Lecture Notes 7: Dynamic Equations
Part D: Differential Equations

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Lecture Outline

First-Order Differential Equations

The typical first-order differential equation in one variable x is

$$\dot{x} = \frac{dx}{dt} = f(x, t)$$

The equation is autonomous just in case f is independent of t, so it can be written as $\dot{x} = f(x)$.

Typically one imposes an initial condition requiring $x(s) = \bar{x}_s$ at time s (not necessarily the earliest time).

Then any solution is a fixed function $t \mapsto x(t)$ that satisfies the corresponding integral equation $x(t) = \bar{x}_s + \int_s^t f(x(u), u)\, du$.

Picard's method of successive approximations starts with an arbitrary function $t \mapsto x^{(0)}(t)$ satisfying $x^{(0)}(s) = \bar{x}_s$.

Then it computes $x^{(n)}(t) = \bar{x}_s + \int_s^t f(x^{(n-1)}(u), u)\, du$ for $n \in \mathbb{N}$.

If convergence occurs, the limit as $n \to \infty$ will be a solution.

Right-Hand Side Independent of x

A special case occurs when the right-hand side $f(x, t)$ is independent of x.

Then the differential equation can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(t)$$

Its solution can be written as the indefinite integral

$$x(t) = \int g(t)\mathrm{d}t$$

Introducing an initial condition $x(s) = \bar{x}_s$
at a particular start time s
allows the solution to be written as the definite integral

$$x(t) = \bar{x}_s + \int_s^t g(\tau)\mathrm{d}\tau$$

CHECK that this alleged solution
satisfies $x(s) = \bar{x}_s$ and $\dot{x}(t) = g(t)$ for all $t \geq s$.

Leibniz's Rule for Differentiating an Integral

Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F(a, b, u) := \int_a^b f(t, u)\mathrm{d}t$$

Its three first-order partial derivatives are:

$$\text{(i) } F'_a = -f(a, u); \text{ (ii) } F'_b = f(b, u); \text{ (iii) } F'_u = \int_a^b \frac{\partial}{\partial u} f(t, u)\mathrm{d}t$$

Applying the chain rule, the total derivative
of the integral function $y \mapsto I(y) := \int_{a(y)}^{b(y)} f(t, y)\mathrm{d}t$ satisfies

$$\begin{aligned} I'(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F(a(y), b(y), y) = a'(y)F'_a + b'(y)F'_b + F'_u \\ &= b'(y)f(b(y), y) - a'(y)f(a(y), y) + \int_{a(y)}^{b(y)} \frac{\partial}{\partial y} f(t, y)\mathrm{d}t \end{aligned}$$

Lecture Outline

Picard's Method of Successive Approximations

The simplest first-order equation with constant coefficients takes the form

$$\dot{x}(t) = ax(t), \text{ with } x(0) \text{ given}$$

It corresponds to the integral equation

$$x(t) - x(0) = \int_0^t ax(u)\,\mathrm{d}u \text{ for all } t \geq 0$$

Starting with even a very crude approximation
such as the constant function $x^{(0)}(t) \equiv x(0)$ for all $t \geq 0$,
we can calculate a sequence $t \mapsto x^{(n)}(t)$ $(n \in N)$
of successive approximations to a solution $[0, \infty) \ni t \mapsto x(t) \in \mathbb{R}$
using, for all $t \geq 0$, the iterative rule

$$x^{(n)}(t) - x(0) = \int_0^t f(x^{(n-1)}(u), u)\,\mathrm{d}u = \int_0^t ax^{(n-1)}(u)\,\mathrm{d}u$$

Initial Three Iterations

Starting from $x^{(0)}(t) \equiv x(0)$, iterating once gives

$$x^{(1)}(t) - x(0) = \int_0^t a\, x^{(0)}(u)\, \mathrm{d}u = a\, x(0)\, t$$

Iterating a second time gives

$$x^{(2)}(t) - x(0) = \int_0^t a\, x(0)(1 + au)\, \mathrm{d}u = a\, x(0)\, t + \tfrac{1}{2} a^2\, x(0)\, t^2$$

Iterating a third time gives

$$\begin{aligned} x^{(3)}(t) - x(0) &= \textstyle\int_0^t [a\, x(0) + a^2\, x(0)\, u + \tfrac{1}{2} a^3\, x(0)\, u^2]\, \mathrm{d}u \\ &= a\, x(0)\, t + \tfrac{1}{2} a^2\, x(0)\, t^2 + \tfrac{1}{6} a^3\, x(0)\, t^3 \end{aligned}$$

Terms of the Sum

Each time we are adding one term to a sum.

So, starting with $y^{(0)}(t) \equiv x(0)$,
define the new incremental variable $y^{(n)}(t) := x^{(n)}(t) - x^{(n-1)}(t)$.
This implies that $x^{(n)}(t) = x(0) + \sum_{k=1}^{n} y^{(k)}(t)$.

Subtract $x^{(n)}(t) - x(0) = \int_0^t a x^{(n-1)}(u)\,\mathrm{d}u$
from $x^{(n+1)}(t) - x(0) = \int_0^t a x^{(n)}(u)\,\mathrm{d}u$
to obtain $y^{(n+1)}(t) = \int_0^t a y^{(n)}(u)\,\mathrm{d}u$.

Now we obtain successively

$$
\begin{aligned}
y^{(1)}(t) &= \int_0^t a\,x(0)\,\mathrm{d}u &&= a\,x(0)\,t \\
y^{(2)}(t) &= \int_0^t a^2\,x(0)\,u\,\mathrm{d}u &&= \tfrac{1}{2}a^2\,x(0)\,t^2 \\
y^{(3)}(t) &= \int_0^t \tfrac{1}{2}a^3\,x(0)\,u^2\,\mathrm{d}u &&= \tfrac{1}{6}a^3\,x(0)\,t^3
\end{aligned}
$$

This suggests the induction hypothesis $y^{(n)}(t) = \dfrac{1}{n!} a^n\,x(0)\,t^n$.

Constructing the Sum

The induction hypothesis $y^{(n)}(t) = \frac{1}{n!} a^n \, x(0) \, t^n$
and the relation $y^{(n+1)}(t) = \int_0^t a y^{(n)}(u) \, \mathrm{d}u$ together imply that

$$\begin{aligned} y^{(n+1)}(t) &= \int_0^t a \frac{1}{n!} a^n \, x(0) \, u^n \, \mathrm{d}u = \frac{1}{n!} a^{n+1} \, x(0) \int_0^t u^n \, \mathrm{d}u \\ &= \frac{1}{n!} a^{n+1} \, x(0) \, \frac{1}{n+1} t^{n+1} = \frac{1}{(n+1)!} a^{n+1} \, x(0) \, t^{n+1} \end{aligned}$$

This confirms the induction hypothesis with n replaced by $n+1$.

It follows that $y^{(n)}(t) = \frac{1}{n!} a^n \, x(0) \, t^n$ for all $n \in \mathbb{N}$
and then that $x^{(n)}(t) = x(0) + \sum_{k=1}^n \frac{1}{k!} a^k \, x(0) \, t^k$.

Euler's Number and the Exponential Function

Euler's number was invented by Jacob Bernoulli in 1683.

Euler chose to denote it by e.

Recall that it is given by

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.718281828$$

My late co-author Knut Sydsæter, as a cultured Norwegian, recognized 1828 as the year when their great playwright Henrik Ibsen was born.

So Knut remembered this 10 digit approximation as "2.7 Ibsen Ibsen".

Full (??) Decimal Expansions of Some Important Numbers

$$1/3 = 0.333333333333333333333333333\ldots$$

$$\sqrt{2} = 1.4142135623730950488016887242096980\ldots$$

$$e = 2.718281828459045235360287471352662497\ldots$$

$$\pi = 3.14159265358979323846264338327950288\ldots$$

The Exponential Function and Exponential Solution

The exponential function, which satisfies $\exp x = e^x$, satisfies

$$\exp x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = 1 + \sum\nolimits_{n=1}^{\infty} \frac{1}{n!} x^n = e^x$$

As $n \to \infty$, the Picard approximate solution $x^{(n)}(t)$
to the differential equation that we found earlier
converges to the infinite series

$$x(0) + \sum\nolimits_{k=1}^{\infty} \frac{1}{k!} a^k \, x(0) \, t^k = x(0) \exp(at) = x(0) \, e^{at}$$

Lecture Outline

General First-Order Affine Equation

The general first-order affine equation takes the form

$$\dot{x}(t) = a(t)x(t) + b(t)$$

for arbitrary integrable functions $t \mapsto a(t)$ and $t \mapsto b(t)$.

In the homogeneous case one has $b(t) \equiv 0$,
and the equation takes the linear form $\dot{x}(t) = a(t)x(t)$.

Assuming that $x > 0$ for all t,
we can take logs and write the equation as

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln x = \frac{\dot{x}}{x} = a(t)$$

After introducing the new variable $y(t) := \ln x(t)$,
the equation becomes $\dot{y} = a(t)$ whose solution is obviously

$$y(t) = y(s) + \int_s^t a(\tau)\mathrm{d}\tau$$

Solution in the Homogenenous Case

Because $x(t) = \exp y(t)$, the solution for x is

$$x(t) = \exp[y(t)] = \exp[y(s)] \exp\left[\int_s^t a(\tau)\mathrm{d}\tau\right] = x(s)\,\alpha_s(t)$$

where $\alpha_s(t)$ denotes the integrating factor $\exp\left[\int_s^t a(\tau)\mathrm{d}\tau\right]$.

In the special case of an autonomous equation
where $a(\tau) = a$ constant, one has $\int_s^t a(\tau)\mathrm{d}\tau = a(t-s)$
and so $\alpha_s(t) = e^{a(t-s)}$.

The Non-Homogenenous Case

The solution $x(t) = x(s)\alpha_s(t)$
to the homogeneous equation $\dot{x}(t) - a(t)x(t) = 0$
can be used to help solve the corresponding
non-homogeneous equation $\dot{x}(t) - a(t)x(t) = f(t)$.

Indeed, consider the result of dividing
each side of this non-homogeneous equation
by the integrating factor $\alpha_s(t) := \exp\left[\int_s^t a(\tau)\mathrm{d}\tau\right]$
whose reciprocal is $1/\alpha_s(t) := \exp\left[-\int_s^t a(\tau)\mathrm{d}\tau\right]$.

Note that $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[-\int_s^t a(\tau)\mathrm{d}\tau\right] = -a(t)$,
implying that $\dfrac{\mathrm{d}}{\mathrm{d}t}[1/\alpha_s(t)] = -a(t)/\alpha_s(t)$ so, by the product rule

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)/\alpha_s(t)] = [1/\alpha_s(t)]\dot{x}(t) - [a(t)/\alpha_s(t)]x(t) = f(t)/\alpha_s(t)$$

for any solution of the equation $\dot{x}(t) - a(t)x(t) = f(t)$.

Solving the Non-Homogenous Equation

Integrating each side of the equation $\frac{\mathrm{d}}{\mathrm{d}t}[x(t)/\alpha_s(t)] = f(t)/\alpha_s(t)$ over the interval from s to t gives us

$$\big|_s^t [x(u)/\alpha_s(u)] = \frac{x(t)}{\alpha_s(t)} - \frac{x(s)}{\alpha_s(s)} = \int_s^t \frac{f(u)}{\alpha_s(u)}\,\mathrm{d}u$$

The definition $\alpha_s(t) = \exp\left[\int_s^t a(\tau)\mathrm{d}\tau\right]$
implies that $\alpha_s(s) = 1$ and also $\alpha_s(t)/\alpha_s(u) = \alpha_u(t)$.

Hence, multiplying each side by $\alpha_s(t)$ gives the solution

$$\begin{aligned} x(t) &= \alpha_s(t)\left[x(s) + \textstyle\int_s^t [1/\alpha_s(u)]\,f(u)\,\mathrm{d}u\right] \\ &= \alpha_s(t)x(s) + \textstyle\int_s^t \alpha_u(t)\,f(u)\,\mathrm{d}u \\ &= \exp\left[\textstyle\int_s^t a(\tau)\mathrm{d}\tau\right]x(s) + \textstyle\int_s^t \exp\left[\textstyle\int_u^t a(\tau)\mathrm{d}\tau\right] f(u)\mathrm{d}u \end{aligned}$$

Linearity in the Forcing Term

Theorem

Suppose that $x^P(t)$ and $y^P(t)$ are particular solutions of the two respective differential equations

$$\dot{x}(t) - a(t)x(t) = d(t) \quad \textit{and} \quad \dot{y}(t) - a(t)y(t) = e(t)$$

Then, for any scalars α and β,
the equation $\dot{z}(t) - a(t)z(t) = f(t) = \alpha d(t) + \beta e(t)$
has as a particular solution
the corresponding linear combination $z^P(t) := \alpha x^P(t) + \beta y^P(t)$.

Consider any equation of the form $\dot{x}(t) - a(t)x(t) = f(t)$
where $f(t)$ is a linear combination $\sum_{k=1}^{n} \alpha_k f^k(t)$
of n forcing terms.

The theorem implies that a particular solution
is the corresponding linear combination $\sum_{k=1}^{n} \alpha_k x^{Pk}(t)$
of particular solutions to the n equations $\dot{x}(t) - a(t)x(t) = f^k(t)$.

Lecture Outline

First-Order Linear Equation with a Constant Coefficient

Next, consider the equation $\dot{x}(t) - ax(t) = f(t)$
where the coefficient a of $x(t)$ has become the constant $a \neq 0$.

The solution we found for the general case was

$$x(t) = \exp\left[\int_s^t a(\tau)\mathrm{d}\tau\right] x(s) + \int_s^t \exp\left[\int_u^t a(\tau)\mathrm{d}\tau\right] f(u)\mathrm{d}u$$

When $a(t) = a$, independent of t, this reduces to

$$x(t) = e^{a(t-s)}x(s) + \int_s^t e^{a(t-u)} f(u)\mathrm{d}u$$

We simplify further by choosing the initial time $s = 0$.

Then

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-u)} f(u)\mathrm{d}u$$

First Special Case

An interesting special case occurs
when the forcing term $f(t)$ is the exponential function $t \mapsto e^{\mu t}$.

Then the solution is

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-u)+\mu u} \mathrm{d}u = e^{at}\left[x(0) + \int_0^t e^{(\mu-a)u} \mathrm{d}u\right]$$

In the degenerate case when $\mu = a$,
one has $\int_0^t e^{(\mu-a)u} \mathrm{d}u = \int_0^t 1 \mathrm{d}u = t$, so the solution collapses to

$$x(t) = e^{at}\left[x(0) + t\right]$$

This solution can be written as $x(t) = x^H(t) + x^P(t)$ where:

1. $x^H(t) = \xi^H e^{at}$ with $\xi^H := x(0)$ is a complementary solution of the homogeneous equation $\dot{x}(t) - ax(t) = 0$;
2. $x^P(t) = \xi^P e^{at} t$ with $\xi^P := 1$ is a particular solution of the inhomogeneous equation $\dot{x}(t) - ax(t) = e^{at}$.

Non-Degenerate Case When $\mu \neq a$

In the non-degenerate case when $\mu \neq a$, one has

$$(\mu - a) \int_0^t e^{(\mu - a)u} \mathrm{d}u = |_0^t e^{(\mu - a)u} = e^{(\mu - a)t} - 1$$

So the solution is

$$x(t) = e^{at} \left[x(0) + \frac{e^{(\mu - a)t} - 1}{\mu - a} \right] = e^{at} x(0) + \frac{e^{\mu t} - e^{at}}{\mu - a}$$

Again, this solution can be written as $x(t) = x^H(t) + x^P(t)$ where:

1. $x^H(t) = \xi^H e^{at}$ with $\xi^H := x(0) - 1/(\mu - a)$ is a solution of the homogeneous equation $\dot{x}(t) - ax(t) = 0$;
2. $x^P(t) = \xi^P e^{\mu t}$ with $\xi^P := 1/(\mu - a)$ is a particular solution of the inhomogeneous equation $\dot{x}(t) - ax(t) = e^{\mu t}$.

Second Special Case

Another interesting special case occurs when $f(t) = t^r e^{\mu t}$ for some $r \in \mathbb{N}$.

Then the solution $x(t) = e^{at}x(0) + \int_0^t e^{a(t-u)} f(u)\mathrm{d}u$ becomes

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-u)} u^r e^{\mu u}\mathrm{d}u = e^{at}\left[x(0) + \int_0^t u^r e^{(\mu-a)u}\mathrm{d}u\right]$$

In the degenerate case when $\mu = a$, the solution collapses to

$$x(t) = e^{at}\left[x(0) + |_0^t (r+1)^{-1}u^{r+1}\right] = e^{at}\left[x(0) + (r+1)^{-1}t^{r+1}\right]$$

This solution can be written as $x(t) = x^H(t) + x^P(t)$ where:

1. $x^H(t) = \xi^H e^{at}$ with $\xi^H := x(0)$ is a solution of the homogeneous equation $\dot{x}(t) - ax(t) = 0$;
2. $x^P(t) = \xi^P e^{at} t^{r+1}$ with $\xi^P := (r+1)^{-1}$ is a particular solution of the inhomogeneous equation $\dot{x}(t) - ax(t) = t^r e^{at}$.

Non-Degenerate Case When $\mu \neq a$

In the non-degenerate case when $\mu \neq a$, the solution is

$$x(t) = e^{at}\left[x(0) + \int_0^t u^r e^{(\mu-a)u} \mathrm{d}u\right] = e^{at}\left[x(0) + I_r(t)\right]$$

where $I_r(t) := \int_0^t u^r e^{(\mu-a)u} \mathrm{d}u$.

In particular, $I_0(t) = \int_0^t e^{(\mu-a)u} \mathrm{d}u = (\mu - a)^{-1}[e^{(\mu-a)t} - 1]$.

Integrating by parts gives the first-order linear difference equation

$$\begin{aligned} I_r(t) &= \int_0^t u^r e^{(\mu-a)u} \mathrm{d}u \\ &= (\mu - a)^{-1}|_0^t u^r e^{(\mu-a)u} - r(\mu - a)^{-1} \int_0^t u^{r-1} e^{(\mu-a)u} \mathrm{d}u \\ &= (a - \mu)^{-1}\left[r\, I_{r-1}(t) - t^r e^{(\mu-a)t}\right] \end{aligned}$$

Solving the First-Order Linear Difference Equation

Let us divide each side of the difference equation

$$I_r(t) = (a-\mu)^{-1}\left[r\, I_{r-1}(t) - t^r e^{(\mu-a)t}\right]$$

by the "summing factor" $\prod_{k=1}^{r} k(a-\mu)^{-1} = r!(a-\mu)^{-r}$ to get

$$\begin{aligned} J_r(t) &:= \frac{1}{r!}(a-\mu)^r I_r(t) \\ &= \frac{1}{r!}\left[r(a-\mu)^{r-1} I_{r-1}(t) - (a-\mu)^{-1} t^r e^{(\mu-a)t}\right] \\ &= \frac{1}{(r-1)!}(a-\mu)^{r-1} I_{r-1}(t) - \frac{1}{r!}(a-\mu)^{r-1} t^r e^{(\mu-a)t} \\ &= J_{r-1}(t) - \frac{1}{r!}(a-\mu)^{r-1} t^r e^{(\mu-a)t} \end{aligned}$$

This obviously implies that

$$J_r(t) = J_0(t) - \sum\nolimits_{k=1}^{r} \frac{1}{k!}(a-\mu)^{k-1} t^k e^{(\mu-a)t}$$

Solving the Differential Equation

Because $J_0(t) = I_0(t) = (\mu - a)^{-1}[e^{(\mu-a)t} - 1]$, this implies that

$$J_r(t) = (\mu - a)^{-1}[e^{(\mu-a)t} - 1] - \sum_{k=1}^{r} \frac{1}{k!}(a-\mu)^{k-1} t^k e^{(\mu-a)t}$$

But $J_r(t) = \frac{1}{r!}(a-\mu)^r I_r(t)$, so

$$\begin{aligned} I_r(t) &:= r!(a-\mu)^{-r} J_r(t) \\ &= -r!(a-\mu)^{-r-1}[e^{(\mu-a)t} - 1] \\ &\quad - \sum_{k=1}^{r} \frac{r!}{k!}(a-\mu)^{k-r-1} t^k e^{(\mu-a)t} \end{aligned}$$

Then

$$\begin{aligned} x(t) &= e^{at}\left[x(0) + I_r(t)\right] \\ &= e^{at}\left[x(0) + r!(a-\mu)^{-r-1}\right] \\ &\quad - r!(a-\mu)^{-r-1} e^{\mu t}\left[1 + \sum_{k=1}^{r} \frac{1}{k!}(a-\mu)^k t^k\right] \end{aligned}$$

Particular and General Solution

For the equation $\dot{x}(t) - ax(t) = t^r e^{\mu t}$ with $\mu \neq a$, the solution

$$x(t) = e^{at}\left[x(0) + r!(a-\mu)^{-r-1}\right] \\ - r!(a-\mu)^{-r-1} e^{\mu t}\left[1 + \sum_{k=1}^{r} \frac{1}{k!}(a-\mu)^k t^k\right]$$

can be written as $x(t) = x^H(t) + x^P(t)$ where:

1. $x^H(t) = \xi^H e^{at}$ with $\xi^H := x(0) + r!(a-\mu)^{-r-1}$
   is a solution of the homogeneous equation $\dot{x}(t) - ax(t) = 0$;
2. $x^P(t) = \xi^P(t) e^{\mu t}$, where the polynomial

   $$t \mapsto \xi^P(t) := -r!(a-\mu)^{-r-1}\left[1 + \sum_{k=1}^{r} \frac{1}{k!}(a-\mu)^k t^k\right]$$

   of degree r in t is a particular solution
   of the inhomogeneous equation $\dot{x}(t) - ax(t) = t^r e^{\mu t}$.

Method of Undetermined Coefficients

A practical issue is finding what polynomial

$$t \mapsto \xi^P(t) = \sum_{k=0}^{r} \xi_k t^k$$

of degree r (the power of t on the right-hand side)
makes $\xi^P(t)e^{\mu t}$ a particular solution
of the inhomogeneous differential equation $\dot{x}(t) - ax(t) = t^r e^{\mu t}$.

The coefficients $(\xi_0, \xi_1, \ldots, \xi_r)$ of the polynomial
are undetermined
till we choose the associated polynomial $t \mapsto \xi^P(t)$
to make $\xi^P(t)e^{\mu t}$ satisfy the differential equation.

Determining the Undetermined Coefficients

For $x^P(t) = e^{\mu t} \sum_{k=0}^{r} \xi_k t^k$ to solve $\dot{x}(t) - ax(t) = t^r e^{\mu t}$, we need

$$\begin{aligned} t^r e^{\mu t} &= \mu e^{\mu t} \sum_{k=0}^{r} \xi_k t^k + e^{\mu t} \sum_{k=1}^{r} \xi_k k t^{k-1} - a e^{\mu t} \sum_{k=0}^{r} \xi_k t^k \\ &= (\mu - a) e^{\mu t} \xi_r t^r + e^{\mu t} \sum_{k=0}^{r-1} \left[(\mu - a)\xi_k + \xi_{k+1}(k+1) \right] t^k \end{aligned}$$

First consider the non-degenerate case $\mu \neq a$.

For $k = r$, this implies that $(\mu - a)\xi_r = 1$, so $\xi_r = (\mu - a)^{-1}$.

For $k = 0, 1, \ldots, r-1$, it implies that $(\mu - a)\xi_k + \xi_{k+1}(k+1) = 0$ or that $\xi_k = (a - \mu)^{-1}(k+1)\xi_{k+1}$, and so

$$\begin{aligned} \xi_k &= \left[\prod_{j=k}^{r-1} (a - \mu)^{-1}(j+1) \right] \xi_r \\ &= \frac{r!}{k!}(a - \mu)^{k-r} \xi_r \; = \; -\frac{r!}{k!}(a - \mu)^{k-r+1} \end{aligned}$$

This matches our previous answer.

Degenerate Case

In the degenerate case when $\mu = a$,
the method of undetermined coefficients
explained on the previous slides does not work.

Instead, to solve $\dot{x}(t) - ax(t) = t^r e^{at}$,
we introduce the new variable $y(t) = e^{-at}x(t)$.

Then $\dot{y}(t) = e^{-at}[\dot{x}(t) - ax(t)] = e^{-at}t^r e^{at} = t^r$.

The solution to this differential equation
is $y(t) = y(0) + \int_0^t u^r \mathrm{d}u = y(0) + (r+1)^{-1}t^{r+1}$.

The solution to the original differential equation
is therefore $x(t) = e^{at}y(t) = e^{at}\left[x(0) + (r+1)^{-1}t^{r+1}\right]$.

The polynomial in t that occurs in this solution
is now of degree $r + 1$ rather than r.

Main Theorem

Theorem

Consider the inhomogeneous first-order linear differential equation

$$\dot{x}(t) - ax(t) = t^r e^{\mu t}, \text{ where } a \neq 0 \text{ and } r \in \mathbb{Z}_+.$$

There exists a particular solution of the form $x^P(t) = Q(t)\, e^{\mu t}$

where the function $t \mapsto Q(t)$ *is a polynomial in* t *of degree:*

- r *in the regular case when* $\mu \neq a$*;*
- $r + 1$ *in the degenerate case when* $\mu = a$*.*

The general solution takes the form $x(t) = x^P(t) + x^C(t)$ *where:*

- $x^P(t)$ *is any particular solution;*
- $x^C(t)$ *is any member of the one-dimensional linear space of complementary solutions to the corresponding homogeneous equation* $\dot{x}(t) - ax(t) = 0$*.*

Lecture Outline

The Autonomous Case

The autonomous case occurs
when the first-order affine equation takes the form

$$\dot{x} = ax + b$$

with the right-hand side independent of t.

The steady state at which $\dot{x}(t) = 0$ occurs when $ax + b = 0$,
and so at $x^* := -b/a$.

Then the deviation $y(t) := x(t) - x^*$ of $x(t)$
from the steady state x^* satisfies the homogeneous equation

$$\dot{y}(t) = \dot{x}(t) = ax(t) + b = a[y(t) + x^*] + b = ay(t)$$

Hence $y(t) = e^{at} y(0)$, implying that $x(t) = x^* + e^{at}[x(0) - x^*]$.

Stability

The steady state $x^* := -b/a$ is stable just in case, for all $x(0)$,
the solution $x(t) = x^* + e^{at}[x(0) - x^*]$
satisfies $x(t) \to x^*$ as $t \to \infty$.

A necessary and sufficient condition for stability
is obviously that $a < 0$.

Lecture Outline

Second-Order Equations with Constant Coefficients

A general second-order differential equation takes the form

$$\ddot{x}(t) = F(\dot{x}(t), x(t), t)$$

To obtain a unique solution (if any solution exists), one typically needs two initial conditions such as $x(s) = x_s$ and $\dot{x}(s) = \dot{x}_s$ at an initial time s.

The equation is autonomous just in case it takes the form $\ddot{x}(t) = F(\dot{x}(t), x(t))$, with F independent of t.

The equation is linear just in case it takes the form $\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0$, with F linear in $(\dot{x}(t), x(t))$.

The equation is linear with constant coefficients just in case it takes the form $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$.

Characteristic Equation

We know that the first-order equation $\dot{x}(t) + ax(t) = 0$
has a solution of the form $x(t) = x(0)e^{\lambda t}$
where λ solves the characteristic equation $\lambda + a = 0$.

So we look for solutions of the form $x(t) = \xi e^{\lambda t}$
to the second-order equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$.

Note that when $x(t) = \xi e^{\lambda t}$,
then $\dot{x}(t) = \lambda \xi e^{\lambda t}$ and $\ddot{x}(t) = \lambda^2 \xi e^{\lambda t}$.

So $x(t) = \xi e^{\lambda t}$ is a non-trivial solution (with $\xi \neq 0$) if and only if

$$0 = \lambda^2 \xi e^{\lambda t} + a\lambda \xi e^{\lambda t} + b\xi e^{\lambda t} = (\lambda^2 + a\lambda + b)\xi e^{\lambda t}$$

and so, given that $\xi e^{\lambda t} \neq 0$, if and only if λ is a root
of the characteristic equation $\lambda^2 + a\lambda + b = 0$.

Characteristic Equation for an Equation of Order n

Definition

A homogeneous linear differential equation of order n
with constant coefficients takes the form

$$\sum_{k=0}^{n} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t) = 0$$

Choose n so that the coefficient of the n derivative satisfies $a_n \neq 0$,
and so can be normalized to take the value $a_n = 1$.

Remark

A similar technique based on roots of the characteristic equation applies to this nth order equation.

It implies that $x(t) = \xi e^{\lambda t}$ is a non-trivial solution if and only if λ is a root of the characteristic equation

$$\sum_{k=0}^{n} a_k \lambda^k = 0$$

Characteristic Roots of a Second-Order Equation

Consider the second-order equation $\ddot{x} + a\dot{x} + b = 0$.

One can factorize the quadratic function $q(\lambda) := \lambda^2 + a\lambda + b$
as $q(\lambda) := (\lambda - \lambda_1)(\lambda - \lambda_2)$
where λ_1 and λ_2 are the two roots of the equation $q(\lambda) = 0$.

As with the corresponding discussion
of second-order difference equations, there are three cases:

1. in case $a^2 > 4b$, there are two distinct real roots λ_1 and λ_2 given by $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.
2. in case $a^2 < 4b$, there are two complex conjugate roots given by $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}i\sqrt{4b - a^2}$.
3. in case $a^2 = 4b$, there are two coincident real roots given by $\lambda = -\frac{1}{2}a = \sqrt{b}$.

Case 1: Two Distinct Real Roots

In this case $a^2 > 4b$, when the two characteristic roots are $\lambda_{1,2} = -\frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.

Because $\lambda_1 \neq \lambda_2$, one has

$$\begin{vmatrix} e^{\lambda_1 0} & e^{\lambda_2 0} \\ e^{\lambda_1 1} & e^{\lambda_2 1} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} \end{vmatrix} = e^{\lambda_2} - e^{\lambda_1} \neq 0$$

and so $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two linearly independent solutions.

So in this case the homogeneous equation $\ddot{x} + a\dot{x} + b = 0$ has the general solution

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

Case 2: Two Complex Conjugate Roots, I

In case $a^2 < 4b$ the two characteristic roots are the complex conjugates $\lambda_{1,2} = -\frac{1}{2}a \pm i\theta$, with $\theta := \frac{1}{2}\sqrt{4b - a^2}$.

Then $x(t) = e^{\lambda_1 t} = e^{-\frac{1}{2}at} e^{i\theta t} = e^{-\frac{1}{2}at}(\cos\theta t + i \sin\theta t)$ and $x(t) = e^{\lambda_2 t} = e^{-\frac{1}{2}at} e^{-i\theta t} = e^{-\frac{1}{2}at}(\cos\theta t - i \sin\theta t)$ are two different solutions, where $\theta \neq 0$.

For any t such that $\sin\theta t \neq 0$, one has

$$\begin{vmatrix} e^{\lambda_1 0} & e^{\lambda_2 0} \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{vmatrix} = e^{\lambda_2 t} - e^{\lambda_1 t} = -2e^{-\frac{1}{2}at} i \sin\theta t \neq 0$$

It follows that $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two linearly independent solutions in the complex plane $\mathbb{C}$.

Case 2: Two Complex Conjugate Roots, II

Focusing on solutions in the real line $\mathbb{R}$,
we can consider $e^{-\frac{1}{2}at}\cos\theta t$ and $e^{-\frac{1}{2}at}\sin\theta t$.

Again, for any t such that $\sin\theta t \neq 0$, one has

$$\begin{aligned}\begin{vmatrix} e^{-\frac{1}{2}a0}\cos\theta 0 & e^{-\frac{1}{2}a0}\sin\theta 0 \\ e^{-\frac{1}{2}at}\cos\theta t & e^{-\frac{1}{2}at}\sin\theta t \end{vmatrix} &= \begin{vmatrix} 1 & 0 \\ -\frac{1}{2}at\cos\theta t & e^{-\frac{1}{2}at}\sin\theta t \end{vmatrix} \\ &= e^{-\frac{1}{2}at}\sin\theta t \neq 0\end{aligned}$$

It follows that $e^{-\frac{1}{2}at}\cos\theta t$ and $e^{-\frac{1}{2}at}\sin\theta t$ are two linearly independent real-valued solutions in the complex plane $\mathbb{C}$.

The general solution of the homogeneous equation
is $x = e^{-\frac{1}{2}at}(A\cos\theta t + B\sin\theta t)$.

Case 3: Two Coincident Real Roots

In this case $a^2 = 4b$, and so

$$q(\lambda) = \lambda^2 + a\lambda + b = (\lambda + \tfrac{1}{2}a)^2 = (\lambda - \sqrt{b})^2$$

The homogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$
has one solution given by $x = e^{\lambda t}$ where $\lambda = -\frac{1}{2}a = \sqrt{b}$.

To find a second linearly independent solution,
introduce the new variable $y(t) := e^{-\lambda t}x(t)$.

Then $\dot{y}(t) = e^{-\lambda t}\dot{x}(t) - \lambda e^{-\lambda t}x(t)$ and so, when $x = e^{\lambda t}$, one has

$$\begin{aligned}\ddot{y}(t) &= e^{-\lambda t}\ddot{x}(t) - 2\lambda e^{-\lambda t}\dot{x}(t) + \lambda^2 e^{-\lambda t}x(t) \\ &= e^{-\lambda t}[\ddot{x}(t) - 2\lambda\dot{x}(t) + \lambda^2 x(t)] \\ &= e^{-\lambda t}[\lambda^2 e^{\lambda t} - 2\lambda \cdot \lambda e^{\lambda t} + \lambda^2 e^{\lambda t}] = 0\end{aligned}$$

The obvious general solution to $\ddot{y}(t) = 0$
satisfies $\dot{y}(t) =$ constant and so $y(t) = A + Bt = e^{-\lambda t}x(t)$.

Hence $x(t) = (A + Bt)e^{\lambda t}$ is the general solution.

Lecture Outline

The Inhomogeneous Equation

Consider next the inhomogeneous equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t)$$

with a non-zero forcing term on the right-hand side.

Suppose that $y(t)$ and $z(t)$ are both solutions, implying that

$$\begin{array}{rrcl} & \ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) & = & f(t) \\ \text{and} & \ddot{z}(t) + a(t)\dot{z}(t) + b(t)z(t) & = & f(t) \end{array}$$

Subtracting the second equation from the first tells us that the function $x_H(t) := y(t) - z(t)$ is a solution of the corresponding homogenenous equation $\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0$.

So the general solution of $\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t)$ is the sum $x_G(t) = x_P(t) + x_H(t)$ of:

- any particular solution $x_P(t)$ of the inhomogeneous equation;
- any function $x_H(t)$ in the two dimensional linear space of solutions to the homogeneous equation.

Linearity in the Forcing Term, I

Theorem

Suppose that $x^P(t)$ and $y^P(t)$ are particular solutions of the two respective differential equations

$$\begin{array}{rrcl} & \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) & = & d(t) \\ \textit{and} & \ddot{y}(t) + a(t)\dot{y}(t) + b(t)y(t) & = & e(t) \end{array}$$

Then, for any scalars α and β, a particular solution of the equation

$$\ddot{z}(t) + a(t)\dot{z}(t) + b(t)z(t) = f(t) = \alpha d(t) + \beta e(t) \qquad (*)$$

is the linear combination $z^P(t) := \alpha x^P(t) + \beta y^P(t)$.

Proof.

Verify the claimed solution
by inserting the specified linear combination $z^P(t)$,
together with its first two derivatives $\dot{z}^P(t)$ and $\ddot{z}^P(t)$,
into the differential equation $(*)$. □

Linearity in the Forcing Term, II

Consider the equation $\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t)$ whose forcing term $f(t)$ is a linear combination $\sum_{k=1}^{n} \alpha_k f^k(t)$ of n forcing terms.

The theorem implies that a particular solution is the corresponding linear combination $\sum_{k=1}^{n} \alpha_k x^{Pk}(t)$ of particular solutions $x^{Pk}(t)$ to the respective n equations

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f^k(t) \quad (k = 1, 2, \ldots, n)$$

Lecture Outline

A Newtonian Example, I

Newton's law: force = mass × acceleration.

A force of 1 Newton, by definition, accelerates a mass of 1 kilogram at the rate of 1 metre per second per second.

So we consider the equation $\ddot{x}(t) = f(t)$ whose solution $t \mapsto x(t)$ is the position (in one dimension) of a 1 kilogram weight that has been subjected to a force function $t \mapsto f(t)$.

Integrating once gives us the equation $\dot{x}(t) = \dot{x}(0) + \int_0^t f(u) \mathrm{d}u$.

Integrating a second time gives us the solution

$$
\begin{aligned}
x(t) = x(0) + \int_0^t \dot{x}(v) \mathrm{d}v &= x(0) + \int_0^t \left[\dot{x}(0) + \int_0^v f(u) \mathrm{d}u \right] \mathrm{d}v \\
&= x(0) + \dot{x}(0)t + \int_0^t \left[\int_0^v f(u) \mathrm{d}u \right] \mathrm{d}v
\end{aligned}
$$

Note that $x(0) + \dot{x}(0)t$ solves the homogeneous equation $\ddot{x}(t) = 0$, whereas the iterated double integral $\int_0^t \left[\int_0^v f(u) \mathrm{d}u \right] \mathrm{d}v$ is a particular solution.

An Important Theorem on Iterated Double Integrals, I

Theorem

For any integrable function $(x, y) \mapsto \phi(x, y) \in \mathbb{R}$
defined on the square domain $[a, b] \times [a, b] \subset \mathbb{R}^2$*, one has*

$$\int_a^b \left[\int_a^y \phi(x, y) \mathrm{d}x \right] \mathrm{d}y = \int_a^b \left[\int_x^b \phi(x, y) \mathrm{d}y \right] \mathrm{d}x$$

Proof.

Define the indicator function $1_{x \le y}(x, y) := \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{if } x > y \end{cases}$. Then

$$\begin{aligned} \int_a^b \left[\int_a^y \phi(x, y) \mathrm{d}x \right] \mathrm{d}y &= \int_a^b \left[\int_a^b 1_{x \le y}(x, y) \phi(x, y) \mathrm{d}x \right] \mathrm{d}y \\ \int_a^b \left[\int_x^b \phi(x, y) \mathrm{d}y \right] \mathrm{d}x &= \int_a^b \left[\int_a^b 1_{x \le y}(x, y) \phi(x, y) \mathrm{d}y \right] \mathrm{d}x \end{aligned}$$

But both right-hand sides equal $\int_a^b \int_a^b 1_{x \le y}(x, y) \phi(x, y) \mathrm{d}x \mathrm{d}y$. □

An Important Theorem on Iterated Double Integrals, II

An alternative simple proof involves noticing that the two integrals

$$\int_a^b \left[\int_a^y \phi(x,y)\mathrm{d}x\right]\mathrm{d}y \quad \text{and} \quad \int_a^b \left[\int_x^b \phi(x,y)\mathrm{d}y\right]\mathrm{d}x$$

are simply two different ways of writing
the integral $\iint_T \phi(x,y)\mathrm{d}x\mathrm{d}y$ of the function ϕ of two variables
over the isosceles right-angled triangle

$$T := \{(x,y) \in [a,b] \times [a,b] \subset \mathbb{R}^2 \mid x \leq y\}$$

Note that T consists of points above and to the left
of the diagonal that joins the two corner points (a,a) and (b,b)
of the square $[a,b] \times [a,b]$.

The set T is also the convex hull
of the three points (a,a), (a,b) and (b,b).

A Newtonian Example, II

Reversing the order of integration allows the particular solution in the form of the iterated double integral $\int_0^t \left[\int_0^v f(u)\mathrm{d}u\right] \mathrm{d}v$ to be rewritten as

$$\int_0^t \left[\int_u^t f(u)\mathrm{d}v\right] \mathrm{d}u = \int_0^t \left[\int_u^t 1\mathrm{d}v\right] f(u)\mathrm{d}u = \int_0^t (t-u)f(u)\mathrm{d}u$$

Ultimately, then, one has

$$x(t) = x(0) + \dot{x}(0)t + \int_0^t (t-u)f(u)\mathrm{d}u$$

Linear Equation with Constant Coefficients, I

Next, consider the equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = f(t)$
where the coefficients a of $\dot{x}(t)$ and b of $\dot{x}(t)$
have both become constants, with $b \neq 0$.

Consider the quadratic function $q(\lambda) := \lambda^2 + a\lambda + b$
that appears in the characteristic equation $\lambda^2 + a\lambda + b = 0$.

One can factorize it as

$$q(\lambda) = \lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

where λ_1 and λ_2 are the two roots of the equation $q(\lambda) = 0$.

Recall that $\lambda_1 + \lambda_2 = -a$ and $\lambda_1\lambda_2 = b$.

Define the new variable $y(t) := \dot{x}(t) - \lambda_1 x(t)$.

Note that, if we could find the function $t \mapsto y(t)$,
then we would have

$$x(t) = e^{\lambda_1 t}x(0) + \int_0^t e^{\lambda_1(t-u)}y(u)\mathrm{d}u$$

Linear Equation with Constant Coefficients, II

We are considering the equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = f(t)$, with $b \neq 0$.

We have introduced the new variable $y(t) := \dot{x}(t) - \lambda_1 x(t)$, implying that $x(t) = e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} y(u) \mathrm{d}u$.

But the characteristic roots satisfy $\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2)$, implying that $\lambda_1 + \lambda_2 = -a$ and $\lambda_1 \lambda_2 = b$, and so

$$\begin{aligned} \dot{y}(t) - \lambda_2 y(t) &= \ddot{x}(t) - \lambda_1 \dot{x}(t) - \lambda_2 \dot{x}(t) + \lambda_1 \lambda_2 x(t) \\ &= \ddot{x}(t) + a\dot{x}(t) + bx(t) \end{aligned}$$

Hence $y(t)$ satisfies the first-order equation $\dot{y}(t) - \lambda_2 y(t) = f(t)$ whose solution is

$$y(t) = e^{\lambda_2 t} y(0) + \int_0^t e^{\lambda_2 (t-v)} f(v) \mathrm{d}v$$

Linear Equation with Constant Coefficients, III

Substituting $y(t) = e^{\lambda_2 t} y(0) + \int_0^t e^{\lambda_2 (t-v)} f(v) \mathrm{d}v$
in the expression $x(t) = e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} y(u) \mathrm{d}u$ gives

$$\begin{aligned} x(t) &= e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} y(u) \mathrm{d}u \\ &= e^{\lambda_1 t} x(0) + \int_0^t e^{\lambda_1 (t-u)} \left[e^{\lambda_2 u} y(0) + \int_0^u e^{\lambda_2 (u-v)} f(v) \mathrm{d}v \right] \mathrm{d}u \end{aligned}$$

We split this form of the solution into two parts:

1. the complementary solution

$$\begin{aligned} t \mapsto x^C(t) &:= e^{\lambda_1 t} x(0) + y(0) \int_0^t e^{\lambda_1 (t-u)} e^{\lambda_2 u} \mathrm{d}u \\ &= e^{\lambda_1 t} \left[x(0) + y(0) \int_0^t e^{(\lambda_2 - \lambda_1) u} \mathrm{d}u \right] \end{aligned}$$

   to the homogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$;

2. a particular solution in the form of the iterated double integral

$$t \mapsto x^P(t) := \int_0^t e^{\lambda_1 (t-u)} \left[\int_0^u e^{\lambda_2 (u-v)} f(v) \mathrm{d}v \right] \mathrm{d}u$$

   to the inhomogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = f(t)$.

Degenerate Case

In the degenerate case when $\lambda_1 = \lambda_2 = \lambda$,

1. the complementary solution takes the form:

$$\begin{aligned} x^C(t) &= e^{\lambda t} x(0) + y(0) \int_0^t e^{\lambda t} \mathrm{d}u \\ &= e^{\lambda t} [x(0) + y(0)t] \end{aligned}$$

2. the particular solution takes the form:

$$\begin{aligned} x^P(t) &= \int_0^t e^{\lambda(t-u)} \left[\int_0^u e^{\lambda(u-v)} f(v) \mathrm{d}v\right] \mathrm{d}u \\ &= e^{\lambda t} \int_0^t \left[\int_0^u e^{-\lambda v} f(v) \mathrm{d}v\right] \mathrm{d}u \\ &= e^{\lambda t} \int_0^t \left[\int_v^t 1 \mathrm{d}u\right] e^{-\lambda v} f(v) \mathrm{d}v \\ &= \int_0^t (t-v) e^{\lambda(t-v)} f(v) \mathrm{d}v \end{aligned}$$

The overall solution is therefore

$$x(t) = e^{\lambda t} \left[x(0) + y(0)t + \int_0^t (t-v) e^{-\lambda v} f(v) \mathrm{d}v\right]$$

Non-Degenerate Case: Complementary Solution

In the non-degenerate case when $\lambda_1 \neq \lambda_2$,
the complementary solution takes the form

$$\begin{aligned} x^C(t) &= e^{\lambda_1 t} \left[x(0) + y(0) \int_0^t e^{(\lambda_2 - \lambda_1)u} \mathrm{d}u \right] \\ &= e^{\lambda_1 t} x(0) + \frac{1}{\lambda_2 - \lambda_1} e^{\lambda_1 t} y(0) \left[e^{(\lambda_2 - \lambda_1)t} - 1 \right] \\ &= x(0) e^{\lambda_1 t} + y(0) \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \end{aligned}$$

After substituting $\dot{x}(0) - \lambda_1 x(0)$ for $y(0)$,
the right-hand side becomes

$$\frac{1}{\lambda_2 - \lambda_1} \left\{ (\lambda_2 - \lambda_1) x(0) e^{\lambda_1 t} + [\dot{x}(0) - \lambda_1 x(0)](e^{\lambda_2 t} - e^{\lambda_1 t}) \right\}$$

and so

$$x^C(t) = \frac{1}{\lambda_2 - \lambda_1} \left[x(0)(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}) + \dot{x}(0)(e^{\lambda_2 t} - e^{\lambda_1 t}) \right]$$

Non-Degenerate Case: Particular Solution

Using our rule for reversing the order of recursive integration, the particular solution takes the form

$$
\begin{aligned}
x^P(t) &= \int_0^t e^{\lambda_1(t-u)} \left[\int_0^u e^{\lambda_2(u-v)} f(v) \mathrm{d}v\right] \mathrm{d}u \\
&= \int_0^t \left[\int_v^t e^{\lambda_1(t-u)} e^{\lambda_2(u-v)} \mathrm{d}u\right] f(v) \mathrm{d}v \\
&= \int_0^t e^{\lambda_1 t - \lambda_2 v} \left[\int_v^t e^{(\lambda_2-\lambda_1)u} \mathrm{d}u\right] f(v) \mathrm{d}v \\
&= \frac{1}{\lambda_2-\lambda_1} \int_0^t e^{\lambda_1 t - \lambda_2 v} \left[e^{(\lambda_2-\lambda_1)t} - e^{(\lambda_2-\lambda_1)v}\right] f(v) \mathrm{d}v \\
&= \frac{1}{\lambda_2-\lambda_1} \int_0^t \left[e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)}\right] f(v) \mathrm{d}v
\end{aligned}
$$

First Special Case

An interesting first special case of the particular solution

$$x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)} \right] f(v) \mathrm{d}v$$

occurs when $f(t)$ is the exponential function $e^{\mu t}$, and so

$$x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[e^{\lambda_2(t-v)} - e^{\lambda_1(t-v)} \right] e^{\mu v} \mathrm{d}v$$

In the degenerate case when $\lambda_2 = \mu \neq \lambda_1$, this reduces to

$$\begin{aligned} x^P(t) &= \frac{1}{\lambda_2 - \lambda_1} \left[e^{\lambda_2 t} t - e^{\lambda_1 t} \int_0^t e^{(\mu - \lambda_1)v} \mathrm{d}v \right] \\ &= \frac{e^{\lambda_2 t} t}{\lambda_2 - \lambda_1} - \frac{e^{\lambda_1 t}(e^{(\lambda_2 - \lambda_1)t} - 1)}{(\lambda_2 - \lambda_1)^2} \\ &= \frac{e^{\lambda_2 t} t}{\lambda_2 - \lambda_1} - \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{(\lambda_2 - \lambda_1)^2} \end{aligned}$$

Non-Degenerate Case

In the non-degenerate case when λ_1, λ_2 and μ are all different, one has the particular solution

$$\begin{aligned} x^P(t) &= \frac{e^{\lambda_2 t}}{\lambda_2 - \lambda_1} \int_0^t e^{(\mu - \lambda_2)v} \mathrm{d}v - \frac{e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \int_0^t e^{(\mu - \lambda_1)v} \mathrm{d}v \\ &= \frac{e^{\lambda_2 t}\left[e^{(\mu - \lambda_2)t} - 1\right]}{(\lambda_2 - \lambda_1)(\mu - \lambda_2)} - \frac{e^{\lambda_1 t}\left[e^{(\mu - \lambda_1)t} - 1\right]}{(\lambda_2 - \lambda_1)(\mu - \lambda_1)} \\ &= \frac{1}{\lambda_2 - \lambda_1} \left(\frac{e^{\mu t} - e^{\lambda_2 t}}{\mu - \lambda_2} - \frac{e^{\mu t} - e^{\lambda_1 t}}{\mu - \lambda_1} \right) \end{aligned}$$

But the multiples of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ can be incorporated in the complementary solution to the homogeneous equation, so this particular solution can be reduced to

$$\tilde{x}^P(t) = \frac{e^{\mu t}}{\lambda_2 - \lambda_1} \left(\frac{1}{\mu - \lambda_2} - \frac{1}{\mu - \lambda_1} \right) = \frac{e^{\mu t}}{(\mu - \lambda_1)(\mu - \lambda_2)}$$

Second Special Case

An interesting second special case of the particular solution

$$x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[e^{\lambda_2 (t-v)} - e^{\lambda_1 (t-v)} \right] f(v) \mathrm{d}v$$

occurs when $f(t)$ is the exponential function $t^r e^{\mu t}$, and so

$$x^P(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t \left[e^{\lambda_2 (t-v)} - e^{\lambda_1 (t-v)} \right] v^r e^{\mu v} \mathrm{d}v$$

In the non-degenerate case when λ_1, λ_2 and μ are all different, this becomes

$$\begin{aligned} x^P(t) &= \frac{1}{\lambda_2 - \lambda_1} \left[e^{\lambda_2 t} \int_0^t v^r e^{(\mu - \lambda_2) v} \mathrm{d}v - e^{\lambda_1 t} \int_0^t v^r e^{(\mu - \lambda_1) v} \mathrm{d}v \right] \\ &= P_2(t) e^{\lambda_2 t} - P_1(t) e^{\lambda_1 t} \end{aligned}$$

for polynomials $t \mapsto P_1(t)$ and $t \mapsto P_2(t)$ of degree r whose coefficients are functions of the parameter triple $(\lambda_1, \lambda_2, \mu)$.

Lecture Outline

The Autonomous Equation

Now consider the autonomous equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = c$$

with a constant right-hand side.

There is a constant solution $x(t) = \bar{x}$
where $\bar{x} = c/b$ is the unique steady state.

The new variable $y(t) := x(t) - \bar{x}$ satisfies
the homogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$.

The associated characteristic equation is

$$\lambda^2 + a\lambda + b = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

A Stability Condition

1. In case there there are two real characteristic roots

$$\lambda_{1,2} = -\tfrac{1}{2}a \pm \tfrac{1}{2}\sqrt{a^2 - 4b}$$

   the general solution $Ae^{\lambda_1 t} + Be^{\lambda_2 t} \to 0$ as $t \to \infty$
   if and only if both λ_1 and λ_2 are negative.

2. In case there are two complex conjugate characteristic roots

$$\lambda_{1,2} = -\tfrac{1}{2}a \pm \tfrac{1}{2}i\sqrt{4b - a^2}$$

   one has $e^{\lambda t} = e^{-\frac{1}{2}at} e^{\pm\frac{1}{2}it\sqrt{4b-a^2}}$.
   The general solution $Ae^{\lambda_1 t} + Be^{\lambda_2 t} \to 0$ as $t \to \infty$
   iff $a > 0$, or iff both λ_1 and λ_2 have negative real parts.

3. In case there are two coincident real characteristic roots,
   the general solution $(A + Bt)e^{\lambda t} \to 0$ as $t \to \infty$ iff $\lambda < 0$.

All these conditions can be subsumed into one: stability holds if and only if each characteristic root has a negative real part.

Lecture Outline

Linear Differential Equation in n Variables

A linear differential equation in n variables specifies the time derivative $\dot{\mathbf{x}}(t)$ of the n-vector $\mathbf{x}(t)$ as an affine function $\mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)$ of $\mathbf{x}(t)$.

That is

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)$$

where

- $t \mapsto \mathbf{A}(t) \in \mathbb{R}^{n \times n}$ is a matrix-valued function of time;
- $t \mapsto \mathbf{b}(t) \in \mathbb{R}^n$ is a vector-valued function of time.

Matrix Differentiation

Consider the $m \times n$ matrix function $t \mapsto \mathbf{A}(t)$ whose elements $(a_{ij}(t))_{m \times n}$ are differentiable functions of t.

For all $h \neq 0$, the Newton quotient matrix $\frac{1}{h}[\mathbf{A}(t+h) - \mathbf{A}(t)]$ has elements equal to the Newton quotients $\frac{1}{h}(a_{ij}(t+h) - a_{ij}(t))_{m \times n}$ of the matrix $(a_{ij}(t))_{m \times n}$.

As $h \to 0$, these converge to the derivatives $\left(\frac{\mathrm{d}}{\mathrm{d}t} a_{ij}(t)\right)_{m \times n}$.

For this reason, the matrix $\mathbf{A}(t)$ is said to be differentiable with derivative $\dot{\mathbf{A}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(t)$ whose elements are $(\frac{\mathrm{d}}{\mathrm{d}t} a_{ij}(t))_{m \times n}$.

Differentiating the Product of Matrices

Suppose that $t \mapsto \mathbf{A}(t)$ and $t \mapsto \mathbf{B}(t)$ are differentiable, where each $\mathbf{A}(t)$ is $\ell \times m$ and each $\mathbf{B}(t)$ is $m \times n$.

Then $t \mapsto \mathbf{C}(t) = \mathbf{A}(t)\,\mathbf{B}(t)$ is well defined as a matrix product with elements given by $c_{ik}(t) = \sum_{j=1}^{m} a_{ij}(t) b_{jk}(t)$ whose time derivatives are

$$\dot{c}_{ik}(t) = \sum\nolimits_{j=1}^{m} [\dot{a}_{ij}(t) b_{jk}(t) + a_{ij}(t) \dot{b}_{jk}(t)]$$

Hence $t \mapsto \mathbf{C}(t)$ is differentiable,

with $\dot{\mathbf{C}}(t) = \dot{\mathbf{A}}(t)\mathbf{B}(t) + \mathbf{A}(t)\dot{\mathbf{B}}(t)$.

Differentiating the Square of a Square Matrix

Suppose that $\mathbf{A}(t)$ is an $n \times n$ matrix for all t,
and that each element is a differentiable function of t.

Then the square matrix $\mathbf{A}^2(t)$ is well defined and differentiable,
with derivative $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^2(t) = \dot{\mathbf{A}}(t)\mathbf{A}(t) + \mathbf{A}(t)\dot{\mathbf{A}}(t)$.

Unless the matrices $\dot{\mathbf{A}}(t)$ and $\mathbf{A}(t)$ happen to commute,
in the sense that $\dot{\mathbf{A}}(t)\mathbf{A}(t) = \mathbf{A}(t)\dot{\mathbf{A}}(t)$,
this will not be equal to $2\dot{\mathbf{A}}(t)\mathbf{A}(t)$ or to $2\mathbf{A}(t)\dot{\mathbf{A}}(t)$.

Example

Note that, even if each $\mathbf{A}(t)$ is square,
it may not commute with $\dot{\mathbf{A}}(t)$.

For example, when $\mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix}$, then $\dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,
implying that $\mathbf{A}(t)\,\dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \neq \dot{\mathbf{A}}(t)\,\mathbf{A}(t) = \begin{pmatrix} 0 & 0 \\ 1 & t \end{pmatrix}$.

Note that in this example $\mathbf{A}$ is symmetric; so therefore is $\dot{\mathbf{A}}$.

Hence $\mathbf{A}(t)\,\dot{\mathbf{A}}(t) = \mathbf{A}(t)^\top\,\dot{\mathbf{A}}^\top(t) = [\dot{\mathbf{A}}(t)\mathbf{A}(t)]^\top$.

Also $\mathbf{A}^2(t) = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix}$ whose derivative satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^2(t) = \dot{\mathbf{A}}(t)\mathbf{A}(t) + \mathbf{A}(t)\,\dot{\mathbf{A}}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 2t \end{pmatrix}$$

This differs from both $2\,\mathbf{A}(t)\,\dot{\mathbf{A}}(t)$ and $2\,\dot{\mathbf{A}}(t)\,\mathbf{A}(t)$.

The Exponential of a Square Matrix

Recall that the exponential function of a scalar is defined so that the solution of the differential equation $\dot{x} = ax$ is $x(t) = e^{at}x(0)$.

Similarly, we define the exponential function of a square matrix so that the solution of the differential equation system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is $\mathbf{x}(t) = \exp(\mathbf{A}t)\,\mathbf{x}(0)$.

The function $t \mapsto \exp(\mathbf{A}t)$ is often called the resolvent.

Recall that, for a scalar, there is the convergent power series

$$e^{at} = 1 + \frac{1}{1!}at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 \ldots = \sum\nolimits_{r=0}^{\infty} \frac{1}{r!}(at)^r$$

with the convention that $0! = 1$.

Similarly, for a square matrix, with the convention that $(\mathbf{A}t)^0 = \mathbf{I}$ one can use a convergent power series to give,

$$\exp(\mathbf{A}t) = \mathbf{I} + \frac{1}{1!}\mathbf{A}t + \frac{1}{2!}(\mathbf{A}t)^2 + \frac{1}{3!}(\mathbf{A}t)^3 \ldots = \sum\nolimits_{r=0}^{\infty} \frac{1}{r!}(\mathbf{A}t)^r$$

The Exponential of a Diagonal Matrix

Dropping the time argument, it follows that we define

$$\exp(\mathbf{C}) := \mathbf{I} + \frac{1}{1!}\mathbf{C} + \frac{1}{2!}(\mathbf{C})^2 + \frac{1}{3!}(\mathbf{C})^3 \ldots = \sum_{r=0}^{\infty} \frac{1}{r!}(\mathbf{C})^r$$

Suppose that $\mathbf{C}$ is the diagonal matrix $\mathbf{diag}(c_1, c_2, \ldots, c_n) = \mathbf{diag}\,\mathbf{c}$ where $\mathbf{c}$ is the vector $(c_1, c_2, \ldots, c_n)$.

Now, each matrix power $(\mathbf{diag}\,\mathbf{c})^r = \mathbf{diag}(c_1^r, c_2^r, \ldots, c_n^r)$ as is readily proved by induction on r.

So, with this notation for the exponential of a matrix, we have

$$\begin{aligned} \exp(\mathbf{C}) &= \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{C}^r = \sum_{r=0}^{\infty} \frac{1}{r!}\,\mathbf{diag}(c_1^r, c_2^r, \ldots, c_n^r) \\ &= \mathbf{diag}\left(e^{c_1}, e^{c_2}, \ldots, e^{c_n}\right) \end{aligned}$$

Also, suppose matrix $\mathbf{C}$ has $\mathbf{C} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ as a diagonalization.

Then each matrix power $\mathbf{C}^r = \mathbf{V}\mathbf{\Lambda}^r\mathbf{V}^{-1}$ implying that $\exp(\mathbf{C}) = \mathbf{V}\exp(\mathbf{\Lambda})\mathbf{V}^{-1}$.

Integrating and Differentiating an Exponential Matrix

From the definition $\exp(\mathbf{A}s) = \sum_{r=0}^{\infty} \frac{1}{r!}(\mathbf{A}s)^r$,
either post- or premultiplying by $\mathbf{A}$ and then integrating gives

$$\int_0^t \exp(\mathbf{A}s)\ \mathbf{A}\,\mathrm{d}s = \int_0^t \mathbf{A}\exp(\mathbf{A}s)\,\mathrm{d}s = \int_0^t \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^{r+1}\,s^r\,\mathrm{d}s$$

Next, integrating term by term, the last expression becomes

$$\sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^{r+1} \int_0^t s^r \mathrm{d}s = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^{r+1} \quad \cdot \quad \Big|_0^t \frac{1}{r+1} s^{r+1}$$

Simplifying converts this to

$$\sum_{r=0}^{\infty} \frac{1}{(r+1)!}\mathbf{A}^{r+1} t^{r+1} = \sum_{r=1}^{\infty} \frac{1}{r!}\mathbf{A}^r t^r = \exp(\mathbf{A}t) - \mathbf{I}$$

So $\int_0^t \exp(\mathbf{A}s)\,\mathbf{A}\,\mathrm{d}s = \int_0^t \mathbf{A}\ \exp(\mathbf{A}s)\,\mathrm{d}s = \exp(\mathbf{A}t) - \mathbf{I}$,
implying that

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(\mathbf{A}t) = \mathbf{A}\exp(\mathbf{A}t) = \exp(\mathbf{A}t)\,\mathbf{A}$$

Affine Equation in n Variables

Consider what happens when we multiply each side of the non-homogeneous affine equation $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}(t)$ by the matrix integrating factor $\exp(-\mathbf{A}t)$.

Because the product rule of differentiation applies to matrices,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[\exp(-\mathbf{A}t)\,\mathbf{x}(t)] &= \exp(-\mathbf{A}t)\,\dot{\mathbf{x}}(t) + \frac{\mathrm{d}}{\mathrm{d}t}[\exp(-\mathbf{A}t)]\;\mathbf{x}(t) \\ &= \exp(-\mathbf{A}t)\,\dot{\mathbf{x}}(t) - \exp(-\mathbf{A}t)\,\mathbf{A}\,\mathbf{x}(t) \\ &= \exp(-\mathbf{A}t)\,\mathbf{b}(t) \end{aligned}$$

if and only if $\mathbf{x}(t)$ solves the equation $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}(t)$.

Hence $\exp(-\mathbf{A}t)\,\mathbf{x}(t) - \exp(-\mathbf{A}s)\,\mathbf{x}(s) = \int_s^t \exp(-\mathbf{A}\tau)\,\mathbf{b}(\tau)\,\mathrm{d}\tau$.

Multiplying each side by $\exp(\mathbf{A}t)$ gives the unique solution

$$\mathbf{x}(t) = \exp[\mathbf{A}(t-s)]\;\mathbf{x}(s) + \int_s^t \exp[\mathbf{A}(t-\tau)]\;\mathbf{b}(\tau)\,\mathrm{d}\tau$$

The Diagonal Case

The diagonal case occurs when $\mathbf{A} = \mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$.

Then the system $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}(t)$ of n coupled equations reduces to the system of n uncoupled equations

$$\dot{x}_i(t) = a_{ii} x_i(t) + b_i(t) = \lambda_i x_i(t) + b_i(t) \ (i = 1, \ldots, n)$$

one in each variable x_i, with respective solutions

$$x_i(t) = e^{\lambda_i t} \, x_i(s) + \int_s^t e^{\lambda_i (t-\tau)} \, b_i(\tau) \, \mathrm{d}\tau$$

The Diagonalizable Case

Suppose that $\mathbf{A}$ has n distinct eigenvalues
— or if not, then n linearly independent eigenvectors
that make up the columns of the matrix $\mathbf{V}$.

Then $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ and $\mathbf{A}t = \mathbf{V}(\mathbf{\Lambda}t)\mathbf{V}^{-1}$
implying that $\exp(\mathbf{A}t) = \mathbf{V}\exp(\mathbf{\Lambda}t)\mathbf{V}^{-1}$.

Hence the solution

$$\mathbf{x}(t) = \exp[\mathbf{A}(t-s)]\ \mathbf{x}(s) + \int_s^t \exp[\mathbf{A}(t-\tau)]\ \mathbf{b}(\tau)\,\mathrm{d}\tau$$

simplifies to

$$\mathbf{x}(t) = \mathbf{V}\exp[\mathbf{\Lambda}(t-s)]\mathbf{V}^{-1}\,\mathbf{x}(s) + \int_s^t \mathbf{V}\exp[\mathbf{\Lambda}(t-\tau)]\mathbf{V}^{-1}\,\mathbf{b}(\tau)\,\mathrm{d}\tau$$

Of course, the transformation $\mathbf{y}(t) := \mathbf{V}^{-1}\,\mathbf{x}(t)$ takes us back
to the diagonal case, with $\dot{\mathbf{y}}(t) := \mathbf{\Lambda}\,\mathbf{y}(t) + \mathbf{V}^{-1}\,\mathbf{b}(t)$.

A Stability Condition

When $\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$,
one has $\exp(\mathbf{\Lambda}) = \mathbf{diag}(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n})$.

Furthermore $\exp(\mathbf{\Lambda} t) = \mathbf{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})$.

This converges to the zero matrix as $t \to \infty$
if and only if each $e^{\lambda_i t} \to 0$,
which is true iff each eigenvalue λ_i has a negative real part.

Similarly, if $\mathbf{A}$ is diagonalizable, with $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$,
then consider the new variables $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$.

The differential equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ becomes transformed to

$$\dot{\mathbf{y}}(t) = \mathbf{V}^{-1}\dot{\mathbf{x}}(t) = \mathbf{V}^{-1}\mathbf{A}\mathbf{x}(t) = \mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{x}(t) = \mathbf{\Lambda}\mathbf{y}(t)$$

Because $\mathbf{V}$ is invertible, one has $\mathbf{x}(t) \to \mathbf{0} \iff \mathbf{y}(t) \to \mathbf{0}$.

Once again, stability holds
iff each eigenvalue of the matrix $\mathbf{A}$ has a negative real part.

This is true even when $\mathbf{A}$ is not diagonalizable.

The Schrödinger Equation in $\mathbb{C}^n$

A wave function is a mapping $\mathbb{R} \ni t \mapsto \psi(t) \in \mathbb{C}^n$.

Schrödinger's wave equation is a linear equation that, in a simple case, can be written in the form $\dot{\psi}(t) = -\mathrm{i}\,\mathbf{H}\psi(t)$ where $\mathbf{H}$ is a Hamiltonian "energy" matrix with complex elements that is self-adjoint.

Because $\mathbf{H}$ is self-adjoint, it can be diagonalized so that, after a change of variables,
one has $\dot{\psi}(t) = -\mathrm{i}\,\mathbf{diag}(h_1, \ldots, h_n)\,\psi(t)$
and so $\dot{\psi}_k(t) = -\mathrm{i}\,h_k \psi_k(t)$ for each $k \in \mathbb{N}_n$.

For each possible initial value $\psi(0) \in \mathbb{C}^n$, and for each $k \in \mathbb{N}_n$, the unique solution is

$$\psi_k(t) = \psi_k(0) e^{-\mathrm{i}\,h_k t} = \psi_k(0)[\cos(-h_k t) + \mathrm{i} \sin(-h_k t)]$$

This is a wave or oscillatory solution with frequency h_k.

Generally, the eigenvalues in the spectrum of $\mathbf{H}$, which are all real, are possible frequencies of oscillatory solutions.

Lecture Outline

Autonomous First-Order Equations

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a general function
that may be non-linear.

Consider the autonomous differential equation $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$.

A solution satisfying the initial condition $\mathbf{x}(s) = \bar{x}$
is a differentiable function $[s, t) \ni t \mapsto \mathbf{x}(t)$
that satisfies $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t))$ for almost all $t \geq s$.

Equivalently, for almost all $t \geq s$, one must have

$$\mathbf{x}(t) = \mathbf{x}(s) + \int_s^t \mathbf{F}(\mathbf{x}(\tau))\,\mathrm{d}\tau$$

Stationary States and Rest Points

A stationary state is a point $\mathbf{x}^* \in \mathbb{R}^n$ with the property that if $\mathbf{x}(s) = \mathbf{x}^*$ at any time s, then $\mathbf{x}(t) = \mathbf{x}^*$ at all times $t \geq s$.

A rest point is a state $\bar{\mathbf{x}} \in \mathbb{R}^n$ with the property that $\mathbf{F}(\bar{\mathbf{x}}) = \mathbf{0}$.

Theorem

Any rest point is a stationary state, and conversely.

Proof.

If $\mathbf{F}(\bar{\mathbf{x}}) = \mathbf{0}$, then the solution of $\mathbf{x}(t) = \mathbf{x}(s) + \int_s^t \mathbf{F}(\mathbf{x}(\tau))\, \mathrm{d}\tau$ with $\mathbf{x}(s) = \bar{\mathbf{x}}$ satisfies $\mathbf{x}(t) = \mathbf{x}(s) = \bar{\mathbf{x}}$ for all $t \geq s$.

Conversely, if that solution satisfies $\mathbf{x}(t) = \mathbf{x}(s) = \mathbf{x}^*$ for all $t \geq s$, then $\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) = \mathbf{F}(\mathbf{x}^*) = \mathbf{0}$ for all $t \geq s$. □

Local Stability of a Stationary State

Let $\mathbf{F}'(\mathbf{x})$ denote the $n \times n$ Jacobian matrix
whose elements are the partial derivatives $\mathbf{F}'_{ij}(\mathbf{x}) = \frac{\partial}{\partial x_j} F_i(\mathbf{x})$
of the different components $(F_i(\mathbf{x}))_{i=1}^n$.

Any particular steady state $\mathbf{x}^*$ is locally asymptotically stable
if and only if all the eigenvalues of $\mathbf{F}'(\mathbf{x}^*)$ have negative real parts.

A System with Two Variables

Consider the coupled pair $\dot{x} = f(x, y), \; \dot{y} = g(x, y)$
of differential equations.

Let (a, b) be any stationary point
satisfying both $f(a, b) = 0$ and $g(a, b) = 0$.

The Jacobian matrix at the stationary point takes the form

$$\mathbf{J}(a, b) = \begin{pmatrix} \dfrac{\partial f}{\partial x}(a, b) & \dfrac{\partial f}{\partial y}(a, b) \\ \dfrac{\partial g}{\partial x}(a, b) & \dfrac{\partial g}{\partial y}(a, b) \end{pmatrix}$$

Local Saddle Point with Two Variables

The product of the two eigenvalues λ_1, λ_2 of $\mathbf{J}(a, b)$ equals its determinant $|\mathbf{J}(a, b)|$.

The two eigenvalues are real and have opposite signs if and only if $|\mathbf{J}(a, b)| < 0$.

This is a sufficient condition for the steady state to be unstable.

But if $(x(0) - a, y(0) - b)^\top$ is an eigenvector corresponding to the negative eigenvalue, then in the case when the equations are linear and so $\mathbf{J}$ is constant, the solution will converge to the steady state.

This is saddle point stability.

The Lotka–Volterra Predator–Prey Model

Foxes are predators; their prey includes rabbits.

Let x denote the expected population of rabbits,
and y denote expected population of foxes.

Assume these populations are linked by the differential equations

$$\begin{aligned} \dot{x} &= x(k - ay) \\ \dot{y} &= y(bx - h) \end{aligned}$$

where a, b, h, k are all positive parameters.

Thus:

1. the rabbit population growth rate $\frac{d}{dt} \ln x = \dot{x}/x$ is a decreasing affine function of the fox population;
2. whereas the fox population growth rate $\frac{d}{dt} \ln y = \dot{y}/y$ is an increasing affine function of the rabbit population.

Lotka–Volterra: Phase Plane Analysis

Given the system $\dot{x} = x(k - ay)$ and $\dot{y} = y(bx - h)$,
the two nullclines where $\dot{x} = 0$ and $\dot{y} = 0$
are given by $y = k/a$ and $x = h/b$ respectively.

So the steady state is at $(x, y) = (h/b, k/a)$.

The Jacobian matrix is $\mathbf{J}(x, y)$ is

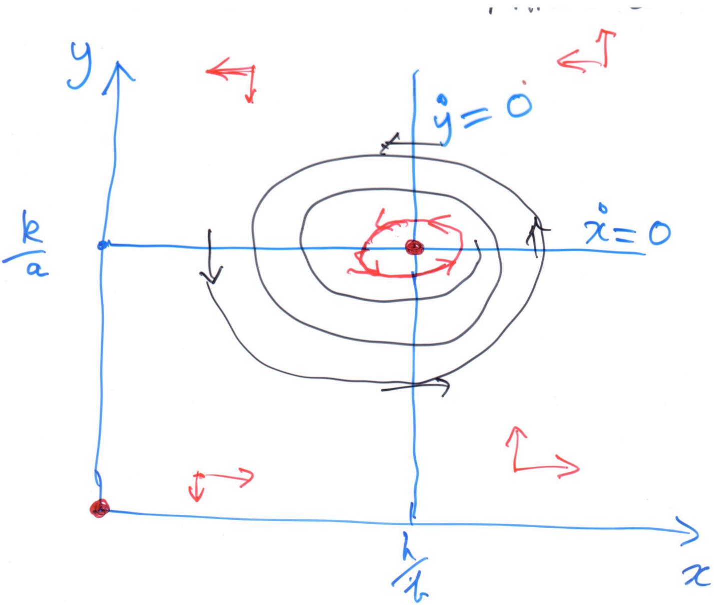
$$\begin{pmatrix} \dfrac{\partial \dot{x}}{\partial x} & \dfrac{\partial \dot{x}}{\partial y} \\[2ex] \dfrac{\partial \dot{y}}{\partial x} & \dfrac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} k - ay & -ax \\ yb & bx - h \end{pmatrix}$$

It reduces to $\begin{pmatrix} 0 & -ah/b \\ bk/a & 0 \end{pmatrix}$ at the steady state $(h/b, k/a)$.

The characteristic equation is $\lambda^2 + hk = 0$,
whose roots are $\pm \mathrm{i}\sqrt{hk}$.

As the following diagram suggests, there can be limit cycles
with $x(t) = \xi \cos \sqrt{hk}t$ and $y(t) = \eta \sin \sqrt{hk}t$.

Lotka–Volterra: Phase Plane Diagram



Saddle Point Example

Consider a macro model where: (i) K denotes capital; (ii) Y denotes output; and (iii) C denotes consumption.

Suppose that net investment $\dot{K} = Y - C$, that $Y = aK - bK^2$, and $\dot{C} = w(a - 2bk)C$, where a, b, k, w are positive constants.

This gives the coupled system with

$$\dot{K} = aK - bK^2 - C \text{ and } \dot{C} = w(a - 2bK)C$$

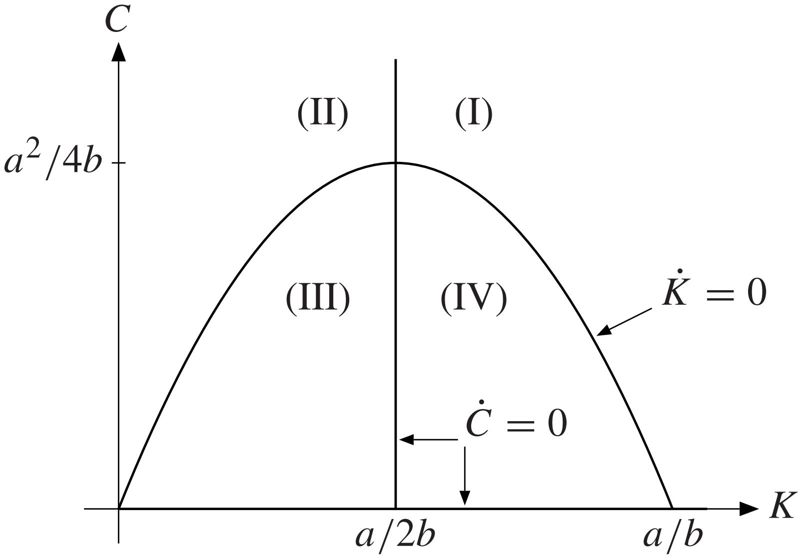
The two nullclines are $C = aK - bK^2$ and $K = a/2b$.

These intersect at the stationary point $(K^*, C^*) = (a/2b, a^2/4b)$. The Jacobian matrix is

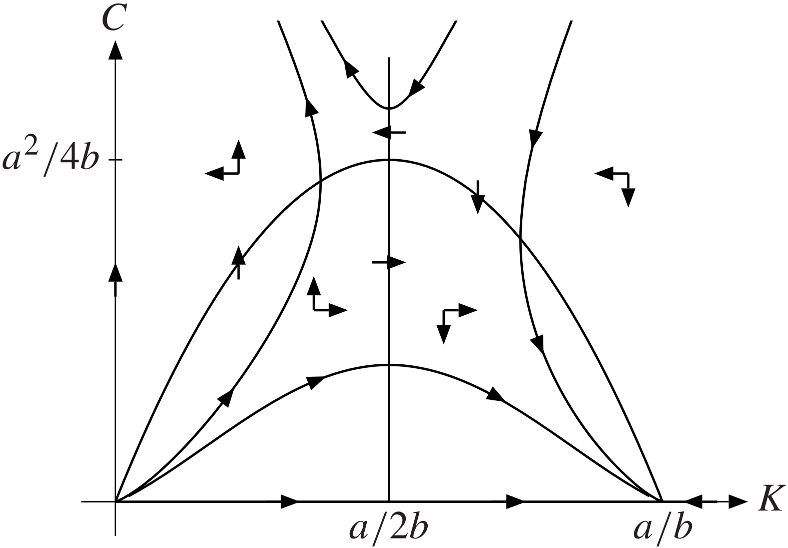
$$\mathbf{J}(K, C) = \begin{pmatrix} \dfrac{\partial \dot{K}}{\partial K} & \dfrac{\partial \dot{K}}{\partial C} \\ \dfrac{\partial \dot{C}}{\partial K} & \dfrac{\partial \dot{C}}{\partial C} \end{pmatrix} = \begin{pmatrix} a - 2bK & -1 \\ -2wbC & w(a - 2bK) \end{pmatrix}$$

This reduces to $\begin{pmatrix} 0 & -1 \\ -\frac{1}{2}a^2 w & 0 \end{pmatrix}$ at the steady state.

Phase Diagram I



Phase Diagram II



Stability Analysis

The Jacobian matrix at the steady state is $\begin{pmatrix} 0 & -1 \\ -\frac{1}{2}a^2 w & 0 \end{pmatrix}$.

This matrix has trace 0 and negative determinant $-\frac{1}{2}a^2 w$.

So the two eigenvalues have sum 0 and product $-\frac{1}{2}a^2 w$.

It follows that the eigenvalues are $\pm\lambda$ where $\lambda^2 = \frac{1}{2}a^2 w$ and so $\lambda = a\sqrt{w/2}$.

The general solution near the steady state takes the form

$$\begin{pmatrix} K - K^* \\ C - C^* \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\lambda t} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} e^{-\lambda t}$$

for arbitrary constant vectors $(A_1, A_2)^\top$ and $(B_1, B_2)^\top$.

This converges to the steady state at $(K^*, C^*) = (a/2b, a^2/4b)$ if and only if $A_1 = A_2 = 0$.

It follows that the steady state is a saddle point.

Existence and Uniqueness Theorem, I

Note: In the following,
we use ordinary Roman rather than bold letters
for vectors in the finite-dimensional space $\mathbb{R}^d$.

Extract from pp. 355–356 in ch. 6 of David Applebaum (2009)
Lévy Processses and Stochastic Calculus, 2nd edn. (Cambridge)

Let $b : \mathbb{R}^d \to \mathbb{R}^d$, so that $b = (b^1, \ldots, b^d)$
where $b^i : \mathbb{R}^d \to \mathbb{R}$ for $1 \leq i \leq d$.

We study the initial value problem posed
by the vector-valued differential equation $\frac{\mathrm{d}}{\mathrm{d}\,t} c(t) = b(c(t))$
with fixed initial condition $c(0) = c_0 \in \mathbb{R}^d$,
whose solution, if it exists, is a curve $(c(t), t \in \mathbb{R})$ in $\mathbb{R}^d$.

Existence and Uniqueness Theorem, II

We say that b is (globally) Lipschitz if there exists $K > 0$ such that, for all $x, y \in \mathbb{R}^d$, $\|b(x) - b(y)\| \leq K\|x - y\|$.

Exercise 6.1.1 Show that if b is differentiable with bounded partial derivatives then it is Lipschitz.

Exercise 6.1.2 Deduce that if b is Lipschitz then it satisfies a linear growth condition $\|b(x)\| \leq L(1 + \|x\|)$ for all $x \in \mathbb{R}^d$, where $L = \max\{K, \|b(0)\|\}$.

Theorem 6.1.3 If $b : \mathbb{R}^d \to \mathbb{R}^d$ is (globally) Lipschitz, then there exists a unique solution $c : \mathbb{R} \to \mathbb{R}^d$ of the initial value problem.

The proof offered by Applebaum does not use a contraction mapping theorem.

Rather, it bounds possible solutions within error bands that are exponential functions that converge to zero.