Lecture Notes: Matrix Algebra
Part E: Quadratic Forms and Their Definiteness

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Outline

Quadratic Forms: Preliminary Exercise

Exercise

Let $\mathbf{A}$ *be any* $n \times n$ *matrix.*

For each $j \in \mathbb{N}_n$*, recall that* $\mathbf{e}^j = (\delta_{ij})_{i=1}^n$ *denotes the* j*th column of the identity matrix* $\mathbf{I}_n$*, and that* $(\mathbf{e}^i)^\top = (\delta_{ij})_{j=1}^n$ *is the* i*th row of* $\mathbf{I}_n$*.*

For each $i, j \in \mathbb{N}_n$*, explain why:*

1. $\mathbf{A}\mathbf{e}^j$ *is the* j*th column* $\mathbf{a}^j$ *of the matrix* $\mathbf{A}$*, whose* i*th component is* $\sum_{k=1}^n a_{ik}\delta_{kj} = a_{ij}$*;*
2. $(\mathbf{e}^i)^\top \mathbf{A}\mathbf{e}^j = \sum_{k=1}^n \delta_{ik} a_{kj}$*, which equals the single element* a_{ij} *of the matrix* $\mathbf{A}$*.*

Definition of Quadratic Form

Definition

A quadratic form on the n-dimensional Euclidean space $\mathbb{R}^n$ is a mapping

$$\mathbb{R}^n \ni \mathbf{x} \mapsto q(\mathbf{x}) = \mathbf{x}^\top \mathbf{Q} \mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i q_{ij} x_j \in \mathbb{R}$$

where $\mathbf{Q}$ is a symmetric $n \times n$ matrix.

The quadratic form $\mathbf{x}^\top \mathbf{Q} \mathbf{x}$ is diagonal just in case the matrix $\mathbf{Q}$ is diagonal, with $\mathbf{Q} = \mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

In this case $\mathbf{x}^\top \mathbf{Q} \mathbf{x}$ reduces to $\mathbf{x}^\top \mathbf{\Lambda} \mathbf{x} = \sum_{i=1}^{n} \lambda_i \, (x_i)^2$.

The Hessian Matrix of a Quadratic Form in Two Variables

Exercise

Given the quadratic form $q(x,y) = (x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$,

show that, even if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is not symmetric, the Hessian matrix of its second-order partial derivatives is the symmetric matrix* $\begin{pmatrix} q''_{xx} & q''_{xy} \\ q''_{yx} & q''_{yy} \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$.

The Hessian Matrix of a Quadratic Form in n Variables

Exercise

Consider the n-variable quadratic form

$$\mathbb{R}^n \ni \mathbf{x} \mapsto q(\mathbf{x}) = \mathbf{x}^\top \mathbf{A} \mathbf{x} \in \mathbb{R}$$

Show that, even if the matrix $\mathbf{A}$ *is not symmetric,*
the $n \times n$ *Hessian matrix* $\mathbf{H}$ *whose elements* h_{ij}
are the constant second-order partial derivatives $\dfrac{\partial^2 q}{\partial x_i \partial x_j}$
is the symmetric matrix $\mathbf{A} + \mathbf{A}^\top$.

Symmetry Loses No Generality

Requiring $\mathbf{Q}$ in $\mathbf{x}^\top \mathbf{Q}\mathbf{x}$ to be symmetric loses no generality.

This is because, given a general non-symmetric $n \times n$ matrix $\mathbf{A}$, repeated transposition implies that

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = (\mathbf{x}^\top \mathbf{A}\mathbf{x})^\top = \tfrac{1}{2}[\mathbf{x}^\top \mathbf{A}\mathbf{x} + (\mathbf{x}^\top \mathbf{A}\mathbf{x})^\top] = \tfrac{1}{2}\mathbf{x}^\top(\mathbf{A} + \mathbf{A}^\top)\mathbf{x}$$

Hence $\mathbf{x}^\top \mathbf{A}\mathbf{x} = \mathbf{x}^\top \mathbf{A}^\top \mathbf{x} = \mathbf{x}^\top \mathbf{Q}\mathbf{x}$
where $\mathbf{Q}$ is the symmetrized matrix $\frac{1}{2}(\mathbf{A} + \mathbf{A}^\top)$.

Note that $\mathbf{Q}$ is indeed symmetric because

$$\mathbf{Q}^\top = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^\top)^\top = \tfrac{1}{2}[\mathbf{A}^\top + (\mathbf{A}^\top)^\top] = \tfrac{1}{2}(\mathbf{A}^\top + \mathbf{A}) = \mathbf{Q}$$

Outline

Definiteness of a Quadratic Form

When $\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^\top \mathbf{Q} \mathbf{x} = 0$. Otherwise:

Definition

The quadratic form $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{x}^\top \mathbf{Q} \mathbf{x} \in \mathbb{R}$,
as well as its associated symmetric $n \times n$ matrix $\mathbf{Q}$, are both:

positive definite just in case $\mathbf{x}^\top \mathbf{Q} \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$;

negative definite just in case $\mathbf{x}^\top \mathbf{Q} \mathbf{x} < 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$;

positive semi-definite just in case $\mathbf{x}^\top \mathbf{Q} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;

negative semi-definite just in case $\mathbf{x}^\top \mathbf{Q} \mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;

indefinite just in case there exist both $\mathbf{x}^+$ and $\mathbf{x}^-$ in $\mathbb{R}^n$
such that $(\mathbf{x}^+)^\top \mathbf{Q} \mathbf{x}^+ > 0$ and $(\mathbf{x}^-)^\top \mathbf{Q} \mathbf{x}^- < 0$. □

Given the domain $\mathcal{Q}_{n \times n}$ of symmetric $n \times n$ matrices,
the sign of each $\mathbf{Q} \in \mathcal{Q}_{n \times n}$ is indicated,
using some obvious abbreviations, by the definiteness function

$$\mathcal{Q}_{n \times n} \ni \mathbf{Q} \mapsto \mathrm{def}(Q) \in \{\mathrm{PD}, \mathrm{ND}, \mathrm{PSD}, \mathrm{NSD}, \mathrm{ID}\}$$

Definiteness of a Diagonal Quadratic Form

Theorem

Suppose that $\mathbf{Q}$ *is the diagonal matrix* $\mathbf{diag}(\lambda_1, \ldots, \lambda_n)$*,*
so that $\mathbf{x}^\top \mathbf{Q} \mathbf{x} = \sum_{i=1}^n \lambda_i (x_i)^2$.

Then the diagonal quadratic form $\mathbf{x} \mapsto \sum_{i=1}^n \lambda_i (x_i)^2 \in \mathbb{R}$ *is:*

positive definite *if and only if* $\lambda_i > 0$ *for* $i = 1, 2, \ldots, n$*;*

negative definite *if and only if* $\lambda_i < 0$ *for* $i = 1, 2, \ldots, n$*;*

positive semi-definite *if and only if* $\lambda_i \geq 0$ *for* $i = 1, 2, \ldots, n$*;*

negative semi-definite *if and only if* $\lambda_i \leq 0$ *for* $i = 1, 2, \ldots, n$*;*

indefinite *if and only if there exist* $i, j \in \{1, 2, \ldots, n\}$
such that $\lambda_i > 0$ *and* $\lambda_j < 0$.

Proof.

The proof is left as an exercise.

The result is obvious if $n = 1$, and straightforward if $n = 2$.

Working out these two cases first suggests the proof for $n > 2$. $\square$

Concavity or Convexity of a Quadratic Form

Exercise

Show that, as a function $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{x}^\top \mathbf{Q}\mathbf{x} \in \mathbb{R}$,
the quadratic form $\mathbf{x}^\top \mathbf{Q}\mathbf{x}$ *is:*

strictly convex if and only if $\mathbf{Q}$ *is positive definite;*

strictly concave if and only if $\mathbf{Q}$ *is negative definite;*

convex if and only if $\mathbf{Q}$ *is positive semi-definite;*

concave if and only if $\mathbf{Q}$ *is negative semi-definite.*

Otherwise $\mathbf{x}^\top \mathbf{Q}\mathbf{x}$ *is neither concave nor convex*
if and only if $\mathbf{Q}$ *is indefinite.*

The solution is more suited to Pablo's lectures than mine!

Outline

Definiteness of a Quadratic Form: Simple Tests

Even if $\mathbf{Q}$ is not a diagonal matrix,
its diagonal elements still provide useful information.

Proposition

The quadratic form $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{x}^\top \mathbf{Q} \mathbf{x}$ *is:*

1. *positive definite only if* $q_{ii} > 0$ *for all* $i \in \mathbb{N}_n$*;*
2. *positive semi-definite only if* $q_{ii} \geq 0$ *for all* $i \in \mathbb{N}_n$*;*
3. *negative definite only if* $q_{ii} < 0$ *for all* $i \in \mathbb{N}_n$*;*
4. *negative semi-definite only if* $q_{ii} \leq 0$ *for all* $i \in \mathbb{N}_n$*;*
5. *indefinite if there exist* $i, j \in \mathbb{N}_n$ *such that* $q_{ii} > 0 > q_{jj}$*.*

Proof.

For each $i \in \mathbb{N}_n$, recall that $(\mathbf{e}^i)^\top \mathbf{Q} \mathbf{e}^i = q_{ii}$
where $\mathbf{e}^i$ denotes column i of $\mathbf{I}_n$.

The result follows from checking the signs
of $(\mathbf{e}^i)^\top \mathbf{Q} \mathbf{e}^i = q_{ii}$ and $(\mathbf{e}^j)^\top \mathbf{Q} \mathbf{e}^j = q_{jj}$ in the 5 different cases. $\square$

Semi-Definiteness of a Quadratic Form

Suppose that the diagonal of the symmetric $n \times n$ matrix $\mathbf{Q}$ has at least one zero element.

By the previous proposition, the quadratic form $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{x}^\top \mathbf{Q} \mathbf{x}$ cannot be either positive definite or negative definite.

But could it still be
either positive semi-definite or negative semi-definite?

Semi-Definiteness of a Quadratic Form: Simple Test

Proposition

Suppose that the diagonal of the symmetric $n \times n$ matrix $\mathbf{Q}$ has two zero elements q_{ii} and q_{jj} with $i \neq j$.

Then the quadratic form $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{x}^\top \mathbf{Q} \mathbf{x}$ is indefinite unless one has $q_{ij} = q_{ji} = 0$.

Proof.

Consider the particular column vector $\mathbf{x} = \alpha \mathbf{e}^i + \beta \mathbf{e}^j \in \mathbb{R}^n$ where α and β are any two real scalars.

Routine calculation shows that, because $\mathbf{Q}$ is symmetric, and $(\mathbf{e}^i)^\top \mathbf{Q} \mathbf{e}^i = (\mathbf{e}^j)^\top \mathbf{Q} \mathbf{e}^j = q_{ii} = q_{jj} = 0$, one has

$$\mathbf{x}^\top \mathbf{Q} \mathbf{x} = \alpha\beta[(\mathbf{e}^i)^\top \mathbf{Q} \mathbf{e}^j + (\mathbf{e}^j)^\top \mathbf{Q} \mathbf{e}^i] = 2 q_{ij} \alpha\beta$$

As both α and β range over all of $\mathbb{R} \setminus \{0\}$, one has $\alpha\beta \gtrless 0$ according as $\operatorname{sgn} \alpha = \pm \operatorname{sgn} \beta$.

So $\mathbf{x}^\top \mathbf{Q} \mathbf{x} = 2 q_{ij} \alpha\beta$ is indefinite unless $q_{ij} = q_{ji} = 0$. $\square$

Outline

The Two Variable Case: Completing the Square

In the 2×2 case, the typical quadratic form is

$$\mathbb{R}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2hxy + by^2$$

Assuming that $a \neq 0$, one can complete the square
by writing $ax^2 + 2hxy + by^2$ as $a\left(x + \frac{h}{a}y\right)^2 + \left(b - \frac{h^2}{a}\right)y^2$,
which can be verified term by term.

First, the quadratic form $ax^2 + 2hxy + by^2$
is neither positive nor negative definite in case:

- $a = 0$, because then $ax^2 + 2hxy + by^2 = 0$ when $x \neq 0$ and $y = 0$;
- $a \neq 0$ but $ab - h^2 = 0$, because then $ax^2 + 2hxy + by^2 = 0$ when $y \neq 0$ and $x = -hy/a$.

Tests Based on Completing the Square

We are considering the quadratic form which, in case $a \neq 0$, after completing the square, becomes

$$ax^2 + 2hxy + by^2 = a\left(x + \frac{h}{a}y\right)^2 + \left(b - \frac{h^2}{a}\right)y^2$$

If $a > 0$, because $b - \frac{h^2}{a} = \frac{1}{a}(ab - h^2)$, the quadratic form is:

positive definite if and only if $ab - h^2 > 0$;

positive semi-definite if and only if $ab - h^2 \geq 0$;

indefinite if and only if $ab - h^2 < 0$.

If $a < 0$, because $b - \frac{h^2}{a} = \frac{1}{a}(ab - h^2)$, the quadratic form is:

negative definite if and only if $ab - h^2 > 0$;

negative semi-definite if and only if $ab - h^2 \geq 0$;

indefinite if and only if $ab - h^2 < 0$.

Completing the Square as Symmetric Pivoting, I

Given the 2×2 matrix $\begin{pmatrix} a & h \\ h & b \end{pmatrix}$, provided that $a \neq 0$,
the downward pivoting operation
involves adding $-h/a$ times row 1 to row 2.

In symbols, this downward pivoting operation is represented by the 2×2 matrix $\mathbf{E}_{11}^{\downarrow} = \mathbf{E}_{2+(-h/a)1} = \begin{pmatrix} 1 & 0 \\ -h/a & 1 \end{pmatrix}$.

Applied to the original matrix, the result is

$$\mathbf{E}_{11}^{\downarrow} \begin{pmatrix} a & h \\ h & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -h/a & 1 \end{pmatrix} \begin{pmatrix} a & h \\ h & b \end{pmatrix} = \begin{pmatrix} a & h \\ 0 & b - h^2/a \end{pmatrix}$$

Completing the Square as Symmetric Pivoting, II

Starting with the equation $\mathbf{E}_{11}^{\downarrow} \begin{pmatrix} a & h \\ h & b \end{pmatrix} = \begin{pmatrix} a & h \\ 0 & b - h^2/a \end{pmatrix}$,

suppose now we post-multiply each side by the transpose $(\mathbf{E}_{11}^{\downarrow})^\top$.

This completes a symmetric pivoting operation whose result is

$$\begin{aligned} \mathbf{E}_{11}^{\downarrow} \begin{pmatrix} a & h \\ h & b \end{pmatrix} (\mathbf{E}_{11}^{\downarrow})^\top &= \begin{pmatrix} a & h \\ 0 & b - h^2/a \end{pmatrix} \begin{pmatrix} 1 & -h/a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & b - h^2/a \end{pmatrix} = \mathbf{diag}(a, b - h^2/a) \end{aligned}$$

The two-part symmetric pivoting operation
converts the original quadratic form $ax^2 + 2hxy + by^2$
to the diagonal form $az^2 + (b - h^2/a)w^2$
where $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{E}_{11}^{\downarrow} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -h/a & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$
and so $\begin{pmatrix} z \\ w \end{pmatrix} = (\mathbf{E}_{11}^{\downarrow})^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ h/a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Example Where Symmetric Pivoting Is Impossible

Example

Let $\mathbf{A}$ be the 2×2 symmetric matrix $\begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}$.

Both diagonal elements are zero.

These zeroes make symmetric pivoting impossible,
so one cannot complete the square in the quadratic form

$$\begin{pmatrix} x & y \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} hy \\ hx \end{pmatrix} = 2hxy$$

Fortunately the definiteness of $\mathbf{A}$ is easy to determine directly.

If $h \neq 0$ then $\mathbf{A}$ is indefinite.

If $h = 0$ then $\mathbf{A} = \mathbf{0}_{2\times 2}$,
the only 2×2 symmetric matrix that is both
positive semi-definite and negative semi-definite.

Outline

The Block Diagonal Case: Proposition

Proposition

Suppose that $\mathbf{A} = \mathbf{diag}(\mathbf{B}, \mathbf{C})$
is a symmetric block diagonal matrix.

Then $\mathbf{A}$ *is positive definite (resp. semi-definite) if and only if both blocks* $\mathbf{B}$ *and* $\mathbf{C}$ *are positive definite (resp. semi-definite).*

Corollary

Suppose that $\mathbf{A} = \mathbf{diag}(\mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(k)})$ *is symmetric.*

Then $\mathbf{A}$ *is positive definite (resp. semi-definite) if and only if each block* $\mathbf{A}^{(i)}$ *(*$i \in \mathbb{N}_k$*) is positive definite (resp. semi-definite).*

The corollary is easily proved by induction on k.

Remark

As usual, the result for a negative (semi-)definite matrix $\mathbf{A}$
follows from the corresponding result
for the positive (semi-)definite matrix $-\mathbf{A}$.

The Block Diagonal Case: Proof

Proof.

Consider the block diagonal quadratic form

$$(\mathbf{y}^\top, \mathbf{z}^\top)\mathbf{A}\begin{pmatrix}\mathbf{y}\\ \mathbf{z}\end{pmatrix} = (\mathbf{y}^\top, \mathbf{z}^\top)\begin{pmatrix}\mathbf{B} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}\end{pmatrix}\begin{pmatrix}\mathbf{y}\\ \mathbf{z}\end{pmatrix} = \mathbf{y}^\top\mathbf{B}\mathbf{y} + \mathbf{z}^\top\mathbf{C}\mathbf{z}$$

If $\mathbf{A}$ is positive definite (resp. semi-definite),

then $(\mathbf{y}^\top, \mathbf{0}^\top)\mathbf{A}\begin{pmatrix}\mathbf{y}\\ \mathbf{0}\end{pmatrix} = \mathbf{y}^\top\mathbf{B}\mathbf{y} > 0$ (resp. ≥ 0) for all $\mathbf{y} \neq \mathbf{0}$,

and $(\mathbf{0}^\top, \mathbf{z}^\top)\mathbf{A}\begin{pmatrix}\mathbf{0}\\ \mathbf{z}\end{pmatrix} = \mathbf{z}^\top\mathbf{C}\mathbf{z} > 0$ (resp. ≥ 0) for all $\mathbf{z} \neq \mathbf{0}$.

So both $\mathbf{B}$ and $\mathbf{C}$ are positive definite (resp. semi-definite).

Conversely, if both $\mathbf{B}$ and $\mathbf{C}$ are positive definite, then so is $\mathbf{A}$ because $\mathbf{y}^\top\mathbf{B}\mathbf{y} + \mathbf{z}^\top\mathbf{C}\mathbf{z} > 0$ unless both $\mathbf{y} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$.

But if both $\mathbf{B}$ and $\mathbf{C}$ are only positive semi-definite, then $\mathbf{y}^\top\mathbf{B}\mathbf{y} + \mathbf{z}^\top\mathbf{C}\mathbf{z} \geq 0$, so $\mathbf{A}$ is positive semi-definite. $\square$

Outline

Quadratic Form Invariance: Statement of Lemma

Lemma

Suppose that $\mathbf{A}$ *and* $\mathbf{B}$ *are* $n \times n$ *symmetric matrices,*
and there exists an invertible $n \times n$ *matrix* $\mathbf{R}$ *such that* $\mathbf{B} = \mathbf{R}\mathbf{A}\mathbf{R}^\top$*.*

Then the definiteness function

$$\mathcal{Q}_{n \times n} \ni \mathbf{Q} \mapsto \mathit{def}(Q) \in \{PD,\ ND,\ PSD,\ NSD,\ ID\}$$

satisfies $\mathrm{def}(\mathbf{B}) = \mathrm{def}(\mathbf{A})$*.*

Quadratic Form Invariance: Proof of Lemma

Proof.

For any $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$ with $\mathbf{x} = \mathbf{R}^\top \mathbf{u}$ and so $\mathbf{u} = (\mathbf{R}^\top)^{-1}\mathbf{x}$, note that

1. $\mathbf{x}^\top \mathbf{A}\mathbf{x} = (\mathbf{R}^\top \mathbf{u})^\top \mathbf{A}\mathbf{R}^\top \mathbf{u} = \mathbf{u}^\top \mathbf{R}\mathbf{A}\mathbf{R}^\top \mathbf{u} = \mathbf{u}^\top \mathbf{B}\mathbf{u}$;
2. $\mathbf{x} = \mathbf{0} \iff \mathbf{u} = \mathbf{0}$ and so $\mathbf{x} \neq \mathbf{0} \iff \mathbf{u} \neq \mathbf{0}$.

From these two statements one can verify
each of the following four equivalences:

$$\forall \mathbf{x} \neq \mathbf{0} : \mathbf{x}^\top \mathbf{A}\mathbf{x} \left\{ \begin{array}{c} > \\ < \\ \geq \\ \leq \end{array} \right\} 0 \iff \forall \mathbf{u} \neq \mathbf{0} : \mathbf{u}^\top \mathbf{B}\mathbf{u} \left\{ \begin{array}{c} > \\ < \\ \geq \\ \leq \end{array} \right\} 0$$

In addition, it also follows from these four equivalences
that $\mathbf{A}$ is indefinite if and only if $\mathbf{B}$ is indefinite. $\square$

Quadratic Form Invariance: Counter Example

Example

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ symmetric matrices, where $\mathbf{A}$ is either positive or negative definite.

Suppose too that there exists a singular $n \times n$ matrix $\mathbf{S}$ such that $\mathbf{B} = \mathbf{S}\mathbf{A}\mathbf{S}^\top$.

Then $|\mathbf{S}| = |\mathbf{S}^\top| = 0$, so $\mathbf{S}^\top$ is also singular.

Hence there exists $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{S}^\top \mathbf{y} = \mathbf{0}$.

Then $\mathbf{y}^\top \mathbf{B} \mathbf{y} = \mathbf{y}^\top \mathbf{S}\mathbf{A}\mathbf{S}^\top \mathbf{y} = 0$.

It follows that $\mathbf{B}$ is neither positive nor negative definite.

Outline

Symmetric Maximal Diagonalization: Definition

Definition

A symmetric maximal diagonalization of an $n \times n$ matrix $\mathbf{A}$ takes the form $\mathbf{R}\mathbf{A}\mathbf{R}^\top = \begin{pmatrix} \mathbf{D}_{r\times r} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(n-r)\times r} & \mathbf{0}_{(n-r)\times(n-r)} \end{pmatrix}$ where:

1. the integer $r \in \mathbb{Z}$ satisfying $0 \leq r \leq \min\{m, n\}$ is the rank;
2. $\mathbf{D}_{r\times r}$ is an $r \times r$ diagonal matrix which is invertible because all its r diagonal elements are non-zero;
3. $\mathbf{R}$ is an invertible $n \times n$ matrix that, because $|\mathbf{R}| = 1$, represents a determinant preserving row operation.

In case $0 < r < n$, the symmetric maximal diagonalization of the $n \times n$ matrix $\mathbf{A}$ needs the full expression for the 2×2 partitioned matrix on the right-hand side.

Otherwise, in case $r = n$, this partitioned matrix reduces to $\mathbf{D}_{n\times n}$.

Straightforward Symmetric Pivoting, First Step

Start with any $n \times n$ symmetric matrix $\mathbf{A}$, also denoted by $\mathbf{A}^{(0)}$, which we can write in partitioned form as $\mathbf{A} = \begin{pmatrix} a_{11} & \mathbf{a}_{>1,1}^\top \\ \mathbf{a}_{>1,1} & \mathbf{A}_{>1,>1} \end{pmatrix}$, where $\mathbf{a}_{>1,1}$ is a column $n-1$-vector.

Provided that $a_{11} \neq 0$, we can pivot symmetrically about a_{11} by:

1. pre-multiplying $\mathbf{A}$
   by the lower triangular downward pivot matrix $\mathbf{E}_{11}^{\downarrow}$;
2. post-multiplying the product $\mathbf{E}_{11}^{\downarrow}\mathbf{A}$
   by the upper triangular transpose $(\mathbf{E}_{11}^{\downarrow})^\top$
   of the downward pivot matrix $\mathbf{E}_{11}^{\downarrow}$.

The combined effect is to transform $\mathbf{A}$ to the symmetric matrix

$$\mathbf{A}^{(1)} = \mathbf{E}_{11}^{\downarrow}\mathbf{A}(\mathbf{E}_{11}^{\downarrow})^\top = \begin{pmatrix} a_{11} & \mathbf{0}_{1,>1} \\ \mathbf{0}_{>1,1} & \mathbf{A}_{>1,>1}^{(1)} \end{pmatrix}$$

Straightforward Symmetric Pivoting, Start of Step k

For each $k \in \mathbb{N}$ with $1 < k < n$,
step k starts with the $n \times n$ matrix $\mathbf{A}^{(k-1)}$ which,
by induction on k, takes the symmetric form

$$\mathbf{A}^{(k-1)} = \begin{pmatrix} \mathbf{D}^{(k-1)}_{<k,<k} & \mathbf{0}_{<k,k} & \mathbf{0}_{<k,>k} \\ \mathbf{0}^\top_{k,<k} & a^{(k-1)}_{kk} & (\mathbf{a}^{(k-1)}_{>k,k})^\top \\ \mathbf{0}_{>k,<k} & \mathbf{a}^{(k-1)}_{>k,k} & \mathbf{A}^{(k-1)}_{>k,>k} \end{pmatrix}$$

where:

1. $\mathbf{D}^{(k-1)}_{<k,<k}$ is a $(k-1) \times (k-1)$ diagonal matrix;
2. $\mathbf{a}^{(k-1)}_{>k,k}$ is a column $n-k$-vector;
3. $\mathbf{A}^{(k-1)}_{>k,>k}$ is an $(n-k) \times (n-k)$ symmetric matrix.

Straightforward Symmetric Pivoting, Step k

Starting from $\mathbf{A}^{(k-1)} = \begin{pmatrix} \mathbf{D}^{(k-1)}_{<k,<k} & \mathbf{0}_{<k,k} & \mathbf{0}_{<k,>k} \\ \mathbf{0}^\top_{k,<k} & a^{(k-1)}_{kk} & (\mathbf{a}^{(k-1)}_{>k,k})^\top \\ \mathbf{0}_{>k,<k} & \mathbf{a}^{(k-1)}_{>k,k} & \mathbf{A}^{(k-1)}_{>k,>k} \end{pmatrix}$,

provided that $a^{(k-1)}_{kk} \neq 0$, we can:

- pre-multiply by the lower triangular pivot matrix $\mathbf{E}^{\downarrow}_{kk}$;
- post-multiply by the upper triangular transpose $(\mathbf{E}^{\downarrow}_{kk})^\top$

The result is $\mathbf{A}^{(k)} = \mathbf{E}^{\downarrow}_{kk}\mathbf{A}^{(k-1)}(\mathbf{E}^{\downarrow}_{kk})^\top$ where

$$\mathbf{A}^{(k)} = \begin{pmatrix} \mathbf{D}^{(k)}_{\leq k,\leq k} & \mathbf{0}_{\leq k,k+1} & \mathbf{0}_{\leq k,>k+1} \\ \mathbf{0}^\top_{k+1,\leq k} & a^{(k)}_{k+1,k+1} & (\mathbf{a}^{(k)}_{>k+1,k+1})^\top \\ \mathbf{0}_{>k+1,\leq k} & \mathbf{a}^{(k)}_{>k+1,k+1} & \mathbf{A}^{(k)}_{>k+1,>k+1} \end{pmatrix}$$

The $k \times k$ diagonal matrix $\mathbf{D}^{(k)}_{\leq k,\leq k}$ is the diagonal matrix $\mathbf{D}^{(k-1)}_{<k,<k}$ with one extra non-zero pivot element $a^{(k)}_{kk}$ on the diagonal.

Conclusion of Straightforward Symmetric Pivoting

Provided that the successive pivot elements $a_{kk}^{(k-1)}$, for $k \in \mathbb{N}_{n-1}$, are all non-zero, straightforward symmetric pivoting can continue until k reaches $n-1$.

After all $n-1$ stages, straightforward symmetric pivoting ends with the $n \times n$ matrix $\mathbf{A}^{(n-1)}$, which equals the diagonal matrix $\mathbf{D}^{(n)}_{\leq n, \leq n}$.

The last diagonal element $a_{nn}^{(n-1)}$ could be zero.

This does not matter because no more pivoting is required.

But like downward pivoting, if $a_{kk}^{(k-1)} = 0$ for some $k < n$, then straightforward symmetric pivoting eventually fails.

Some adjustment of at least one pivot element is needed.

Outline

Adjusted Symmetric Pivoting: The Matrix Sequence, I

Like straightforward downward pivoting,
straightforward symmetric pivoting works provided
each successive pivot element $a_{kk}^{(k-1)}$ $(k = 1, 2, \ldots, n-1)$
that is relevant because $k < n$ is non-zero.

Adjusted symmetric pivoting allows for the possibility
that at least one relevant prospective pivot $a_{kk}^{(k-1)}$ with $k < n$ is 0.

The adjusted symmetric pivoting process that lasts at least r steps
will generate, for each $k \in \mathbb{N}_r$, an $n \times n$ symmetric matrix $\tilde{\mathbf{A}}^{(k)}$

that takes the partitioned form $\tilde{\mathbf{A}}^{(k)} = \begin{pmatrix} \tilde{\mathbf{D}}_{k \times k}^{(k)} & \mathbf{0}_{k \times (n-k)} \\ \mathbf{0}_{(n-k) \times k} & \tilde{\mathbf{A}}_{>k,>k}^{(k)} \end{pmatrix}$.

Here $\tilde{\mathbf{D}}_{k \times k}^{(k)}$ is the diagonal matrix in which, for each $p \in \mathbb{N}_k$,
the non-zero diagonal element $\tilde{u}_{pp}^{(k)}$
is the adjusted pivot element $\tilde{a}_{pp}^{(p-1)}$ that was used at stage p.

Adjusted Symmetric Pivoting: The Matrix Sequence, II

Suppose that the adjusted symmetric pivoting process lasts exactly r steps, where $r \in \mathbb{N}_n$.

Starting from $\tilde{\mathbf{A}}^{(0)} = \mathbf{A}$, each successive step $k \in \mathbb{N}_r$, after adjusting any non-zero pivots, makes double symmetric uses of the determinant preserving downward pivoting operation $\tilde{\mathbf{E}}^{\downarrow}_{kk}$.

The double symmetric application of $\tilde{\mathbf{E}}^{\downarrow}_{kk}$ to the matrix $\tilde{\mathbf{A}}^{(k-1)}$ leads to the symmetric matrix $\tilde{\mathbf{A}}^{(k)} = \tilde{\mathbf{E}}^{\downarrow}_{kk} \tilde{\mathbf{A}}^{(k-1)} (\tilde{\mathbf{E}}^{\downarrow}_{kk})^\top$.

By induction on k, for each $k \in \mathbb{N}_r$ one has $\tilde{\mathbf{A}}^{(k)} = \mathbf{R}^{(k)} \mathbf{A} (\mathbf{R}^{(k)})^\top$ where $\mathbf{R}^{(k)} = \prod_{q=0}^{k-1} \tilde{\mathbf{E}}^{\downarrow}_{k-q,k-q} = \tilde{\mathbf{E}}^{\downarrow}_{kk} \tilde{\mathbf{E}}^{\downarrow}_{k-1,k-1} \cdots \tilde{\mathbf{E}}^{\downarrow}_{22} \tilde{\mathbf{E}}^{\downarrow}_{11}$, multiplied in that specific order.

Pivoting ceases with the matrix $\tilde{\mathbf{A}}^{(r)} = \mathbf{R}^{(r)} \mathbf{A} (\mathbf{R}^{(r)})^\top$.

Note that $\mathbf{R}^{(k)}$ is invertible as the product of invertible matrices.

From quadratic form invariance,
it follows that $\operatorname{def}(\mathbf{A}) = \operatorname{def}(\tilde{\mathbf{A}}^{(k)}) = \operatorname{def}(\tilde{\mathbf{A}}^{(r)})$ for all $k \in \mathbb{N}_r$.

Adjusted Symmetric Pivoting: The End, Case 1

The $(k+1)$th step starts from $\tilde{\mathbf{A}}^{(k)} = \begin{pmatrix} \tilde{\mathbf{D}}^{(k)}_{k\times k} & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(n-k)\times k} & \tilde{\mathbf{A}}^{(k)}_{>k,>k} \end{pmatrix}$.

Case 1: If the bottom right submatrix $\tilde{\mathbf{A}}^{(k)}_{>k,>k} = \mathbf{0}_{(n-k)\times(n-k)}$, then the $(k+1)$th pivot step is impossible.

All the r pivoting steps that are possible have been completed.

The final matrix takes the form $\begin{pmatrix} \mathbf{D}_{r\times r} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(n-r)\times(n-r)} \end{pmatrix}$,
where $\mathbf{D}_{r\times r}$ is an invertible $r \times r$ diagonal matrix with $r < n$.

Because $r < n$, quadratic form invariances implies
that the original symmetric matrix $\mathbf{A}$ cannot be definite.

It is positive or negative semi-definite according as the (non-zero) diagonal elements of $\mathbf{D}_{r\times r}$ are all positive or all negative
— that is, according as $\mathbf{D}_{r\times r}$ is positive or negative definite.

But if the diagonal of $\mathbf{D}_{r\times r}$
has both positive and negative elements, then $\mathbf{A}$ is indefinite.

Adjusted Symmetric Pivoting: The End, Case 2

The $(k+1)$th step starts from $\tilde{\mathbf{A}}^{(k)} = \begin{pmatrix} \tilde{\mathbf{D}}^{(k)}_{k \times k} & \mathbf{0}_{k \times (n-k)} \\ \mathbf{0}_{(n-k) \times k} & \tilde{\mathbf{A}}^{(k)}_{>k,>k} \end{pmatrix}$.

Case 2: Suppose that in the bottom right submatrix $\tilde{\mathbf{A}}^{(k)}_{>k,>k}$, at least two of the diagonal elements $(\tilde{a}^{(k)}_{qq})^n_{q=k+1}$ are zero, even though $\tilde{\mathbf{A}}^{(k)}_{>k,>k} \neq \mathbf{0}_{(n-k)\times(n-k)}$.

Then there exist a pair $p, q \in \mathbb{N}$ with $k < p < q \leq n$ such that $\tilde{a}^{(k)}_{pp} = \tilde{a}^{(k)}_{qq} = 0$ and yet $\tilde{a}^{(k)}_{pq} = \tilde{a}^{(k)}_{qp} \neq 0$.

So the simple test for semi-definiteness implies that the symmetric matrix $\tilde{\mathbf{A}}^{(k)}$ is indefinite.

By quadratic form invariance, so is the original matrix $\mathbf{A}$.

Adjusted Symmetric Pivoting: How to Adjust

The $(k+1)$th step starts from $\tilde{\mathbf{A}}^{(k)} = \begin{pmatrix} \tilde{\mathbf{D}}^{(k)}_{k\times k} & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(n-k)\times k} & \tilde{\mathbf{A}}^{(k)}_{>k,>k} \end{pmatrix}$.

Case 3: Suppose that $\tilde{a}^{(k)}_{k+1,k+1} = 0$ but the matrix $\tilde{\mathbf{A}}^{(k)}_{>k,>k}$ has at least one non-zero diagonal element $\tilde{a}^{(k)}_{qq}$ with $q > k+1$.

That is, there exists at least non-zero element $\tilde{a}^{(k)}_{qq}$ on the part of the diagonal below and to the right of $\tilde{a}^{(k)}_{k+1,k+1}$.

We adjust the pivot symmetrically along the diagonal by applying one sign corrected swap matrix along with its transpose:

1. first, we pre-multiply $\tilde{\mathbf{A}}^{(k)}$ by the $n \times n$ matrix $\hat{\mathbf{T}}^{q,k+1}_{n\times n}$, which first swaps rows q and $k+1$, then corrects the sign;
2. then we post-multiply $\hat{\mathbf{T}}^{q,k+1}_{n\times n}\tilde{\mathbf{A}}^{(k)}$ by the $n \times n$ transposed matrix $(\hat{\mathbf{T}}^{q,k+1}_{n\times n})^\top$, which first swaps columns q and $k+1$, then corrects the sign.

Adjusted Symmetric Pivoting: The Next Step

The $(k+1)$th step starts from $\tilde{\mathbf{A}}^{(k)} = \begin{pmatrix} \tilde{\mathbf{D}}^{(k)}_{k\times k} & \mathbf{0}_{k\times(n-k)} \\ \mathbf{0}_{(n-k)\times k} & \tilde{\mathbf{A}}^{(k)}_{>k,>k} \end{pmatrix}$.

Together, the two sign corrected swaps $\hat{\mathbf{T}}^{q,k+1}_{n\times n}$ and $(\hat{\mathbf{T}}^{q,k+1}_{n\times n})^\top$ move the original non-zero element $\tilde{a}^{(k)}_{qq}$ in $\tilde{\mathbf{A}}^{(k)}$ up left
to the $k+1, k+1$ position
in the adjusted matrix $\hat{\mathbf{T}}^{q,k+1}_{n\times n}\tilde{\mathbf{A}}^{(k)}(\hat{\mathbf{T}}^{q,k+1}_{n\times n})^\top$.

These prior sign corrected swaps
of both rows and columns q and $k+1$ allow us to apply
the standard symmetric pivoting operation based on $\mathbf{E}^{\downarrow}_{k+1,k+1}$
to this new version of the matrix $\tilde{\mathbf{A}}^{(k)}$

The result of this $(k+1)$th adjusted symmetric pivot step
is the next matrix $\tilde{\mathbf{A}}^{(k+1)} = \tilde{\mathbf{E}}^{\downarrow}_{k+1,k+1}\tilde{\mathbf{A}}^{(k)}(\tilde{\mathbf{E}}^{\downarrow}_{k+1,k+1})^\top$
where $\tilde{\mathbf{E}}^{\downarrow}_{k+1,k+1}$ is the adjusted pivot matrix $\mathbf{E}^{\downarrow}_{k+1,k+1}\hat{\mathbf{T}}^{q,k+1}_{n\times n}$.

How Adjusted Symmetric Pivoting Ends: Case A

Given an $n \times n$ symmetric matrix $\mathbf{A}$, adjusted symmetric pivoting can go on through steps $k = 1, 2, \ldots, r$
until it reaches a terminal symmetric matrix $\tilde{\mathbf{A}}^{(r)}$ with $r \leq n$.

There are two possible cases.

Case A: Symmetric pivoting may end after r steps with

$$\tilde{\mathbf{A}}^{(r)} = \begin{pmatrix} \mathbf{D}_{r \times r} & \mathbf{0}_{r \times (n-r)} \\ \mathbf{0}_{(n-r) \times r} & \mathbf{0}_{(n-r) \times (n-r)} \end{pmatrix} = \mathbf{diag}(a_{11}^{(r)}, \ldots, a_{rr}^{(r)}, \mathbf{0}_{n-r})$$

where $a_{kk}^{(r)}$ is the non-zero kth pivot element, for all $k \in \mathbb{N}_r$,
and so $\mathbf{D}_{r \times r}$ is an invertible $r \times r$ diagonal matrix.

Then the definiteness $\operatorname{def}(\mathbf{A})$ of the original matrix
is the same as the definiteness $\operatorname{def}(\tilde{\mathbf{A}}^{(r)})$ of the diagonal matrix,
which is easy to determine.

How Adjusted Symmetric Pivoting Ends: Case B

Case B: Alternatively symmetric pivoting may end after r steps with $\tilde{\mathbf{A}}^{(r)} = \begin{pmatrix} \mathbf{D}_{r\times r} & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(n-r)\times r} & \mathbf{S}_{(n-r)\times(n-r)} \end{pmatrix}$
where $\mathbf{D}_{r\times r} = \mathbf{diag}(a_{11}^{(r)}, \ldots, a_{rr}^{(r)})$ whose element $a_{kk}^{(r)}$ is the non-zero kth pivot element, for each $k \in \mathbb{N}_r$,
and $\mathbf{S}_{(n-r)\times(n-r)}$ is a non-zero symmetric matrix whose diagonal elements are all zero.

In this case too the definiteness $\text{def}(\mathbf{A})$ of the original matrix equals the definiteness of the non-diagonal matrix $\text{def}(\tilde{\mathbf{A}}^{(r)})$.

But in this case $\tilde{\mathbf{A}}^{(r)}$ is always indefinite.

Curtailing the Symmetric Pivoting

After k steps, symmetric pivoting reaches a partial diagonalization of the form $\tilde{\mathbf{A}}^{(k)} = \begin{pmatrix} \tilde{\mathbf{D}}^{(k)}_{k \times k} & \mathbf{0}_{k \times (n-k)} \\ \mathbf{0}_{(n-k) \times k} & \tilde{\mathbf{A}}^{(k)}_{>k,>k} \end{pmatrix}$.

The simple tests of definiteness we discussed earlier imply that, in case the matrix $\tilde{\mathbf{D}}^{(k)}_{k \times k}$ has:

1. two elements of different signs,
   both it and the original symmetric matrix are indefinite;
2. any zero element, neither it nor the original symmetric matrix can be either positive definite or negative definite.

These properties may allow a test of definiteness to be curtailed before the symmetric pivoting process has been completed.

Outline

Notation for Relevant Principal Minors

Recall the earlier definitions
of the principal and leading principal minors of a determinant.

Given any $n \times n$ symmetric matrix $\mathbf{A}$
and any non-empty subset $K \subseteq \mathbb{N}_n$ with $k = \#K$, let:

1. $\mathbf{A}_{K \times K}$ denote the $k \times k$ matrix
   whose elements form the symmetric submatrix $(a_{ij})_{(i,j) \in K \times K}$
   made up of the rows $i \in K$ and columns $j \in K$;
2. let $\Delta_k^K = |\mathbf{A}_{K \times K}|$ denote
   the corresponding principal minor of order k.

In case $K = \mathbb{N}_k = \{1, 2, \ldots, k\}$, let D_k denote $\Delta_k^{\mathbb{N}_k}$,
which is the unique leading principal minor of order k.

Sylvester's Criterion: General Statement

Theorem (Sylvester's criterion)

*Any $n \times n$ symmetric matrix $\mathbf{A}$
and associated quadratic form $\mathbf{x}^\top \mathbf{A} \mathbf{x}$ are both:*

$$
\begin{aligned}
\text{positive definite} &\iff D_k > 0 \text{ for all } k = 1, \ldots, n \\
\text{positive semidefinite} &\iff \Delta_k^K \geq 0 \text{ for all } \Delta_k^K \text{ of any order } k \\
\text{negative definite} &\iff (-1)^k D_k > 0 \text{ for all } k = 1, \ldots, n \\
\text{negative semidefinite} &\iff (-1)^k \Delta_k^K \geq 0 \text{ for all } \Delta_k^K \text{ of any order } k
\end{aligned}
$$

Otherwise the quadratic form $\mathbf{x}^\top \mathbf{A} \mathbf{x}$ and matrix $\mathbf{A}$ are indefinite.

Note that the conditions for $\mathbf{A}$ to be negative (semi-) definite
are exactly those for $-\mathbf{A}$ to be positive (semi-) definite.

The Case of a Quadratic Form in Two Variables

The general quadratic form in 2 variables is

$$(x, y) \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2hxy + by^2$$

If it is positive definite, it is positive whenever $x \neq 0$ and $y = 0$.

This implies that $ax^2 > 0$ whenever $x \neq 0$,
which holds if and only if the first leading principal minor $a > 0$.

But if $a > 0$, then completing the square implies that

$$ax^2 + 2hxy + by^2 = a(x + hy/a)^2 + (b - h^2/a)y^2$$

Given that $a > 0$, this is positive definite if and only if $b > h^2/a$,
or iff the second leading principal minor $ab - h^2 = \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$.

For the case of 2 variables, this proves
that the real-symmetric matrix $\mathbf{A}$ is positive definite
if and only if all the leading principal minors of $\mathbf{A}$ are positive.

Quadratic Form in Two Variables: Exercise

Exercise

For the case of a quadratic form in two variables, prove the other cases of Sylvester's criterion.

The Case of a Diagonal Quadratic Form

The general diagonal quadratic form in n variables is $\mathbf{x}^\top \Lambda \mathbf{x}$
where $\mathbf{x}$ is an n-vector
and Λ is an $n \times n$ diagonal matrix $\mathbf{diag}(\lambda_1, \ldots, \lambda_n)$.

Then the quadratic form $\mathbf{x}^\top \Lambda \mathbf{x} = \sum_{i=1}^n \lambda_i (x_i)^2$ and matrix Λ are:

1. positive definite if and only if $\lambda_i > 0$ for $i = 1, 2, \ldots, n$.

   This is true if and only if the k-fold product $\prod_{i=1}^k \lambda_i$
   is positive for each $k = 1, 2, \ldots, n$.

   But $\prod_{i=1}^k \lambda_i = |\, \mathbf{diag}(\lambda_1, \ldots, \lambda_k)|$
   is the leading principal minor D_k of order k for Λ.

2. positive semi-definite if and only if $\lambda_i \geq 0$ for $i = 1, 2, \ldots, n$.

   This is true if and only if the product $\prod_{i \in K} \lambda_i$ is nonnegative
   for every nonempty $K \subseteq \mathbb{N}_n = \{1, 2, \ldots, n\}$.

   But each product $\prod_{i \in K} \lambda_i$ equals
   the determinant $|\Lambda_{K \times K}|$ of the diagonal submatrix $\Lambda_{K \times K}$,
   which is the particular principal minor Δ_k^K of order $k = \# K$.

Toward the General Case

The formal proof of Sylvester's criterion
for a general $n \times n$ symmetric matrix $\mathbf{A}$
to be positive or negative definite will rely on:

1. showing that unadjusted symmetric pivoting, while it works, preserves each leading principal minor of $\mathbf{A}$;
2. using unadjusted symmetric pivoting to reduce the general case to the case when $\mathbf{A}$ is diagonal.

A similar argument allowing for adjusted symmetric pivoting
will treat the case when $\mathbf{A}$ is positive or negative semi-definite.

For large n ($n > 3$?), the best way to compute those minors,
however, may well be to use symmetric pivoting …

Outline

Symmetric Pivoting Preserves Leading Principal Minors

Given any $n \times n$ symmetric matrix $\mathbf{A}$ and any $k \in \mathbb{N}_n$,
let $\mathbf{A}_{\leq k, \leq k}$ denote the $k \times k$ matrix whose determinant $|\mathbf{A}_{\leq k, \leq k}|$ is the kth order leading principal minor.

Whenever $p < q \leq k$, the elementary row operation $\mathbf{A} \mapsto \mathbf{E}_{q+\alpha p}\mathbf{A}$ of adding α times row p to row q of $\mathbf{A}$ preserves not only $|\mathbf{A}|$,
but also each leading principal minor $|\mathbf{A}_{\leq k, \leq k}|$
when $\mathbf{E}_{q+\alpha p}$ is restricted to the $k \times k$ matrix $\mathbf{A}_{\leq k, \leq k}$.

The same property of leading principal minor preservation applies to each elementary column operation $\mathbf{A} \mapsto \mathbf{A}\mathbf{E}^\top_{q+\alpha p}$.

From this, it follows that leading principal minor preservation also applies to the symmetric pivoting operation $\mathbf{A} \mapsto \mathbf{E}^\downarrow_{pp}\mathbf{A}(\mathbf{E}^\downarrow_{pp})^\top$ when it is restricted to $\mathbf{A}_{\leq k, \leq k}$, where $k > p$.

Proof by Induction: Key Ideas

We will prove Sylvester's criterion
for a general $n \times n$ symmetric matrix $\mathbf{A}$.

Actually, we prove a superficially stronger necessary condition
for $\mathbf{A}$ to be positive definite:
all its principal minors, whether leading or not, must be positive.

The proof of this modified form of Sylvester's criterion
will be by induction on n.

The result is trivial when $n = 1$ and $\mathbf{A} = (a_{11})$,
whose only minor is $\det(a_{11}) = a_{11}$.

The induction hypothesis will be that Sylvester's modified criterion
is valid for any $m \times m$ symmetric matrix $\mathbf{A}$.

The induction step will be to prove
that if Sylvester's modified criterion is valid
for every $(n-1) \times (n-1)$ symmetric matrix,
then it is valid for every $n \times n$ symmetric matrix.

Proof by Induction in Four Parts

To repeat, the induction step will be to prove
that if Sylvester's modified criterion is valid
for every $(n-1) \times (n-1)$ symmetric matrix,
then it is valid for every $n \times n$ symmetric matrix.

This induction step has to be proved four times for Sylvester's:

1. modified necessary condition for a positive definite matrix;
2. sufficient condition for a positive definite matrix;
3. necessary condition for a positive semi-definite matrix;
4. sufficient condition for a positive semi-definite matrix.

Each of the four proofs will occupy two slides.

Recall that the criterion
for a negative definite or semi-definite symmetric matrix $\mathbf{A}$
is equivalent to the same criterion
for the positive definite or semi-positive symmetric matrix $-\mathbf{A}$.

1. Proving Necessity for a Positive Definite Matrix, I

(a) Suppose the $n \times n$ symmetric matrix $\mathbf{A}$ is positive definite.

(b) We have already argued that $a_{11} > 0$, as a diagonal element.

(c) So the downward pivoting matrix $\mathbf{E}_{11}^{\downarrow}$
is well defined and invertible.

(d) Because $\mathbf{E}_{11}^{\downarrow}$ is invertible and $\mathbf{A}$ is positive definite,
so is the block diagonal matrix $\mathbf{E}_{11}^{\downarrow}\mathbf{A}(\mathbf{E}_{11}^{\downarrow})^{\top} = \mathbf{diag}(a_{11}, \mathbf{B})$
where $\mathbf{B}$ is the $(n-1) \times (n-1)$ symmetric submatrix
that results from one round of symmetric pivoting.

(e) It follows from (d) that the block $\mathbf{B}$ is positive definite.

(f) Because $\mathbf{B}$ is positive definite, the induction hypothesis
implies that each principal minor Δ_k^K of $|\mathbf{B}|$ is positive.

1. Proving Necessity for a Positive Definite Matrix, II

(g) From (f) it follows that every principal minor of $\mathbf{diag}(a_{11}, \mathbf{B})$ which does not include the diagonal element a_{11} must be positive.

(h) But apart from a_{11} by itself,
all the other principal minors of $\mathbf{diag}(a_{11}, \mathbf{B})$
which do include the element a_{11} take the form $a_{11}\Delta_k^K$
where Δ_k^K is a principal minor of $|\mathbf{B}|$.

(i) Because $a_{11} > 0$, it follows from (f), (g) and (h)
that every principal minor of $\mathbf{diag}(a_{11}, \mathbf{B})$ must be positive.

(j) But the matrix $\mathbf{E}_{11}^{\downarrow}$ is determinant preserving,
so $\mathbf{E}_{11}^{\downarrow}\mathbf{A}(\mathbf{E}_{11}^{\downarrow})^\top = \mathbf{diag}(a_{11}, \mathbf{B})$ has the same principal minors as $\mathbf{A}$,
implying that all the principal minors of $\mathbf{A}$ are also positive. $\square$

2. Proving Sufficiency for a Positive Definite Matrix, I

(a) Suppose that every leading principal minor
of the $n \times n$ symmetric matrix $\mathbf{A}$ is positive.

(b) Note that (a) implies in particular
that the first leading principal minor satisfies $a_{11} > 0$.

(c) So the downward pivoting matrix $\mathbf{E}_{11}^{\downarrow}$
is well defined and determinant preserving.

(d) But (c) implies that $\mathbf{A}$ has the same leading principal minors
as the block diagonal matrix $\mathbf{E}_{11}^{\downarrow}\mathbf{A}(\mathbf{E}_{11}^{\downarrow})^{\top} = \mathbf{diag}(a_{11}, \mathbf{B})$
where $\mathbf{B}$ is the $(n-1) \times (n-1)$ symmetric submatrix
that results from one round of symmetric pivoting.

(e) Evidently, the leading principal minors of $|\,\mathbf{diag}(a_{11}, \mathbf{B})|$
take the form $a_{11}, a_{11}D_1, \ldots, a_{11}D_{n-1}$
where each D_k denotes the kth leading principal minor of $|\mathbf{B}|$.

2. Proving Sufficiency for a Positive Definite Matrix, II

(f) By the induction hypothesis, because (e) implies that all the leading principal minors of $|\mathbf{B}|$ are positive, the $(n-1) \times (n-1)$ symmetric matrix $\mathbf{B}$ is positive definite.

(g) Then, because (b) implies that $a_{11} > 0$, it follows from (f) that $\mathbf{diag}(a_{11}, \mathbf{B})$ is positive definite.

(h) Finally, because $\mathbf{E}_{11}^{\downarrow}$ is invertible and (g) implies that $\mathbf{diag}(a_{11}, \mathbf{B}) = \mathbf{E}_{11}^{\downarrow}\mathbf{A}(\mathbf{E}_{11}^{\downarrow})^{\top}$ is positive definite, it follows from quadratic form invariance that $\mathbf{A}$ is also positive definite.

3. Proving Necessity for a Positive Semi-Definite Matrix, I

(a) Suppose the $n \times n$ symmetric matrix $\mathbf{A}$ is positive semi-definite.

(b) In case all the diagonal elements of $\mathbf{A}$ are zero,
we must have $\mathbf{A} = \mathbf{0}_{n \times n}$, otherwise $\mathbf{A}$ would be indefinite.

(c) In the trivial case when $\mathbf{A} = \mathbf{0}_{n \times n}$, all minors of $|\mathbf{A}|$ are zero.

(d) Otherwise there exists a diagonal element $a_{pp} \neq 0$,
which is positive because $\mathbf{A}$ is positive semi-definite.

(e) Let $\hat{\mathbf{T}}_{1p}$ denote the sign adjusted swap of rows 1 and p.

Use it to define an adjusted symmetric pivot operation that gives
the symmetric matrix $\mathbf{E}_{11}^{\downarrow} \hat{\mathbf{T}}_{1p} \mathbf{A} (\mathbf{E}_{11}^{\downarrow} \hat{\mathbf{T}}_{1p})^{\top} = \mathbf{diag}(a_{pp}, \tilde{\mathbf{B}})$,
where $\tilde{\mathbf{B}}$ is an $(n-1) \times (n-1)$ symmetric matrix.

(f) Because $\mathbf{E}_{11}^{\downarrow} \hat{\mathbf{T}}_{1p}$ is invertible,
it follows from quadratic form invariance
that positive semi-definiteness of $\mathbf{A}$
implies the same for $\mathbf{diag}(a_{pp}, \tilde{\mathbf{B}})$, and so also for $\tilde{\mathbf{B}}$.

3. Proving Necessity for a Positive Semi-Definite Matrix, II

(g) Because $\tilde{\mathbf{B}}$ is positive semi-definite, the induction hypothesis implies that, for each $k \in \mathbb{N}_{n-1}$ and each $K \subseteq \mathbb{N}_{n-1}$ with $\#K = k$, the principal minor Δ_k^K of $|\tilde{\mathbf{B}}|$ is non-negative.

(h) Now each principal minor of $|\,\mathbf{diag}(a_{pp}, \tilde{\mathbf{B}})|$
that is not a principal minor of $|\tilde{\mathbf{B}}|$
must take the form $a_{pp}\,\Delta_k^K$ for some principal minor Δ_k^K of $|\tilde{\mathbf{B}}|$.

(i) But then $a_{pp} > 0$ by (d) and $\Delta_k^K \geq 0$ by (g),
so (h) implies that every principal minor of $|\,\mathbf{diag}(a_{pp}, \tilde{\mathbf{B}})|$ is non-negative.

(j) Now $\mathbf{diag}(a_{11}, \tilde{\mathbf{B}}) = \mathbf{E}_{11}^{\downarrow} \hat{\mathbf{T}}_{1p} \mathbf{A} (\mathbf{E}_{11}^{\downarrow} \hat{\mathbf{T}}_{1p})^{\top}$
where $\hat{\mathbf{T}}_{1p}$ is a sign-preserving swap of two rows
and the downward pivot matrix $\mathbf{E}_{11}^{\downarrow}$ is determinant preserving.

It follows that there is an obvious bijection
between each of the $2^n - 1$ principal minors of $|\,\mathbf{diag}(a_{pp}, \tilde{\mathbf{B}})|$
and a unique corresponding principal minor of $\mathbf{A}$.

(k) From (i) and (j), each principal minor of $\mathbf{A}$ is non-negative. $\square$

4. Proving Sufficiency for a Positive Semi-Definite Matrix, I

(a) Suppose that every principal minor
of the $n \times n$ symmetric matrix $\mathbf{A}$ is non-negative.

(b) In case all the diagonal elements of $\mathbf{A}$ are zero,
we must have $\mathbf{A} = \mathbf{0}_{n \times n}$, otherwise at least one principal minor
of the symmetric $\mathbf{A}$ would be negative.

(c) In the trivial case when $\mathbf{A} = \mathbf{0}_{n \times n}$,
the matrix $\mathbf{A}$ is evidently positive semi-definite.

(d) Otherwise there exists a non-zero diagonal element a_{pp},
which is positive because every principal minor of $\mathbf{A}$ is ≥ 0.

(e) Let $\hat{\mathbf{T}}_{1p}$ denote the sign adjusted swap of rows 1 and p.

Use it to define an adjusted symmetric pivot operation that gives
the $n \times n$ symmetric matrix $\mathbf{E}_{11}^{\downarrow} \hat{\mathbf{T}}_{1p} \mathbf{A} (\mathbf{E}_{11}^{\downarrow} \hat{\mathbf{T}}_{1p})^\top = \mathbf{diag}(a_{pp}, \tilde{\mathbf{B}})$,
where $a_{pp} > 0$ and $\tilde{\mathbf{B}}$ is an $(n-1) \times (n-1)$ symmetric matrix.

4. Proving Sufficiency for a Semi-Definite Matrix, II

(f) Because $\hat{\mathbf{T}}_{1p}$ is a sign-preserving swap of two rows whereas $\mathbf{E}_{11}^{\downarrow}$ is determinant preserving, there exists an obvious bijection between each of the principal minors of $|\mathbf{E}_{11}^{\downarrow}\hat{\mathbf{T}}_{1p}\mathbf{A}(\mathbf{E}_{11}^{\downarrow}\hat{\mathbf{T}}_{1p})^{\top}| = |\,\mathbf{diag}(a_{pp}, \tilde{\mathbf{B}})|$ and a unique corresponding principal minor of $\mathbf{A}$.

(g) Together (a) and (f) imply that each principal minor of $|\,\mathbf{diag}(a_{pp}, \tilde{\mathbf{B}})|$ is non-negative. So therefore is each principal minor of $|\tilde{\mathbf{B}}|$.

(h) By the induction hypothesis, (g) implies that the $(n-1) \times (n-1)$ matrix $\tilde{\mathbf{B}}$ is positive semi-definite.

(i) Because $a_{pp} > 0$, (h) implies that the $n \times n$ matrix $\mathbf{diag}(a_{pp}, \tilde{\mathbf{B}})$ is positive semi-definite.

(j) But $\mathbf{diag}(a_{pp}, \tilde{\mathbf{B}}) = \mathbf{E}_{11}^{\downarrow}\hat{\mathbf{T}}_{1p}\mathbf{A}(\mathbf{E}_{11}^{\downarrow}\hat{\mathbf{T}}_{1p})^{\top}$ where $\mathbf{E}_{11}^{\downarrow}\hat{\mathbf{T}}_{1p}$ is invertible.

(k) By quadratic form invariance, together (i) and (j) imply that $\mathbf{A}$ is positive semi-definite. □

Envoi

Though Sylvester's Criterion has been proved, remember it is here only because it is in various textbooks, including ours.

To establish the definiteness of a symmetric matrix,
especially if it is larger than 3×3,
one can and should use symmetric pivoting first.

Key reference for idea of symmetric pivoting:
Paul Binding (1991) "Simple Tests for Classifying Critical Points of Quadratics with Linear Constraints"
American Mathematical Monthly 98 (10): 949–954.

This paper also considers conditions for a quadratic form $\mathbf{x}^\top \mathbf{A} \mathbf{x}$
to be positive (semi-)definite subject to a constraint $\mathbf{K}\mathbf{x} = \mathbf{0}$,
— in the sense that $\mathbf{x}^\top \mathbf{A} \mathbf{x} > (\geq)\ 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$
that satisfy $\mathbf{K}\mathbf{x} = \mathbf{0}$.

The relevant tests involves "bordered Hessians".

We can finally move on at last!