Lecture Notes 9: Measure and Probability
Part C: Probability as Measure

Peter Hammond
(loosely based on early notes by Andrés Carvajal)

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Outline

Probability Measure

Fix a measurable space (S, Σ),
where S is a set of unknown states of the world.

Then Σ is a σ-algebra of unknown events.

A probability measure on (S, Σ) is a measure $\mathbb{P} : \Sigma \to \bar{\mathbb{R}}_+$
satisfying the additional requirement that $\mathbb{P}(S) = 1$.

Countable additivity (or just additivity) of the measure $\mathbb{P}$
implies that, for every event $E \in \Sigma$,
one has $\mathbb{P}(E) + \mathbb{P}(E^c) = 1$ where $E^c := S \setminus E$.

For all $E \in \Sigma$, because $\mathbb{P}(E) \geq 0$, it follows that $\mathbb{P}(E) \in [0, 1]$.

Probability Space

Following Kolmogorov (1933), a probability space
is a triple $(S, \Sigma, \mathbb{P})$ where:

1. S is the state space;
2. Σ is a σ-algebra of measurable events,
   making (S, Σ) a measurable space;
3. $\Sigma \ni E \mapsto \mathbb{P}(E) \in [0, 1]$ is a probability measure on (S, Σ),

Properties of Probability

Theorem

Let $(S, \Sigma, \mathbb{P})$ be a probability space.

Then the following hold for all Σ-measurable sets E, E' etc.

1. $\mathbb{P}(E) \leq 1$ *and* $\mathbb{P}(S \setminus E) = 1 - \mathbb{P}(E)$;
2. $\mathbb{P}(E \setminus E') = \mathbb{P}(E) - \mathbb{P}(E \cap E')$ *and*
   $\mathbb{P}(E \cup E') = \mathbb{P}(E) + \mathbb{P}(E') - \mathbb{P}(E \cap E')$;
3. *for every partition* $\{E_n\}_{n=1}^m$ *of* S
   into m *pairwise disjoint* Σ*-measurable sets,*
   one has $\mathbb{P}(E) = \sum_{n=1}^m \mathbb{P}(E \cap E_n)$;
4. $\mathbb{P}(E \cap E') \geq \mathbb{P}(E) + \mathbb{P}(E') - 1$.
5. $\mathbb{P}(\cup_{n=1}^\infty E_n) \leq \sum_{n=1}^\infty \mathbb{P}(E_n)$.

Proof.

We leave the routine proof as an exercise. □

Trivial and Minimal Probability Spaces

Exercise

Given any non-empty set S, show that the triple $(S, \{\emptyset, S\}, P)$ in which Σ consists of only two sets is a probability space, called the trivial *probability space, just in case $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(S) = 1$.*

Exercise

Let $(S, \Sigma, \mathbb{P})$ be any probability space.

Given any event $E \in \Sigma$ with $\emptyset \subsetneqq E \subsetneqq S$, show that the σ-algebra $\sigma(\{E\})$ generated by $\{E\}$ is the Boolean algebra $\{\emptyset, E, S \setminus E, S\}$.

Show too that $(S, \sigma(\{E\}), \mathbb{P}_E)$ is a probability space, called a minimal non-trivial *probability space, provided that, for any $\mathbb{P}_E(E) \in [0, 1]$, we take:*

$$\mathbb{P}_E(\emptyset) = 0, \quad \mathbb{P}_E(S \setminus E) = 1 - \mathbb{P}_E(E), \quad \textit{and} \quad \mathbb{P}_E(S) = 1$$

Two Limiting Properties

Theorem

Let $(S, \Sigma, \mathbb{P})$ be a probability space,
and $(E_n)_{n=1}^\infty$ an infinite sequence of Σ-measurable sets.

1. *If $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$,*
   then $\mathbb{P}(\cup_{n=1}^\infty E_n) = \lim_{n\to\infty} \mathbb{P}(E_n) = \sup_n \mathbb{P}(E_n)$.
2. *If $E_n \supseteq E_{n+1}$ for all $n \in \mathbb{N}$,*
   then $\mathbb{P}(\cap_{n=1}^\infty E_n) = \lim_{n\to\infty} \mathbb{P}(E_n) = \inf_n \mathbb{P}(E_n)$.

Proving the Two Limiting Properties

Proof.

1. Because $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, it follows that

$$\begin{array}{rrcl} & E_n & = & E_1 \cup [\bigcup_{k=2}^n (E_k \setminus E_{k-1})] \\ \text{and} & \cup_{n=1}^\infty E_n & = & E_1 \cup [\bigcup_{k=2}^\infty (E_k \setminus E_{k-1})] \end{array}$$

   Note that E_1 and the sets $E_k \setminus E_{k-1}$ for $k = 2, 3, \ldots$ are all pairwise disjoint, implying that

$$\begin{array}{rcl} \mathbb{P}(E_n) & = & \mathbb{P}(E_1) + \sum_{k=2}^n \mathbb{P}(E_k \setminus E_{k-1}) \\ \mathbb{P}(\cup_{n=1}^\infty E_n) & = & \mathbb{P}(E_1) + \sum_{k=2}^\infty \mathbb{P}(E_k \setminus E_{k-1}) \\ & = & \lim_{n\to\infty} [\mathbb{P}(E_1) + \sum_{k=2}^n \mathbb{P}(E_k \setminus E_{k-1})] \\ & = & \lim_{n\to\infty} \mathbb{P}(E_n) \end{array}$$

2. Apply part 1 to the complements $S \setminus E_n$ of the sets E_n. □

Conditional Probability and an Extension

Let $(S, \Sigma, \mathbb{P})$ be any probability space with σ-algebra $\Sigma \subseteq 2^S$.

Let $E^* \in \Sigma$ be any measurable event satisfying $\mathbb{P}(E^*) > 0$.

Exercise

Define $\Sigma(E^*) := \{E \cap E^* \mid E \in \Sigma\} = \{E \in \Sigma \mid E \subseteq E^*\}$

and the mapping $\Sigma(E^*) \ni E \mapsto \mathbb{P}(E|E^*) := \dfrac{\mathbb{P}(E)}{\mathbb{P}(E^*)} \in [0, 1]$.

Prove that:

1. $\Sigma(E^*)$ *is a* σ*-algebra;*
2. *the mapping* $E \mapsto \mathbb{P}(E|E^*)$ *is a probability measure, called the* *conditional probability measure* *given the event* E^**, defined on the measurable space* $(S, \Sigma(E^*))$*;*
3. *the mapping* $\Sigma \ni E \mapsto \mathbb{P}(E|E^*) := \dfrac{\mathbb{P}(E \cap E^*)}{\mathbb{P}(E^*)} \in [0, 1]$
   is an *extended conditional probability measure* *given* E^**, defined on the whole of* Σ *while satisfying* $\mathbb{P}(E^*|E^*) = 1$ *as well as* $\mathbb{P}(E \setminus E^*|E^*) = 0$ *for all* $E \in \Sigma$.

Bayes' Rule

The formula $\mathbb{P}(E|E^*) = \dfrac{\mathbb{P}(E \cap E^*)}{\mathbb{P}(E^*)}$
for all $E, E^* \in \Sigma$ with $\mathbb{P}(E^*) > 0$
is sometimes known as Bayes' Rule.

Conditional Probability: The Law of Total Probability

Proposition

Provided that $\mathbb{P}(E) \in (0,1)$, *one has*

$$\mathbb{P}(E') = \mathbb{P}(E)\mathbb{P}(E'|E) + (1 - \mathbb{P}(E))\mathbb{P}(E'|E^c)$$

Proof.

The extended definition of conditional probability implies that

$$\begin{aligned} &\mathbb{P}(E)\mathbb{P}(E'|E) + (1 - \mathbb{P}(E))\mathbb{P}(E'|E^c) \\ &= \mathbb{P}(E)\frac{\mathbb{P}(E' \cap E)}{\mathbb{P}(E)} + \mathbb{P}(E^c)\frac{\mathbb{P}(E' \cap E^c)}{\mathbb{P}(E^c)} \\ &= \mathbb{P}(E' \cap E) + \mathbb{P}(E' \cap E^c) \end{aligned}$$

But $(E' \cap E) \cap (E' \cap E^c) = \emptyset$ and $(E' \cap E) \cup (E' \cap E^c) = E'$,
so $\mathbb{P}(E' \cap E) + \mathbb{P}(E' \cap E^c) = \mathbb{P}(E')$. □

Conditional Probability: Multiplicative Rule

Proposition

Let $(E_k)_{k=1}^n$ be any finite list of events in the probability space $(S, \Sigma, \mathbb{P})$.

Provided that $\mathbb{P}(\cap_{k=1}^{n-1} E_k) > 0$, one has

$$\mathbb{P}(\cap_{k=1}^n E_k) = \mathbb{P}(E_1)\,\mathbb{P}(E_2|E_1)\,\mathbb{P}(E_3|E_1 \cap E_2)\,\ldots\,\mathbb{P}(E_n| \cap_{k=1}^{n-1} E_k)$$

Proof.

By induction,
using the extended definition of conditional probability.

Details are left as an exercise. □

Independent Events

The finite or countably infinite family $\{E_k\}_{k \in K}$ of events in the probability space $(S, \Sigma, \mathbb{P})$ is:

- pairwise independent
  if $\mathbb{P}(E \cap E') = \mathbb{P}(E) \, \mathbb{P}(E')$ whenever $E \neq E'$;
- independent if for any finite subfamily $\{E_k\}_{k=1}^n$,
  one has $\mathbb{P}(\cap_{k=1}^n E_k) = \prod_{k=1}^n \mathbb{P}(E_k)$.

Pairwise Independence Does Not Imply Independence

Example

Consider the probability space $(S, 2^S, \mathbb{P})$ where $S = \mathbb{N}_9$ and $\mathbb{P}(\{s\}) = 1/9$ for all $s \in S$.

Consider the three events

$$E_1 = \{1, 2, 7\}, \; E_2 = \{3, 4, 7\} \text{ and } E_3 = \{5, 6, 7\}$$

which all have probability $\frac{1}{3}$.

Note that for each pair $i, j \in \mathbb{N}_3$ with $i \neq j$ one has $E_i \cap \mathbb{E}_j = \{7\}$ and so $\mathbb{P}(E_i \cap E_j) = \mathbb{P}(\{7\}) = \frac{1}{9} = \mathbb{P}(E_i)\,\mathbb{P}(E_j)$.

Thus, the three events are pairwise independent.

Yet $E_1 \cap E_2 \cap E_3 = \{7\}$
so $\mathbb{P}(E_1 \cap E_2 \cap E_3) = \frac{1}{9} \neq \mathbb{P}(E_1)\,\mathbb{P}(E_2)\,\mathbb{P}(E_3) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$,
implying that the three events are not independent.

Implications of Independence

Let $(S, \Sigma, \mathbb{P})$ be any probability space.

Notation

Given any $E \subseteq S$, let $E^c := S \setminus E$ denote the complementary event.

Exercise

Show that, if the two events E and $\tilde{E}$ in Σ are independent, then:

1. *the pairs $\{E^c, \tilde{E}\}$ and $\{E, \tilde{E}^c\}$ are both independent;*
2. *provided that $\mathbb{P}(E)$ and $\mathbb{P}(\tilde{E})$ are both positive, the conditional probabilities satisfy:*
   - $\mathbb{P}(E|\tilde{E}) = \mathbb{P}(E \cap \tilde{E})/\mathbb{P}(\tilde{E}) = \mathbb{P}(E)$, *independent of $\tilde{E}$;*
   - $\mathbb{P}(\tilde{E}|E) = \mathbb{P}(E \cap \tilde{E})/\mathbb{P}(E) = \mathbb{P}(\tilde{E})$, *independent of E.*

Exercise

Prove that if E and E' are independent, then so are E^c and E'.

The Measurable Product Space

Definition

Let $\langle (S_k, \Sigma_k) \rangle_{k=1}^n$ be a finite list of n measurable spaces.

Then the measurable space (S, Σ)
is the **product** of these n measurable spaces just in case:

1. the state space S is the Cartesian product $\prod_{k=1}^n S_k$
   of the individual state spaces;
2. the σ-algebra Σ on $S = \prod_{k=1}^n S_k$
   is the measurable product $\bigotimes_{k=1}^n \Sigma_k$
   of the individual σ-algebras,
   defined as the σ-algebra $\sigma\left(\prod_{k=1}^n \Sigma_k\right)$
   generated by all **measurable rectangles** $\prod_{k=1}^n E_k$
   satisfying $E_k \in \Sigma_k$ for all $k \in \mathbb{N}_n$.

The Product of a Finite List of Probability Spaces

Definition

Let $\langle (S_k, \Sigma_k, \mathbb{P}_k) \rangle_{k=1}^n$ be a finite list of n probability spaces.

Then the probability space $(S, \Sigma, \mathbb{P})$
is the product of these n probability spaces just in case:

1. the measurable space (S, Σ)
   is the measurable product $(\prod_{k=1}^n S_k, \bigotimes_{k=1}^n \Sigma_k)$
   of the n measurable spaces (S_k, Σ_k);
2. the probability measure $\mathbb{P}$ is the product measure $\otimes_{k=1}^n \mathbb{P}_k$,
   defined as the unique extension
   to the product σ-algebra $\bigotimes_{k=1}^n \Sigma_k$ of the function that,
   for each product $\prod_{k=1}^n E_k$ of measurable rectangles,
   satisfies $\otimes_{k=1}^n \mathbb{P}_k (\prod_{k=1}^n E_k) = \prod_{k=1}^n \mathbb{P}_k(E_k)$.

Independence for Marginal Probabilities

Given the product probability space $(\prod_{k=1}^{n} S_k, \bigotimes_{k=1}^{n} \Sigma_k, \otimes_{k=1}^{n} \mathbb{P}_k)$, any $k \in \mathbb{N}_n$, and any event $E_k \in \Sigma_k$, there is a corresponding product measurable marginal event $\prod_{i=1}^{k-1} S_i \times E_k \times \prod_{j=k+1}^{n} S_j$ whose probability is $\mathbb{P}_k(E_k)$,
which equals the marginal probability $[\operatorname{marg}_{S_k} \mathbb{P}](E_k)$.

The above definitions imply that,
whenever $k, \ell \in \mathbb{N}_n$ with $k < \ell$ and also $E_k \in \Sigma_k$, $E_\ell \in \Sigma_\ell$,
then the two marginal events $\prod_{i=1}^{k-1} S_i \times E_k \times \prod_{j=k+1}^{n} S_j$
and $\prod_{i=1}^{\ell-1} S_i \times E_\ell \times \prod_{j=\ell+1}^{n} S_j$ are independent,
because their intersection

$$\prod_{h=1}^{k-1} S_h \times E_k \times \prod_{i=k+1}^{\ell-1} S_i \times E_\ell \times \prod_{j=\ell+1}^{n} S_j$$

has probability $\mathbb{P}_k(E_k)\,\mathbb{P}_\ell(E_\ell)$.

The Product of a Sequence of Probability Spaces

Definition

Let $\langle (S_k, \Sigma_k) \rangle_{k \in \mathbb{N}}$ be an infinite sequence of probability spaces.

Then the measurable space $(S, \Sigma, \mathbb{P})$ is the product

$$\left(\prod_{k \in \mathbb{N}} S_k, \bigotimes_{k \in \mathbb{N}} \Sigma_k, \otimes_{k \in \mathbb{N}} \mathbb{P}_k \right)$$

of all these probability spaces just in case:

1. the state space S is the Cartesian product $\prod_{k \in \mathbb{N}} S_k$ of all the individual state spaces;
2. the σ-algebra Σ on $S = \prod_{k \in \mathbb{N}} S_k$ is the σ-algebra $\sigma\left(\cup_{n \in \mathbb{N}} \bigotimes_{k=1}^{n} \Sigma_k \right)$ generated by the union of all the finite product σ-algebras $\bigotimes_{k=1}^{n} \Sigma_k$;
3. the probability measure $\mathbb{P} = \otimes_{k \in \mathbb{N}} \mathbb{P}_k$, defined as the unique measure that, for each infinite product $\prod_{k \in \mathbb{N}} E_k$ of measurable rectangles, satisfies $\otimes_{k \in \mathbb{N}} \mathbb{P}_k \left(\prod_{k \in \mathbb{N}} E_k \right) = \inf_{n \in \mathbb{N}} \otimes_{k=1}^{n} \mathbb{P}_k \left(\prod_{k=1}^{n} E_k \right)$.

Outline

Random Variables

Definition

Let $(S, \Sigma, \mathbb{P})$ be a fixed probability space.

- The function $X : S \to \mathbb{R}$ is Σ-measurable just in case for every $x \in \mathbb{R}$ one has

$$X^{-1}((-\infty, x]) := \{s \in S \mid X(s) \leq x\} \in \Sigma$$

- A random variable (with values in $\mathbb{R}$) is a Σ-measurable function $S \ni s \mapsto X(s) \in \mathbb{R}$.
- The distribution function or cumulative distribution function (cdf) of X is the mapping $F_X : \mathbb{R} \to [0, 1]$ defined by

$$x \mapsto F_X(x) = \mathbb{P}(\{s \in S \mid X(s) \leq x\}) = \mathbb{P}\left(X^{-1}((-\infty, x])\right)$$

Properties of Distribution Functions, I

Theorem

The CDF of any random variable $s \mapsto X(s)$ satisfies:

1. $\lim_{x \to -\infty} F_X(x) = 0$ *and* $\lim_{x \to +\infty} F_X(x) = 1$.
2. $x \geq x'$ *implies* $F_X(x) \geq F_X(x')$.
3. $\lim_{h \downarrow 0} F_X(x + h) = F_X(x)$.
4. $\mathbb{P}(\{s \in S \mid X(s) > x\}) = 1 - F_X(x)$.
5. $\mathbb{P}(\{s \in S : x < X(s) \leq x'\}) = F_X(x') - F_X(x)$ *whenever* $x < x'$,
6. $\mathbb{P}(\{s \in S : X(s) = x\}) = F_X(x) - \lim_{h \uparrow 0} F_X(x + h)$.

CDFs are sometimes said to be càdlàg,
which is a French acronym for *continue à droite, limite à gauche*
(continuous on the right, limit on the left).

Properties of Distribution Functions, II

Definition

A continuity point of the CDF $F_X : \mathbb{R} \to [0,1]$
is an $\bar{x} \in \mathbb{R}$ at which the mapping $x \mapsto F_X(x)$ is continuous.

Is it always true that $\lim_{h \uparrow 0} F_X(x+h) = F_X(x)$?

Exercise

Let $F_X : \mathbb{R} \to [0,1]$ be the CDF of any random variable $S \ni s \mapsto X(s) \to \mathbb{R}$, and $\bar{x} \in \mathbb{R}$ any point.

Prove that the following three conditions are equivalent:

1. *$\bar{x}$ is a continuity point of F_X;*
2. $\mathbb{P}(\{s \in S \mid X(s) = \bar{x}\}) = 0$;
3. $\lim_{h \uparrow 0} F_X(\bar{x}+h) = F_X(\bar{x})$.

Continuous Random Variables

Definition

- A random variable $S \ni s \mapsto X(s) \to \mathbb{R}$ is
  1. continuously distributed just in case $x \mapsto F_X(x)$ is continuous;
  2. absolutely continuous just in case there exists a density function $\mathbb{R} \ni x \mapsto f_X(x) \to \mathbb{R}_+$ such that $F_X(x) = \int_{-\infty}^{x} f_X(u) \mathrm{d}u$ for all $x \in \mathbb{R}$.
- The support of the random variable $S \ni s \mapsto X(s) \to \mathbb{R}$ is the closure of the set on which F_X is strictly increasing.

Example

The uniform distribution on a closed interval $[a, b]$ of $\mathbb{R}$ has density function f and distribution function F given by

$$f_X(x) := \frac{1}{b-a} 1_{[a,b]}(x) \quad \text{and} \quad F_X(x) := \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } x \in [a, b] \\ 1 & \text{if } x > b \end{cases}$$

The Standard Normal or Gaussian Distribution

Example

The standard normal distribution on $\mathbb{R}$
has density function f given by $f_X(x) := ke^{-\frac{1}{2}x^2}$
where the normalizing constant k must be chosen
so that $\int_{-\infty}^{+\infty} ke^{-\frac{1}{2}x^2} \, \mathrm{d}\, x = 1$.

Make the substitution $y = x/\sqrt{2}$,
implying that $y^2 = \frac{1}{2}x^2$ and $\mathrm{d}\, x = \sqrt{2}\, \mathrm{d}\, y$.

Using the rule for integration by substitution, for each $b \in \mathbb{R}$
one has $\int_{-b}^{+b} ke^{-\frac{1}{2}x^2} \, \mathrm{d}\, x = \int_{-b/\sqrt{2}}^{+b/\sqrt{2}} k\sqrt{2}e^{-y^2} \, \mathrm{d}\, y$.

Taking limits as $b \to \infty$, we see that $\int_{-\infty}^{+\infty} ke^{-\frac{1}{2}x^2} \, \mathrm{d}\, x = 1$
only if $\int_{-\infty}^{+\infty} k\sqrt{2}e^{-y^2} \, \mathrm{d}\, y = 1$.

But the Gaussian integral is $\int_{-\infty}^{+\infty} e^{-y^2} \, \mathrm{d}\, y = \sqrt{\pi}$ and so $k\sqrt{2\pi} = 1$
implying that $k = 1/\sqrt{2\pi}$ and so $f_X(x) := (1/\sqrt{2\pi})e^{-\frac{1}{2}x^2}$.

Outline

Expectation w.r.t. a Probability Measure

Given the probability space $(S, \Sigma, \mathbb{P})$,
consider the Σ-measurable random variable $S \ni s \mapsto X(s) \in \mathbb{R}$.

Provided that $S \ni s \mapsto |X(s)| \in \mathbb{R}_+$ is integrable,
with $\int_S |X(s)| \, \mathbb{P}(\mathrm{d}\, s) < +\infty$,
we can define the expectation or expected value
of the random variable $S \ni s \mapsto X(s) \in \mathbb{R}$
as the Lebesgue integral $\int_S X(s) \, \mathbb{P}(\mathrm{d}\, s)$.

Expectation w.r.t. a Density Function

Let $g : \mathbb{R} \to \mathbb{R}$ be any Borel measurable function,
and $x \mapsto f_X(x)$ the density function of the random variable X.
Whenever the integral $\int_{-\infty}^{\infty} |g(x)| f_X(x) \mathrm{d}x$ exists,
the expectation of $g \circ X$ is defined as

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) \mathrm{d}x$$

Theorem

Let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ and $a, b, c \in \mathbb{R}$. Then:

1. $\mathbb{E}(a g_1(X) + b g_2(X) + c) = a\mathbb{E}(g_1(X)) + b\mathbb{E}(g_2(X)) + c$.
2. *If $g_1 \geq 0$, then $\mathbb{E}(g_1(X)) \geq 0$.*
3. *If $g_1 \geq g_2$, then $\mathbb{E}(g_1(X)) \geq \mathbb{E}(g_2(X))$.*

Chebychev's Inequality: Statement

Theorem

For any random variable $S \ni s \mapsto X(s) \in Z$*,*
fix any measurable function $g : Z \to \mathbb{R}_+$ *with* $\mathbb{E}[g(X(s))] < +\infty$*.*
Then for all $r > 0$ *one has* $\mathbb{P}(g(X) \geq r) \leq \frac{1}{r}\mathbb{E}[g(X)]$*.*

Chebychev's Inequality: Proof

Proof.
The two indicator functions $s \mapsto 1_{g(X)\geq r}(s)$ and $s \mapsto 1_{g(X)<r}(s)$ satisfy $1_{g(X)\geq r}(s) + 1_{g(X)<r}(s) = 1$ for all $s \in S$.

Because $g(X(s)) \geq 0$ for all $s \in S$, one has

$$\begin{aligned} \mathbb{E}[g(X)] &= \mathbb{E}[\{1_{g(X)\geq r}(s) + 1_{g(X)<r}(s)\}\, g(X(s))] \\ &= \mathbb{E}[1_{g(X)\geq r}(s)\; g(X(s))] + \mathbb{E}[1_{g(X)<r}(s)\; g(X(s))] \\ &\geq \mathbb{E}[1_{g(X)\geq r}(s)\; g(X(s))] \\ &\geq r\, \mathbb{E}[1_{g(X)\geq r}(s)] \;=\; r\, \mathbb{P}(g(X) \geq r) \end{aligned}$$

Dividing by r, which is positive,
it follows that $\frac{1}{r}\mathbb{E}[g(X)] \geq \mathbb{P}(g(X) \geq r)$. □

Moments and Central Moments

For a random variable X and any $k \in \mathbb{N}$:

- its kth (noncentral) moment is $\mathbb{E}[X^k]$ (where $X^k(s)$ denotes the kth power of the random variable $X(s)$);
- its kth central moment is $\mathbb{E}[(X - \mathbb{E}[X])^k]$, assuming that $\mathbb{E}[X]$ exists in $\mathbb{R}$;
- its variance, $\operatorname{Var} X$, is its second central moment.

Odd Central Moments of the Gaussian Distribution

Given any $n \in \mathbb{N}$ and any $a > 0$,
define $m_n(a) := \int_{-a}^{+a} \frac{1}{\sqrt{2\pi}} x^n e^{-\frac{1}{2}x^2} dx$.

When n is odd, one has $(-x)^n = -x^n$, so

$$\begin{aligned}
m_n(a) &= \int_{-a}^{+a} \frac{1}{\sqrt{2\pi}} x^n e^{-\frac{1}{2}x^2} dx \\
&= \int_{-a}^{0} \frac{1}{\sqrt{2\pi}} x^n e^{-\frac{1}{2}x^2} dx + \int_{0}^{+a} \frac{1}{\sqrt{2\pi}} x^n e^{-\frac{1}{2}x^2} dx \\
&= -\int_{0}^{+a} \frac{1}{\sqrt{2\pi}} x^n e^{-\frac{1}{2}x^2} dx + \int_{0}^{+a} \frac{1}{\sqrt{2\pi}} x^n e^{-\frac{1}{2}x^2} dx \\
&= 0
\end{aligned}$$

This allows us to define $m_n := \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} x^n e^{-\frac{1}{2}x^2} dx$
as the nth central moment of the standard Gaussian distribution,
and to assert that $m_n = 0$ when n is odd.

Even Central Moments of the Gaussian Distribution

Now suppose $n = 2r$, where $r \in \mathbb{N}$.

Because $\frac{d}{dx} e^{-\frac{1}{2}x^2} = -x\, e^{-\frac{1}{2}x^2}$, integrating by parts gives

$$\begin{aligned}
\int_{-a}^{+a} \frac{1}{\sqrt{2\pi}} x^n e^{-\frac{1}{2}x^2} dx \;&=\; -\int_{-a}^{+a} \frac{1}{\sqrt{2\pi}} x^{n-1} \left(\frac{d}{dx} e^{-\frac{1}{2}x^2} \right) dx \\
= -\Big|_{-a}^{+a} \frac{1}{\sqrt{2\pi}} x^{n-1} e^{-\frac{1}{2}x^2} &+ \int_{-a}^{+a} \frac{1}{\sqrt{2\pi}} (n-1) x^{n-2} e^{-\frac{1}{2}x^2} dx \\
= -\frac{1}{\sqrt{2\pi}} [a^{n-1} - (-a)^{n-1}] e^{-\frac{1}{2}a^2} &+ \int_{-a}^{+a} \frac{1}{\sqrt{2\pi}} (n-1) x^{n-2} e^{-\frac{1}{2}x^2} dx
\end{aligned}$$

Taking the limit as $a \to \infty$, the first non-integral term tends to 0, so one obtains $m_n = (n-1) m_{n-2}$.

Note that $m_0 = 1$, so when n is an even integer $2r$, one has

$$\begin{aligned}
m_{2r} \;&=\; (2r-1)(2r-3)\cdots 5 \cdot 3 \cdot 1 \\
&=\; \frac{2r(2r-1)(2r-2)(2r-3)\cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2r(2r-2)(2r-4)\cdots 6 \cdot 4 \cdot 2} \;=\; \frac{(2r)!}{2^r\, r!}
\end{aligned}$$

Outline

Bivariate Distribution

Let $(S, \Sigma, \mathbb{P})$ be a probability space.

Suppose that $S \ni s \mapsto (X(s), Y(s)) \in \mathbb{R}^2$
is a pair of Σ-measurable functions.

The bivariate probability distribution function is the mapping
defined by $\mathbb{R}^2 \ni (x, y) \mapsto F_{X,Y}(x, y) \in [0, 1]$
where $F_{X,Y}(x, y) := \mathbb{P}(\{s \in S \mid X(s) \leq x \text{ and } Y(s) \leq y\})$.

There are two separate
marginal distributions $x \mapsto F_X(x)$ and $y \mapsto F_Y(y)$
of the two random variables $X(s)$ and $Y(s)$ given by

$$F_X(x) := \mathbb{P}(\{s \in S \mid X(s) \leq x\}) = \lim_{y \to \infty} F_{X,Y}(x, y)$$

$$F_Y(y) := \mathbb{P}(\{s \in S \mid Y(s) \leq y\}) = \lim_{x \to \infty} F_{X,Y}(x, y)$$

Multiple Random Variables

Let $S \ni s \mapsto \mathbf{X}(s) = (X_n(s))_{n=1}^N$
be an N-dimensional vector of random variables
defined on the probability space $(S, \Sigma, \mathbb{P})$.

- Its joint distribution function is the mapping defined by

$$\mathbb{R}^N \ni \mathbf{x} \mapsto F_{\mathbf{X}}(\mathbf{x}) := \mathbb{P}(\{s \in S \mid \mathbf{X}(s) \leqq \mathbf{x}\}) \in [0, 1]$$

- The random vector $\mathbf{X}$ is absolutely continuous
just in case there exists a density function $f_{\mathbf{X}} : \mathbb{R}^N \to \mathbb{R}_+$
such that

$$F_{\mathbf{X}}(\mathbf{x}) = \int_{\mathbf{u} \leqq \mathbf{x}} f_{\mathbf{X}}(\mathbf{u})\, d\mathbf{u} \quad \text{for all } \mathbf{x} \in \mathbb{R}^N$$

Independent Random Variables

Let $\mathbf{X}$ be an N-dimensional vector valued random variable.

- If $\mathbf{X}$ is absolutely continuous,
  the marginal density $\mathbb{R} \ni x \mapsto f_{X_n}(x)$ of its nth component X_n is defined as the $N-1$-dimensional iterated integral

  $$f_{X_n}(x) = \int \cdots \int f_{\mathbf{X}}(x_1, \ldots, x_{n-1}, x, x_{n+1}, \ldots, x_N)\, \mathrm{d}x_1 \ldots \mathrm{d}x_N$$

  in which every random variable except X_n gets "integrated out".
- The N components of $\mathbf{X}$ are independent just in case:
  1. the joint density $f_{\mathbf{X}}$ is the product $\prod_{n=1}^N f_{X_n}$ of the marginal densities;
  2. the joint CDF $F_{\mathbf{X}}$ is the product $\prod_{n=1}^N F_{X_n}$ of the marginal CDFs.
- The infinite sequence $(X_n)_{n=1}^\infty$ of random variables is independent just in case every finite subsequence $(X_n)_{n \in K}$ (K finite) is independent.

Expectations of a Function of N Random Variables

Let $\mathbf{X}$ be an N-dimensional vector valued random variable, and $g : \mathbb{R}^N \to \mathbb{R}$ a measurable function.

The expectation of $g(\mathbf{X})$ is defined as the N-dimensional integral

$$\mathbb{E}[g(\mathbf{X})] := \int_{\mathbb{R}^N} g(\mathbf{u}) f_{\mathbf{X}}(\mathbf{u})\, d\mathbf{u}$$

when this integral exists.

Theorem

If the collection $(X_n)_{n=1}^N$ of random variables is independent, then $\mathbb{E}\left[\prod_{n=1}^N X_n\right] = \prod_{n=1}^N \mathbb{E}(X_n)$.

Exercise

Prove that if the pair (X_1, X_2) of r.v.s is independent, then its covariance satisfies

$$\text{Cov}(X_1, X_2) := \mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2])] = 0$$

Zero Covariance Does Not Imply Independence

Example

1. Suppose that X and Y are two independent random variables that induce on $\mathbb{R}$ the respective measures

$$\xi = \tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_1 \quad \text{and} \quad \eta = \tfrac{1}{2}\delta_{-1} + \tfrac{1}{2}\delta_1$$

2. Suppose that Z is the random variable defined by $Z = XY$.
   The measure it induces on $\mathbb{R}$ is $\zeta = \frac{1}{4}\delta_{-1} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_1$, with $\mathbb{E}Z = 0$.
3. The joint measure that (X, Z) induces on $\mathbb{R}^2$ is

$$\tfrac{1}{2}\delta_{(0,0)} + \tfrac{1}{4}\delta_{(1,-1)} + \tfrac{1}{4}\delta_{(1,1)}$$

Evidently $\text{Cov}(X, Z) := \mathbb{E}[(X - \frac{1}{2})Z] = \mathbb{E}[XZ] = 0$.

Yet the conditional distributions of Z are δ_0 given $X = 0$ but $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ given $X = 1$.

Marginal and Conditional Density

Fix the pair (X_1, X_2) of random variables.

- The marginal density of X_1 is

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{(X_1,X_2)}(x_1, x_2)\mathrm{d}x_2.$$

- At points x_1 where $f_{X_1}(x_1) > 0$,
  the conditional density of X_2 given that $X_1 = x_1$ is

$$f_{X_2|X_1}(x_2|x_1) = \frac{f_{(X_1,X_2)}(x_1, x_2)}{f_{X_1}(x_1)}$$

Theorem

If the pair (X_1, X_2) is independent and $f_{X_1}(x_1) > 0$, then

$$f_{X_2|X_1}(x_2|x_1) = f_{X_2}(x_2)$$

Conditional Expectations

Fix the pair (X_1, X_2) of random variables.

- The conditional expectation of $g(X_2)$ given that $X_1 = x_1$ is

$$\mathbb{E}[g(X_2)|X_1 = x_1] = \int_{-\infty}^{\infty} g(x_2) f_{X_2|X_1}(x_2|x_1) \mathrm{d}x_2.$$

- Given any measurable function $(x_1, x_2) \mapsto g(x_1, x_2)$, the law of iterated expectations states that

$$\mathbb{E}_{f_{(X_1,X_2)}}[g((X_1, X_2)(s))] = \mathbb{E}_{f_{X_1}}[\mathbb{E}_{f_{X_2|X_1}}[g((X_1, X_2)(s))]]$$

Proof.

$$\begin{aligned}
\mathbb{E}_{f_{(X_1,X_2)}}[g] &= \int_{\mathbb{R}^2} g(x_1, x_2)\ f_{(X_1,X_2)}(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(x_1, x_2)\ f_{X_2|X_1}(x_2|x_1)\, \mathrm{d}x_2\right]\ f_{X_1}(x_1)\, \mathrm{d}x_1 \\
&= \mathbb{E}_{f_{X_1}}[\mathbb{E}_{f_{X_2|X_1}} g(x_1, x_2)]] \quad \square
\end{aligned}$$

Outline

Convergence of Random Variables

The sequence $(X_n)_{n=1}^{\infty}$ of random variables:

- converges in probability to X (written as $X_n \xrightarrow{p} X$) just in case, for all $\epsilon > 0$, one has

$$\lim_{n \to \infty} \mathbb{P}(|X_n - X| < \epsilon) = 1.$$

- converges in distribution to X (written as $X_n \xrightarrow{d} X$) just in case, for all x at which F_X is continuous,

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

Definition of Weak Convergence

Definition

Let $(X, \Sigma, \mathbb{P})$ be any probability space.

Then a continuity set of $(X, \Sigma, \mathbb{P})$
is any set $B \in \Sigma$ whose boundary ∂B satisfies $\mathbb{P}(\partial B) = 0$.

Definition

Let (X, d) be a metric space with its Borel σ-algebra Σ.

A sequence $(\mathbb{P}_n)_{n \in \mathbb{N}}$ of probability measures
on the measurable space (X, Σ) converges weakly
to the probability measure $\mathbb{P}$, written $\mathbb{P}_n \Rightarrow \mathbb{P}$,
just in case $\mathbb{P}_n(B) \to \mathbb{P}(B)$ as $n \to \infty$
for any continuity set B of $(X, \Sigma, \mathbb{P})$.

Portmanteau Theorem

Theorem

Let $\mathbb{P}$ and $(\mathbb{P}_n)_{n \in \mathbb{N}}$ be probability measures on the measurable space (X, Σ).

Then $\mathbb{P}_n \Rightarrow \mathbb{P}$ if and only if:

1. *for all bounded continuous functions $f : X \to \mathbb{R}$, one has:*

$$\int_X f(x) \mathbb{P}_n(\mathrm{d}\, x) \to \int_X f(x) \mathbb{P}(\mathrm{d}\, x)$$

2. $\limsup_{n \to \infty} \mathbb{P}_n(C) \leq \mathbb{P}(C)$ *for every closed subset $C \subset X$;*
3. $\liminf_{n \to \infty} \mathbb{P}_n(U) \geq \mathbb{P}(U)$ *for every open set $U \subset X$.*

Convergence of Distribution Functions

Theorem

Let F and $(F_n)_{n \in \mathbb{N}}$ be cumulative distribution functions on $\mathbb{R}$ with associated probability measures $\mathbb{P}$ and $(\mathbb{P}_n)_{n \in \mathbb{N}}$ on the Lebesgue real line that satisfy

$$F(x) = \mathbb{P}((-\infty, x]) \quad \textit{and} \quad F_n(x) = \mathbb{P}_n((-\infty, x]) \quad (n \in \mathbb{N})$$

on the measurable space (X, Σ).

Then $\mathbb{P}_n \Rightarrow \mathbb{P}$ if and only if $F_n(x) \to F(x)$ for all x at which F is continuous.

Convergence of Probabilities: Warning

The following example shows that it is not very sensible to say that the sequence of probability measures $\mathbb{P}_n$ $(n \in \mathbb{N})$ on a measurable space (X, Σ) converges to $\mathbb{P}$ just in case $\mathbb{P}_n(E) \to \mathbb{P}(E)$ for all $E \in \Sigma$, even when E is not a continuity set.

Example

Suppose that for each $n \in \mathbb{N}$ the probability measure $\mathbb{P}_n$ on the Borel real line corresponds to the uniform distribution on the interval $I_n := (-\frac{1}{n}, \frac{1}{n})$.

Then $\mathbb{P}_n \Rightarrow \delta_0$,
the degenerate probability measure that satisfies $\delta_0(\{0\}) = 1$,
even though $\mathbb{P}_n(\{0\}) = 0$ for all $n \in \mathbb{N}$.

Verify that 0 is not a continuity point of δ_0.

The Weak Law of Large Numbers

- The sequence $(X_n)_{n=1}^{\infty}$ of random variables is i.i.d.
  — i.e., independently and identically distributed
  — just in case
  1. it is independent, and
  2. for every Borel set $D \subseteq \mathbb{R}$, one has $\mathbb{P}(X_n \in D) = \mathbb{P}(X_{n'} \in D)$.
- The weak law of large numbers:
  Let $(X_n)_{n=1}^{\infty}$ be i.i.d. with $\mathbb{E}(X_n) = \mu$.
  Define the sequence

$$(\bar{X}_n)_{n=1}^{\infty} := \left(\frac{1}{n} \sum\nolimits_{k=1}^{n} X_k \right)_{n=1}^{\infty}$$

of sample means. Then, $\bar{X}_n \xrightarrow{p} \mu$.

A “Frequentist” Interpretation of Probability

Prove the following:

Let $\gamma = p(X \in \Omega) \in (0, 1)$.

Consider the following experiment:
“n realizations of X are taken independently.”

Let G_n be the relative frequency with which a realization in Ω is obtained in the experiment.

Then, $G_n \stackrel{p}{\to} \gamma$.

The Central Limit Theorem

- The central limit theorem:
  Let $(X_k)_{k=1}^{\infty}$ be an infinite sequence of i.i.d. random variables with common mean $\mathbb{E}(X_k) = \mu$ and variance $\mathsf{V}(X_k) = \sigma^2$.
  For each $n \in \mathbb{N}$, define $\bar{X}_n := \frac{1}{n} \sum_{k=1}^{n} X_k$ as the sample average of n observations. Then:
  1. $\mathbb{E}(\bar{X}_n) = \mu$ and $\mathsf{V}(\bar{X}_n) = \dfrac{1}{n^2} \sum_{k=1}^{n} \mathsf{V}(X_k) = \dfrac{n\sigma^2}{n^2} = \dfrac{\sigma^2}{n}$;
  2. For each $n \in \mathbb{N}$, the random variable $Z_n := \sqrt{n}\dfrac{\bar{X}_n - \mu}{\sigma}$ is standardized in the sense that $\mathbb{E}[Z_n] = 0$ and $\mathbb{E}[Z_n^2] = 1$.
  3. One has $Z_n \xrightarrow{d} Y$ where Y has the standard normal cdf given by $F_Y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} \mathrm{d}u$ for all $x \in \mathbb{R}$.

     In particular, $\mathbb{E}(Y) = 0$ and $\mathbb{E}(Y^2) = 1$.

The Fundamental Theorems

Let $(X_n)_{n=1}^{\infty}$ be i.i.d., with $\mathbb{E}[X_n] = \mu$ and $\mathbf{V}(X_n) = \sigma^2$. Then:

- by the law of large numbers,

$$\bar{X}_n \xrightarrow{p} \mu;$$

  so

$$\bar{X}_n \xrightarrow{d} \mu;$$

- but by the central limit theorem,

$$Z_n := \frac{\bar{X}_n - \mu}{(\sigma/\sqrt{n})} \xrightarrow{d} Z \text{ where } F_Z(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} \mathrm{d}u$$

Example

In case each X_n is Gaussian, it can be shown that the linear combination Z_n is Gaussian.

But $\mathbb{E}[Z_n] = 0$ and $\mathbf{V}(Z_n) = 1$,
so each Z_n is exactly Gaussian with mean 0 and variance 1.

Concepts of Convergence, I

Definition

Say that the sequence X_n of random variables converges almost surely or with probability 1 or strongly towards X just in case, for every $\epsilon > 0$, one has

$$\liminf_{n \to \infty} \mathbb{P}(\{\omega \in \Omega \mid |X_n(\omega) - X(\omega)| < \epsilon\}) = 1$$

Hence, the values of X_n approach those of X, in the sense that the event that $X_n(\omega)$ does not converge to $X(\omega)$ has probability 0.

Almost sure convergence is often denoted by $X_n \xrightarrow[P-\text{a.s.}]{} X$, with "$P$-a.s." under the arrow that indicates convergence.

Of course, the concept of almost sure convergence depends on the probability measure being used.

Concepts of Convergence, II

For generic random elements X_n on a general metric space (S, d), almost sure convergence is defined similarly, replacing the absolute value $|X_n(\omega) - X(\omega)|$ by the distance $d(X_n(\omega), X(\omega))$.

Almost sure convergence implies convergence in probability, and *a fortiori* convergence in distribution.

It is the notion of convergence used in the strong law of large numbers.

The Strong Law

Definition

The strong law of large numbers (or SLLN)
states that the sample average $\bar{X}_n$
converges almost surely to the expected value $\mu = \mathbb{E}X$.

It is this law (rather than the weak LLN)
that justifies the intuitive interpretation
of the expected value of a random variable
as its "long-term average when sampling repeatedly."

Differences Between the Weak and Strong Laws

The weak law states that for a specified large n, the average $\bar{X}_n$ is likely to be near μ.

This leaves open the possibility that $|\bar{X}_n - \mu| \geq \epsilon$ happens an infinite number of times, although at infrequent intervals.

The strong law shows that this almost surely will not occur.

In particular, it implies that with probability 1,
for any $\epsilon > 0$ there exists n_ϵ
such that $|\bar{X}_n - \mu| < \epsilon$ holds for all $n > n_\epsilon$.

Outline

Moment-Generating Functions

Definition

The nth moment about the origin is defined as $m_n := \mathbb{E}[X^n]$.

This may not exist for large n unless the random variable X is essentially bounded both above and below, meaning that there exists an upper bound $\bar{x}$ on the modulus such that $\mathbb{P}(\{\omega \in \Omega \mid |X(\omega)| \leq \bar{x}\}) = 1$.

Definition

The moment-generating function of a random variable X is

$$\mathbb{R} \ni t \mapsto M_X(t) := \mathbb{E}[e^{tX}]$$

wherever this expectation exists.

At $t = 0$, of course, $M_X(0) = 1$.

For $t \neq 0$, however, unless X is essentially bounded, the expectation may not exist because e^{tX} can be unbounded.

The Gaussian Case

For a normal or Gaussian distribution $N(\mu, \sigma^2)$,
even though the random variable is unbounded,
the tails of the distribution are thin enough to ensure
that the moment generating function exists and is given by

$$\begin{aligned} M(t;\, \mu, \sigma^2) = \mathbb{E}[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{tx-(x-\mu)^2/2\sigma^2} dx \end{aligned}$$

Now make the substitution $y = (x - \mu - \sigma^2 t)/\sigma$,
implying that $dx = \sigma dy$ and that

$$tx - \frac{(x-\mu)^2}{2\sigma^2} = -\tfrac{1}{2}y^2 + \mu t + \tfrac{1}{2}\sigma^2 t^2$$

This transforms the integral to

$$M(t;\, \mu, \sigma^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} e^{\mu t + \frac{1}{2}\sigma^2 t^2} dy = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

From Moment-Generating Functions to Moments

Note that

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots + \frac{t^nX^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{t^nX^n}{n!}$$

Taking the expectation term by term
and then using the definition of the moments of the distribution,
one obtains

$$\begin{aligned} M_X(t) &= \mathbb{E}[e^{tX}] \\ &= 1 + t\mathbb{E}[X] + \frac{t^2}{2!}\mathbb{E}[X^2] + \cdots + \frac{t^n}{n!}\mathbb{E}[X^n] + \cdots \\ &= 1 + tm_1 + \frac{t^2}{2!}m_2 + \cdots + \frac{t^n}{n!}m_n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!}m_n \end{aligned}$$

Derivatives of the Moment-Generating Function

Suppose we find the nth derivative with respect to t
of $M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} m_k$.

One can easily proved by induction on n that

$$\frac{d^n}{dt^n} t^k = k(k-1)(k-2)\dots(k-n+1)t^{k-n} = \frac{k!}{(k-n)!} t^{k-n}$$

So differentiating $t \mapsto M_X(t)$ term by term n times, one obtains

$$\begin{aligned} M_X^{(n)}(t) \; = \; \mathbb{E}\left[\frac{d^n}{dt^n} e^{tX}\right] \; &= \; \sum\nolimits_{k=n}^{\infty} \frac{k!}{(k-n)!} \frac{t^{k-n}}{k!} m_k \\ &= \; \sum\nolimits_{k=n}^{\infty} \frac{t^{k-n}}{(k-n)!} m_k \end{aligned}$$

Putting $t = 0$ yields the equality $M_X^{(n)}(0) = \frac{t^0}{0!} m_n = m_n$.

In this sense, the moment-generating function
does "exponentially generate"
the moments of the probability distribution.

Definition of Characteristic Functions

The moment-generating function may not exist
because the expectation need not converge absolutely.

By contrast, the expectation of the bounded function e^{itX}
always lies in the unit disc of the complex plane $\mathbb{C}$.

So the characteristic function
that we are about to introduce always exists,
which makes it more useful in many contexts.

Definition

For a scalar random variable X with CDF $x \mapsto F_X(x)$,
the characteristic function is defined
as the (complex) expected value of $e^{itX} = \cos tX + i \sin tX$,
where $i = \sqrt{-1}$ is the imaginary unit,
and $t \in \mathbb{R}$ is the argument of the characteristic function:

$$\mathbb{R} \ni t \mapsto \phi_X(t) = \mathbb{E}e^{itx} = \int_{-\infty}^{+\infty} e^{itx} dF_X(x) \in \mathbb{C}$$

Gaussian Case

Consider a normally distributed random variable X with mean μ and variance σ^2.

Its characteristic function can be found by replacing t by it in the expression for the moment

$$M(t;\,\mu,\sigma^2) = \mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Recalling that $(it)^2 = -t^2$, the result is

$$\varphi(t;\,\mu,\sigma^2) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} dx = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$$

In the standard normal or $N(0,1)$ case, when $\mu = 0$ and $\sigma^2 = 1$, one has $\varphi(t;\,0,1) = e^{-\frac{1}{2}t^2}$.

Linear Combinations of Gaussian Random Variables

Suppose that the two independent random variables X and Y have respective means m_X, m_Y and variances v_X, v_Y.

Consider the linear combination $Z := \alpha X + \beta Y$ where $\alpha, \beta \in \mathbb{R}$.

Exercise

Show that Z has mean $m_Z = \alpha m_X + \beta m_Y$ and variance $v_Z = \alpha^2 v_X + \beta^2 v_Y$.

Proposition

If X and Y are both Gaussian, then so is Z.

Proofs

Proof of Proposition.

Because X and Y are independent and Gaussian,
the char. function $\varphi_Z(t) = \mathbb{E}[e^{itZ}]$ of $Z = \alpha X + \beta Y$ satisfies

$$\varphi_Z(t) = \mathbb{E}[e^{it(\alpha X + \beta Y)}] = \mathbb{E}[e^{i(\alpha t)X}]\,\mathbb{E}[e^{i(\beta t)Y}] = \varphi_X(\alpha t)\,\varphi_Y(\beta t)$$

But X and Y are Gaussian
with respective means m_X, m_Y and variances v_X, v_Y.

So $\varphi_X(t) = e^{im_X t - \frac{1}{2} v_X t^2}$ and $\varphi_Y(t) = e^{im_Y t - \frac{1}{2} v_Y t^2}$, implying that

$$\varphi_X(\alpha t) = e^{i\alpha m_X t - \frac{1}{2}\alpha^2 v_X t^2} \text{ and } \varphi_Y(\beta t) = e^{i\beta m_Y t - \frac{1}{2}\beta^2 v_Y t^2}$$

It follows that $\varphi_Z(t) = \varphi_X(\alpha t)\,\varphi_Y(\beta t)$
takes the form $\exp[i(\alpha m_X + \beta m_Y)t - \frac{1}{2}(\alpha^2 v_X + \beta^2 v_Y)t^2]$,
which is the characteristic function $e^{im_Z t - \frac{1}{2} v_Z t^2}$
of a Gaussian random variable with mean $m_Z = \alpha m_X + \beta m_Y$
and variance $v_Z = \alpha^2 v_X + \beta^2 v_Y$. □

Use of Characteristic Functions

Characteristic functions can be used
to give superficially simple proofs
of both the LLN and the classical central limit theorems.

The following merely sketches the argument.

For much more careful detail,
see Richard M. Dudley's major text, *Real Analysis and Probability*.

A key tool is Lévy's continuity theorem.

For a sequence of random variables,
this connects convergence in distribution
to pointwise convergence of their characteristic functions.

Statement of Lévy's Continuity Theorem

Theorem

Suppose $(X_n)_{n=1}^\infty$ *is a sequence of random variables, not necessarily sharing a common probability space, with the corresponding sequence*

$$\mathbb{R} \ni t \mapsto \varphi_n(t) = \mathbb{E}e^{itX_n} \in \mathbb{C} \quad (n \in \mathbb{N})$$

of complex-valued characteristic functions.

If X_n *converges in distribution to the random variable* X*, then* $t \mapsto \varphi_n(t)$ *converges pointwise to* $t \mapsto \varphi(t) = \mathbb{E}e^{itX}$*, the characteristic function of* X*.*

Conversely, if $t \mapsto \varphi_n(t)$ *converges pointwise to a function* $t \mapsto \varphi(t)$ *which is continuous at* $t = 0$*, then* $t \mapsto \varphi(t)$ *is the characteristic function* $\mathbb{E}e^{itX}$ *of a random variable* X*, and* X_n *converges in distribution to* X*.*

Linear Approximation to the Characteristic Function

Suppose that the random variable X
has a mean $\mu_X := \mathbb{E}X = \int_{-\infty}^{\infty} x dF(x)$.

One can then differentiate within the expectation to obtain

$$\frac{d}{dt}\mathbb{E}e^{itX} = \mathbb{E}\left[\frac{d}{dt}e^{itX}\right] = \mathbb{E}[iXe^{itX}]$$

Consider the linear approximation

$$\mathbb{E}e^{ihX} = 1 + i[\mu + \xi(h)]h \quad \text{where} \quad \xi(h) := (\mathbb{E}e^{ihX} - 1 - ih\mu)/h$$

By l'Hôpital's rule, one has

$$\lim_{h \to 0} \xi(h) = \text{“}0/0\text{”} = \lim_{h \to 0} (\mathbb{E}[iXe^{ihX}] - i\mu)/1 = \mathbb{E}[iX] - i\mu = 0$$

Quadratic Approximation to the Characteristic Function

Next, suppose that the random variable X
has not only a mean $\mu_X := \int_{-\infty}^{\infty} x dF(x)$,
but also a variance $\sigma_X^2 := \int_{-\infty}^{\infty} (x-\mu)^2 dF(x)$.

One can then differentiate twice within the expectation to obtain

$$\frac{d^2}{dt^2}\mathbb{E}e^{itX} = \mathbb{E}\left[\frac{d^2}{dt^2}e^{itX}\right] = \mathbb{E}[(iX)^2 e^{itX}] = -\mathbb{E}[X^2 e^{itX}]$$

Consider the quadratic approximation

$$\mathbb{E}e^{ihX} = 1 + i\mu h - \tfrac{1}{2}[\sigma^2 + \mu^2 + \eta(h)]h^2$$

where $\eta(h) := (1/h^2)[\mathbb{E}e^{ihX} - 1 - ih\mu] + \frac{1}{2}(\sigma^2 + \mu^2)$.

Applying l'Hôpital's rule twice, one has

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h^2}[\mathbb{E}e^{ihX} - 1 - ih\mu] = \text{“}0/0\text{”} &= \lim_{h\to 0} \frac{1}{2h}(\mathbb{E}[iXe^{ihX}] - i\mu) \\ = \text{“}0/0\text{”} = \lim_{h\to 0} \tfrac{1}{2}\mathbb{E}[(iX)^2 e^{ihX}] &= -\tfrac{1}{2}\mathbb{E}X^2 = -\tfrac{1}{2}(\sigma^2 + \mu^2)\end{aligned}$$

implying that $\eta(h) \to 0$ as $h \to 0$.

A Helpful Lemma

Lemma

Suppose that $\mathbb{R} \ni h \mapsto \zeta(h) \in \mathbb{C}$ *satisfies* $\zeta(h) \to 0$ *as* $h \to 0$.

Then for all $z \in \mathbb{C}$, *one has* $\{1 + \frac{1}{n}[z + \zeta(1/n)]\}^n \to e^z$ *as* $n \to \infty$.

For a sketch proof, first one can show that

$$\lim_{n\to\infty} \{1 + \frac{1}{n}[z + \zeta(1/n)]\}^n = \lim_{n\to\infty} (1 + \frac{1}{n}z)^n$$

Second, in case $z \in \mathbb{R}$, putting $h = 1/n$ and taking logs gives

$$\begin{aligned} \ln\left[\lim_{n\to\infty}(1 + \frac{1}{n}z)^n\right] &= \ln\left[\lim_{h\to 0}(1 + hz)^{1/h}\right] \\ &= \lim_{h\to 0}\frac{1}{h}[\ln(1 + hz) - \ln 1] = \frac{d}{dh}\ln(1 + hz)\Big|_{h=0} = z \end{aligned}$$

implying that $(1 + \frac{1}{n}z)^n \to e^z$ as $n \to \infty$.

Dealing with the case when z is complex is more tricky.

Sketch Proof of the Weak LLN, I

Consider now any infinite sequence $X_1, X_2, \ldots$ of observations of IID random variables drawn from a common CDF $F(x)$ on $\mathbb{R}$, with common characteristic function $t \mapsto \varphi_X(t) = \mathbb{E}[e^{itX}]$.

For each $n \in N$, let $\bar{X}_n := \frac{1}{n} \sum_{j=1}^n X_j$ denote the random variable whose value is the sample mean of the first n observations.

This sample mean has its own characteristic function

$$\varphi_{\bar{X}_n}(t) := \mathbb{E}[e^{it\bar{X}_n}] = \mathbb{E}\left[\prod_{j=1}^n e^{itX_j/n}\right]$$

Then

$$\varphi_{\bar{X}_n}(t) = \prod_{j=1}^n \mathbb{E}[e^{itX_j/n}] = \left(\mathbb{E}[e^{itX/n}]\right)^n$$

because the random variables X_j
are respectively independently and identically distributed.

Sketch Proof of the Weak LLN, II

Suppose we take the linear approximation

$$\mathbb{E}e^{ihX} = 1 + i[\mu + \xi(h)]h \quad \text{where} \quad \xi(h) \to 0 \quad \text{as} \quad h \to 0$$

and replace h by t/n to obtain

$$\mathbb{E}[e^{it\bar{X}_n}] = \{1 + (it/n)[\mu + \xi(t/n)]\}^n$$

Because $\xi(t/n) \to 0$ as $n \to \infty$ and so $h = t/n \to 0$, one has

$$\lim_{n\to\infty} \{1 + \frac{1}{n} it[\mu + \xi(t/n)]\}^n = \lim_{n\to\infty} (1 + \frac{1}{n} it\mu)^n = e^{it\mu} = \mathbb{E}[e^{itY}]$$

where $\mathbb{E}[e^{itY}]$ is the characteristic function
of a degenerate random variable Y
which is equal to μ with probability 1.

Using the Lévy theorem, it follows that the distribution of $\bar{X}_n$
converges to this degenerate distribution,
implying that $\bar{X}_n$ converges to μ in probability.

Sketch Proof of the Classical CLT, I

For each $j \in \mathbb{N}$, let Z_j denote the standardized value $(X_j - \mu)/\sigma$ of X_j, defined to have the property that $\mathbb{E}Z_j = 0$ and $\mathbb{E}Z_j^2 = 1$.

Now define $\bar{Z}_n := \sum_{j=1}^{n} \frac{Z_j}{\sqrt{n}}$.

This is called the standardized mean because:

1. linearity implies that $\mathbb{E}\bar{Z}_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mathbb{E}Z_j = 0$;
2. independence implies that $\mathbb{E}\bar{Z}_n^2 = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}Z_j^2 = 1$.

Putting $\mu = 0$ and $\sigma^2 = 1$ in the quadratic approximation

$$\mathbb{E}e^{ihX} = 1 + i\mu h - \tfrac{1}{2}[\sigma^2 + \mu^2 + \eta(h)]h^2$$

implies $\mathbb{E}e^{ihZ} = 1 - \frac{1}{2}[1 + \eta(h)]h^2$ where $\eta(h) \to 0$ as $h \to 0$.

Replacing hX by $tZ_j/\sqrt{n}$ in this quadratic approximation yields

$$\mathbb{E}[e^{itZ_j/\sqrt{n}}] = 1 - \tfrac{1}{2}\frac{t^2}{n}[1 + \eta(t/n)]$$

Sketch Proof of the Classical CLT, II

Now independence implies that

$$\mathbb{E}[e^{it\bar{Z}_n}] = \mathbb{E}\left[\exp\left(it\frac{1}{\sqrt{n}}\sum\nolimits_{j=1}^{n} Z_j\right)\right] = \prod\nolimits_{j=1}^{n} \mathbb{E}[e^{itZ_j}/\sqrt{n}]$$

Hence, another careful limiting argument shows that

$$\mathbb{E}[e^{it\bar{Z}_n}] = \left\{1 - \tfrac{1}{2}\frac{t^2}{n}[1 + \eta(t/n)]\right\}^n \to e^{-\frac{1}{2}t^2} \text{ as } n \to \infty$$

But we showed that this limit $e^{-\frac{1}{2}t^2}$
is precisely the characteristic function
of a standard normal distribution $N(0,1)$.

Because $t \mapsto e^{-\frac{1}{2}t^2}$ is continuous at $t = 0$,
the central limit theorem follows from the Lévy continuity theorem,
which confirms that the convergence of characteristic functions
implies convergence in distribution.

Outline

A Continuum of Independent Random Variables

Hotelling had a continuum of ice cream buyers
distributed along a finite beach on a hot day.

Vickrey and Mirrlees considered optimal income taxation
when a population of workers have continuously distributed skills.

Aumann and Hildenbrand modelled
a perfectly competitive market system
with a continuum of traders
who each have negligible individual influence over market prices.

Market clearing in the economy as a whole requires,
for each separate commodity, equality between:
(i) mean demand per trader; and (ii) mean endowment per trader.

Bewley considered a continuum of consumers
with "independently fluctuating" random endowments.

Then, is mean endowment in the population even defined?

A Process with a Continuum of IID Random Variables

Let $\mathcal{L}$ denote the Lebesgue σ-field on $\mathbb{R}$,
and let I denote the unit interval $[0, 1]$.

Definition

A process with a continuum of iid random variables
on the Lebesgue unit interval $(I, \mathcal{L}, \lambda)$ involves:

- a sample probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- a mapping $I \times \Omega \ni (i, \omega) \mapsto f(i, \omega) \in \mathbb{R}$ satisfying

$$\mathbb{P}\left(\cap_{n\in\mathbb{N}}\{\omega \in \Omega \mid f(i_n, \omega) \in B_n\}\right) = \prod_{n\in\mathbb{N}} \mathbb{P}\left(\{\omega \in \Omega \mid f(i_n, \omega) \in B_n\}\right)$$

for every countable collection $(i^{\mathbb{N}}, B^{\mathbb{N}})$
of pairs $(i, B) \in I \times \mathcal{B}$. □

Difficulty Illustrated

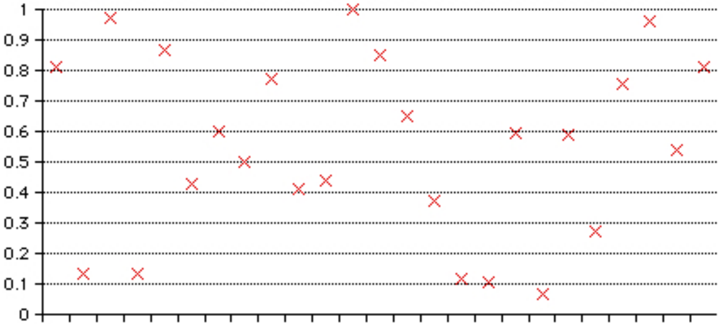
In the following graphs, think of the horizontal axis as the Lebesgue unit interval, indicating something like a U.S. social security number (SSN) ($\times 10^{-9}$).

Think of the vertical axis as the Lebesgue unit interval, indicating something like an individual's height, measured as a percentile.

Assume that SSN gives no information about height.

Then the heights of, approximately, a continuum of individuals may be regarded as statistically IID random variables.

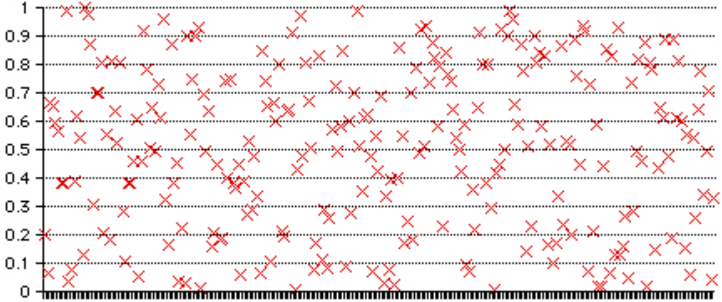
Difficulty Illustrated: 25 Random Draws



The points have $x \in \left\{(1/25)(n - \frac{1}{2}) \mid n \in \{1, \ldots, 25\}\right\}$
and y pseudo-randomly drawn from a uniform distribution on $[0, 1]$.

Finding the mean of y is becoming messy.

Difficulty Illustrated: 200 Random Draws



The points have $x \in \left\{(1/200)(n - \frac{1}{2}) \mid n \in \{1, \ldots, 200\}\right\}$ and y pseudo-randomly drawn from a uniform distribution on $[0, 1]$.

Finding the mean of y is becoming impossible.

Essential Supremum and Infimum

Recall that the supremum is the least upper bound,
and the infimum is the greatest lower bound.

Define the essential supremum and infimum
of each random variable $\omega \mapsto f(i, \omega)$ as:

$$\begin{aligned} \operatorname{ess\,sup} f(i, \omega) &:= \inf \{ b \in \mathbb{R} \mid \mathbb{P}(\{\omega \in \Omega \mid f(i, \omega) \leq b\}) = 1 \} \\ \operatorname{ess\,inf} f(i, \omega) &:= \sup \{ a \in \mathbb{R} \mid \mathbb{P}(\{\omega \in \Omega \mid f(i, \omega) \geq a\}) = 1 \} \end{aligned}$$

These differ from the supremum and infimum
by allowing one to disregard an event which has probability zero.

A Theorem on Non-Measurable Sample Paths

For all the continuum of IID random variables $\omega \mapsto f(i, \omega)$, let $a \in \mathbb{R}$ denote the common value of $\operatorname{ess\,inf} f(i, \omega)$ and $b \in \mathbb{R}$ the common value of $\operatorname{ess\,sup} f(i, \omega)$.

Theorem

Whenever $a < b$,
the sample path $I \ni i \mapsto f(i, \omega)$ is $\mathbb{P}$-a.s. non measurable.

Of course, when $a = b$, then $\mathbb{P}$-a.s. one has $f(i, \omega) = a = b$ — i.e., the process $(i, \omega) \mapsto f(i, \omega)$ is essentially constant.

The key idea in the proof is to show that the sample path has a lower integral a and an upper integral b.

Monte Carlo Integration

Because of the strong law of large numbers, here is one way to approximate numerically the integral $\int_K f(\mathbf{x})\,\mu(d\mathbf{x})$ of a complicated function of ℓ variables, where $K \subset \mathbb{R}^\ell$ has an ℓ-dimensional Lebesgue measure $\mu(K) < +\infty$.

1. First, choose a large sample $\langle \mathbf{x}^r \rangle_{r=1}^n$ of n points that are independent and identically distributed random draws from the set K, with common probability measure π satisfying $\pi(B) = \mu(B)/\mu(K)$ for all Borel sets $B \subseteq K$.
2. Second, calculate the sample average function value

$$M^n(\langle \mathbf{x}^r \rangle_{r=1}^n) := \frac{1}{n} \sum\nolimits_{r=1}^n f(\mathbf{x}^r)$$

3. Third, observe that, by the strong law of large numbers, the sample average $M^n(\langle \mathbf{x}^r \rangle_{r=1}^n)$ converges almost surely as $n \to \infty$ to the theoretical mean

$$\mathbb{E}_\pi[f(\mathbf{x}^k)] = \int_K f(\mathbf{x})\,\pi(d\mathbf{x}) = \frac{1}{\mu(K)} \int_K f(\mathbf{x})\,\mu(d\mathbf{x})$$

The Monte Carlo Integral: Rescuing Macroeconomics

Definition

Given the process $I \times \Omega \ni (i, \omega) \mapsto f(i, \omega) \in \mathbb{R}$
with a continuum of iid random variables,
define the Monte Carlo integral as the random variable

$$\Omega \ni \omega \mapsto {}_{\text{MC}}\int_I f(i, \omega) \lambda(di) \in \mathbb{R}$$

as the almost sure limit as $n \to \infty$ of the average $\frac{1}{n} \sum_{k=1}^{n} f(i_k, \omega)$
when the n points $\langle i_k \rangle_{k=1}^{n}$ are independent draws
from the Lebesgue unit interval $(I, \mathcal{L}, \lambda)$.

Then, even though the Lebesgue integral $\int_I f(i, \omega) \lambda(di)$
is almost surely undefined, the strong law of large numbers
implies that the Monte Carlo integral ${}_{\text{MC}}\int_I f(i, \omega) \lambda(di)$
is well defined as a degenerate random variable $\Omega \ni \omega \mapsto \delta_m(\omega)$
that attaches probability one
to the common theoretical mean $m := \int_\Omega f(i, \omega) \, P(d\omega)$.