Lecture Notes 8: Dynamic Optimization
Part 2: Optimal Control

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2024 September 18th; typeset from `optControl24.tex`

Outline

Statement of Basic Optimal Growth Problem

A consumption path $\mathbf{C}$ is a mapping $[t_0, t_1] \ni t \mapsto C(t) \in \mathbb{R}_+$.

A capital path $\mathbf{K}$ is a mapping $[t_0, t_1] \ni t \mapsto K(t) \in \mathbb{R}_+$.

Given $K(t_0)$ at the intial time t_0, the benevolent planner's objective is to choose the pair $(\mathbf{C}, \mathbf{K})$ in order to maximize

$$J(\mathbf{C}, \mathbf{K}) := \int_{t_0}^{t_1} e^{-rt} u(C(t)) \mathrm{d}t$$

subject to the continuum of equality constraints

$$C(t) = f(K(t)) - \dot{K}(t)$$

Introduce the Lagrange multiplier path $\mathbf{p}$
as a mapping $[t_0, t_1] \ni t \mapsto p(t) \in \mathbb{R}_+$.

Use it to define the Lagrangian integral

$$\mathcal{L}_{\mathbf{p}}(\mathbf{C}, \mathbf{K}) = \int_{t_0}^{t_1} e^{-rt} u(C(t)) \mathrm{d}t - \int_{t_0}^{t_1} p(t)[C(t) - f(K(t)) + \dot{K}(t)] \mathrm{d}t$$

Integrate by Parts

So we have the "Lagrangian"

$$\mathcal{L}_{\mathbf{p}}(\mathbf{C},\mathbf{K}) = \int_{t_0}^{t_1} e^{-rt} u(C(t)) \mathrm{d}t - \int_{t_0}^{t_1} p(t)[C(t) - f(K(t)) + \dot{K}(t)] \mathrm{d}t$$

Integrating the last term by parts yields

$$-\int_{t_0}^{t_1} p(t)\dot{K}(t)\mathrm{d}t = -\Big|_{t_0}^{t_1} p(t)K(t) + \int_{t_0}^{t_1} \dot{p}(t)\, K(t)\mathrm{d}t$$

Hence

$$\mathcal{L}_{\mathbf{p}}(\mathbf{C},\mathbf{K}) = \int_{t_0}^{t_1} \left[e^{-rt}u(C) + \dot{p}\,K - p\,C + p\,f(K)\right] \mathrm{d}t - \Big|_{t_0}^{t_1} p(t)K(t)$$

For the moment we ignore the last "endpoint terms",
and consider just the integral

$$\mathcal{I}_{\mathbf{p}}(\mathbf{C},\mathbf{K}) := \int_{t_0}^{t_1} \left[e^{-rt}u(C) + \dot{p}\,K - p\,C + p\,f(K)\right] \mathrm{d}t$$

Maximizing the Integrand

Evidently the two paths $t \mapsto C(t)$ and $t \mapsto K(t)$
jointly maximize the integral

$$\mathcal{I}_{\mathbf{p}}(\mathbf{C}, \mathbf{K}) = \int_{t_0}^{t_1} \left[e^{-rt} u(C) + \dot{p}\, K - p\, C + p\, f(K) \right] \mathrm{d}t$$

with $\mathbf{p}$ fixed, if and only if,
for almost all $t \in (t_0, t_1)$, the pair $(C(t), K(t))$
jointly maximizes w.r.t. C and K the integrand

$$e^{-rt} u(C) + \dot{p}\, K - p\, C + p\, f(K)$$

The first-order conditions for maximizing this integrand,
at any time $t \in (t_0, t_1)$, are found by differentiating partially:

1. w.r.t. $C(t)$ to obtain $e^{-rt} u'(C(t)) = p(t)$;
2. w.r.t. $K(t)$ to obtain $\dot{p}(t) = -p(t)\, f'(K(t))$;

There is also the equality constraint $\dot{K}(t) = f(K(t)) - C(t)$.

Outline

Statement of Sufficient Conditions

Consider the static problem of maximizing
the objective function $\mathbb{R}^n \supseteq D \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R}$
subject to the vector constraint $\mathbf{g}(\mathbf{x}) \leqq \mathbf{a} \in \mathbb{R}^m$
where $\mathbb{R}^n \supseteq D \ni \mathbf{x} \mapsto \mathbf{g}(\mathbf{x}) \in \mathbb{R}^m$.

Definition

The pair $(\mathbf{p}, \mathbf{x}^*) \in \mathbb{R}^m \times \mathbb{R}^n$
jointly satisfies complementary slackness just in case:

$$\text{(i) } \mathbf{p}^\top \geqq 0; \quad \text{(ii) } \mathbf{g}(\mathbf{x}^*) \leqq a; \quad \text{(iii) } \mathbf{p}^\top[\mathbf{g}(\mathbf{x}^*) - \mathbf{a}] = 0$$

These are generally summarized as $\mathbf{p}^\top \geqq 0,\ \mathbf{g}(\mathbf{x}^*) \leqq \mathbf{a}$ (comp). □

Theorem

Suppose that $\mathbf{x}^* \in \mathbb{R}^n$ is a global maximum over the domain D of the Lagrangian function $\mathcal{L}_\mathbf{p}(\mathbf{x}) = f(\mathbf{x}) - \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{a}]$ where $(\mathbf{p}, \mathbf{x}^*) \in \mathbb{R}^m \times \mathbb{R}^n$ jointly satisfy the complementary slackness conditions.

Then $\mathbf{x}^*$ is a global maximum of $f(\mathbf{x})$ subject to $\mathbf{g}(\mathbf{x}) \leqq \mathbf{a}$.

Proof of Sufficient Conditions

Proof.

By definition of the Lagrangian $\mathcal{L}_{\mathbf{p}}(\mathbf{x}) = f(\mathbf{x}) - \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - a]$, for every $\mathbf{x} \in D$ one has

$$f(\mathbf{x}) - f(\mathbf{x}^*) = \mathcal{L}_{\mathbf{p}}(\mathbf{x}) + \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{a}] - \mathcal{L}_{\mathbf{p}}(\mathbf{x}^*) - \mathbf{p}^\top[\mathbf{g}(\mathbf{x}^*) - \mathbf{a}]$$

By hypothesis one has $\mathcal{L}_{\mathbf{p}}(\mathbf{x}) \leq \mathcal{L}_{\mathbf{p}}(\mathbf{x}^*)$ for all $\mathbf{x} \in D$, so

$$f(\mathbf{x}) - f(\mathbf{x}^*) \leq \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{a}] - \mathbf{p}^\top[\mathbf{g}(\mathbf{x}^*) - \mathbf{a}] = \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)]$$

But the complementary slackness conditions

$$\mathbf{p}^\top \geqq \mathbf{0}, \ \mathbf{g}(\mathbf{x}^*) \leqq \mathbf{a} \text{ (comp)}$$

imply that for any $\mathbf{x} \in D$ satisfying the constraint $\mathbf{g}(\mathbf{x}) \leqq \mathbf{a}$ one has $\mathbf{p}^\top\mathbf{g}(\mathbf{x}) \leq \mathbf{p}^\top\mathbf{a}$, whereas $\mathbf{p}^\top\mathbf{g}(\mathbf{x}^*) = \mathbf{p}^\top\mathbf{a}$.

Hence $f(\mathbf{x}) - f(\mathbf{x}^*) \leq \mathbf{p}^\top[\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)] \leq \mathbf{p}^\top\mathbf{a} - \mathbf{p}^\top\mathbf{a} = 0$. $\square$

A Cheap Result on Necessary Conditions

Recall that we are considering the problem of choosing $\mathbf{x} \in D \subseteq \mathbb{R}^n$ in order to maximize $f(\mathbf{x})$ subject to $\mathbf{g}(\mathbf{x}) \leqq \mathbf{a}$.

Suppose we know that any solution $\mathbf{x}^* \in D$ must be unique.

This will be the case, for example, if:

1. the common domain D of the functions $D \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R}$ and $D \ni \mathbf{x} \mapsto \mathbf{g}(\mathbf{x}) \in \mathbb{R}^m$ is a convex subset of $\mathbb{R}^n$;
2. the objective function $D \ni \mathbf{x} \mapsto f(\mathbf{x}) \in \mathbb{R}$ is strictly concave;
3. each component function $D \ni \mathbf{x} \mapsto g_j(\mathbf{x}) \in \mathbb{R}$ of the vector function $D \ni \mathbf{x} \mapsto \mathbf{g}(\mathbf{x}) \in \mathbb{R}^m$ is convex.

Suppose that the pair $(\mathbf{p}, \mathbf{x}^*) \in \mathbb{R}^m \times D$ jointly satisfy the sufficient conditions for maximizing the Lagrangian while also meeting the complementary slackness conditions.

Then it is necessary that the only possible maximum satisfy these sufficient conditions!

Outline

Statement of General Problem

Given the time interval $[t_0, t_1] \subset \mathbb{R}$,
consider the general one-variable optimal control problem of choosing paths:

1. $[t_0, t_1] \ni t \mapsto x(t) \in \mathbb{R}$ of states;
2. $[t_0, t_1] \ni t \mapsto u(t) \in \mathbb{R}$ of controls.

The objective functional is taken to be the integral

$$\int_{t_0}^{t_1} f(t, x(t), u(t))\, \mathrm{d}t$$

We fix the initial state $x(t_0) = x_0$, where x_0 is given.

We leave the terminal state $x(t_1)$ free.

Finally, we impose the dynamic constraint $\dot{x} = g(t, x, u)$ at every time $t \in [t_0, t_1]$.

The Lagrangian Integral

Consider the path $[t_0, t_1] \ni t \mapsto p(t) \in \mathbb{R}$
of a single costate variable or shadow price p.

Here $p(t)$ is the Lagrange multiplier
associated with the dynamic constraint at time t.

Then, after dropping the time argument from p, x and u,
the associated "Lagrangian integral" is

$$\mathcal{L} = \int_{t_0}^{t_1} f(t, x, u)\, \mathrm{d}t - \int_{t_0}^{t_1} p[\dot{x} - g(t, x, u)]\, \mathrm{d}t$$

Because $\frac{\mathrm{d}}{\mathrm{d}t} p\, x = \dot{p}\, x + p\, \dot{x}$, integrating by parts
gives $\int_{t_0}^{t_1} p\, \dot{x}\, \mathrm{d}t = - \int_{t_0}^{t_1} \dot{p}\, x\, \mathrm{d}t + \big|_{t_0}^{t_1} p\, x$ and so

$$\mathcal{L} = \int_{t_0}^{t_1} [f(t, x, u) + \dot{p}\, x + p\, g(t, x, u)]\, \mathrm{d}t - \big|_{t_0}^{t_1} p\, x$$

The Hamiltonian

Definition

For the problem of maximizing $\int_{t_0}^{t_1} f(t, x, u)\,\mathrm{d}t$
subject to $\dot{x} = g(t, x, u)$,
the Hamiltonian function is defined as

$$H(t, x, u, p) := f(t, x, u) + p\, g(t, x, u) \quad \square$$

With this definition, the integral part of the Lagrangian, which is

$$\int_{t_0}^{t_1} [f(t, x, u) + \dot{p}\, x + p\, g(t, x, u)]\, \mathrm{d}t$$

can be written as $\int_{t_0}^{t_1} [H(t, x, u, p) + \dot{p}\, x]\, \mathrm{d}t$.

The Maximum Principle

Recall the definition $H(t, x, u, p) := f(t, x, u) + p\, g(t, x, u)$.

Definition

According to the maximum principle, for a.e. $t \in [t_0, t_1]$, an optimal control should satisfy

$$u^*(t) \in \arg\max_u H(t, x, u, p) \text{ where } x = x(t) \text{ and } p = p(t)$$

Moreover the co-state variable $p(t)$ should evolve according to the vector differential equation

$$\dot{p} = -H'_x(t, x, u, p)$$

where $H'_x(t, x, u, p)$ denotes the partial derivative of the Hamiltonian H w.r.t. the state x.

An Extended Maximum Principle

Definition

We add an extra term $\dot{p}\,x$ to the Hamiltonian $H(t, x, u, p)$ in order to give the extended Hamiltonian

$$\tilde{H}(t, x, u, p) := H(t, x, u, p) + \dot{p}\,x = f(t, x, u) + p\,g(t, x, u) + \dot{p}\,x$$

According to the extended maximum principle, for a.e. (almost every) time $t \in [t_0, t_1]$, one should have

$$(u^*(t), x^*(t)) \in \arg\max_{(u,x)} \tilde{H}(t, x, u, p(t))$$

Remark

The first-order conditions for maximizing $\tilde{H}(t, x, u, p)$ *include*

$$\dot{p} = -f'_x(t, x, u) - pg'_x(t, x, u) = -H'_x(t, x, u, p)$$

as required by the maximum principle.

Outline

A Macroeconomic Quadratic Control Problem: Statement

Let $c > 0$ denote an adjustment cost parameter.

Consider the problem of choosing the path $t \mapsto (u(t), x(t)) \in \mathbb{R}^2$ in order to minimize the quadratic integral $\int_0^T (x^2 + cu^2)\, \mathrm{d}t$ subject to the dynamic constraint $\dot{x} = u$, as well as the initial condition $x(0) = x_0$ and the terminal condition allowing $x(T)$ to be chosen freely.

The associated Hamiltonian is

$$H = -x^2 - cu^2 + p\, u$$

with a minus sign to convert the minimization problem into a maximization problem.

The associated extended Hamiltonian, including the extra term $\dot{p}\, x$, is

$$\tilde{H} = -x^2 - cu^2 + p\, u + \dot{p}\, x$$

First-Order Conditions

Consider the problem of maximizing, at any time $t \in [0, T]$,
either the Hamiltonian $H = -x^2 - cu^2 + p\,u$,
or the extended Hamiltonian $\tilde{H} = -x^2 - cu^2 + p\,u + \dot{p}\,x$

The first-order conditions include $0 = H'_u = \tilde{H}'_u = -2c\,u + p$.

Either of these two equivalent conditions implies that $u^* = p/2c$.

A second first-order condition for maximizing w.r.t. x
the extended Hamiltonian $\tilde{H}$
is the co-state differential equation $\dot{p} = -H'_x(t, x, u, p) = 2x$.

Combining this co-state differential equation
with the dynamic constraint $\dot{x} = u$
leads to the following coupled pair of differential equations:

$$\dot{p} = -H'_x = 2x \quad \text{and} \quad \dot{x} = u^* = p/2c$$

Example: Solving the Coupled Pair

In order to solve the coupled pair

$$\dot{p} = 2x \quad \text{and} \quad \dot{x} = p/2c$$

- differentiate the first equation w.r.t. t to obtain $\ddot{p} = 2\dot{x}$;
- in this last equation $\ddot{p} = 2\dot{x}$,
  substitute $\dot{x} = p/2c$ which leads to $\ddot{p} = 2\dot{x} = p/c$.

We need to consider the second-order differential equation

$$\ddot{p} = p/c$$

in p, whose associated characteristic equation is $\lambda^2 - 1/c = 0$.

The two roots are $\lambda_{1,2} = \pm c^{-1/2} = \pm r$ where $r := c^{-1/2}$.

The general solution of the homogeneous equation $\ddot{p} = p/c$
is $p = Ae^{rt} + Be^{-rt}$ for arbitrary constants A and B.

Explicit Solution

In addition to $p = Ae^{rt} + Be^{-rt}$ with $r := c^{-1/2}$ and $\dot{p} = 2x$, we also have $\dot{x} = p/2c$, along with the initial condition $x(0) = x_0$ and the terminal condition $p(T) = 0$.

This terminal condition implies $Ae^{rT} + Be^{-rT} = 0$, from which one obtains $B = -Ae^{2rT}$.

Also differentiating $p = Ae^{rt} + Be^{-rt}$ w.r.t. t implies $\dot{p} = r(Ae^{rt} - Be^{-rt})$.

At time $t = 0$ one has $\dot{p}(0) = 2x_0$ and so $r(A - B) = 2x_0$.

Substituting $B = -Ae^{2rT}$ gives $r(A + Ae^{2rT}) = 2x_0$,
so $A = 2x_0/r(1 + e^{2rT}) = 2x_0 e^{-rT}/r(e^{-rT} + e^{rT})$
implying that $B = -2x_0 e^{rT}/r(e^{-rT} + e^{rT})$.

So $p = Ae^{rt} + Be^{-rt} = 2x_0(e^{-r(T-t)} - e^{r(T-t)})/r(e^{-rT} + e^{rT})$

and $x = \dot{p}/2 = x_0(e^{-r(T-t)} + e^{r(T-t)})/(e^{-rT} + e^{rT})$.

Also $u = \dot{x} = rx_0(e^{-r(T-t)} - e^{r(T-t)})/(e^{-rT} + e^{rT})$.

The Case of an Infinite Horizon

Multiply both numerator and denominator of the right-hand side of each equation by e^{-rT}, leading to the explicit solution:

$$\begin{aligned}
p(t) &= \frac{2x_0\left[e^{-r(T-t)} - e^{r(T-t)}\right]}{r\left[e^{-rT} + e^{rT}\right]} &= \frac{2x_0\left[e^{-r(2T-t)} - e^{-rt}\right]}{r(e^{-2rT} + 1)} \\
x(t) &= \frac{x_0\left[e^{-r(T-t)} + e^{r(T-t)}\right]}{r(e^{-rT} + e^{rT})} &= \frac{x_0\left[e^{-r(2T-t)} + e^{-rt}\right]}{r(e^{-2rT} + 1)} \\
u(t) &= \frac{x_0\left[e^{-r(T-t)} - e^{r(T-t)}\right]}{e^{-rT} + e^{rT}} &= \frac{x_0\left[e^{-r(2T-t)} - e^{-rt}\right]}{e^{-2rT} + 1}
\end{aligned}$$

Taking the limit as $T \to \infty$, one has $p(t) \to -2x_0 e^{-rt}/r$.

Similarly $x(t) = \frac{1}{2}\dot{p} \to x_0 e^{-rt}$, and $u(t) = \dot{x}(t) \to -x_0 e^{-rt}$.

Finally, $(p(t), x(t), u(t)) \to (0, 0, 0)$ as $t \to \infty$.

See page 311 of FMEA.

Outline

Mangasarian and Arrow's Sufficient Conditions

At any fixed time t, let $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ be a stationary point w.r.t. $(\mathbf{x}, \mathbf{u})$ of the extended Hamiltonian

$$\tilde{H}(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t)) := H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t)) + \dot{\mathbf{p}}^\top(t)\,\mathbf{x}$$

That is, suppose that the respective partial gradients satisfy

$$H'_{\mathbf{u}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t)) = 0 \quad \text{and} \quad \dot{\mathbf{p}}(t) = -H'_{\mathbf{x}}(t, \mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t))$$

Here are two alternative sufficient conditions for $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ to maximize the extended Hamiltonian.

1. See FMEA Theorem 9.7.1, due to Mangasarian.
   Suppose that $(\mathbf{x}, \mathbf{u}) \mapsto H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t))$ is concave,
   which implies that $(\mathbf{x}, \mathbf{u}) \mapsto \tilde{H}(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t))$ is also concave.
2. See FMEA Theorem 9.7.2, due to Arrow.
   Define $\hat{H}(t, \mathbf{x}, \mathbf{p}(t)) := \max_{\mathbf{u}} H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}(t))$,
   and suppose that $\mathbf{x} \mapsto \hat{H}(t, \mathbf{x}, \mathbf{p}(t))$ is concave.

Sufficient Conditions

Consider the single variable problem
of choosing the paths $t \mapsto (x(t), u(t)) \in \mathbb{R}^2$
in order to maximize $\int_0^T f(t, x, u) \, dt$
subject to $\dot{x} \leq g(t, x, u)$ (all $t \in [0, T]$)
as well as $x(0) \leq x_0$, $x(T) \geq x_T$.

Including the extra term $\dot{p} \, x$, the extended Hamiltonian is

$$\tilde{H}(t, x, u, p) = f(t, x, u) + p \, g(t, x, u) + \dot{p} \, x$$

Suppose that for all $t \in [0, T]$ the path $t \mapsto (x^*(t), u^*(t)) \in \mathbb{R}^2$
satisfies the extended maximization condition

$$(x^*(t), u^*(t)) \in \arg\max_{x,u} \tilde{H}(t, x, u, p(t))$$

as well as the three complementary slackness conditions:

1. $p(t) \geq 0$, $\dot{x}^*(t) \leq g(t, x^*(t), u^*(t))$ (comp) (all $t \in [0, T]$);
2. $p(0) \geq 0$, $x^*(0) \leq x_0$ (comp);
3. $p(T) \geq 0$, $x^*(T) \geq x_T$ (comp).

Proof of Sufficiency, I

Consider any alternative feasible path $t \mapsto (x(t), u(t))$ satisfying all the constraints.

Define $D(\mathbf{x}, \mathbf{u}) := \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))] \, \mathrm{d}t$.

After dropping the time arguments from $x(t), u(t), x^*(t), u^*(t)$, the definition $\tilde{H} = f + pg + p\dot{x}$ implies that

$$D(\mathbf{x}, \mathbf{u}) = \int_0^T \Big\{ \Big[\tilde{H}(t, x, u, p) - p\, g(t, x, u) - \dot{p}\, x \Big] - \Big[\tilde{H}(t, x^*, u^*, p) - p\, g(t, x^*, u^*) - \dot{p}\, x^* \Big] \Big\} \, \mathrm{d}t$$

The maximization hypothesis implies that, for all $t \in (0, T)$, one has $\tilde{H}(t, x(t), u(t), p(t)) \leq \tilde{H}(t, x^*(t), u^*(t), p(t))$.

From this it follows that

$$D(\mathbf{x}, \mathbf{u}) \leq \int_0^T \{ [-p\, g(t, x, u) - \dot{p}\, x] - [-p\, g(t, x^*, u^*) - \dot{p}\, x^*] \} \, \mathrm{d}t$$

Proof of Sufficiency, II

We have shown that

$$D(\mathbf{x}, \mathbf{u}) \leq \int_0^T \{[-p\, g(t, x, u) - \dot{p}\, x] - [-p\, g(t, x^*, u^*) - \dot{p}\, x^*]\}\, \mathrm{d}t$$

But feasibility implies that $\dot{x}(t) \leq g(t, x, u)$
and prices satisfy $p(t) \geq 0$, so $p(t)\, \dot{x}(t) \leq p(t)\, g(t, x, u)$.

Furthermore, the complementary slackness conditions for optimality imply that $p(t)\, g(t, x^*(t), u^*(t)) = p(t)\, \dot{x}^*(t)$.

It follows that

$$\begin{aligned} D(\mathbf{x}, \mathbf{u}) &\leq \int_0^T [-p\, \dot{x} - \dot{p}\, x + p\, \dot{x}^* + \dot{p}\, x^*]\, \mathrm{d}t \\ &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} [-p(t)\, x(t) + p(t)\, x^*(t)]\, \mathrm{d}t \\ &= -p(T)\, [x(T) - x^*(T)] + p(0)\, [x(0) - x^*(0)] \end{aligned}$$

Proof of Sufficiency, III

So far, we have shown that

$$D(\mathbf{x}, \mathbf{u}) \leq -p(T)\,[x(T) - x^*(T)] + p(0)\,[x(0) - x^*(0)]$$

But, together with feasibility and non-negativity of prices, the second and third complementary slackness conditions regarding the endpoints at times $t = 0$ and $t = T$ imply that

$$\begin{array}{rcl} p(T)\,x(T) & \geq & p(T)\,x_T \quad ; \quad p(T)\,x^*(T) \;=\; p(T)\,x_T\,; \\ p(0)\,x(0) & \leq & p(0)\,x_0 \quad ; \quad p(0)\,x^*(0) \;=\; p(0)\,x_0\,. \end{array}$$

It follows that

$$p(T)\,x(T) \geq p(T)\,x^*(T) \quad \text{and} \quad p(0)\,x(0) \leq p(0)\,x^*(0)$$

which together imply that $D(\mathbf{x}, \mathbf{u}) \leq 0$.

Finally, after recalling the definition

$$D(\mathbf{x}, \mathbf{u}) := \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))]\, \mathrm{d}t$$

one concludes that the path $t \mapsto (x^*(t), u^*(t))$ is optimal. $\square$

Outline

The Infinite Horizon Problem

We consider the problem of choosing $[0, \infty) \ni t \mapsto (x(t), u(t))$ to maximize the infinite horizon objective functional

$$\int_0^\infty f(t, x(t), u(t))\, \mathrm{d}t$$

subject to $\dot{x} = g(t, x, u)$ at every time $t \in [0, \infty)$, as well as $x(0) = x_0$, where x_0 is given.

As before, the extended maximum principle suggests looking for a path $[0, \infty) \ni t \mapsto p(t)$ of co-state variables, as well as a path $[0, \infty) \ni t \mapsto (x^*(t), u^*(t))$ of the state and control variables which maximizes the extended Hamiltonian

$$\tilde{H}(t, x, u, p) := f(t, x, u) + p(t)\, g(t, x, u) + \dot{p}(t)\, x$$

— i.e., for (almost) all $t \in [0, \infty)$ one has

$$(x^*(t), u^*(t)) \in \arg\max_{(u,x)} \tilde{H}(t, x, u, p)$$

Implications of the Extended Maximum Principle, I

Consider any alternative feasible path $t \mapsto (x(t), u(t))$ satisfying all the constraints.

We start by repeating our earlier argument for a finite horizon.

Define $D^T(\mathbf{x}, \mathbf{u}) := \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))] \, dt$.

After dropping the time arguments from $x(t), u(t), x^*(t), u^*(t)$, this difference $D^T(\mathbf{x}, \mathbf{u})$ equals

$$\int_0^T \left\{ \left[\tilde{H}(t, x, u, p) - p \, g(t, x, u) - \dot{p} \, x \right] \right.$$
$$\left. - \left[\tilde{H}(t, x^*, u^*, p) - p \, g(t, x^*, u^*) - \dot{p} \, x^* \right] \right\} dt$$

The extended maximum principle implies that for all $t \in [0, T]$ one has

$$\tilde{H}(t, x(t), u(t), p(t)) \leq \tilde{H}(t, x^*(t), u^*(t), p(t))$$

Implications of the Extended Maximum Principle, II

Arguing as before, from $(x^*(t), u^*(t)) \in \arg\max_{(u,x)} \tilde{H}(t, x, u, p)$
where $\tilde{H}(t, x, u, p) := f(t, x, u) + p(t)\, g(t, x, u) + \dot{p}(t)\, x$,
it follows that for all finite T the difference $D^T(\mathbf{x}, \mathbf{u})$ satisfies

$$\begin{aligned}
D^T(\mathbf{x}, \mathbf{u}) &:= \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))]\, \mathrm{d}t \\
&= \int_0^T \Big\{ \Big[\tilde{H}(t, x, u, p) - p\, g(t, x, u) - \dot{p}\, x \Big] \\
&\qquad - \Big[\tilde{H}(t, x^*, u^*, p) - p\, g(t, x^*, u^*) - \dot{p}\, x^* \Big] \Big\}\, \mathrm{d}t \\
&= \int_0^T \Big[\tilde{H}(t, x, u, p) - \tilde{H}(t, x^*, u^*, p) \Big]\, \mathrm{d}t \\
&\quad - \int_0^T [p\, g(t, x, u) + \dot{p}\, x - p\, g(t, x^*, u^*) - \dot{p}\, x^*]\, \mathrm{d}t \\
&\leq -\int_0^T [p\, \dot{x} + \dot{p}\, x - p\, \dot{x}^* - \dot{p}\, x^*]\, \mathrm{d}t \\
&= -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} [p\, x - p\, x^*]\, \mathrm{d}t \\
&= -p(T)\, [x(T) - x^*(T)] + p(0)\, [x(0) - x^*(0)] \\
&= p(T)\, [x^*(T) - x(T)] \text{ given that } x(0) = x^*(0) = x_0
\end{aligned}$$

A Transversality Condition

Consider the transversality condition

$$\limsup_{T\to\infty} p(T)\,[x^*(T) - x(T)] = 0$$

If this were satisfied, it would imply that

$$\begin{aligned} 0 &\geq \limsup_{T\to\infty} D^T(\mathbf{x}, \mathbf{u}) \\ &= \limsup_{T\to\infty} \int_0^T [f(t, x(t), u(t)) - f(t, x^*(t), u^*(t))]\, \mathrm{d}t \end{aligned}$$

In the case when

$$\int_0^T f(t, x^*(t), u^*(t))\, \mathrm{d}t \to \int_0^\infty f(t, x^*(t), u^*(t))\, \mathrm{d}t$$

as $T \to \infty$, it would imply that

$$\limsup_{T\to\infty} \int_0^T f(t, x(t), u(t))\, \mathrm{d}t \leq \int_0^\infty f(t, x^*(t), u^*(t))\, \mathrm{d}t$$

Malinvaud's Transversality Condition

Edmond Malinvaud (1953) "Capital Accumulation and Efficient Allocation of Resources" *Econometrica* 21: 233–268.

In many economic contexts, feasibility requires that, for all t, one has both $x(t) \geq 0$ and $\dot{x}(t) \leq g(t, x(t), u(t))$.

Then, since $p(t) \geq 0$,
for any alternative feasible path $x(t)$ and any terminal time T,
one has $p(T)\,[x^*(T) - x(T)] \leq p(T)\,x^*(T)$.

Definition

The Malinvaud transversality condition
is that $p(T)\,x^*(T) \to 0$ as $T \to \infty$.

When this Malinvaud transversality condition is satisfied, evidently

$$\limsup_{T\to\infty} p(T)\;[x^*(T) - x(T)] \leq \limsup_{T\to\infty} p(T)\,x^*(T) = 0$$

Hence, the general transversality condition is also satisfied.

Outline

A Problem with Exponential Discounting

Consider the general problem of choosing paths:

1. $[t_0, t_1] \ni t \mapsto x(t) \in \mathbb{R}$ of states;
2. $[t_0, t_1] \ni t \mapsto u(t) \in \mathbb{R}$ of controls.

The objective functional is taken to be the integral

$$\int_{t_0}^{t_1} e^{-rt} f(x(t), u(t)) \, \mathrm{d}t$$

where: (i) f is independent of t;
(ii) there is a constant discount rate r
and associated exponential discount factor e^{-rt}.

Assume too that the dynamic constraint is $\dot{x} = g(x, u)$,
at every time $t \in [t_0, t_1]$, where g is independent of t.

Fix the initial state $x(t_0) = x_0$, where x_0 is given.

But leave the terminal state $x(t_1)$ free.

Present versus Current Value Hamiltonian

Up to now, we have considered the present value Hamiltonian

$$H(t, x, u, p) := e^{-rt} f(x, u) + p\, g(x, u)$$

We remove the discount factor e^{-rt}
by defining the current value Hamiltonian

$$H^C(x, u, q) := f(x, u) + q\, g(x, u)$$

with the current value co-state variable $q := e^{rt}\, p$.

These definitions imply that

$$H(t, x, u, p) = e^{-rt}[f(x, u) + e^{rt}\, p\, g(x, u)] = e^{-rt} H^C(x, u, q)$$

where $q = e^{rt} p$, so $\dot{q} = re^{rt} p + e^{rt} \dot{p} = rq + e^{rt} \dot{p}$.

Present and Current Value Maximum Principles

The (present value) maximum principle
states that for (almost) all $t \in [0, \infty)$ one has

$$u^*(t) \in \arg\max_u H(t, x, u, p) \quad \text{and} \quad \dot{p} = -H'_x(t, x, u, p)$$

By definition, one has $H(t, x, u, p) = e^{-rt} H^C(x, u, q)$
where $q = e^{rt} p$.

Because e^{-rt} is independent of u,
it follows that $u^*(t) \in \arg\max_u H^C(x, u, q)$.

Also $\dot{q} - rq = e^{rt} \dot{p} = -e^{rt} H'_x(t, x, u, p) = -{H^C}'_x(x, u, q)$.

We have derived the current value maximum principle
states that for (almost) all $t \in [0, \infty)$ one has

$$u^*(t) \in \arg\max_u H^C(x, u, q) \quad \text{and} \quad \dot{q} - rq = -{H^C}'_x(x, u, q)$$

Outline

Statement of the Problem

The problem will be to choose:

1. a consumption stream $\mathbb{R}_+ \ni t \mapsto C(t) \in \mathbb{R}_{++}$;
2. a stream $\mathbb{R}_+ \ni t \mapsto K(t) \in \mathbb{R}_{++}$ of capital stocks.

At any time t, given capital K, output will be $Y = a\,K - b\,K^2$, where $a, b \in \mathbb{R}$ are positive parameters, with $a > r > 0$.

Output is divided between consumption C and investment $\dot{K}$, so $\dot{K} = Y - C$; there is no depreciation.

The planner's objective is to maximize the utility integral $\int_0^\infty e^{-rt}\, u(C(t))\, \mathrm{d}t$.

We assume that the utility function $\mathbb{R}_{++} \ni C \mapsto u(C)$ takes the isoelastic form with $u'(C) = C^{-\epsilon}$.

The constant elasticity parameter $\epsilon > 0$ is a constant degree of relative fluctuation aversion.

The Current Value Maximum Principle

The optimal growth problem is to maximize $\int_0^\infty e^{-rt}\, u(C(t))\, \mathrm{d}t$ subject to $\dot{K} = a\, K - b\, K^2 - C$ where $u'(C) = C^{-\epsilon}$.

With λ as the co-state variable, the current value Hamiltonian is

$$H^C(K, C) := u(C) + \lambda(a\, K - b\, K^2 - C)$$

The first-order condition for maximizing $(K, C) \mapsto H^C(K, C)$ w.r.t. C is $u'(C) = \lambda$, which implies $C^{-\epsilon} = \lambda$ and so $C = \lambda^{-1/\epsilon}$.

Because $C \mapsto u(C)$ is strictly concave, this is the unique maximum.

The co-state variable evolves according to the equation

$$\dot{\lambda} - r\, \lambda = -H^{C\,\prime}_K(K, C) = -\lambda\, (a - 2\, b\, K)$$

Finally, therefore, we have the coupled differential equations

$$\dot{K} = a\, K - b\, K^2 - \lambda^{-1/\epsilon} \quad \text{and} \quad \dot{\lambda} = \lambda\, (r - a + 2\, b\, K)$$

Steady State of Coupled Differential Equations

The coupled differential equations

$$\dot{K} = a\,K - b\,K^2 - \lambda^{-1/\epsilon} \quad \text{and} \quad \dot{\lambda} = \lambda\,(r - a + 2\,b\,K)$$

have a steady state at any point satisfying $\dot{K} = 0$ and $\dot{\lambda} = 0$.

There is a unique steady state at the point $(K, \lambda) = (K^*, \lambda^*)$ with $K^* = (a - r)/2b$ and $\lambda^* = [K^*(a - bK^*)]^{-\epsilon}$.

Phase Diagram Analysis of Coupled Differential Equations

We have the coupled differential equations

$$\dot{K} = a\,K - b\,K^2 - \lambda^{-1/\epsilon} \quad \text{and} \quad \dot{\lambda} = \lambda\,(r - a + 2\,b\,K)$$

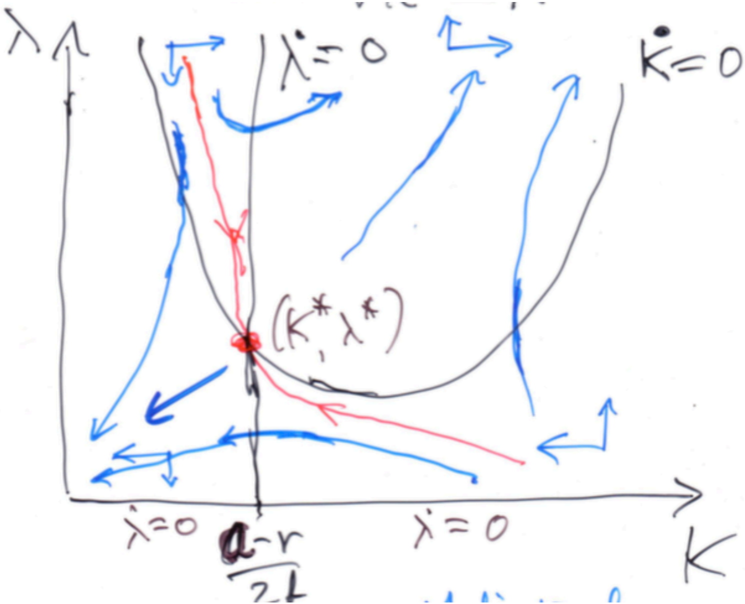
with a unique steady state at

$$K^* = (a - r)/2b, \quad \lambda^* = [K^*(a - bK^*)]^{-\epsilon}$$

The phase diagram on the next slide shows:

1. the two "isoclines" where $\dot{K} = 0$ and $\dot{\lambda} = 0$ respectively;
2. the intersection of these two isoclines at the unique stationary point (K^*, λ^*);
3. the division of the plane of (K, λ) values into four different "phases" according as $\dot{K} \gtrless 0$ and $\dot{\lambda} \gtrless 0$, marked by blue arrows pointing in the relevant direction;
4. six possible different solutions of the coupled equations, which are marked by blue curves.

Phase Diagram



Suboptimal Solutions to the Differential Equations

Paths of pairs (K, λ) where λ starts out too low,
and so $C = \lambda^{-1/\epsilon}$ starts out too high:

1. pass below and to the left of the steady state (K^*, λ^*);
2. eventually reach the phase where $\dot{K} < 0$ and $\dot{\lambda} < 0$;
3. in that profligate phase, where $K(t)$ reaches 0 in finite time, after which there is no output and so $C = K = 0$ for ever thereafter.

Such paths could be optimal for a suitable finite horizon,
but with an infinite horizon, they end in disaster.

Paths of pairs (K, λ) where λ starts out too high,
and so $C = \lambda^{-1/\epsilon}$ starts out too low:

1. pass above and to the right of the steady state (K^*, λ^*);
2. eventually reach the phase where $\dot{K} > 0$ and $\dot{\lambda} > 0$;
3. in that phase of wasteful over-accumulation one has $K(t) \to \infty$ yet $C(t) \to 0$ as $t \to \infty$.

Optimal Solutions to the Differential Equations

The red curve in the phase diagram
shows the unique solution curve
that passes through the steady state (K^*, λ^*).

Along this solution curve where $(K, \lambda) \to (K^*, \lambda^*)$ as $t \to \infty$
lies the happy medium between:

1. profligacy, where $K(t)$ reaches 0 in finite time;
2. wasteful over-accumulation,
   where $K(t) \to \infty$ yet $C(t) \to 0$ as $t \to \infty$.

Furthermore, the present discounted value $e^{-rt} \lambda(t) K(t)$
of the capital stock converges to zero.

So the Malinvaud transversality condition is satisfied.

This completes the proof
that the path whose graph is the red curve
solves the infinite-horizon optimal growth problem.