# Lecture Notes 8: Dynamic Optimization
# Part 2: Optimal Control

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# Outline

# Statement of Basic Optimal Growth Problem

A consumption path $\mathbf{C}$ is a mapping $[t_0, t_1] \ni t \mapsto C(t) \in \mathbb{R}_+$.

A capital path $\mathbf{K}$ is a mapping $[t_0, t_1] \ni t \mapsto K(t) \in \mathbb{R}_+$.

Given $K(0)$ at time 0, the benevolent planner's objective
is to choose the pair $(\mathbf{C}, \mathbf{K})$ in order to maximize

$$J(\mathbf{C}, \mathbf{K}) := \int_{t_0}^{t_1} e^{-rt} u(C(t)) \mathrm{d}t$$

subject to the continuum of equality constraints

$$C(t) = f(K(t)) - \dot{K}(t)$$

Introduce the Lagrange multiplier path $\mathbf{p}$
as a mapping $[t_0, t_1] \ni t \mapsto p(t) \in \mathbb{R}_+$.

Use it to define the Lagrangian integral

$$\mathcal{L}_{\mathbf{p}}(\mathbf{C}, \mathbf{K}) = \int_{t_0}^{t_1} e^{-rt} u(C(t)) \mathrm{d}t - \int_{t_0}^{t_1} p(t)[C(t) - f(K(t)) + \dot{K}(t)] \mathrm{d}t$$

# Integrate by Parts

So we have the "Lagrangian"

$$\mathcal{L}_{\mathbf{p}}(\mathbf{C},\mathbf{K}) = \int_{t_0}^{t_1} e^{-rt} u(C(t)) \mathrm{d}t - \int_{t_0}^{t_1} p(t)[C(t) - f(K(t)) + \dot{K}(t)] \mathrm{d}t$$

Integrating the last term by parts yields

$$-\int_{t_0}^{t_1} p(t)\dot{K}(t)\mathrm{d}t = -\Big|_{t_0}^{t_1} p(t)K(t) + \int_{t_0}^{t_1} \dot{p}(t)\, K(t)\mathrm{d}t$$

Hence

$$\mathcal{L}_{\mathbf{p}}(\mathbf{C},\mathbf{K}) = \int_{t_0}^{t_1} \left[e^{-rt}u(C) + \dot{p}\, K - p\, C + p\, f(K)\right] \mathrm{d}t - \Big|_{t_0}^{t_1} p(t)K(t)$$

For the moment we ignore the last "endpoint terms",
and consider just the integral

$$\mathcal{I}_{\mathbf{p}}(\mathbf{C},\mathbf{K}) := \int_{t_0}^{t_1} \left[e^{-rt}u(C) + \dot{p}\, K - p\, C + p\, f(K)\right] \mathrm{d}t$$

# Maximizing the Integrand

Evidently the two paths $t \mapsto C(t)$ and $t \mapsto K(t)$ jointly maximize the integral

$$\mathcal{I}_{\mathbf{p}}(\mathbf{C}, \mathbf{K}) = \int_{t_0}^{t_1} \left[ e^{-rt} u(C) + \dot{p}\, K - p\, C + p\, f(K) \right] \mathrm{d}t$$

with $\mathbf{p}$ fixed, if and only if,
for almost all $t \in (t_0, t_1)$, the pair $(C(t), K(t))$
jointly maximizes w.r.t. $C$ and $K$ the integrand

$$e^{-rt} u(C) + \dot{p}\, K - p\, C + p\, f(K)$$

The first-order conditions for maximizing this integrand,
at any time $t \in (t_0, t_1)$, are found by differentiating partially:

1. w.r.t. $C(t)$ to obtain $e^{-rt} u'(C(t)) = p(t)$;
2. w.r.t. $K(t)$ to obtain $\dot{p}(t) = -p(t)\, f'(K(t))$;

There is also the equality constraint $\dot{K}(t) = f(K(t)) - C(t)$.

# Outline

# Statement of Sufficient Conditions

Consider the static problem of maximizing
the objective function $\mathbb{R}^n \supseteq D \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R}$
subject to the vector constraint $\mathbf{g}(\mathbf{x}) \leqq \mathbf{a} \in \mathbb{R}^m$
where $\mathbb{R}^n \supseteq D \ni \mathbf{x} \mapsto \mathbf{g}(\mathbf{x}) \in \mathbb{R}^m$.

## Definition

The pair $(\mathbf{p}, \mathbf{x}^*) \in \mathbb{R}^m \times \mathbb{R}^n$
jointly satisfies complementary slackness just in case:

$$\text{(i) } \mathbf{p}^\top \geqq 0; \quad \text{(ii) } \mathbf{g}(\mathbf{x}^*) \leqq a; \quad \text{(iii) } \mathbf{p}^\top[\mathbf{g}(\mathbf{x}^*) - \mathbf{a}] = 0$$

These are generally summarized as $\mathbf{p}^\top \geqq 0, \; \mathbf{g}(\mathbf{x}^*) \leqq \mathbf{a}$ (comp). □

## Theorem

*Suppose that $\mathbf{x}^* \in \mathbb{R}^n$ is a global maximum over the domain $D$ of the Lagrangian function $\mathcal{L}_\mathbf{p}(\mathbf{x}) = f(\mathbf{x}) - \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{a}]$ where $(\mathbf{p}, \mathbf{x}^*) \in \mathbb{R}^m \times \mathbb{R}^n$ jointly satisfy the complementary slackness conditions.*

*Then $\mathbf{x}^*$ is a global maximum of $f(\mathbf{x})$ subject to $\mathbf{g}(\mathbf{x}) \leqq \mathbf{a}$.*

# Proof of Sufficient Conditions

## Proof.

By definition of the Lagrangian $\mathcal{L}_\mathbf{p}(\mathbf{x}) = f(\mathbf{x}) - \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - a]$,
for every $\mathbf{x} \in D$ one has

$$f(\mathbf{x}) - f(\mathbf{x}^*) = \mathcal{L}_\mathbf{p}(\mathbf{x}) + \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{a}] - \mathcal{L}_\mathbf{p}(\mathbf{x}^*) - \mathbf{p}^\top[\mathbf{g}(\mathbf{x}^*) - \mathbf{a}]$$

By hypothesis one has $\mathcal{L}_\mathbf{p}(\mathbf{x}) \leq \mathcal{L}_\mathbf{p}(\mathbf{x}^*)$ for all $\mathbf{x} \in D$, so

$$f(\mathbf{x}) - f(\mathbf{x}^*) \leq \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{a}] - \mathbf{p}^\top[\mathbf{g}(\mathbf{x}^*) - \mathbf{a}] = \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)]$$

But the complementary slackness conditions

$$\mathbf{p}^\top \geqq \mathbf{0}, \ \mathbf{g}(\mathbf{x}^*) \leqq \mathbf{a} \text{ (comp)}$$

imply that for any $\mathbf{x} \in D$ satisfying the constraint $\mathbf{g}(\mathbf{x}) \leqq \mathbf{a}$
one has $\mathbf{p}^\top\mathbf{g}(\mathbf{x}) \leq \mathbf{p}^\top\mathbf{a}$, whereas $\mathbf{p}^\top\mathbf{g}(\mathbf{x}^*) = \mathbf{p}^\top\mathbf{a}$.

Hence $f(\mathbf{x}) - f(\mathbf{x}^*) \leq \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)] \leq \mathbf{p}^\top\mathbf{a} - \mathbf{p}^\top\mathbf{a} = 0.$ □

# A Cheap Result on Necessary Conditions

Recall that we are considering the problem of choosing $\mathbf{x} \in D \subseteq \mathbb{R}^n$ in order to maximize $f(\mathbf{x})$ subject to $\mathbf{g}(\mathbf{x}) \leqq \mathbf{a}$.

Suppose we know that any solution $\mathbf{x}^*$ must be unique.

This will be the case, for example, if:

1. the common domain $D$ of the functions $D \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R}$ and $D \ni \mathbf{x} \mapsto \mathbf{g}(\mathbf{x}) \in \mathbb{R}^m$ is a convex subset of $\mathbb{R}^n$;
2. the objective function $D \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R}$ is strictly concave;
3. each component function $D \ni \mathbf{x} \mapsto g_j(\mathbf{x}) \in \mathbb{R}$ of the vector function $D \ni \mathbf{x} \mapsto \mathbf{g}(\mathbf{x}) \in \mathbb{R}^m$ is convex.

Suppose that the pair $(\mathbf{p}, \mathbf{x}^*) \in \mathbb{R}^m \times D$ jointly satisfy the sufficient conditions for maximizing the Lagrangian while also meeting the complementary slackness conditions.

Then it is necessary that the only possible maximum satisfy these sufficient conditions!

# Outline

# Statement of General Problem

Given the time interval $[t_0, t_1] \subset \mathbb{R}$,
consider the general one-variable optimal control problem
of choosing paths:

1. $[t_0, t_1] \ni t \mapsto x(t) \in \mathbb{R}$ of states;
2. $[t_0, t_1] \ni t \mapsto u(t) \in \mathbb{R}$ of controls.

The objective functional is taken to be the integral

$$\int_{t_0}^{t_1} f(t, x(t), u(t))\, \mathrm{d}t$$

We fix the initial state $x(t_0) = x_0$, where $x_0$ is given.

We leave the terminal state $x(t_1)$ free.

Finally, we impose the dynamic constraint $\dot{x} = g(t, x, u)$
at every time $t \in [t_0, t_1]$.

# The Lagrangian Integral

Consider the path $[t_0, t_1] \ni t \mapsto p(t) \in \mathbb{R}$
of a single costate variable or shadow price $p$.

Here $p(t)$ is the Lagrange multiplier
associated with the dynamic constraint at time $t$.

Then, after dropping the time argument from $p$, $x$ and $u$,
the associated "Lagrangian integral" is

$$\mathcal{L} = \int_{t_0}^{t_1} f(t, x, u)\,\mathrm{d}t - \int_{t_0}^{t_1} p[\dot{x} - g(t, x, u)]\,\mathrm{d}t$$

Because $\frac{\mathrm{d}}{\mathrm{d}t} p\,x = \dot{p}\,x + p\,\dot{x}$, integrating by parts
gives $\int_{t_0}^{t_1} p\,\dot{x}\,\mathrm{d}t = -\int_{t_0}^{t_1} \dot{p}\,x\,\mathrm{d}t + \Big|_{t_0}^{t_1} p\,x$ and so

$$\mathcal{L} = \int_{t_0}^{t_1} [f(t, x, u) + \dot{p}\,x + p\,g(t, x, u)]\,\mathrm{d}t - \Big|_{t_0}^{t_1} p\,x$$

# The Hamiltonian

## Definition

For the problem of maximizing $\int_{t_0}^{t_1} f(t, x, u)\, \mathrm{d}t$
subject to $\dot{x} = g(t, x, u)$,
the Hamiltonian function is defined as

$$H(t, x, u, p) := f(t, x, u) + p\, g(t, x, u) \quad \square$$

With this definition, the integral part of the Lagrangian, which is

$$\int_{t_0}^{t_1} [f(t, x, u) + \dot{p}\, x + p\, g(t, x, u)]\, \mathrm{d}t$$

can be written as $\int_{t_0}^{t_1} [H(t, x, u, p) + \dot{p}\, x]\, \mathrm{d}t$.

# The Maximum Principle

Recall the definition $H(t, x, u, p) := f(t, x, u) + p\, g(t, x, u)$.

Definition

According to the maximum principle, for a.e. $t \in [t_0, t_1]$,
an optimal control should satisfy

$$u^*(t) \in \arg\max_u H(t, x, u, p) \text{ where } x = x(t) \text{ and } p = p(t)$$

Moreover the co-state variable $p(t)$ should evolve
according to the vector differential equation

$$\dot{p} = -H'_x(t, x, u, p)$$

where $H'_x(t, x, u, p)$ denotes the partial derivative
of the Hamiltonian $H$ w.r.t. the state $x$.

# An Extended Maximum Principle

## Definition

Add an extra term $\dot{p}\,x$ to the Hamiltonian $H(t, x, u, p)$
in order to give the extended Hamiltonian

$$\tilde{H}(t, x, u, p) := H(t, x, u, p) + \dot{p}\,x = f(t, x, u) + p\,g(t, x, u) + \dot{p}\,x$$

According to the extended maximum principle,
for a.e. (almost every) time $t \in [t_0, t_1]$, one should have

$$(u^*(t), x^*(t)) \in \arg\max_{(u,x)} \tilde{H}(t, x, u, p(t))$$

## Remark

*The first-order conditions for maximizing $\tilde{H}(t, x, u, p)$ include*

$$\dot{p} = -f'_x(t, x, u) - p g'_x(t, x, u) = -H'_x(t, x, u, p)$$

*as required by the maximum principle.*

# Outline

# A Macroeconomic Quadratic Control Problem: Statement

Let $c > 0$ denote an adjustment cost parameter.

Consider the problem of choosing the path $t \mapsto (u(t), x(t)) \in \mathbb{R}^2$ in order to minimize the quadratic integral $\int_0^T (x^2 + cu^2)\,\mathrm{d}t$ subject to the dynamic constraint $\dot{x} = u$, as well as the initial condition $x(0) = x_0$ and the terminal condition allowing $x(T)$ to be chosen freely.

The associated Hamiltonian is

$$H = -x^2 - cu^2 + p\,u$$

with a minus sign to convert the minimization problem into a maximization problem.

The associated extended Hamiltonian, including the extra term $\dot{p}\,x$, is

$$\tilde{H} = -x^2 - cu^2 + p\,u + \dot{p}\,x$$

# First-Order Conditions

Consider the problem of maximizing, at any time $t \in [0, T]$,
either the Hamiltonian $H = -x^2 - cu^2 + p\,u$,
or the extended Hamiltonian $\tilde{H} = -x^2 - cu^2 + p\,u + \dot{p}\,x$

The first-order conditions include $0 = H'_u = \tilde{H}'_u = -2c\,u + p$.

Either of these two equivalent conditions implies that $u^* = p/2c$.

A second first-order condition for maximizing w.r.t. $x$
the extended Hamiltonian $\tilde{H}$ is $\dot{p} = -H'_x = 2x$.

This coincides with the co-state differential equation.

Combining this with the dynamic constraint $\dot{x} = u$
leads to the following coupled pair of differential equations:

$$\dot{p} = -H'_x = 2x \quad \text{and} \quad \dot{x} = u^* = p/2c$$

# Example: Solving the Coupled Pair

In order to solve the coupled pair

$$\dot{p} = 2x \quad \text{and} \quad \dot{x} = p/2c$$

- differentiate the first equation w.r.t. $t$ to obtain $\ddot{p} = 2\dot{x}$;
- substitute in the second equation to obtain $\ddot{p} = 2\dot{x} = p/c$.

We need to consider the second-order differential equation

$$\ddot{p} = p/c$$

in $p$, whose associated characteristic equation is $\lambda^2 - 1/c = 0$.

The two roots are $\lambda_{1,2} = \pm c^{-1/2} = \pm r$ where $r := c^{-1/2}$.

The general solution of this homogeneous equation
is $p = Ae^{rt} + Be^{-rt}$ for arbitrary constants $A$ and $B$.

# Explicit Solution

In addition to $p = Ae^{rt} + Be^{-rt}$ with $r := c^{-1/2}$ and $\dot{p} = 2x$, we also have $\dot{x} = p/2c$, along with the initial condition $x(0) = x_0$ and the terminal condition $p(T) = 0$.

This terminal condition implies $Ae^{rT} + Be^{-rT} = 0$, from which one obtains $B = -Ae^{2rT}$.

Also differentiating $p = Ae^{rt} + Be^{-rt}$ w.r.t. $t$ implies $\dot{p} = r(Ae^{rt} - Be^{-rt})$.

At time $t = 0$ one has $\dot{p}(0) = 2x_0$ and so $r(A - B) = 2x_0$.

Substituting $B = -Ae^{2rT}$ gives $r(A + Ae^{2rT}) = 2x_0$,
so $A = 2x_0/r(1 + e^{2rT}) = 2x_0 e^{-rT}/r(e^{-rT} + e^{rT})$
implying that $B = -2x_0 e^{rT}/r(e^{-rT} + e^{rT})$.

So $p = Ae^{rt} + Be^{-rt} = 2x_0(e^{-r(T-t)} - e^{r(T-t)})/r(e^{-rT} + e^{rT})$

and $x = \dot{p}/2 = x_0(e^{-r(T-t)} + e^{r(T-t)})/(e^{-rT} + e^{rT})$.

Also $u = \dot{x} = rx_0(e^{-r(T-t)} - e^{r(T-t)})/(e^{-rT} + e^{rT})$.

# The Case of an Infinite Horizon

Multiply both numerator and denominator of the right-hand side of each equation by $e^{-rT}$, leading to the explicit solution:

$$
\begin{aligned}
p(t) &= \frac{2x_0\left[e^{-r(T-t)} - e^{r(T-t)}\right]}{r\left[e^{-rT} + e^{rT}\right]} &= \frac{2x_0\left[e^{-r(2T-t)} - e^{-rt}\right]}{r(e^{-2rT} + 1)} \\
x(t) &= \frac{x_0\left[e^{-r(T-t)} + e^{r(T-t)}\right]}{r(e^{-rT} + e^{rT})} &= \frac{x_0\left[e^{-r(2T-t)} + e^{-rt}\right]}{r(e^{-2rT} + 1)} \\
u(t) &= \frac{x_0\left[e^{-r(T-t)} - e^{r(T-t)}\right]}{e^{-rT} + e^{rT}} &= \frac{x_0\left[e^{-r(2T-t)} - e^{-rt}\right]}{e^{-2rT} + 1}
\end{aligned}
$$

Taking the limit as $T \to \infty$, one has $p(t) \to -2x_0 e^{-rt}/r$.

Similarly $x(t) = \frac{1}{2}\dot{p} \to x_0 e^{-rt}$, and $u(t) = \dot{x}(t) \to -x_0 e^{-rt}$.

Finally, $(p(t), x(t), u(t)) \to (0, 0, 0)$ as $t \to \infty$.

See page 311 of FMEA.

# Outline

# Mangasarian and Arrow's Sufficient Conditions

At any fixed time $t$, let $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ be a stationary point w.r.t. $(\mathbf{x}, \mathbf{u})$ of the extended Hamiltonian

$$\tilde{H}(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t)) := H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t)) + \dot{\mathbf{p}}^\top(t)\,\mathbf{x}$$

That is, suppose that the respective partial gradients satisfy

$$H'_{\mathbf{u}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t)) = 0 \quad \text{and} \quad \dot{\mathbf{p}}(t) = -H'_{\mathbf{x}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t))$$

Here are two alternative sufficient conditions for $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ to maximize the extended Hamiltonian.

1. See FMEA Theorem 9.7.1, due to Mangasarian.
   Suppose that $(\mathbf{x}, \mathbf{u}) \mapsto H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t))$ is concave,
   which implies that $(\mathbf{x}, \mathbf{u}) \mapsto \tilde{H}(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t))$ is also concave.
2. See FMEA Theorem 9.7.2, due to Arrow.
   Define $\hat{H}(t, \mathbf{x}, \mathbf{p}(t)) := \max_{\mathbf{u}} H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t))$,
   and suppose that $\mathbf{x} \mapsto \hat{H}(t, \mathbf{x}, \mathbf{p}(t))$ is concave.

# Sufficient Conditions

Consider the single variable problem
of choosing the paths $t \mapsto (x(t), u(t)) \in \mathbb{R}^2$
in order to maximize $\int_0^T f(t, x, u)\, dt$
subject to $\dot{x} \leq g(t, x, u)$ (all $t \in [0, T]$)
as well as $x(0) \leq x_0$, $x(T) \geq x_T$.

Including the extra term $\dot{p}\, x$, the extended Hamiltonian is

$$\tilde{H}(t, x, u, p) = f(t, x, u) + p\, g(t, x, u) + \dot{p}\, x$$

Suppose that for all $t \in [0, T]$ the path $t \mapsto (x^*(t), u^*(t)) \in \mathbb{R}^2$
satisfies the extended maximization condition

$$(x^*(t), u^*(t)) \in \arg\max_{x,u} \tilde{H}(t, x, u, p(t))$$

as well as the three complementary slackness conditions:

1. $p(t) \geq 0, \; \dot{x}^*(t) \leq g(t, x^*(t), u^*(t))$ (comp) (all $t \in [0, T]$);
2. $p(0) \geq 0, \; x^*(0) \leq x_0$ (comp);
3. $p(T) \geq 0, \; x^*(T) \geq x_T$ (comp).

# Proof of Sufficiency, I

Consider any alternative feasible path $t \mapsto (x(t), u(t))$ satisfying all the constraints.

Define $D(\mathbf{x}, \mathbf{u}) := \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))] \, \mathrm{d}t$.

After dropping the time arguments from $x(t), u(t), x^*(t), u^*(t)$, the definition $\tilde{H} = f + pg + p\dot{x}$ implies that

$$D(\mathbf{x}, \mathbf{u}) = \int_0^T \left\{ \left[ \tilde{H}(t, x, u, p) - p\, g(t, x, u) - \dot{p}\, x \right] - \left[ \tilde{H}(t, x^*, u^*, p) - p\, g(t, x^*, u^*) - \dot{p}\, x^* \right] \right\} \mathrm{d}t$$

The maximization hypothesis implies that, for all $t \in (0, T)$, one has $\tilde{H}(t, x(t), u(t), p(t)) \leq \tilde{H}(t, x^*(t), u^*(t), p(t))$.

From this it follows that

$$D(\mathbf{x}, \mathbf{u}) \leq \int_0^T \{[-p\, g(t, x, u) - \dot{p}\, x] - [-p\, g(t, x^*, u^*) - \dot{p}\, x^*]\} \, \mathrm{d}t$$

# Proof of Sufficiency, II

We have shown that

$$D(\mathbf{x}, \mathbf{u}) \leq \int_0^T \{[-p\, g(t, x, u) - \dot{p}\, x] - [-p\, g(t, x^*, u^*) - \dot{p}\, x^*]\}\, \mathrm{d}t$$

But feasibility implies that $\dot{x}(t) \leq g(t, x, u)$
and prices satisfy $p(t) \geq 0$, so $p(t)\, \dot{x}(t) \leq p(t)\, g(t, x, u)$.

Furthermore, the complementary slackness conditions for optimality imply that $p(t)\, g(t, x^*(t), u^*(t)) = p(t)\, \dot{x}^*(t)$.

It follows that

$$\begin{aligned} D(\mathbf{x}, \mathbf{u}) &\leq \int_0^T [-p\, \dot{x} - \dot{p}\, x + p\, \dot{x}^* + \dot{p}\, x^*]\, \mathrm{d}t \\ &= \int_0^T \tfrac{\mathrm{d}}{\mathrm{d}t} [-p(t)\, x(t) + p(t)\, x^*(t)]\, \mathrm{d}t \\ &= -p(T)\, [x(T) - x^*(T)] + p(0)\, [x(0) - x^*(0)] \end{aligned}$$

# Proof of Sufficiency, III

So far, we have shown that

$$D(\mathbf{x}, \mathbf{u}) \leq -p(T)\,[x(T) - x^*(T)] + p(0)\,[x(0) - x^*(0)]$$

But, together with feasibility and non-negativity of prices, the second and third complementary slackness conditions regarding the endpoints at times $t = 0$ and $t = T$ imply that

$$\begin{array}{rclcrcl} p(T)\,x(T) & \geq & p(T)\,x_T & ; & p(T)\,x^*(T) & = & p(T)\,x_T\ ; \\ p(0)\,x(0) & \leq & p(0)\,x_0 & ; & p(0)\,x^*(0) & = & p(0)\,x_0\ . \end{array}$$

It follows that

$$p(T)\,x(T) \geq p(T)\,x^*(T) \quad \text{and} \quad p(0)\,x(0) \leq p(0)\,x^*(0)$$

which together imply that $D(\mathbf{x}, \mathbf{u}) \leq 0$.

Finally, after recalling the definition

$$D(\mathbf{x}, \mathbf{u}) := \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))]\, dt$$

one conclues that the path $t \mapsto (x^*(t), u^*(t))$ is optimal. □

# Outline

# The Infinite Horizon Problem

We consider the problem of choosing $[0, \infty) \ni t \mapsto (x(t), u(t))$ to maximize the infinite horizon objective functional

$$\int_0^\infty f(t, x(t), u(t))\, dt$$

subject to $\dot{x} = g(t, x, u)$ at every time $t \in [0, \infty)$, as well as $x(0) = x_0$, where $x_0$ is given.

As before, the extended maximum principle suggests looking for a path $[0, \infty) \ni t \mapsto p(t)$ of co-state variables, as well as a path $[0, \infty) \ni t \mapsto (x^*(t), u^*(t))$ of the state and control variables which maximizes the extended Hamiltonian

$$\tilde{H}(t, x, u, p) := f(t, x, u) + p(t)\, g(t, x, u) + \dot{p}(t)\, x$$

— i.e., for (almost) all $t \in [0, \infty)$ one has

$$(x^*(t), u^*(t)) \in \arg\max_{(u,x)} \tilde{H}(t, x, u, p)$$

# Implications of the Extended Maximum Principle, I

Consider any alternative feasible path $t \mapsto (x(t), u(t))$ satisfying all the constraints.

We start by repeating our earlier argument for a finite horizon.

Define $D^T(\mathbf{x}, \mathbf{u}) := \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))] \, dt$.

After dropping the time arguments from $x(t), u(t), x^*(t), u^*(t)$, this difference $D^T(\mathbf{x}, \mathbf{u})$ equals

$$\int_0^T \Big\{ \Big[ \tilde{H}(t, x, u, p) - p\, g(t, x, u) - \dot{p}\, x \Big] - \Big[ \tilde{H}(t, x^*, u^*, p) - p\, g(t, x^*, u^*) - \dot{p}\, x^* \Big] \Big\} \, dt$$

The extended maximum principle implies that for all $t \in [0, T]$ one has

$$\tilde{H}(t, x(t), u(t), p(t)) \leq \tilde{H}(t, x^*(t), u^*(t), p(t))$$

# Implications of the Extended Maximum Principle, II

Arguing as before, from $(x^*(t), u^*(t)) \in \arg\max_{(u,x)} \tilde{H}(t, x, u, p)$
where $\tilde{H}(t, x, u, p) := f(t, x, u) + p(t)\, g(t, x, u) + \dot{p}(t)\, x$,
it follows that for all finite $T$ the difference $D^T(\mathbf{x}, \mathbf{u})$ satisfies

$$
\begin{aligned}
D^T(\mathbf{x}, \mathbf{u}) &:= \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))]\, \mathrm{d}t \\
&= \int_0^T \left\{ \left[ \tilde{H}(t, x, u, p) - p\, g(t, x, u) - \dot{p}\, x \right] \right. \\
&\qquad \left. - \left[ \tilde{H}(t, x^*, u^*, p) - p\, g(t, x^*, u^*) - \dot{p}\, x^* \right] \right\} \mathrm{d}t \\
&= \int_0^T \left[ \tilde{H}(t, x, u, p) - \tilde{H}(t, x^*, u^*, p) \right] \mathrm{d}t \\
&\quad - \int_0^T [p\, g(t, x, u) + \dot{p}\, x - p\, g(t, x^*, u^*) - \dot{p}\, x^*]\, \mathrm{d}t \\
&\leq - \int_0^T [p\, \dot{x} + \dot{p}\, x - p\, \dot{x}^* - \dot{p}\, x^*]\, \mathrm{d}t \\
&= - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} [p\, x - p\, x^*]\, \mathrm{d}t \\
&= -p(T)\, [x(T) - x^*(T)] + p(0)\, [x(0) - x^*(0)] \\
&= p(T)\, [x^*(T) - x(T)] \text{ given that } x(0) = x^*(0) = x_0
\end{aligned}
$$

# A Transversality Condition

Consider the transversality condition

$$\limsup_{T\to\infty} p(T)\ [x^*(T) - x(T)] = 0$$

If this were satisfied, it would imply that

$$\begin{aligned} 0 &\geq \limsup_{T\to\infty} D^T(\mathbf{x}, \mathbf{u}) \\ &= \limsup_{T\to\infty} \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))]\, dt \end{aligned}$$

In the case when

$$\int_0^T f(t, x^*(t), u^*(t))\, dt \to \int_0^\infty f(t, x^*(t), u^*(t))\, dt$$

as $T \to \infty$, it would imply that

$$\limsup_{T\to\infty} \int_0^T f(t, x(t), u(t))\, dt \leq \int_0^\infty f(t, x^*(t), u^*(t))\, dt$$

# Malinvaud's Transversality Condition

Edmond Malinvaud (1953) "Capital Accumulation and Efficient Allocation of Resources" *Econometrica* 21: 233–268.

In many economic contexts, feasibility requires that, for all $t$, one has both $x(t) \geq 0$ and $\dot{x}(t) \leq g(t, x(t), u(t))$.

Then, since $p(t) \geq 0$,
for any alternative feasible path $x(t)$ and any terminal time $T$, one has $p(T)\,[x^*(T) - x(T)] \leq p(T)\,x^*(T)$.

## Definition

The Malinvaud transversality condition
is that $p(T)\,x^*(T) \to 0$ as $T \to \infty$.

When this Malinvaud transversality condition is satisfied, evidently

$$\limsup_{T\to\infty} p(T)\ [x^*(T) - x(T)] \leq \limsup_{T\to\infty} p(T)\,x^*(T) = 0$$

Hence, the general transversality condition is also satisfied.

# Outline

# A Problem with Exponential Discounting

Consider the general problem of choosing paths:

1. $[t_0, t_1] \ni t \mapsto x(t) \in \mathbb{R}$ of states;
2. $[t_0, t_1] \ni t \mapsto u(t) \in \mathbb{R}$ of controls.

The objective functional is taken to be the integral

$$\int_{t_0}^{t_1} e^{-rt} f(x(t), u(t)) \, dt$$

where: (i) $f$ is independent of $t$;
(ii) there is a constant discount rate $r$
and associated exponential discount factor $e^{-rt}$.

Assume too that the dynamic constraint is $\dot{x} = g(x, u)$,
at every time $t \in [t_0, t_1]$, where $g$ is independent of $t$.

Fix the initial state $x(t_0) = x_0$, where $x_0$ is given.

But leave the terminal state $x(t_1)$ free.

# Present versus Current Value Hamiltonian

Up to now, we have considered the present value Hamiltonian

$$H(t, x, u, p) := e^{-rt} f(x, u) + p\, g(x, u)$$

We remove the discount factor $e^{-rt}$
by defining the current value Hamiltonian

$$H^C(x, u, q) := f(x, u) + q\, g(x, u)$$

with the current value co-state variable $q := e^{rt}\, p$.

These definitions imply that

$$H(t, x, u, p) = e^{-rt}[f(x, u) + e^{rt}\, p\, g(x, u)] = e^{-rt} H^C(x, u, q)$$

where $q = e^{rt} p$, so $\dot{q} = re^{rt} p + e^{rt} \dot{p} = rq + e^{rt} \dot{p}$.

# Present and Current Value Maximum Principles

The (present value) maximum principle
states that for (almost) all $t \in [0, \infty)$ one has

$$u^*(t) \in \arg\max_u H(t, x, u, p) \quad \text{and} \quad \dot{p} = -H'_x(t, x, u, p)$$

By definition, one has $H(t, x, u, p) = e^{-rt} H^C(x, u, q)$
where $q = e^{rt} p$.

Because $e^{-rt}$ is independent of $u$,
it follows that $u^*(t) \in \arg\max_u H^C(x, u, q)$.

Also $\dot{q} - rq = e^{rt}\dot{p} = -e^{rt} H'_x(t, x, u, p) = -H^{C\prime}{}_x(x, u, q)$.

We have derived the current value maximum principle
states that for (almost) all $t \in [0, \infty)$ one has

$$u^*(t) \in \arg\max_u H^C(x, u, q) \quad \text{and} \quad \dot{q} - rq = -H^{C\prime}{}_x(x, u, q)$$

# Outline

# Statement of the Problem

The problem will be to choose:

1. a consumption stream $\mathbb{R}_+ \ni t \mapsto C(t) \in \mathbb{R}_{++}$;
2. a stream $\mathbb{R}_+ \ni t \mapsto K(t) \in \mathbb{R}_{++}$ of capital stocks.

At any time $t$, given capital $K$, output will be $Y = a\,K - b\,K^2$, where $a, b \in \mathbb{R}$ are positive parameters, with $a > r > 0$.

Output is divided between consumption $C$ and investment $\dot{K}$, so $\dot{K} = Y - C$; there is no depreciation.

The planner's objective is to maximize the utility integral $\int_0^\infty e^{-rt}\, u(C(t))\, \mathrm{d}t$.

We assume that the utility function $\mathbb{R}_{++} \ni C \mapsto u(C)$ takes the isoelastic form with $u'(C) = C^{-\epsilon}$.

The constant elasticity parameter $\epsilon > 0$ is a constant degree of relative fluctuation aversion.

# The Current Value Maximum Principle

The optimal growth problem is to maximize $\int_0^\infty e^{-rt}\, u(C(t))\, dt$ subject to $\dot{K} = a\,K - b\,K^2 - C$ where $u'(C) = C^{-\epsilon}$.

With $\lambda$ as the co-state variable, the current value Hamiltonian is

$$H^C(K, C) := u(C) + \lambda(a\,K - b\,K^2 - C)$$

The first-order condition for maximizing $(K, C) \mapsto H^C(K, C)$ w.r.t. $C$ is $u'(C) = \lambda$, which implies $C^{-\epsilon} = \lambda$ and so $C = \lambda^{-1/\epsilon}$.

Because $C \mapsto u(C)$ is strictly concave, this is the unique maximum.

The co-state variable evolves according to the equation

$$\dot{\lambda} - r\,\lambda = -H_K^{C\prime}(K, C) = -\lambda\,(a - 2\,b\,K)$$

Finally, therefore, we have the coupled differential equations

$$\dot{K} = a\,K - b\,K^2 - \lambda^{-1/\epsilon} \quad \text{and} \quad \dot{\lambda} = \lambda\,(r - a + 2\,b\,K)$$

# Steady State of Coupled Differential Equations

The coupled differential equations

$$\dot{K} = a\,K - b\,K^2 - \lambda^{-1/\epsilon} \quad \text{and} \quad \dot{\lambda} = \lambda\,(r - a + 2\,b\,K)$$

have a steady state at any point satisfying $\dot{K} = 0$ and $\dot{\lambda} = 0$.

There is a unique steady state at the point $(K, \lambda) = (K^*, \lambda^*)$ with $K^* = (r - a)/2b$ and $\lambda^* = [K^*(a - bK^*)]^{-\epsilon}$.

# Phase Diagram Analysis of Coupled Differential Equations

We have the coupled differential equations

$$\dot{K} = a\,K - b\,K^2 - \lambda^{-1/\epsilon} \quad \text{and} \quad \dot{\lambda} = \lambda\,(r - a + 2\,b\,K)$$

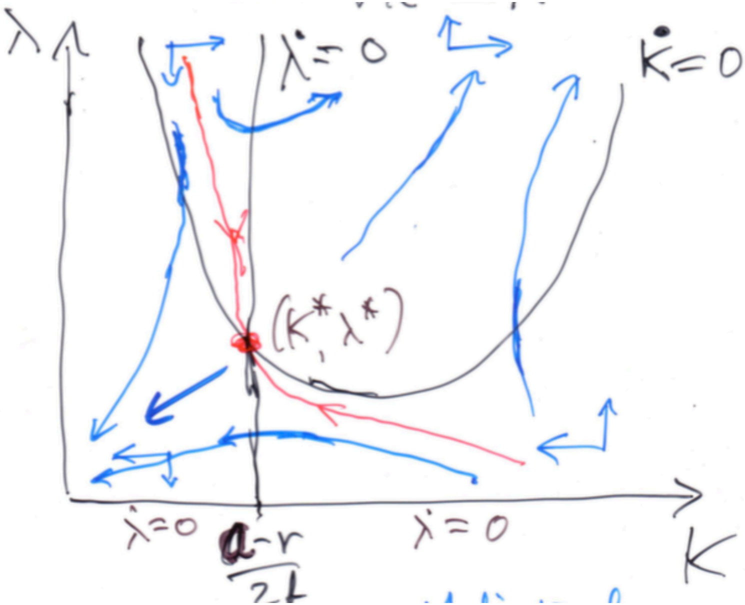
with a unique steady state at

$$K^* = (a - r)/2b, \quad \lambda^* = [K^*(a - bK^*)]^{-\epsilon}$$

The phase diagram on the next slide shows:

1. the two "isoclines" where $\dot{K} = 0$ and $\dot{\lambda} = 0$ respectively;
2. the intersection of these two isoclines at the unique stationary point $(K^*, \lambda^*)$;
3. the division of the plane of $(K, \lambda)$ values into four different "phases" according as $\dot{K} \gtrless 0$ and $\dot{\lambda} \gtrless 0$, marked by blue arrows pointing in the relevant direction;
4. six possible different solutions of the coupled equations, which are marked by blue curves.

# Phase Diagram



# Suboptimal Solutions to the Differential Equations

Paths of pairs $(K, \lambda)$ where $\lambda$ starts out too low,
and so $C = \lambda^{-1/\epsilon}$ starts out too high:

1. pass below and to the left of the steady state $(K^*, \lambda^*)$;
2. eventually reach the phase where $\dot{K} < 0$ and $\dot{\lambda} < 0$;
3. in that profligate phase, where $K(t)$ reaches 0 in finite time, after which there is no output and so $C = K = 0$ for ever thereafter.

Such paths could be optimal for a suitable finite horizon,
but with an infinite horizon, they end in disaster.

Paths of pairs $(K, \lambda)$ where $\lambda$ starts out too high,
and so $C = \lambda^{-1/\epsilon}$ starts out too low:

1. pass above and to the right of the steady state $(K^*, \lambda^*)$;
2. eventually reach the phase where $\dot{K} > 0$ and $\dot{\lambda} > 0$;
3. in that phase of wasteful over-accumulation one has $K(t) \to \infty$ yet $C(t) \to 0$ as $t \to \infty$.

# Optimal Solutions to the Differential Equations

The red curve in the phase diagram
shows the unique solution curve
that passes through the steady state $(K^*, \lambda^*)$.

Along this solution curve where $(K, \lambda) \to (K^*, \lambda^*)$ as $t \to \infty$
lies the happy medium between:

1. profligacy, where $K(t)$ reaches 0 in finite time;
2. wasteful over-accumulation,
   where $K(t) \to \infty$ yet $C(t) \to 0$ as $t \to \infty$.

Furthermore, the present discounted value $e^{-rt} \lambda(t) K(t)$
of the capital stock converges to zero.

So the Malinvaud transversality condition is satisfied.

This completes the proof
that the path whose graph is the red curve
solves the infinite-horizon optimal growth problem.