EC9A0: Pre-sessional Advanced Mathematics Course

Comparative Statics

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September 2023

Lecture Outline

Introduction

- Comparative statics results describe what happens to an optimal solution in response to changes in exogenous parameters such as prices, wealth or taxes.
- For example,
  1. What happens with the cost/profit function and input demands of a competitive firm when wages change?
  2. What happens with the agent's utility and Walrasian demand when income changes?
- In particular, will small changes in these parameters lead to only small changes in the objective function? And to small changes in the optimal solution?
- The purpose of this section is to establish some of these results.

Preliminaries

- $X \subset \mathbb{R}^L$ is the set of exogenous parameters, $Y \subset R^K$ is the set of choice variables.
- $f : X \times Y \mapsto \mathbb{R}$ is a function.
- $\Gamma : X \mapsto Y$ is a non-empty correspondence.
- We are interested in the following problem:

$$\sup_y f(x, y)$$

$$\text{s.t. } y \in \Gamma(x)$$

  where the $\Gamma : X \mapsto Y$ describe the feasibility constraints.
- If $\Gamma(x)$ is nonempty and compact valued, Weirstrass theorem implies $v : X \mapsto \mathbb{R}$

$$v(x) \equiv \sup_{y \in \Gamma(x)} f(x, y) \tag{1}$$

  is well defined.
- $G : X \mapsto Y$ defined by

$$G(x) = \{y \in \Gamma(x) : f(x, y) = v(x)\} \tag{2}$$

  is the set of values of y that solve the problem for each x.

Examples

Example 1

- The profit maximisation problem is:

$$\sup_{(z,q)\in\mathbb{R}_+^{L+1}} pq - w'z \qquad \text{s.t. } q \leq f(z)$$

- $X = \mathbb{R}_+^{L+1}$ is the price space and $Y \subset \mathbb{R}_+^{L+1}$ is the commodity space.
- $f(w,p,z,q) = pq - w'z$, where $p, q \in \mathbb{R}$ and $(w,z) \in \mathbb{R}_+^L$, is the profit function.
- $\Gamma(w,p) = \{(z,q) \in Y : q \leq f(z)\}$ is the set of technologically feasible plans.

Example 2

- The utility maximisation problem is:

$$\sup_{c\in\mathbb{R}_+^{L}} u(c) \qquad \text{s.t. } p'c \leq w$$

- $X = \mathbb{R}^{L+1}$ is the space of income and prices and $Y \subset \mathbb{R}_+^L$ is the consumption set.
- $f(w,p,c) = u(c)$, where $(w,p) \in \mathbb{R}^{L+1}$ and $c \in \mathbb{R}_+^L$, is the utility function.
- $\Gamma(w,p) = \{c \in Y : p'c \leq w\}$ is the budget set.

Lower- and Upper- Hemicontinuity

Definition

A correspondence $\Gamma : X \mapsto Y$ is lower hemi-continuous (l.h.c.) at x if $\Gamma(x)$ is nonempty and if, for every sequence $x_n \to x$ and for every $y \in \Gamma(x)$, there exists $N \geq 1$ and a sequence $\{y_n\}_{n=N}^{\infty}$ such that $y_n \in \Gamma(x_n)$, all $n \geq N$, and $y_n \to y$.

Definition

A compact valued correspondence $\Gamma : X \mapsto Y$ is upper hemi-continuous (u.h.c.) at x if $\Gamma(x)$ is nonempty and if, for every sequence $x_n \to x$ and every sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in \Gamma(x_n)$, all n, there exists a convergent subsequence of $\{y_n\}_{n=1}^{\infty}$ whose limit point y is in $\Gamma(x)$.

Definition

A correspondence $\Gamma : X \mapsto Y$ is continuous at $x \in X$ if it is both u.h.c. and l.h.c. at x. A correspondence $\Gamma : X \to Y$ is called l.h.c, u.h.c., or continuous if it has that property at every point $x \in X$.

Lower- and Upper- Hemicontinuity

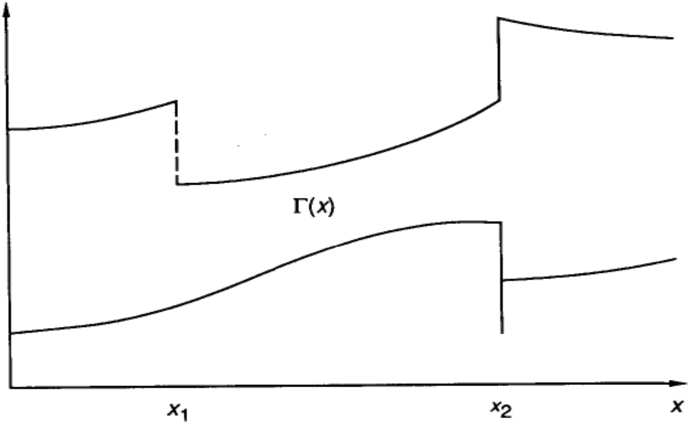

Figure: Lower- and hemi - continuity

The correspondence is l.h.c but not u.h.c at x_1 and u.h.c but not l.h.c at x_2.

Lower- and Upper- Hemicontinuity: Examples

Example

Show that:

a. if Γ is single valued and u.h.c., then it is continuous.

b. if Γ is single valued and l.h.c., then it is continuous.

Example

a. Let $\Gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be defined by $\Gamma(x) = [0, x]$. Show that Γ is continuous.

b. Let $f_i : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$, be a continuous functions and define the correspondence $\Gamma : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$ by $\Gamma(x) = [0, f(x)]$. Show that Γ is continuous.

Theorem of the Maximum

Theorem of the Maximum

Let $X \subset \mathbb{R}^L$ and $Y \subset R^K$, let $f : X \times Y \mapsto \mathbb{R}$ be a continuous function and $\Gamma : X \mapsto Y$ be a compact-valued and continuous correspondence. Then the function $v : X \mapsto \mathbb{R}$ defined in (1) is continuous, and the correspondence $G : X \mapsto Y$ defined in (2) is nonempty, compact valued, and u.h.c.

Proof: Let $x \in X$.

- First we show $G(x)$ is nonempty and compact.
  1. $\Gamma(x)$ is nonempty and compact, and $f(x, \cdot)$ is continuous.
  2. By Weierstrass Theorem, $v(x)$ is well defined and $G(x)$ is nonempty.
  3. Since $G(x) \subset \Gamma(x)$ and $\Gamma(x)$ is bounded, $G(x)$ is bounded.
  4. Let $y_n \rightarrow y$ where $y_n \in G(x) \subset \Gamma(x)$. Since $\Gamma(x)$ is closed, $y \in \Gamma(x)$.
  5. Since $v(x) = f(x, y_n)$ for all n and f is continuous, $f(x, y) = v(x)$.
  6. Then, $y \in G(x)$. Thus, $G(x)$ is closed.
- Next we show $G(x)$ is u.h.c.
  1. Let $x_n \rightarrow x$. Choose $y_n \in G(x_n)$.
  2. Since Γ is u.h.c., there is $y_{n_k} \rightarrow y \in \Gamma(x)$.
  3. Let $z \in \Gamma(x)$. To show $y \in G(x)$, we need to show $f(x, y) \geq f(x, z)$.
  4. Since $\Gamma(x)$ is l.h.c., there is $z_{n_k} \rightarrow z$, with $z_{n_k} \in \Gamma(x_{n_k})$.
  5. Since $f(x_{n_k}, y_{n_k}) \geq f(x_{n_k}, z_{n_k})$ and f is continuous, $f(x, y) \geq f(x, z)$.
- Continuity of v is left as an exercise.

Q.E.D.

Theorem of the Maximum: Example

Example

- Let $X = \mathbb{R}$, $\Gamma(x) = Y = [-1, 1]$, all $x \in X$ and $f : X \times Y \mapsto \mathbb{R}$ where $f(x, y) = xy^2$.
- Then,

$$G(x) = \begin{cases} \{-1, 1\} & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ \{0\} & \text{if } x < 0 \end{cases}$$

- We show $G(x)$ is u.h.c. at $x = 0$.
  1. $G(0)$ is nonempty and compact valued. Let $x_n \to 0$ and $y_n \in G(x_n)$.
  2. Suppose there is $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} < 0$ for all k. Then $y_{n_k} = 0$ for all k and so there is a subsequence of $\{y_n\}$ with $y_{n_k} \to 0 \in G(0)$.
  3. Suppose there is $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} > 0$ for all k. Then there is a convergent subsequence of $\{y_{n_k}\}_{k=1}^{\infty}$. Thus, $y_{n_k} \to 1 \in G(0)$ or $y_{n_k} \to -1 \in G(0)$.
- We show $G(x)$ is not l.h.c at $x = 0$.
  1. Choose $y = 0.5 \in G(0)$ and $x_n \to 0$ such that $x_n < 0$ for all $n \in \mathbb{N}$.
  2. Hence, $y_n = 0$ for all $n \in \mathbb{N}$.
  3. Hence it cannot be the case that $y_n \to y = 0.5$.

Envelope Theorem

- Suppose $Y \subseteq \mathbb{R}^K$ and $X \subseteq \mathbb{R}^L$ are open.
- $f : X \times Y \to \mathbb{R}$ and $g : X \times Y \to \mathbb{R}^J$,
$$v(x) = \max_{y \in Y} f(x, y) : g(x, y) = 0.$$
- To learn how the value of the problem changes with x_i, we need $\frac{\partial v(x)}{\partial x_i}$.
- Suppose there are differentiable functions $h : X \to Y$ and $\lambda : X \to \mathbb{R}^J$, given by the solution of the problem and the associated multiplier, for all x.
- Of course, we could use brute force
$$\frac{\partial v(x)}{\partial x_i} = \frac{\partial f(x, h(x))}{\partial x_i} + \sum_{k=1}^{K} \frac{\partial f(x, h(x))}{\partial y_k} \frac{\partial h_k(x)}{\partial x_i}$$
but $\frac{\partial h_k(x)}{\partial x_i}$ might be hard to compute.
- Suppose $h(x)$ solves this maximisation problem if and only if there is a $\lambda(x) \in \mathbb{R}^J$ such that $D_y \mathcal{L}(x, h(x), \lambda(x)) = 0$.

Theorem

If v is continuously differentiable at $\bar{x}$, $D_x v(\bar{x}) = D_x \mathcal{L}(\bar{x}, h(\bar{x}), \lambda(\bar{x}))$.

Proof of the Envelope Theorem

Proof.

Note that:

$$v(x) \equiv f\,(x, h(x)) = \mathcal{L}(x, h(x), \lambda(x)) \text{ for all } x \in X$$

Thus

$$\begin{aligned}
\frac{\partial v(x)}{\partial x_i} &= \frac{\partial \mathcal{L}(x,h(x),\lambda(x))}{\partial x_i} \\
&= \frac{\partial \mathcal{L}(x,h(x),\lambda(x))}{\partial x_i} + \sum_{k=1}^{K} \frac{\partial \mathcal{L}(x,h(x),\lambda(x))}{\partial y_k} \frac{\partial h_k(x)}{\partial x_i} + \sum_{j=1}^{J} \frac{\partial \mathcal{L}(x,h(x),\lambda(x))}{\partial \lambda_j} \frac{\partial \lambda_j(x)}{\partial x_i}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v(\bar{x})}{\partial x_i} &= \frac{\partial \mathcal{L}(x,h(\bar{x}),\lambda(\bar{x}))}{\partial x_i} + \sum_{k=1}^{K} \frac{\partial \mathcal{L}(x,h(\bar{x}),\lambda(\bar{x}))}{\partial y_k} \frac{\partial h_k(\bar{x})}{\partial x_i} + \sum_{j=1}^{J} \frac{\partial \mathcal{L}(x,h(\bar{x}),\lambda(\bar{x}))}{\partial \lambda_j} \frac{\partial \lambda_j(\bar{x})}{\partial x_i} \\
&= \frac{\partial \mathcal{L}(x,h(\bar{x}),\lambda(\bar{x}))}{\partial x_i}
\end{aligned}$$

□