$x_t - a_t x_{t-1} = f_t \quad \text{for } t = 1, 2, \ldots, T$$

for non-zero constants $a_t \in \mathbb{R}$ and a forcing term $\mathbb{N} \ni t \mapsto f_t \in \mathbb{R}$.

When this equation holds for $t = 1, 2, \ldots, T$, where $T \geq 6$, this equation can be written in the following matrix form:

$$\begin{pmatrix} -a_1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -a_2 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -a_3 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -a_{T-1} & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -a_T & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \\ x_T \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{T-1} \\ f_T \end{pmatrix}$$

# Matrix Form

The matrix form of the difference equation is $\mathbf{Cx} = \mathbf{f}$, where:

1. $\mathbf{C}$ is the $T \times (T+1)$ coefficient matrix whose elements are

$$c_{st} = \begin{cases} -a_s & \text{if } t = s \\ 1 & \text{if } t = s+1 \\ 0 & \text{otherwise} \end{cases}$$

   for $s = 1, 2, \ldots, T$ and $t = 0, 1, 2, \ldots, T$;
2. $\mathbf{x}$ is the $T+1$-dimensional column vector $(x_t)_{t=0}^T$ of endogenous unknowns, to be determined;
3. $\mathbf{f}$ is the $T$-dimensional column vector $(f_t)_{t=1}^T$ of exogenous shocks.

# Partitioned Matrix Form

The matrix equation $\mathbf{C}\mathbf{x} = \mathbf{f}$ can be written in partitioned form as

$$\begin{pmatrix} \mathbf{U} & \mathbf{e}_T \end{pmatrix} \begin{pmatrix} \mathbf{x}^{T-1} \\ x_T \end{pmatrix} = \mathbf{f}$$

where:

1. $\mathbf{U}$ is an upper triangular $T \times T$ matrix;
2. $\mathbf{e}_T = (0, 0, 0, \ldots, 0, 1)^\top$ is the $T$th column vector of the canonical basis of the vector space $\mathbb{R}^T$;
3. $\mathbf{x}^{T-1}$ denotes the column vector which is the transpose of the row $T$-vector $(x_0, x_1, x_2, \ldots, x_{T-2}, x_{T-1})$.

In fact the matrix $\mathbf{U}$ satisfies

$$(\mathbf{U}, \mathbf{e}_T) = (-\,\mathbf{diag}(a_1, a_2, \ldots, a_T), \mathbf{e}_T) + (\mathbf{0}_{T\times 1}, \mathbf{I}_{T\times T})$$

Hence there are $T$ independent equations in $T + 1$ unknowns, leaving one degree of freedom in the solution.

# An Initial Condition

Consider the difference equation $x_t - a_t x_{t-1} = f_t$,
or $\mathbf{C}\mathbf{x} = \mathbf{f}$ in matrix form.

An initial condition specifies an exogenous value $\bar{x}_0$ for the value $x_0$ at time 0.

This removes the only degree of freedom in the system of $T$ equations in $T+1$ unknowns.

Consider the special case when $a_t = 1$ for all $t \in \mathbb{N}$.

The obvious unique solution of $x_t - x_{t-1} = f_t$ is then that each $x_t$ is the forward sum

$$x_t = \bar{x}_0 + \sum\nolimits_{s=1}^{t} f_s$$

of the initial state $\bar{x}_0$, and of the $t$ exogenously specified succeeding differences $f_s$ $(s = 1, 2, \ldots, t)$.

# A Terminal Condition

Alternatively, a terminal condition
for the difference equation $x_t - x_{t-1} = f_t$
specifies an exogenous value $\bar{x}_T$ for the value $x_T$
at the terminal time $T$.

It leads to a unique solution as a backward sum

$$x_t = \bar{x}_T - \sum_{s=0}^{T-t-1} f_{T-s}$$

of the exogenously specified

- terminal state $\bar{x}_T$;
- preceding backward differences $-f_{T-s}$
  $(s = 0, 1, \ldots, T - t - 1)$.

# Lecture Outline

# Particular and General Solutions

We are interested in solving the system $\mathbf{C}\mathbf{x} = \mathbf{f}$
of $T$ equations in $T + 1$ unknowns,
where $\mathbf{C}$ is a $T \times (T + 1)$ matrix.

When the rank of $\mathbf{C}$ is $T$, there is one degree of freedom.

The associated homogeneous equation $\mathbf{C}\mathbf{x} = \mathbf{0}$
will have a one-dimensional space of solutions $x_t^H = \xi \bar{x}_t^H$ $(\xi \in \mathbb{R})$.

Given any particular solution $x_t^P$ satisfying $\mathbf{C}\mathbf{x}^P = \mathbf{f}$
for the particular time series $\mathbf{f}$ of forcing terms,
the general solution $x_t^G$ must also satisfy $\mathbf{C}\mathbf{x}^G = \mathbf{f}$.

Simple subtraction leads to $\mathbf{C}(\mathbf{x}^G - \mathbf{x}^P) = \mathbf{0}$, so $\mathbf{x}^G - \mathbf{x}^P = \mathbf{x}^H$
for some solution $\mathbf{x}^H$ of the homogeneous equation $\mathbf{C}\mathbf{x} = \mathbf{0}$.

So $\mathbf{x}$ solves the equation $\mathbf{C}\mathbf{x} = \mathbf{f}$
iff there exists a scalar $\xi \in \mathbb{R}$ such that $\mathbf{x} = \mathbf{x}^P + \xi\, \mathbf{x}^H$,
which leads to the formula $\mathbf{x}^G = \mathbf{x}^P + \xi\, \mathbf{x}^H$ for the general solution.

# Complementary Solutions

Consider again the general first-order linear equation which takes the inhomogeneous form $x_t - a_t x_{t-1} = f_t$.

The associated homogeneous equation takes the form

$$x_t - a_t x_{t-1} = 0 \quad (\text{for all } t \in \mathbb{N})$$

with a zero right-hand side.

The associated complementary solutions make up the one-dimensional linear subspace $L$ of solutions to this homogeneous equation.

The space $L$ consists of functions $\mathbb{Z}_+ \ni t \mapsto x_t \in \mathbb{R}$ satisfying

$$x_t = x_0 \prod_{s=1}^{t} a_s \quad (\text{for all } t \in \mathbb{N})$$

where $x_0$ is an arbitrary scaling constant.

# From Particular to General Solutions

Consider again the inhomogeneous equation

$$x_t - a_t x_{t-1} = f_t$$

for a general RHS $f_t$.

The associated homogeneous equation takes the form

$$x_t - a_t x_{t-1} = 0$$

Let $x_t^P$ denote a particular solution,
and $x_t^G$ any alternative general solution,
of the inhomogeneous equation.

Our assumptions imply that, for each $t = 1, 2, \ldots$, one has

$$\begin{aligned} x_t^P - a_t x_{t-1}^P &= f_t \\ x_t^G - a_t x_{t-1}^G &= f_t \end{aligned}$$

Subtracting the first equation from the second implies that

$$x_t^G - x_t^P - a_t(x_{t-1}^G - x_{t-1}^P) = 0$$

This shows that $x_t^H := x_t^G - x_t^P$ solves the homogeneous equation.

# Characterizing the General Solution

### Theorem

*Consider the inhomogeneous equation* $x_t - a_t x_{t-1} = f_t$ *with forcing term* $f_t$.

*Its general solution* $x_t^G$ *is the sum* $x_t^P + x_t^H$ *of*

- *any particular solution* $x_t^P$ *of the inhomogeneous equation;*
- *the general complementary solution* $x_t^H$ *of the corresponding homogeneous equation* $x_t - a_t x_{t-1} = 0$.

# Linearity in the Forcing Term, I

**Theorem**

*Suppose that $x_t^P$ and $y_t^P$ are particular solutions of the two respective difference equations*

$$x_t - a_t x_{t-1} = d_t \quad \textit{and} \quad y_t - a_t y_{t-1} = e_t$$

*Then, for any scalars $\alpha$ and $\beta$, the equation $z_t - a_t z_{t-1} = \alpha d_t + \beta e_t$ has as a particular solution the corresponding linear combination $z_t^P := \alpha x_t^P + \beta y_t^P$.*

**Proof.**

Routine algebra. □

# Linearity in the Forcing Term, II

Consider any equation of the form $x_t - a_t x_{t-1} = f_t$
where $f_t$ is a linear combination $\sum_{k=1}^n \alpha_k f_t^k$
of $n$ forcing terms $\langle f_t^k \rangle_{k=1}^n$.

The theorem implies that a particular solution
is the corresponding linear combination $\sum_{k=1}^n \alpha_k x_t^{Pk}$
of particular solutions $\langle x_t^{Pk} \rangle_{k=1}^n$
to the respective $n$ equations $x_t - a_t x_{t-1} = f_t^k$ $(k = 1, 2, \ldots, n)$.

# Lecture Outline

# Solving the General Linear Equation

Consider a first-order linear difference equation

$$x_{t+1} = a_t x_t + f_t$$

for a process $T \ni t \mapsto x_t \in \mathbb{R}$, where each $a_t \neq 0$
(to avoid trivialities).

We will prove by induction on $t$ that for $t = 0, 1, 2, \ldots$
there exist suitable non-zero constants $p_{t,k}$ $(k = 0, 1, 2, \ldots, t)$
such that, given any possible value of the initial state $x_0$
and of the forcing terms $f_t$ $(t = 0, 1, 2, \ldots)$,
the unique solution can be expressed as

$$x_t = p_{t,0} x_0 + \sum\nolimits_{k=1}^{t} p_{t,k} f_{k-1}$$

The proof, of course, will also involve deriving
a recurrence relation for the constants $p_{t,k}$ $(k = 0, 1, 2, \ldots, t)$.

# Early Terms of the Solution

Because $x_0 = p_{0,0} x_0 = x_0$,
the first term is obviously $p_{0,\,0} = 1$ when $t = 0$.

Next $x_1 = a_0 x_0 + f_0$ when $t = 1$
implies that $p_{1,\,0} = a_0$ and $p_{1,\,1} = 1$.

Next, the solution for $t = 2$ is

$$x_2 = a_1 x_1 + f_1 = a_1 a_0 x_0 + a_1 f_0 + f_1$$

This formula matches the formula

$$x_t = p_{t,0} x_0 + \sum_{k=1}^{t} p_{t,k} f_{k-1}$$

when $t = 2$ provided that:

- $p_{2,\,0} = a_1 a_0$;
- $p_{2,\,1} = a_1$;
- $p_{2,\,2} = 1$.

# Explicit Solution, I

Now, substituting the two expansions

$$\begin{aligned} x_t &= p_{t,0} x_0 + \textstyle\sum_{k=1}^{t} p_{t,k} f_{k-1} \\ \text{and} \quad x_{t+1} &= p_{t+1,0} x_0 + \textstyle\sum_{k=1}^{t+1} p_{t+1,k} f_{k-1} \end{aligned}$$

into both sides of the original equation $x_{t+1} = a_t x_t + f_t$ gives

$$p_{t+1,0} x_0 + \sum_{k=1}^{t+1} p_{t+1,k} f_{k-1} = a_t \left( p_{t,0} x_0 + \sum_{k=1}^{t} p_{t,k} f_{k-1} \right) + f_t$$

Equating the coefficients of $x_0$ and of each $f_{k-1}$
in this equation implies that for general $t$ one has

$$p_{t+1,k} = a_t p_{t,k} \text{ for } k = 0, 1, \ldots, t, \text{ with } p_{t+1,t+1} = 1$$

# Explicit Solution, II

The equation $p_{t+1,k} = a_t p_{t,k}$ for $k = 0, 1, \ldots, t$ implies that

$$\begin{aligned} p_{t,\,0} &= a_{t-1} \cdot a_{t-2} \cdots a_0 \quad \text{when } k = 0 \\ p_{t,\,k} &= a_{t-1} \cdot a_{t-2} \cdots a_k \quad \text{when } k = 1, 2, \ldots, t \end{aligned}$$

or, after defining the product of the empty set of real numbers as 1,

$$p_{t,\,k} = \prod\nolimits_{s=1}^{t-k} a_{t-s}$$

Inserting these into our formula $x_t = p_{t,0} x_0 + \sum_{k=1}^{t} p_{t,k} f_{k-1}$ gives the explicit solution

$$x_t = \left( \prod\nolimits_{s=1}^{t} a_{t-s} \right) x_0 + \sum\nolimits_{k=1}^{t} \left( \prod\nolimits_{s=1}^{t-k} a_{t-s} \right) f_{k-1}$$

Putting $x_0 = 0$ gives one particular solution of $x_{t+1} = a_t x_t + f_t$, namely

$$x_t^P = \sum\nolimits_{k=1}^{t} \left( \prod\nolimits_{s=1}^{t-k} a_{t-s} \right) f_{k-1}$$

# Lecture Outline

# First-Order Linear Equation with a Constant Coefficient

Next, consider the equation $x_t - ax_{t-1} = f_t$,
where the coefficient $a_t$ has become the constant $a \neq 0$.

Evidently one has $x_1 = ax_0 + f_1$,
then $x_2 = ax_1 + f_2 = a(ax_0 + f_1) + f_2 = a^2 x_0 + af_1 + f_2$, then

$$x_3 = ax_2 + f_3 = a(a^2 x_0 + af_1 + f_2) + f_2 = a^3 x_0 + a^2 f_1 + af_2 + f_3$$

etc. One can easily verify by induction the explicit formula

$$x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} f_k$$

In the very special case when $a = 1$,
this also accords with our earlier sum solution $x_t = x_0 + \sum_{k=1}^{t} f_k$.

# First Special Case

An interesting special case occurs when there exists some $\mu \neq 0$ such that $f_t$ is the discrete exponential function $\mathbb{N} \ni t \mapsto \mu^t$.

Then the solution is $x_t = a^t x_0 + S_t$ where $S_t := \sum_{k=1}^t a^{t-k} \mu^k$.

Note that

$$(a - \mu) S_t = \sum_{k=1}^t a^{t-k+1} \mu^k - \sum_{k=1}^t a^{t-k} \mu^{k+1} = a^t \mu - \mu^{t+1}$$

In the non-degenerate case when $\mu \neq a$,
it follows that $S_t = \mu \left( \frac{a^t - \mu^t}{a - \mu} \right)$ and so $x_t = a^t x_0 + \mu \left( \frac{a^t - \mu^t}{a - \mu} \right)$.

This solution can be written as $x_t = x_t^H + x_t^P$ where:

1. $x_t^H = \xi^H a^t$ with $\xi^H := x_0 + \mu/(a - \mu)$ is a solution of the homogeneous equation $x_t - a x_{t-1} = 0$;
2. $x_t^P = \xi^P \mu^t$ with $\xi^P := -\mu/(a - \mu)$ is a particular solution of the inhomogeneous equation $x_t - a x_{t-1} = \mu^t$.

# Degenerate Case When $\mu = a$

In the degenerate case when $\mu = a$, the solution collapses to

$$x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} a^k = a^t x_0 + \sum_{k=1}^{t} a^t = a^t (x_0 + t)$$

Again, this solution can be written as $x_t = x_t^H + x_t^P$ where:

1. $x_t^H = \xi^H a^t$ with $\xi^H := x_0$ is a solution of the homogeneous equation $x_t - a x_{t-1} = 0$;
2. $x_t^P = \xi^P a^t t$ with $\xi^P := 1$ is a particular solution of the inhomogeneous equation $x_t - a x_{t-1} = a^t$.

Note carefully the term in $a^t t$, where $a^t$ is multiplied by $t$.

# Second Special Case

Another interesting special case is when $f_t = t^r \mu^t$ for some $r \in \mathbb{N}$.

Then the explicit solution we found previously
takes the form $x_t = a^t x_0 + S_t$ where $S_t := \sum_{k=1}^t a^{t-k} k^r \mu^k$.

We aim to simplify this expression for $S_t$.

We first restrict attention to the non-degenerate case when $\mu \neq a$.

The solution can still be written as $x_t = x_t^H + x_t^P$ where:

1. $x_t^H = \xi^H a^t$ with scalar $\xi^H \in \mathbb{R}$
   solves the homogeneous equation $x_t - a x_{t-1} = 0$;
2. $x_t^P = \xi^P(t)\, \mu^t$ is any particular solution
   of the inhomogeneous equation $x_t - a x_{t-1} = t^r \mu^t$.

The issue is finding a useful form of the function $t \mapsto \xi^P(t)$
that makes $\xi^P(t)\, \mu^t$ a solution of $x_t - a x_{t-1} = t^r \mu^t$.

# Method of Undetermined Coefficients

We will find a particular solution $x_t^P = \xi^P(t)\mu^t$
of the inhomogeneous difference equation $x_t - a x_{t-1} = t^r \mu^t$
where $\xi^P(t) = \sum_{k=0}^r \xi_k t^k$ is a polynomial in $t$ of degree $r$,
the power of $t$ on the right-hand side.

The coefficients $(\xi_0, \xi_1, \ldots, \xi_r)$ of the polynomial
are undetermined till we consider the difference equation itself.

Note that, by the binomial theorem,

$$\begin{aligned}
\xi^P(t-1) &= \sum_{k=0}^r \xi_k \cdot (t-1)^k = \sum_{k=0}^r \xi_k \sum_{j=0}^k \binom{k}{j} t^j (-1)^{k-j} \\
&= \sum_{j=0}^r \sum_{k=0}^r 1_{j \le k} \xi_k \binom{k}{j} t^j (-1)^{k-j} \\
&= \sum_{j=0}^r \sum_{k=j}^r \xi_k \binom{k}{j} (-1)^{k-j} t^j
\end{aligned}$$

where $1_{j \le k}$ denotes 1 if $j \le k$, but 0 if $j > k$.

# Determining the Undetermined Coefficients

For $x_t^P = \mu^t \sum_{k=0}^r \xi_k t^k$ to solve $x_t - a x_{t-1} = t^r \mu^t$, we need

$$\mu^t \sum\nolimits_{j=0}^r \xi_j t^j - a\mu^{t-1} \sum\nolimits_{j=0}^r \sum\nolimits_{k=j}^r \xi_k \binom{k}{j} (-1)^{k-j} t^j = t^r \mu^t$$

First consider the non-degenerate case $\mu \neq a$.

Equating coefficients of $t^r$ implies that $\mu^t \xi_r - a\mu^{t-1}\xi_r = \mu^t$.

Dividing by $\mu^{t-1}$ gives $\mu\xi_r - a\xi_r = \mu$, and so $\xi_r = \mu(\mu - a)^{-1}$.

For $j = 0, 1, \ldots, r-1$, equating coefficients of $t^j$ implies that

$$\mu^t \xi_j - a\mu^{t-1} \sum\nolimits_{k=j}^r \xi_k \binom{k}{j} (-1)^{k-j} = 0$$

so $\xi_j = (\mu - a)^{-1} \sum_{k=j+1}^r \xi_k \binom{k}{j} (-1)^{k-j}$.

In principle one can solve this system of $r+1$ equations
in the $r+1$ unknowns $(\xi_r, \xi_{r-1}, \xi_{r-2}, \ldots, \xi_0)$
by backward recursion, starting with $\xi_r = \mu(\mu - a)^{-1}$, ending at $\xi_0$.

# Degenerate Case

But in the degenerate case $\mu = a$,
the equation $\mu\xi_r - a\xi_r = \mu$ for $\xi_r$ has no solution,
so the method does not work.

Instead, to solve $x_t - ax_{t-1} = t^r a^t$,
we introduce the new variable $y_t = a^{-t} x_t$.

Then $y_t = a^{-t}(ax_{t-1} + t^r a^t) = y_{t-1} + t^r$.

The solution is $y_t = y_0 + S_r(t)$ where $S_r(n) := \sum_{k=1}^{n} j^r$
is the much studied sum of $r$th powers of the first $n$ integers.

Hence $x_t = a^t \left[x_0 + S_r(t)\right]$.

### Theorem

*The sums $S_r(n)$ satisfy the recurrence relation*

$$(r+1)S_r(n) = (n+1)^{r+1} - 1 - \sum_{k=0}^{r-1} \binom{r+1}{k} S_k(n)$$

# Proof of Theorem

For $j = 1, 2, \ldots n$, the binomial theorem implies that

$$(j+1)^{r+1} = \sum_{k=0}^{r} \binom{r+1}{k} j^k + j^{r+1}$$

Summing over $j$, then interchanging the order of summation, gives

$$\begin{aligned}(n+1)^{r+1} - 1 &= \sum_{j=1}^{n} [(j+1)^{r+1} - j^{r+1}] = \sum_{j=1}^{n} \sum_{k=0}^{r} \tbinom{r+1}{k} j^k \\ &= \sum_{k=0}^{r} \tbinom{r+1}{k} S_k(n)\end{aligned}$$

Isolating the last term gives

$$(n+1)^{r+1} - 1 = \sum_{k=0}^{r-1} \binom{r+1}{k} S_k(n) + \binom{r+1}{r} S_r(n)$$

and so, after rearranging

$$(r+1) S_r(n) = (n+1)^{r+1} - 1 - \sum_{k=0}^{r-1} \binom{r+1}{k} S_k(n)$$

$\square$

# Important Corollary

## Corollary

*Each sum* $S_r(n) := \sum_{j=1}^{n} j^r$
*of the* $r$*th powers of the first* $n$ *natural numbers*
*equals a polynomial* $\sum_{i=0}^{r+1} a_{ri} n^i$ *of degree* $r+1$ *in* $n$*,*
*whose coefficients are rational,*
*with constant term* $a_{r0} = 0$
*and leading coefficient* $a_{r,r+1} = 1/(r+1)$.

Using the theorem, we prove the corollary by induction,
starting with the obvious $S_0(n) := \sum_{j=1}^{n} j^0 = n$
and the well known $S_1(n) := \sum_{j=1}^{n} j = \frac{1}{2} n(n+1) = \frac{1}{2} n + \frac{1}{2} n^2$.

# Proof by Induction

Indeed, consider the induction hypothesis
that $S_q(n)$ satisfies the corollary for $q = 0, 1, 2, \ldots, r-1$.

This hypothesis implies in particular
that each $S_k(n)$ is a polynomial of degree $k+1$ in $n$.

It follows that the right-hand side of the equation

$$(r+1)S_r(n) = (n+1)^{r+1} - 1 - \sum\nolimits_{k=0}^{r-1} \binom{r+1}{k} S_k(n)$$

is obviously a polynomial of degree at most $r+1$ in $n$.

Moreover, it has rational coefficients, a constant term 0,
and a leading coefficient 1 attached to the highest power $n^{r+1}$. $\square$

# Main Theorem

## Theorem

*Consider the inhomogeneous first-order linear difference equation*

$$x_t - a x_{t-1} = t^r \mu^t, \text{ where } a \neq 0 \text{ and } r \in \mathbb{Z}_+.$$

*Then there exists a particular solution of the form* $x_t^P = Q(t)\,\mu^t$ *where the function* $t \mapsto Q(t)$ *is a polynomial which:*

- *in the regular case when* $\mu \neq a$, *has degree* $r$;
- *in the degenerate case when* $\mu = a$, *has degree* $r + 1$.

*The general solution takes the form* $x_t = x_t^P + x_t^C$ *where:*

- $x_t^P$ *is any particular solution of the inhomogeneous equation;*
- $x_t^C$ *is any member of the one-dimensional linear space of complementary solutions to the corresponding homogeneous equation* $x_t - a x_{t-1} = 0$.

# Lecture Outline

# Stationary States, I

The general first-order equation $x_{t+1} = f_t(x_t)$ is non-autonomous;
for it to become autonomous,
there should be a mapping $x \mapsto f(x)$ that is independent of $t$.

Given an autonomous equation $x_{t+1} = f(x_t)$,
a stationary state is a fixed point $x^* \in \mathbb{R}$ of the mapping $x \mapsto f(x)$.

It earns its name because if $x_s = x^*$ for any finite $s$,
then $x_t = x^*$ for all $t = s, s+1, \ldots$.

# Stationary States, II

Wherever it exists, the solution of the autonomous equation can be written as a function $x_t = \Phi_{t-s}(x_s)$ $(t = s, s+1, \ldots)$ of the state $x_s$ at the initial time $s$, as well as of the number of periods $t - s$ that the function $x \mapsto f(x)$ must be iterated in order to determine the state $x_t$ at time $t$.

Indeed, the sequence of functions $\Phi_k : \mathbb{R} \to \mathbb{R}$ $(k \in \mathbb{N})$ is defined iteratively by $\Phi_k(x) = f(\Phi_{k-1}(x))$ for all $x$.

Note that any stationary state $x^*$ is a fixed point of each mapping $\Phi_k$ in the sequence, as well as of $\Phi_1 \equiv f$.

# Local and Global Stability

The stationary state $x^*$ is:

- globally stable if $\Phi_k(x_0) \to x^*$ as $k \to \infty$, regardless of the initial state $x_0$;
- locally stable if there is an (open) neighbourhood $N \subset \mathbb{R}$ of $x^*$ such that whenever $x_0 \in N$ one has $\Phi_k(x_0) \to x^*$ as $k \to \infty$.

Generally, global stability implies local stability, but not conversely.

Global stability also implies that the steady state $x^*$ is unique.

We begin by studying stability for linear equations,
where local stability is equivalent to global stability.

Later, we will consider the local stability of non-linear equations.

# Stationary States of a Linear Equation

Consider the linear (or rather, affine) equation $x_{t+1} = ax_t + f$ for a fixed forcing term $f$.

A stationary state $x^*$ has the defining property that $x_t = x^* \Longrightarrow x_{t+1} = x^*$, which is satisfied if and only $x^* = ax^* + f$.

In case $a = 1$, there is:

- no stationary state unless $f = 0$;
- the whole real line $\mathbb{R}$ of stationary states if $f = 0$.

Otherwise, if $a \neq 1$, the only stationary state is $x^* = (1 - a)^{-1} f$.

# Stability of a Linear Equation

If $a \neq 1$, let us denote by $y_t := x_t - x^*$
the deviation of state $x_t$ from the stationary state $x^* = (1-a)^{-1} f$.

Then $y_{t+1} = x_{t+1} - x^* = a x_t + f - x^* = a(y_t + x^*) + f - x^* = a y_t$.

This equation has the obvious solution $y_t = y_0 a^t$,
or equivalently $x_t = x^* + (x_0 - x^*) a^t$.

The solution is evidently both locally and globally stable
if and only if $a^t \to 0$ as $t \to \infty$,
which is true if and only if $|a| < 1$.

# Lecture Outline

# Stationary States

Let $I$ be an open interval of the real line,
and $I \ni x \mapsto f(x) \in I$ a general, possibly nonlinear, function.

A fixed point $x \in I$ of $f$ satisfies $f(x) = x$.

Consider the autonomous difference equation $x_{t+1} = f(x_t)$.

For each natural number $n \in \mathbb{N}$,
define the iterated function $I \ni x \mapsto f^n(x) \in I$
so that $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for $n = 2, 3, \ldots$.

A stationary state is any $x^* \in I$
with the property that $f^n(x^*) = x^*$ for all $n \in \mathbb{N}$.

This implies in particular that $x^* = x_{s+1} = f(x_s) = f(x^*)$,
so any stationary state must be a fixed point of $f$.

Conversely, it is obvious that any fixed point of $f$
must be a stationary state.

# Local Stability

The steady state $x^*$ is locally stable
just in case there is a neighbourhood $N$ of $x^*$
such that whenever $x \in N$ one has $f^n(x) \to x^*$ as $n \to \infty$.

## Theorem

*Let $x^*$ be an equilibrium state of $x_{t+1} = f(x_t)$.*

*Suppose that $x \mapsto f(x)$ is continuously differentiable in an open interval $I \subset \mathbb{R}$ that includes $x^*$.*

*Then*

1. $|f'(x^*)| < 1$ *implies that $x^*$ is locally stable;*
2. $|f'(x^*)| > 1$ *implies that $x^*$ is locally unstable.*

The proof below uses the fact that, by the mean value theorem,
if $x_t \in I$, then there exists a $c_t \in I$ between $x_t$ and $x^*$
such that $f'(c_t)$ is a mean value of $f'(x)$ in the sense that

$$x_{t+1} - x^* = f(x_t) - f(x^*) = f'(c_t)(x_t - x^*)$$

# Proof of Local Stability

By the hypotheses that $f'$ is continuous on $I$ and $|f'(x^*)| < 1$, there exist an $\epsilon > 0$ and a $k \in (0, 1)$ such that:

1. $(x^* - \epsilon, x^* + \epsilon) \subseteq I$;
2. $|f'(x)| \leq k$ for all $x \in (x^* - \epsilon, x^* + \epsilon)$.

Suppose that $|x_t - x^*| < \epsilon$.

Then the $c_t$ between $x_t$ and $x^*$
where $|x_{t+1} - x^*| = |f'(c_t)(x_t - x^*)|$ must satisfy $|c_t - x^*| < \epsilon$.

Hence $|f'(c_t)| \leq k$, implying that

$$|x_{t+1} - x^*| = |f'(c_t)(x_t - x^*)| \leq k|x_t - x^*| < k\epsilon < \epsilon$$

By induction on $t$, if $|x_0 - x^*| < \epsilon$, it follows that $|x_t - x^*| \leq \epsilon$ and in fact $|x_t - x^*| \leq k^t|x_0 - x^*|$ for $t = 1, 2, \ldots$.

Hence $|x_t - x^*| \to 0$ as $t \to \infty$. $\square$

# Proof of Local Instability

By the hypotheses that $f'$ is continuous on $I$ and $|f'(x^*)| > 1$, there exist an $\epsilon > 0$ and a $K > 1$ such that:

1. $(x^* - \epsilon, x^* + \epsilon) \subseteq I$;
2. $|f'(x)| \geq K$ for all $x \in (x^* - \epsilon, x^* + \epsilon)$.

Suppose that there exist $s, r \in \mathbb{N}$ such that $x_t \in I$ and $0 < |x_t - x^*| < \epsilon$ for all $t \in T_{s,r} := \{s, s+1, \ldots, s+r-1\}$, the set of $r$ successive times starting from time $s$.

Then for each $t \in T_{s,r}$, any $c_t \in I$ between $x_t$ and $x^*$ where $|x_{t+1} - x^*| = |f'(c_t)(x_t - x^*)|$ must satisfy $|c_t - x^*| < \epsilon$.

This implies that $|x_{t+1} - x^*| \geq K|x_t - x^*|$ for all $t \in T_{s,r}$.

By induction on $r$, it follows that $|x_{s+r} - x^*| \geq K^r|x_s - x^*|$.

So $|x_{s+r} - x^*| \geq K^r|x_s - x^*| > \epsilon$ for $r$ large enough.

This proves there is no $s \in \mathbb{N}$ such that $|x_t - x^*| < \epsilon$ for all $t \geq s$.

It follows that $x_t \not\to x^*$ as $t \to \infty$. $\square$