

Lecture Notes 7: Dynamic Equations Part A: First-Order Difference Equations in One Variable

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Lecture Outline

Introduction: Difference vs. Differential Equations

First-Order Difference Equations

First-Order Linear Difference Equations: Introduction

General First-Order Linear Equation

Particular, General, and Complementary Solutions

Explicit Solution as a Sum

Constant and Undetermined Coefficients

Stationary States and Stability for Linear First-Order Equations

Local Stability of Nonlinear First-Order Equations

Walking as a Simple Difference Equation

What is the difference
between difference and differential equations?

It is relatively common to indicate by:

a **subscript** a discrete time function like $m \mapsto x_m$;

parentheses a continuous time function like $t \mapsto x(t)$.

Walking on two feet can be modelled as a **discrete time process**,
with **time domain** $T = \{0, 1, 2, \dots\} = \mathbb{Z}_+ = \{0\} \cup \mathbb{N}$
that counts the number of completed steps.

After m steps, the respective positions $\ell, r \in \mathbb{R}^2$
of the left and right feet on the ground
can be described by function $T \ni m \mapsto (\ell_m, r_m) \in \mathbb{R}^2$.

Walking as a More Complicated Difference Equation

Athletics rules limit a walking step to be no longer than a stride.

So a walking process that starts with the left foot might be described by the two coupled equations

$$\ell_m = \begin{cases} \lambda(r_{m-1}) & \text{if } m \text{ is odd} \\ \ell_{m-1} & \text{if } m \text{ is even} \end{cases} \quad \text{and} \quad r_m = \begin{cases} \rho(\ell_{m-1}) & \text{if } m \text{ is even} \\ r_{m-1} & \text{if } m \text{ is odd} \end{cases}$$

for $m = 0, 1, 2, \dots$

Or, if the length and direction of each pace are affected by the length and direction of its predecessor, by

$$\ell_m = \begin{cases} \lambda(r_{m-1}, \ell_{m-2}) & \text{if } m \text{ is odd} \\ \ell_{m-1} & \text{if } m \text{ is even} \end{cases}$$
$$\text{and } r_m = \begin{cases} \rho(\ell_{m-1}, r_{m-2}) & \text{if } m \text{ is even} \\ r_{m-1} & \text{if } m \text{ is odd} \end{cases}$$

for $m = 0, 1, 2, \dots$

Walking as a Differential Equation

Newtonian physics implies that a walker's centre of mass must be a **continuous function of time**, described by a 3-vector valued mapping $\mathbb{R}_+ \ni t \mapsto (x(t), y(t), z(t)) \in \mathbb{R}^3$.

The time domain is therefore $T := \mathbb{R}_+$.

The same will be true for the position of, for instance, the extreme end of the walker's left big toe.

Newtonian physics requires that the **acceleration** 3-vector described by the second derivative $\frac{d^2}{dt^2}(x(t), y(t), z(t)) \in \mathbb{R}^3$ should be well defined for all t .

The biology of survival requires it to be bounded.

Actually, the motion becomes seriously uncomfortable unless the acceleration (or deceleration) is continuous — as a driving instructor taught me more than 60 years ago!

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Basic Definition

Let $T = \mathbb{Z}_+ \ni t \mapsto x_t \in X$ describe a discrete time process, with $X = \mathbb{R}$ (or $X = \mathbb{R}^m$) as the **state space**.

Its **difference** at time t is defined as

$$\Delta x_t := x_{t+1} - x_t$$

A standard **first-order difference equation** takes the form

$$x_{t+1} - x_t = \Delta x_t = d_t(x_t)$$

where each $d_t : X \rightarrow X$, or equivalently,

$$T \times X \ni (t, x) \mapsto d_t(x)$$

Equivalent Recurrence Relations

Obviously, the difference equation $x_{t+1} - x_t = \Delta x_t = d_t(x_t)$ is equivalent to the **recurrence relation** $x_{t+1} = r_t(x_t)$ where $T \times X \ni (t, x) \mapsto r_t(x) = x + d_t(x)$, or equivalently, $d_t(x) = r_t(x) - x$.

Thus difference equations and recurrence relations are **entirely equivalent**.

We follow standard mathematical practice in using the notation for recurrence relations, even when discussing difference equations.

We may even write “difference equation” when actually we are considering a recurrence relation.

Existence of Solutions

Example

Consider the difference equation $x_t = \sqrt{x_{t-1} - 1}$ with $x_0 = 5$.

Evidently $x_1 = \sqrt{5 - 1} = 2$, then $x_2 = \sqrt{2 - 1} = 1$,

and next $x_3 = \sqrt{1 - 1} = 0$,

leaving $x_4 = \sqrt{0 - 1}$ undefined as a real number.

The domain of $(t, x) \mapsto \sqrt{x - 1}$ is limited to

$D := \mathbb{Z}_+ \times [1, \infty)$. □

Generally, consider a mapping $D \ni (t, x) \mapsto r_t(x)$
whose domain is restricted to a subset $D \subset \mathbb{Z}_+ \times X$.

For the difference equation $x_{t+1} = r_t(x_t)$ to have a solution
for all $t \in \mathbb{Z}_+$, one must ensure that

$$(t, x) \in D \implies (t + 1, r_t(x_t)) \in D \quad \text{for all } t \in \mathbb{Z}_+$$

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Application: Wealth Accumulation in Discrete Time

Consider a consumer who, in discrete time $t = 0, 1, 2, \dots$:

- ▶ starts each period t with an amount w_t of accumulated wealth;
- ▶ receives income y_t ;
- ▶ spends an amount e_t ;
- ▶ earns interest on the residual wealth $w_t + y_t - e_t$ at the rate r_t .

The process of wealth accumulation is then described by any of the three equivalent equations

$$w_{t+1} = (1 + r_t)(w_t + y_t - e_t) = \rho_t(w_t - x_t) = \rho_t(w_t + s_t)$$

where, at each time t ,

- ▶ $\rho_t := 1 + r_t$ is the **interest factor**;
- ▶ $x_t = e_t - y_t$ denotes **net expenditure**;
- ▶ $s_t = y_t - e_t = -x_t$ denotes **net saving**.

Compound Interest

Define the **compound interest factor**

$$R_t := \prod_{k=0}^{t-1} (1 + r_k) = \prod_{k=0}^{t-1} \rho_k$$

with the convention that the product of zero terms equals 1 — just as the sum of zero terms equals 0.

This compound interest factor is the unique solution to the recurrence relation $R_{t+1} = (1 + r_t)R_t$ that satisfies the initial condition $R_0 = 1$.

In the special case when $r_t = r$ (all t), it reduces to $R_t = (1 + r)^t = \rho^t$.

Present Discounted Value (PDV)

We transform the difference equation $w_{t+1} = \rho_t(w_t - x_t)$ by using the compound interest factor $R_t = \prod_{k=0}^{t-1} \rho_k$ in order to discount both future wealth and future expenditure.

To do so, define new variables ω_t, ξ_t which denote the present discounted values (PDVs) of, respectively:

1. wealth w_t at time t as $\omega_t := (1/R_t)w_t$;
2. net expenditure x_t at time t as $\xi_t := (1/R_t)x_t$.

With these new variables, the wealth equation $w_{t+1} = \rho_t(w_t - x_t)$ becomes

$$R_{t+1}\omega_{t+1} = \rho_t R_t(\omega_t - \xi_t)$$

But $R_{t+1} = \rho_t R_t$, so eliminating this common factor reduces the equation to $\omega_{t+1} = \omega_t - \xi_t$, with the evident solution $\omega_t = \omega_0 - \sum_{k=0}^{t-1} \xi_k$ for $k = 1, 2, \dots$

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General First-Order Linear Equation

The general first-order linear difference equation can be written in the form

$$x_t - a_t x_{t-1} = f_t \quad \text{for } t = 1, 2, \dots, T$$

for non-zero constants $a_t \in \mathbb{R}$ and a forcing term $\mathbb{N} \ni t \mapsto f_t \in \mathbb{R}$.

When this equation holds for $t = 1, 2, \dots, T$, where $T \geq 6$, this equation can be written in the following matrix form:

$$\begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -a_2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -a_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -a_{T-1} & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -a_T & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{T-2} \\ x_{T-1} \\ x_T \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{T-1} \\ f_T \end{pmatrix}$$

Matrix Form

The matrix form of the difference equation is $\mathbf{C}\mathbf{x} = \mathbf{f}$, where:

1. \mathbf{C} is the $T \times (T + 1)$ **coefficient matrix** whose elements are

$$c_{st} = \begin{cases} -a_s & \text{if } t = s \\ 1 & \text{if } t = s + 1 \\ 0 & \text{otherwise} \end{cases}$$

for $s = 1, 2, \dots, T$ and $t = 0, 1, 2, \dots, T$;

2. \mathbf{x} is the $T + 1$ -dimensional column vector $(x_t)_{t=0}^T$ of endogenous unknowns, to be determined;
3. \mathbf{f} is the T -dimensional column vector $(f_t)_{t=1}^T$ of exogenous shocks.

Partitioned Matrix Form

The matrix equation $\mathbf{C}\mathbf{x} = \mathbf{f}$ can be written in partitioned form as

$$(\mathbf{U} \quad \mathbf{e}_T) \begin{pmatrix} \mathbf{x}^{T-1} \\ x_T \end{pmatrix} = \mathbf{f}$$

where:

1. \mathbf{U} is an upper triangular $T \times T$ matrix;
2. $\mathbf{e}_T = (0, 0, 0, \dots, 0, 1)^\top$ is the T th column vector of the canonical basis of the vector space \mathbb{R}^T ;
3. \mathbf{x}^{T-1} denotes the column vector which is the transpose of the row T -vector $(x_0, x_1, x_2, \dots, x_{T-2}, x_{T-1})$.

In fact the matrix \mathbf{U} satisfies

$$(\mathbf{U}, \mathbf{e}_T) = (-\mathbf{diag}(a_1, a_2, \dots, a_T), \mathbf{e}_T) + (\mathbf{0}_{T \times 1}, \mathbf{I}_{T \times T})$$

Hence there are T independent equations in $T + 1$ unknowns, leaving one degree of freedom in the solution.

An Initial Condition

Consider the difference equation $x_t - a_t x_{t-1} = f_t$,
or $\mathbf{C}\mathbf{x} = \mathbf{f}$ in matrix form.

An **initial condition** specifies an exogenous value \bar{x}_0
for the value x_0 at time 0.

This removes the only degree of freedom
in the system of T equations in $T + 1$ unknowns.

Consider the special case when $a_t = 1$ for all $t \in \mathbb{N}$.

The obvious unique solution of $x_t - x_{t-1} = f_t$
is then that each x_t is the **forward sum**

$$x_t = \bar{x}_0 + \sum_{s=1}^t f_s$$

of the initial state \bar{x}_0 , and of the t exogenously specified
succeeding differences f_s ($s = 1, 2, \dots, t$).

A Terminal Condition

Alternatively, a **terminal condition**

for the difference equation $x_t - x_{t-1} = f_t$
specifies an exogenous value \bar{x}_T for the value x_T
at the **terminal time** T .

It leads to a unique solution as a **backward sum**

$$x_t = \bar{x}_T - \sum_{s=0}^{T-t-1} f_{T-s}$$

of the exogenously specified

- ▶ terminal state \bar{x}_T ;
- ▶ preceding backward differences $-f_{T-s}$
($s = 0, 1, \dots, T - t - 1$).

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Particular and General Solutions

We are interested in solving the system $\mathbf{C}\mathbf{x} = \mathbf{f}$ of T equations in $T + 1$ unknowns, where \mathbf{C} is a $T \times (T + 1)$ matrix.

When the rank of \mathbf{C} is T , there is one degree of freedom.

The associated **homogeneous equation** $\mathbf{C}\mathbf{x} = \mathbf{0}$ will have a one-dimensional space of solutions $\mathbf{x}_t^H = \xi \bar{\mathbf{x}}_t^H$ ($\xi \in \mathbb{R}$).

Given any **particular solution** \mathbf{x}_t^P satisfying $\mathbf{C}\mathbf{x}^P = \mathbf{f}$ for the particular time series \mathbf{f} of forcing terms, the **general solution** \mathbf{x}_t^G must also satisfy $\mathbf{C}\mathbf{x}^G = \mathbf{f}$.

Simple subtraction leads to $\mathbf{C}(\mathbf{x}^G - \mathbf{x}^P) = \mathbf{0}$, so $\mathbf{x}^G - \mathbf{x}^P = \mathbf{x}^H$ for some solution \mathbf{x}^H of the homogeneous equation $\mathbf{C}\mathbf{x} = \mathbf{0}$.

So \mathbf{x} solves the equation $\mathbf{C}\mathbf{x} = \mathbf{f}$

iff there exists a scalar $\xi \in \mathbb{R}$ such that $\mathbf{x} = \mathbf{x}^P + \xi \mathbf{x}^H$, which leads to the formula $\mathbf{x}^G = \mathbf{x}^P + \xi \mathbf{x}^H$ for the general solution.

Complementary Solutions

Consider again the general first-order linear equation which takes the **inhomogeneous** form $x_t - a_t x_{t-1} = f_t$.

The associated **homogeneous equation** takes the form

$$x_t - a_t x_{t-1} = 0 \quad (\text{for all } t \in \mathbb{N})$$

with a zero right-hand side.

The associated **complementary solutions** make up the one-dimensional linear subspace L of solutions to this homogeneous equation.

The space L consists of functions $\mathbb{Z}_+ \ni t \mapsto x_t \in \mathbb{R}$ satisfying

$$x_t = x_0 \prod_{s=1}^t a_s \quad (\text{for all } t \in \mathbb{N})$$

where x_0 is an arbitrary scaling constant.

From Particular to General Solutions

Consider again the **inhomogeneous equation**

$$x_t - a_t x_{t-1} = f_t$$

for a general RHS f_t .

The associated **homogeneous equation** takes the form

$$x_t - a_t x_{t-1} = 0$$

Let x_t^P denote a **particular solution**,
and x_t^G any alternative **general solution**,
of the inhomogeneous equation.

Our assumptions imply that, for each $t = 1, 2, \dots$, one has

$$\begin{aligned}x_t^P - a_t x_{t-1}^P &= f_t \\x_t^G - a_t x_{t-1}^G &= f_t\end{aligned}$$

Subtracting the first equation from the second implies that

$$x_t^G - x_t^P - a_t(x_{t-1}^G - x_{t-1}^P) = 0$$

This shows that $x_t^H := x_t^G - x_t^P$ solves the homogeneous equation.

Characterizing the General Solution

Theorem

Consider the inhomogeneous equation $x_t - a_t x_{t-1} = f_t$ with *forcing term* f_t .

Its general solution x_t^G is the sum $x_t^P + x_t^H$ of

- ▶ any particular solution x_t^P of the inhomogeneous equation;
- ▶ the general complementary solution x_t^H of the corresponding homogeneous equation $x_t - a_t x_{t-1} = 0$.

Linearity in the Forcing Term, I

Theorem

Suppose that x_t^P and y_t^P are particular solutions of the two respective difference equations

$$x_t - a_t x_{t-1} = d_t \quad \text{and} \quad y_t - a_t y_{t-1} = e_t$$

Then, for any scalars α and β , the equation $z_t - a_t z_{t-1} = \alpha d_t + \beta e_t$ has the corresponding linear combination $z_t^P := \alpha x_t^P + \beta y_t^P$ as a particular solution.

Proof.

Routine algebra. □

Linearity in the Forcing Term, II

Consider any equation of the form $x_t - a_t x_{t-1} = f_t$
where f_t is a linear combination $\sum_{k=1}^n \alpha_k f_t^k$
of n forcing terms $\langle f_t^k \rangle_{k=1}^n$.

The theorem implies that a particular solution
is the corresponding linear combination $\sum_{k=1}^n \alpha_k x_t^{Pk}$
of particular solutions $\langle x_t^{Pk} \rangle_{k=1}^n$
to the respective n equations $x_t - a_t x_{t-1} = f_t^k$ ($k = 1, 2, \dots, n$).

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Solving the General Linear Equation

Consider a first-order linear difference equation

$$x_{t+1} = a_t x_t + f_t$$

for a **process** $T \ni t \mapsto x_t \in \mathbb{R}$, where each $a_t \neq 0$ (to avoid trivialities).

We will prove by induction on t that for $t = 0, 1, 2, \dots$ there exist suitable non-zero constants $p_{t,k}$ ($k = 0, 1, 2, \dots, t$) such that, given any possible value of the **initial state** x_0 and of the **forcing terms** f_t ($t = 0, 1, 2, \dots$), the unique solution can be expressed as

$$x_t = p_{t,0} x_0 + \sum_{k=1}^t p_{t,k} f_{k-1}$$

The proof, of course, will also involve deriving a recurrence relation for the constants $p_{t,k}$ ($k = 0, 1, 2, \dots, t$).

Early Terms of the Solution

Because $x_0 = p_{0,0}x_0 = x_0$,
the first term is obviously $p_{0,0} = 1$ when $t = 0$.

Next $x_1 = a_0x_0 + f_0$ when $t = 1$
implies that $p_{1,0} = a_0$ and $p_{1,1} = 1$.

Next, the solution for $t = 2$ is

$$x_2 = a_1x_1 + f_1 = a_1a_0x_0 + a_1f_0 + f_1$$

This formula matches the formula

$$x_t = p_{t,0}x_0 + \sum_{k=1}^t p_{t,k}f_{k-1}$$

when $t = 2$ provided that:

- ▶ $p_{2,0} = a_1a_0$;
- ▶ $p_{2,1} = a_1$;
- ▶ $p_{2,2} = 1$.

Explicit Solution, I

Now, substituting the two expansions

$$x_t = p_{t,0}x_0 + \sum_{k=1}^t p_{t,k}f_{k-1}$$

and

$$x_{t+1} = p_{t+1,0}x_0 + \sum_{k=1}^{t+1} p_{t+1,k}f_{k-1}$$

into both sides of the original equation $x_{t+1} = a_t x_t + f_t$ gives

$$p_{t+1,0}x_0 + \sum_{k=1}^{t+1} p_{t+1,k}f_{k-1} = a_t \left(p_{t,0}x_0 + \sum_{k=1}^t p_{t,k}f_{k-1} \right) + f_t$$

Equating the coefficients of x_0 and of each f_{k-1} in this equation implies that for general t one has

$$p_{t+1,k} = a_t p_{t,k} \text{ for } k = 0, 1, \dots, t, \text{ with } p_{t+1,t+1} = 1$$

Explicit Solution, II

The equation $p_{t+1,k} = a_t p_{t,k}$ for $k = 0, 1, \dots, t$ implies that

$$\begin{aligned} p_{t,0} &= a_{t-1} \cdot a_{t-2} \cdots a_0 & \text{when } k = 0 \\ p_{t,k} &= a_{t-1} \cdot a_{t-2} \cdots a_k & \text{when } k = 1, 2, \dots, t \end{aligned}$$

or, after defining the product of the empty set of real numbers as 1,

$$p_{t,k} = \prod_{s=1}^{t-k} a_{t-s}$$

Inserting these into our formula $x_t = p_{t,0}x_0 + \sum_{k=1}^t p_{t,k}f_{k-1}$ gives the explicit solution

$$x_t = \left(\prod_{s=1}^t a_{t-s} \right) x_0 + \sum_{k=1}^t \left(\prod_{s=1}^{t-k} a_{t-s} \right) f_{k-1}$$

Putting $x_0 = 0$ gives one particular solution of $x_{t+1} = a_t x_t + f_t$, namely

$$x_t^P = \sum_{k=1}^t \left(\prod_{s=1}^{t-k} a_{t-s} \right) f_{k-1}$$

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First-Order Linear Equation with a Constant Coefficient

Next, consider the equation $x_t - ax_{t-1} = f_t$,
where the coefficient a_t has become the **constant** $a \neq 0$.

Evidently one has $x_1 = ax_0 + f_1$,
then next $x_2 = ax_1 + f_2 = a(ax_0 + f_1) + f_2 = a^2x_0 + af_1 + f_2$,
followed by

$$x_3 = ax_2 + f_3 = a(a^2x_0 + af_1 + f_2) + f_3 = a^3x_0 + a^2f_1 + af_2 + f_3$$

etc. One can easily verify by induction the explicit formula

$$x_t = a^t x_0 + \sum_{k=1}^t a^{t-k} f_k$$

In the very special case when $a = 1$,
this also accords with our earlier sum solution $x_t = x_0 + \sum_{k=1}^t f_k$.

First Special Case

An interesting special case occurs when there exists some $\mu \neq 0$ such that f_t is the discrete exponential function $\mathbb{N} \ni t \mapsto \mu^t$.

Then the solution is $x_t = a^t x_0 + S_t$ where $S_t := \sum_{k=1}^t a^{t-k} \mu^k$.

Note that

$$(a - \mu)S_t = \sum_{k=1}^t a^{t-k+1} \mu^k - \sum_{k=1}^t a^{t-k} \mu^{k+1} = a^t \mu - \mu^{t+1}$$

In the **non-degenerate case** when $\mu \neq a$,

it follows that $S_t = \mu \left(\frac{a^t - \mu^t}{a - \mu} \right)$ and so $x_t = a^t x_0 + \mu \left(\frac{a^t - \mu^t}{a - \mu} \right)$.

This solution can be written as $x_t = x_t^H + x_t^P$ where:

1. $x_t^H = \xi^H a^t$ with $\xi^H := x_0 + \mu/(a - \mu)$ is a solution of the **homogeneous** equation $x_t - ax_{t-1} = 0$;
2. $x_t^P = \xi^P \mu^t$ with $\xi^P := -\mu/(a - \mu)$ is a **particular** solution of the **inhomogeneous** equation $x_t - ax_{t-1} = \mu^t$.

Degenerate Case When $\mu = a$

In the **degenerate case** when $\mu = a$, the solution collapses to

$$x_t = a^t x_0 + \sum_{k=1}^t a^{t-k} a^k = a^t x_0 + \sum_{k=1}^t a^t = a^t (x_0 + t)$$

Again, this solution can be written as $x_t = x_t^H + x_t^P$ where:

1. $x_t^H = \xi^H a^t$ with $\xi^H := x_0$ is a solution of the **homogeneous** equation $x_t - ax_{t-1} = 0$;
2. $x_t^P = \xi^P a^t t$ with $\xi^P := 1$ is a **particular** solution of the **inhomogeneous** equation $x_t - ax_{t-1} = a^t$.

Note carefully the term in $a^t t$, where a^t is multiplied by t .

Second Special Case

Another interesting special case is when $f_t = t^r \mu^t$ for some $r \in \mathbb{N}$.

Then the explicit solution we found previously takes the form $x_t = a^t x_0 + S_t$ where $S_t := \sum_{k=1}^t a^{t-k} k^r \mu^k$.

We aim to simplify this expression for S_t .

We first restrict attention to the **non-degenerate case** when $\mu \neq a$.

The solution can still be written as $x_t = x_t^H + x_t^P$ where:

1. $x_t^H = \xi^H a^t$ with scalar $\xi^H \in \mathbb{R}$
solves the **homogeneous** equation $x_t - ax_{t-1} = 0$;
2. $x_t^P = \xi^P(t) \mu^t$ is any **particular** solution
of the **inhomogeneous** equation $x_t - ax_{t-1} = t^r \mu^t$.

The issue is finding a useful form of the function $t \mapsto \xi^P(t)$ that makes $\xi^P(t) \mu^t$ a solution of $x_t - ax_{t-1} = t^r \mu^t$.

Method of Undetermined Coefficients

We will find a particular solution $x_t^P = \xi^P(t)\mu^t$ of the inhomogeneous difference equation $x_t - ax_{t-1} = t^r\mu^t$ where $\xi^P(t) = \sum_{k=0}^r \xi_k t^k$ is a polynomial in t of degree r , the power of t on the right-hand side.

The coefficients $(\xi_0, \xi_1, \dots, \xi_r)$ of the polynomial are **undetermined** till we consider the difference equation itself.

Let $1_{j \leq k}$ denote 1 if $j \leq k$, but 0 if $j > k$.

Then, by the binomial theorem,

$$\begin{aligned}\xi^P(t-1) &= \sum_{k=0}^r \xi_k \cdot (t-1)^k \\ &= \sum_{k=0}^r \xi_k \sum_{j=0}^k \binom{k}{j} t^j (-1)^{k-j} \\ &= \sum_{j=0}^r \sum_{k=0}^r 1_{j \leq k} \xi_k \binom{k}{j} t^j (-1)^{k-j} \\ &= \sum_{j=0}^r \sum_{k=j}^r \xi_k \binom{k}{j} (-1)^{k-j} t^j\end{aligned}$$

Determining the Undetermined Coefficients

For $x_t^P = \mu^t \sum_{k=0}^r \xi_k t^k$ to solve $x_t - ax_{t-1} = t^r \mu^t$, we need

$$\mu^t \sum_{j=0}^r \xi_j t^j - a\mu^{t-1} \sum_{j=0}^r \sum_{k=j}^r \xi_k \binom{k}{j} (-1)^{k-j} t^j = t^r \mu^t$$

First consider the **non-degenerate case** $\mu \neq a$.

Equating coefficients of t^r implies that $\mu^t \xi_r - a\mu^{t-1} \xi_r = \mu^t$.

Dividing by μ^{t-1} gives $\mu \xi_r - a \xi_r = \mu$, and so $\xi_r = \mu(\mu - a)^{-1}$.

For $j = 0, 1, \dots, r-1$, equating coefficients of t^j implies that

$$\mu^t \xi_j - a\mu^{t-1} \sum_{k=j}^r \xi_k \binom{k}{j} (-1)^{k-j} = 0$$

so $\xi_j = (\mu - a)^{-1} \sum_{k=j+1}^r \xi_k \binom{k}{j} (-1)^{k-j}$.

In principle one can solve this system of $r+1$ equations

in the $r+1$ unknowns $(\xi_r, \xi_{r-1}, \xi_{r-2}, \dots, \xi_0)$

by backward recursion, starting with $\xi_r = \mu(\mu - a)^{-1}$, ending at ξ_0 .

Degenerate Case

But in the **degenerate case** $\mu = a$,
the equation $\mu\xi_r - a\xi_r = \mu$ for ξ_r has no solution,
so the method does not work.

Instead, to solve $x_t - ax_{t-1} = t^r a^t$,
we introduce the new variable $y_t = a^{-t}x_t$.

Then $y_t = a^{-t}(ax_{t-1} + t^r a^t) = y_{t-1} + t^r$.

The solution is $y_t = y_0 + S_r(t)$ where $S_r(n) := \sum_{k=1}^n j^r$
is the much studied sum of r th powers of the first n integers.

Hence $x_t = a^t [x_0 + S_r(t)]$.

Theorem

The sums $S_r(n)$ satisfy the recurrence relation

$$(r+1)S_r(n) = (n+1)^{r+1} - 1 - \sum_{k=0}^{r-1} \binom{r+1}{k} S_k(n)$$

Proof of Theorem

For $j = 1, 2, \dots, n$, the binomial theorem implies that

$$(j+1)^{r+1} = \sum_{k=0}^r \binom{r+1}{k} j^k + j^{r+1}$$

Summing over j , then interchanging the order of summation, gives

$$\begin{aligned}(n+1)^{r+1} - 1 &= \sum_{j=1}^n [(j+1)^{r+1} - j^{r+1}] \\ &= \sum_{j=1}^n \sum_{k=0}^r \binom{r+1}{k} j^k = \sum_{k=0}^r \binom{r+1}{k} S_k(n)\end{aligned}$$

Isolating the last term gives

$$(n+1)^{r+1} - 1 = \sum_{k=0}^{r-1} \binom{r+1}{k} S_k(n) + \binom{r+1}{r} S_r(n)$$

and so, after rearranging

$$(r+1)S_r(n) = (n+1)^{r+1} - 1 - \sum_{k=0}^{r-1} \binom{r+1}{k} S_k(n)$$



Important Corollary

Corollary

*Each sum $S_r(n) := \sum_{j=1}^n j^r$
of the r th powers of the first n natural numbers
equals a polynomial $\sum_{i=0}^{r+1} a_{ri} n^i$ of degree $r + 1$ in n ,
whose coefficients are rational,
with constant term $a_{r0} = 0$
and leading coefficient $a_{r,r+1} = 1/(r + 1)$.*

Using the theorem, we prove the corollary by induction,
starting with the obvious $S_0(n) := \sum_{j=1}^n j^0 = n$
and the well known $S_1(n) := \sum_{j=1}^n j = \frac{1}{2}n(n + 1) = \frac{1}{2}n + \frac{1}{2}n^2$.

Proof by Induction

Indeed, consider the induction hypothesis that $S_q(n)$ satisfies the corollary for $q = 0, 1, 2, \dots, r - 1$.

This hypothesis implies in particular that each $S_k(n)$ is a polynomial of degree $k + 1$ in n .

It follows that the right-hand side of the equation

$$(r + 1)S_r(n) = (n + 1)^{r+1} - 1 - \sum_{k=0}^{r-1} \binom{r+1}{k} S_k(n)$$

is obviously a polynomial of degree at most $r + 1$ in n .

Moreover, it has rational coefficients, a constant term 0, and a leading coefficient 1 attached to the highest power n^{r+1} . \square

Main Theorem

Theorem

Consider the inhomogeneous first-order linear difference equation

$$x_t - ax_{t-1} = t^r \mu^t, \text{ where } a \neq 0 \text{ and } r \in \mathbb{Z}_+.$$

Then there exists a particular solution of the form $x_t^P = Q(t) \mu^t$ where the function $t \mapsto Q(t)$ is a polynomial which:

- ▶ in the regular case when $\mu \neq a$, has degree r ;
- ▶ in the degenerate case when $\mu = a$, has degree $r + 1$.

The general solution takes the form $x_t = x_t^P + x_t^C$ where:

- ▶ x_t^P is any particular solution of the inhomogeneous equation;
- ▶ x_t^C is any member of the one-dimensional linear space of complementary solutions to the corresponding homogeneous equation $x_t - ax_{t-1} = 0$.

Lecture Outline

Introduction: Difference vs. Differential Equations

First-Order Difference Equations

First-Order Linear Difference Equations: Introduction

General First-Order Linear Equation

Particular, General, and Complementary Solutions

Explicit Solution as a Sum

Constant and Undetermined Coefficients

Stationary States and Stability for Linear First-Order Equations

Local Stability of Nonlinear First-Order Equations

Stationary States, I

The general first-order equation $x_{t+1} = f_t(x_t)$ is **non-autonomous**; for it to become **autonomous**, there should be a mapping $x \mapsto f(x)$ that is independent of t .

Given an autonomous equation $x_{t+1} = f(x_t)$, a **stationary state** is a fixed point $x^* \in \mathbb{R}$ of the mapping $x \mapsto f(x)$.

It earns its name because if $x_s = x^*$ for any finite s , then $x_t = x^*$ for all $t = s, s + 1, \dots$

Stationary States, II

Wherever it exists, the solution of the autonomous equation can be written as a function $x_t = \Phi_{t-s}(x_s)$ ($t = s, s + 1, \dots$) of the state x_s at the initial time s , as well as of the number $t - s$ of times that the function $x \mapsto f(x)$ must be iterated in order to determine the state x_t at time t .

Indeed, the sequence of functions $\Phi_k : \mathbb{R} \rightarrow \mathbb{R}$ ($k \in \mathbb{N}$) is defined iteratively by $\Phi_k(x) = f(\Phi_{k-1}(x))$ for all x .

Note that any stationary state x^* is a fixed point of each mapping Φ_k in the sequence, as well as of $\Phi_1 \equiv f$.

Local and Global Stability

The stationary state x^* is:

- ▶ **globally stable** if $\Phi_k(x_0) \rightarrow x^*$ as $k \rightarrow \infty$, regardless of the initial state x_0 ;
- ▶ **locally stable** if there is an (open) neighbourhood $N \subset \mathbb{R}$ of x^* such that whenever $x_0 \in N$ one has $\Phi_k(x_0) \rightarrow x^*$ as $k \rightarrow \infty$.

Generally, global stability implies local stability, but not conversely.

Global stability also implies that the steady state x^* is unique.

We begin by studying stability for linear equations, where local stability is equivalent to global stability.

Later, we will consider the local stability of non-linear equations.

Stationary States of a Linear Equation

Consider the linear (or rather, affine) equation $x_{t+1} = ax_t + f$ for a fixed **forcing term** f .

A stationary state x^* has the defining property that $x_t = x^* \implies x_{t+1} = x^*$, which is satisfied if and only if $x^* = ax^* + f$.

In case $a = 1$, there is:

- ▶ no stationary state unless $f = 0$;
- ▶ the whole real line \mathbb{R} of stationary states if $f = 0$.

Otherwise, if $a \neq 1$, the only stationary state is $x^* = (1 - a)^{-1}f$.

Stability of a Linear Equation

If $a \neq 1$, let us denote by $y_t := x_t - x^*$

the **deviation** of state x_t from the stationary state $x^* = (1 - a)^{-1}f$.

Then $y_{t+1} = x_{t+1} - x^* = ax_t + f - x^* = a(y_t + x^*) + f - x^* = ay_t$.

This equation has the obvious solution $y_t = y_0 a^t$,
or equivalently $x_t = x^* + (x_0 - x^*)a^t$.

The solution is evidently both locally and globally stable
if and only if $a^t \rightarrow 0$ as $t \rightarrow \infty$,
which is true if and only if $|a| < 1$.

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Stationary States

Let I be an open interval of the real line,
and $I \ni x \mapsto f(x) \in I$ a general, possibly nonlinear, function.

A **fixed point** $x \in I$ of f satisfies $f(x) = x$.

Consider the autonomous difference equation $x_{t+1} = f(x_t)$.

For each natural number $n \in \mathbb{N}$,

define the **iterated function** $I \ni x \mapsto f^n(x) \in I$

so that $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for $n = 2, 3, \dots$

A **stationary state** is any $x^* \in I$

with the property that $f^n(x^*) = x^*$ for all $n \in \mathbb{N}$.

This implies in particular that $x^* = x_{s+1} = f(x_s) = f(x^*)$,
so any stationary state must be a fixed point of f .

Conversely, it is obvious that any fixed point of f
must be a stationary state.

Local Stability

The steady state x^* is **locally stable** just in case there is a neighbourhood N of x^* such that whenever $x \in N$ one has $f^n(x) \rightarrow x^*$ as $n \rightarrow \infty$.

Theorem

Let x^* be an equilibrium state of $x_{t+1} = f(x_t)$.

Suppose that $x \mapsto f(x)$ is continuously differentiable in an open interval $I \subset \mathbb{R}$ that includes x^* .

Then

1. $|f'(x^*)| < 1$ implies that x^* is locally stable;
2. $|f'(x^*)| > 1$ implies that x^* is locally unstable.

The proof below uses the fact that, by the mean value theorem, if $x_t \in I$, then there exists a $c_t \in I$ between x_t and x^* such that $f'(c_t)$ is a **mean value** of $f'(x)$ in the sense that

$$x_{t+1} - x^* = f(x_t) - f(x^*) = f'(c_t)(x_t - x^*)$$

Proof of Local Stability

By the hypotheses that f' is continuous on I and $|f'(x^*)| < 1$, there exist an $\epsilon > 0$ and a $k \in (0, 1)$ such that:

1. $(x^* - \epsilon, x^* + \epsilon) \subseteq I$;
2. $|f'(x)| \leq k$ for all $x \in (x^* - \epsilon, x^* + \epsilon)$.

Suppose that $|x_t - x^*| < \epsilon$.

Then the c_t between x_t and x^*

where $|x_{t+1} - x^*| = |f'(c_t)(x_t - x^*)|$ must satisfy $|c_t - x^*| < \epsilon$.

Hence $|f'(c_t)| \leq k$, implying that

$$|x_{t+1} - x^*| = |f'(c_t)(x_t - x^*)| \leq k|x_t - x^*| < k\epsilon < \epsilon$$

By induction on t , if $|x_0 - x^*| < \epsilon$, it follows that $|x_t - x^*| \leq \epsilon$ and in fact $|x_t - x^*| \leq k^t|x_0 - x^*|$ for $t = 1, 2, \dots$

Hence $|x_t - x^*| \rightarrow 0$ as $t \rightarrow \infty$. □

Proof of Local Instability

By the hypotheses that f' is continuous on I and $|f'(x^*)| > 1$, there exist an $\epsilon > 0$ and a $K > 1$ such that:

1. $(x^* - \epsilon, x^* + \epsilon) \subseteq I$;
2. $|f'(x)| \geq K$ for all $x \in (x^* - \epsilon, x^* + \epsilon)$.

Suppose that there exist $s, r \in \mathbb{N}$ such that $x_t \in I$ and $0 < |x_t - x^*| < \epsilon$ for all $t \in T_{s,r} := \{s, s+1, \dots, s+r-1\}$, the set of r successive times starting from time s .

Then for each $t \in T_{s,r}$, any $c_t \in I$ between x_t and x^* where $|x_{t+1} - x^*| = |f'(c_t)(x_t - x^*)|$ must satisfy $|c_t - x^*| < \epsilon$.

This implies that $|x_{t+1} - x^*| \geq K|x_t - x^*|$ for all $t \in T_{s,r}$.

By induction on r , it follows that $|x_{s+r} - x^*| \geq K^r|x_s - x^*|$.

So $|x_{s+r} - x^*| \geq K^r|x_s - x^*| > \epsilon$ for r large enough.

This proves there is no $s \in \mathbb{N}$ such that $|x_t - x^*| < \epsilon$ for all $t \geq s$.

It follows that $x_t \not\rightarrow x^*$ as $t \rightarrow \infty$.

