EC9A0: Pre-sessional Advanced Mathematics Course

Unconstrained Optimisation

Pablo F. Beker
Department of Economics
University of Warwick

September 2020

Lecture Outline

1. Infimum and Supremum
   - Definitions
   - Properties
2. Maximisers
   - Definitions
   - Existence
3. Local Maxima
   - Necessary Conditions in $\mathbb{R}$
   - Sufficient Conditions in $\mathbb{R}$
   - Necessary Conditions in $\mathbb{R}^K$
   - Sufficient Conditions in $\mathbb{R}^K$
4. When is a Local max also a Global Max?
   - Functions in $\mathbb{R}$ with only one critical point
   - Concavity and Quasi-Concavity
5. Uniqueness

Infimum and Supremum: Definitions

Definition

Fix a set $Y \subseteq \mathbb{R}$. A number $\alpha \in \mathbb{R}$ is *an upper bound of Y* if $y \leq \alpha$ for all $y \in Y$, and is *a lower bound of Y* if the opposite inequality holds.

Definition

$\alpha \in \mathbb{R}$ is *the least upper bound of Y*, denoted $\alpha = \sup Y$, if:

1. α is an upper bound of Y; and
2. $\gamma \geq \alpha$ for any other upper bound γ of Y.

Definition

$\beta \in \mathbb{R}$ is *the greatest lower bound of Y*, denoted $\beta = \inf Y$, if:

1. β is a lower bound of Y; and
2. if γ is a lower bound of Y, then $\gamma \leq \beta$.

Properties of Infimum and Supremum

Theorem 1

$\alpha = \sup Y$ *if and only if for every* $\varepsilon > 0$,
(a) $y < \alpha + \varepsilon$ *for all* $y \in Y$; *and*
(b) there is some $y \in Y$ *such that* $\alpha - \varepsilon < y$.

Corollary 1

Let $Y \subseteq \mathbb{R}$ *and let* $\alpha \equiv \sup Y$. *Then there exists a sequence* $\{y_n\}_{n=1}^{\infty}$ *in* Y *that converges to* α.

We need a stronger concept of extremum, in particular one that implies that the extremum lies within the set.

Maximisers

Definition

A point $x \in \mathbb{R}$ is *the maximum of set $Y \subseteq \mathbb{R}$*, denoted $x = \max A$, if $x \in Y$ and $y \leq x$ for all $y \in Y$.

- Typically, it is of more interest in economics to find extrema of functions, rather than extrema of sets.

Definition

$\bar{x} \in D$ is *a global maximizer of $f : D \to \mathbb{R}$* if $f(x) \leq f(\bar{x})$ for all $x \in D$.

Definition

$\bar{x} \in D$ is *a local maximizer of $f : D \to \mathbb{R}$* if there exists some $\varepsilon > 0$ such that $f(x) \leq f(\bar{x})$ for all $x \in B_\varepsilon(\bar{x}) \cap D$.

- When $\bar{x} \in D$ is a local (global) maximizer of $f : D \to \mathbb{R}$, the number $f(\bar{x})$ is said to be *a local (the global) maximum of f*.

Existence

Theorem (Weierstrass)

Let $C \subseteq D$ be nonempty and compact. If $f : D \to \mathbb{R}$ is continuous, then there are $\bar{x}, \underline{x} \in C$ such that $f(\underline{x}) \leq f(x) \leq f(\bar{x})$ for all $x \in C$.

Proof: It follows from 5 steps:

1. Since C is compact and f is continuous, then $f[C]$ is compact.
2. By Corollary 1, there is $\{y_n\}_{n=1}^{\infty}$ in $f[C]$ s.t. $y_n \to \sup f[C]$.
3. Since $f[C]$ is compact, then it is closed. Therefore, $\sup f[C] \in f[C]$.
4. Thus, there is $\overline{x} \in C$ such that $f(\overline{x}) = \sup f[C]$.
5. By def. of sup, $f(\overline{x}) \geq f(x)$ for all $x \in C$.

Q.E.D.

Characterising Maximisers in $\mathbb{R}$

Lemma 1

Suppose $D \subset \mathbb{R}$ is open and $f : D \to \mathbb{R}$ is differentiable. Let $\bar{x} \in int(D)$. If $f'(\bar{x}) > 0$, then there is some $\delta > 0$ such that for each $x \in B_\delta(\bar{x}) \cap D$:

1. *$f(x) > f(\bar{x})$ if $x > \bar{x}$.*
2. *$f(x) < f(\bar{x})$ if $x < \bar{x}$.*

Proof: $\varepsilon \equiv \frac{f'(\bar{x})}{2} > 0$. Then, $f'(\bar{x}) - \varepsilon > 0$. By def. of f', $\exists \delta > 0$ s.t.,

$$\left|\frac{f(x)-f(\bar{x})}{x-\bar{x}} - f'(\bar{x})\right| < \varepsilon, \ \forall x \in B'_\delta(\bar{x}) \cap D.$$

Hence, $\frac{f(x)-f(\bar{x})}{x-\bar{x}} > f'(\bar{x}) - \varepsilon > 0.$ *Q.E.D.*

Corollary 2

Suppose $D \subset \mathbb{R}$ is open and $f : D \to \mathbb{R}$ is differentiable. Let $\bar{x} \in D$. If $f'(\bar{x}) < 0$, then there is $\delta > 0$ such that for every $x \in B_\delta(\bar{x}) \cap D$:

1. *$f(x) < f(\bar{x})$ if $x > \bar{x}$.*
2. *$f(x) > f(\bar{x})$ if $x < \bar{x}$.*

Characterising Maximisers in $\mathbb{R}$: FO Necessary Conditions

Theorem (FONC)

Suppose that $f : D \to \mathbb{R}$ is differentiable. If $\bar{x} \in \text{int}(D)$ is a local maximiser of f then $f'(\bar{x}) = 0$.

Proof: Suppose $f'(\bar{x}) \neq 0$. WLOG, suppose $f'(\bar{x}) > 0$.

1. By Lemma 1, $\exists \delta > 0$ such that $f(x) > f(\bar{x})$ for all $x \in B_\delta(\bar{x}) \cap D$ satisfying $x > \bar{x}$.
2. Since $\bar{x}$ is a local maximizer of f, $\exists \varepsilon > 0$ such that $f(x) \leq f(\bar{x})$ for all $x \in B_\varepsilon(\bar{x}) \cap D$.
3. Since $\bar{x} \in \text{int}(D)$, $\exists \gamma > 0$ such that $B_\gamma(\bar{x}) \subseteq D$.
4. Let $\beta = \min\{\varepsilon, \delta, \gamma\} > 0$.
5. Clearly, $(\bar{x}, \bar{x} + \beta) \subset B'_\beta(\bar{x}) \subseteq D$. Moreover, $B'_\beta(\bar{x}) \subseteq B_\delta(\bar{x}) \cap D$ and $B'_\beta(\bar{x}) \subseteq B_\varepsilon(\bar{x}) \cap D$.
6. $\exists x$ such that $f(x) > f(\bar{x})$ and $f(x) \leq f(\bar{x})$, a contradiction.

Q.E.D.

Characterising Maximisers in $\mathbb{R}$: SO Necessary Conditions

Theorem (SONC)

Let $f : D \to \mathbb{R}$ be $\mathbb{C}^2$. If $\bar{x} \in \text{int}(D)$ is a local max of f, then $f''(\bar{x}) \leq 0$.

Proof: Since $\bar{x} \in \text{int}(D)$, there is a $\varepsilon > 0$ such that $B_\varepsilon(\bar{x}) \subseteq D$.

1. Let $h \in B_\varepsilon(0)$. Since f is $\mathbb{C}^2$, Taylor's Theorem implies $\exists x_h^* \in [\bar{x}, \bar{x} + h]$ such that
$$f(\bar{x} + h) = f(\bar{x}) + f'(\bar{x})h + \frac{1}{2}f''(x_h^*)h^2$$
2. $\exists \delta > 0$ such that $f(x) \leq f(\bar{x})$ for all $x \in B_\delta(\bar{x}) \cap D$.
3. Let $\beta = \min\{\varepsilon, \delta\} > 0$. By construction, for any $h \in B'_\beta(0)$
$$f'(\bar{x})h + \frac{1}{2}f''(x_h^*)h^2 = f(\bar{x} + h) - f(\bar{x}) \leq 0.$$
4. By Theorem FONC, $f'(\bar{x}) = 0$ and so $f'(\bar{x})h = 0$.
5. Hence, $f''(x_h^*)h^2 \leq 0 \implies f''(x_h^*) \leq 0$.
6. $\lim_{h\to 0} f''(x_h^*) \leq 0$, and hence that $f''(\bar{x}) \leq 0$, since f'' is continuous and each x_h lies in the interval joining $\bar{x}$ and $\bar{x} + h$. *Q.E.D.*

Characterising Maximisers in $\mathbb{R}$: Sufficient Conditions

Theorem (FOSC & SOSC)

Suppose that $f : D \to \mathbb{R}$ is $\mathbb{C}^2$. Let $\bar{x} \in \text{int}(D)$. If $f'(\bar{x}) = 0$ and $f''(\bar{x}) < 0$, then $\bar{x}$ is a local maximizer.

Proof: Since $f : D \to \mathbb{R}$ is $\mathbb{C}^2$ & $f''(\bar{x}) < 0$, by Corollary 2 $\exists \delta > 0$ s.t.
(a)$f'(x) < f'(\bar{x}) = 0$, for all $x \in B_\delta(\bar{x}) \cap D$ for which $x > \bar{x}$; and
(b)$f'(x) > f'(\bar{x}) = 0$, for all $x \in B_\delta(\bar{x}) \cap D$ for which $x < \bar{x}$.

1. Since $x \in \text{int}(D)$, there is $\varepsilon > 0$ such that $B_\varepsilon(\bar{x}) \subseteq D$.
2. Let $\beta = \min\{\delta, \varepsilon\} > 0$. By the MV Theorem, $\exists x^* \in [\bar{x}, x]$ s.t.

$$f(x) = f(\bar{x}) + f'(x^*)(x - \bar{x}) \text{ for all } x \in B_\beta(\bar{x})$$

3. $x > \bar{x} \Rightarrow x^* \geq \bar{x}$. Hence, (a) $\Rightarrow f'(x^*)(x - \bar{x}) \leq 0 \Rightarrow f(x) \leq f(\bar{x})$.
4. $x < \bar{x} \Rightarrow x^* \leq \bar{x}$. Hence, (b) $\Rightarrow f'(x^*)(x - \bar{x}) \leq 0 \Rightarrow f(x) \leq f(\bar{x})$. ■

- We use $f''(\overline{x}) < 0$ to show $f'(x^*)(x - \bar{x}) \leq 0$. Why $f''(\overline{x}) \leq 0$ is not enough?

Example in $\mathbb{R}$

- Consider $f(x) = x^4 - 4x^3 + 4x^2 + 4$.
- Note that

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x-1)(x-2).$$

- Hence, $f'(x) = 0 \iff x \in \{0, 1, 2\}$.
- Since $f''(x) = 12x^2 - 24x + 8$,

$$f''(0) = 8 > 0, f''(1) = -4 < 0, \text{ and } f''(2) = 8 > 0$$

- $x = 0$ and $x = 2$ are local min of f and $x = 1$ is a local max.
- $x = 0$ and $x = 2$ are global min but $x = 1$ is not a global max.

Characterising Maximisers in $\mathbb{R}^K$: Necessary Conditions

Suppose $D \subset \mathbb{R}^K$

Theorem

If $f : D \to \mathbb{R}$ is differentiable and $x^* \in int(D)$ is a local maximizer of f, then $Df(x^*) = 0$.

Theorem

If $f : D \to \mathbb{R}$ is $\mathbb{C}^2$ and $x^* \in int(D)$ is a local maximizer of f, then $D^2 f(x^*)$ is negative semidefinite.

Characterising Maximisers in $\mathbb{R}^K$: Sufficient Conditions

Suppose $D \subset \mathbb{R}^K$

Theorem

Suppose that $f : D \to \mathbb{R}$ is $\mathbb{C}^2$ and let $\bar{x} \in int(D)$. If $Df(\bar{x}) = 0$ and $D^2 f(\bar{x})$ is negative definite, then $\bar{x}$ is a local maximizer.

Example in $\mathbb{R}^2$

- Consider $f(x,y) = x^3 - y^3 + 9xy$.
- Note that

$$\begin{aligned} f'_x(x,y) &= 3x^2 + 9y \\ f'_y(x,y) &= -3y^2 + 9x \end{aligned}$$

- Hence,
  $f'_x(x,y) = 0$ and $f'_y(x,y) = 0 \iff (x,y) \in \{(0,0),(3,-3)\}$.

$$D^2 f(x) = \begin{pmatrix} f''_{xx} & f''_{yx} \\ f''_{xy} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 6x & 9 \\ 9 & -6y \end{pmatrix}.$$

- $f''_{xx} = 6x$ and $|D^2 f(x,y)| = -36xy - 81$.
- At $(0,0)$ the two minors are 0 and -81. Hence, $D^2 f(0,0)$ is indef.
- At $(3,-3)$ the two minors are 18 and 243. Hence, $D^2 f(3,-3)$ is positive definite and $(3,-3)$ is a local min.
- $(3,-3)$ is not a global min since $f(0,n) = -n^3 \to -\infty$ as $n \to \infty$.

Functions in $\mathbb{R}$ with only one critical point

First let's note that any derivative has the intermediate value property, a result due to Darboux.

Theorem (Darboux)

If a function $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b), then for every γ between $f'(a)$ and $f'(b)$ there exists an $x \in [a, b]$ for which $f'(x) = \gamma$.

Theorem

Suppose that $f : D \to \mathbb{R}$ is differentiable in the interior of $D \subset \mathbb{R}$ and:

1. *the domain of f is an interval in $\mathbb{R}$.*
2. *x is a local maximum of f, and*
3. *x is the only solution to $f'(x) = 0$ on D.*

Then, x is the global maximum of f.

Concavity and Quasi-Concavity: Definitions

Definition

Let D be a convex subset of $\mathbb{R}^K$. Then, $f : D \to \mathbb{R}$ is

- *concave* if for all $x, y \in D$, and for all $\theta \in [0, 1]$,

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta) f(y)$$

- *strictly concave* if for all $x, y \in D$, $x \neq y$, and for all $\theta \in (0, 1)$,

$$f(\theta x + (1-\theta)y) > \theta f(x) + (1-\theta) f(y)$$

- *quasi-concave* if for all $x, y \in D$, and for all $\theta \in [0, 1]$,

$$f(x) \geq f(y) \implies f(\theta x + (1-\theta)y) \geq f(y)$$

- *strictly quasi-concave* if for all $x, y \in D$, $x \neq y$, and for all $\theta \in (0, 1)$,

$$f(x) \geq f(y) \implies f(\theta x + (1-\theta)y) > f(y)$$

Ordinal Properties

Theorem

Suppose $f : D \to \mathbb{R}$ is quasi-concave and $g : f(D) \to \mathbb{R}$ is nondecreasing. Then $g \circ f : D \to \mathbb{R}$ is quasi-concave. If f is strictly quasi-concave and g is strictly increasing, then $g \circ f$ is strictly quasi-concave.

Proof: Since f is quasi-concave, $f(\theta x + (1-\theta)y) \geq \min\{f(x), f(y)\}$. Since g is nondecreasing,

$$g(f(\theta x + (1-\theta)y)) \geq g(\min\{f(x), f(y)\}) = \min\{g(f(x)), g(f(y))\}.$$

If f is strictly quasi-concave, $x \neq y$, $f(\theta x + (1-\theta)y) > \min\{f(x), f(y)\}$. Since g is strictly increasing,

$$g(f(\theta x + (1-\theta)y)) > g(\min\{f(x), f(y)\}) = \min\{g(f(x)), g(f(y))\}.$$

Q.E.D.

When is a Local Max also a Global Max? - Concavity

Theorem

Suppose that $D \subset \mathbb{R}^K$ is convex and $f : D \to \mathbb{R}$ is a concave function. If $\bar{x} \in D$ is a local maximizer of f, then it is also a global maximizer.

Proof: Suppose that $\bar{x} \in D$ is a local but not a global maximizer of f.

- $\exists \varepsilon > 0$ such that $f(x) \leq f(\bar{x})$ for all $x \in B_\varepsilon(\bar{x}) \cap D$ and
- $\exists x^* \in D$ such that $f(x^*) > f(\bar{x})$.

1. $x^* \notin B_\varepsilon(\bar{x})$, which implies that $\|x^* - \bar{x}\| \geq \varepsilon$.
2. Since D is convex and f is concave, we have that for $\theta \in [0,1]$,
$$f(\theta x^* + (1-\theta)\bar{x}) \geq \theta f(x^*) + (1-\theta) f(\bar{x}).$$
3. Since $f(x^*) > f(\bar{x})$, $\theta f(x^*) + (1-\theta) f(\bar{x}) > f(\bar{x})$ for all $\theta \in (0,1]$.
4. Hence, $f(\theta x^* + (1-\theta)\bar{x}) > f(\bar{x})$.
5. Let $\theta^* \in (0, \varepsilon / \|x^* - \bar{x}\|)$. $\theta^* \in (0,1)$ & $f(\theta^* x^* + (1-\theta^*)\bar{x}) > f(\bar{x})$.
6. $\|(\theta^* x^* + (1-\theta^*)\bar{x}) - \bar{x}\| = \theta^* \|x^* - \bar{x}\| < \left(\frac{\varepsilon}{\|x^* - \bar{x}\|}\right) \|x^* - \bar{x}\| = \varepsilon$,
7. By convexity of D, $(\theta^* x^* + (1-\theta^*)\bar{x}) \in B_\varepsilon(\bar{x}) \cap D$. This contradicts the fact that $f(x) \leq f(\bar{x})$ for all $x \in B_\varepsilon(\bar{x}) \cap D$. *Q.E.D.*

When is a Local Max also a Global Max?-Quasi-Concavity

Theorem

Suppose that $D \subset \mathbb{R}^K$ is convex and $f : D \to \mathbb{R}$ is strictly quasi-concave. If $\bar{x} \in D$ is a local maximizer of f, then it is also a global maximizer.

- Can we prove the last theorem assuming only quasi-concavity?

Uniqueness

Suppose $D \subset \mathbb{R}^K$.

Theorem

Suppose $f : D \to \mathbb{R}$ attains a maximum.
(a) If f is quasi-concave, then the set of maximisers is convex.
(b) If f is strictly quasi-concave, then the maximiser of f is unique.