Aggregate Uncertainty in Runoff Elections

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Abstract

In this paper I develop a model of strategic entry by candidates for office in runoff elections. The main contribution of the paper is that I introduce aggregate uncertainty over the distribution of policy preferences in the electorate. I model aggregate uncertainty over voter preferences in the second round of a runoff election by allowing voters to have state-dependent policy preferences, whereas candidates have complete information over the distribution of voter preferences in the first round of the election. The set of equilibria with three candidates expands and equilibrium configurations become more diverse after adding aggregate uncertainty, providing a theoretical basis for Duverger's Hypothesis. Three candidate equilibria also predict three empirical phenomena that were heretofore unexplained: some candidates who reach the second round of the election receive fewer votes than they receive in the first round (an apparent violation of WARP), electoral "reversals," where candidates who obtain a plurality in the first round do not necessarily win in the second round, and candidates who lose with certainty still choose to run.

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1 Introduction

In a runoff election, if no candidate obtains a predetermined fraction of the votes, the electorate is asked to vote for a second time over the two candidates who obtained the largest vote totals on the first ballot. The candidate who obtains a majority of votes in the second round is the winner of the election. Runoff elections are a widespread feature of political competition, as a majority of countries that directly elect a president do so by a runoff rule and the popularity of runoffs is growing over time (Blais et al 1997, Golder 2005).[1] Many elections at the state and local level within the United States also use runoff rules (Bullock III and Johnson 1992, Engstrom and Engstrom 2008). Despite their popularity, however, we have a limited understanding of the incentives faced by prospective candidates in constituencies governed by runoff rules.

In this paper I synthesize two common approaches used in the literature to consider the question of entry incentives in runoff elections with aggregate uncertainty. Despite the shift towards utilizing runoff rules, few papers have considered the question of candidate incentives under runoff rules. Moreover, the few models of candidate behavior in runoff elections assume complete information about both the distribution of preferences in the electorate and what subset of the electorate will exercise their vote (Myerson 1993, Osborne and Slivinski 1996, Lizzeri and Persico 2005, Callander 2005). Assuming that candidates have complete information about the electorate, and that the electorate does not vary across rounds of the election, does not closely approximate reality. Debate performances are often credited as the reason for "bounces" in polls, and candidates regularly compete for third party endorsements. In some cases, third party endorsements which occur between rounds are viewed to have substantial, election-changing effects.[2] The assumptions of complete information and constant preferences are also costly; the set of equilibria generated rarely matches vote shares observed in runoff elections. Standard models, for example, predict that all (strategic) candidates obtain equal vote shares in the first round of the election. I show that the addition of aggregate uncertainty has significant effects on candidate entry incentives and substantially improves the explanatory power of standard models.

Three main results arise from my model. First, I show that adding aggregate uncertainty over voter preferences generates equilibria which feature candidates receiving fewer votes in the second round than in the first round, sometimes causing a *reversal* (the candidate who obtains a plurality in the first round is not the winner in the second round). Reversals are

[1] In the 1960s approximately 30% of presidential elections were runoffs, as compared to 70% in the 1990s. Additionally, 61 of 91 countries which directly elect a president do so through runoff elections.

[2] See, e.g., Sheriff Harry Lee's endorsement of Rep. William Jefferson and attack campaign against State Rep. Karen Carter between rounds of the 2006 election for Louisiana's Second US House District.

a common feature of runoff elections, occurring in approximately 30% of runoff elections in the US (Bullock III and Johnson 1992). Additionally, runoff elections relatively frequently feature candidates obtaining fewer total votes in the second round of the election than they did in the first round of the election. This is also the case in some runoff elections where voter turnout increases or remains constant over the elections, suggesting that the distribution of voter preferences differs across rounds.

Second, in some equilibria, candidates still have an incentive to enter despite losing with certainty. By choosing to enter the race, a sure loser may generate a lottery over the two competing candidates by forcing a second round of the election, thus inducing a preferable expected policy outcome for the entrant. To my knowledge, this is the first strategic explanation of the presence of non-competitive candidates in runoff elections. In some equilibria, the sure loser is the Condorcet winner and one potential winner is the Condorcet loser (both defined with respect to the distribution of preferences in the first round of the election). The possibility of a Condorcet loser obtaining office in equilibrium is also empirically relevant; the victory of Ted Cruz in the Texas Republican primary election for US Senate in 2012 is one example (discussed in more detail below).

Third, I show that under aggregate uncertainty the set of three-candidate equilibria expands and is substantially more diverse than in models with constant preferences. In the citizen-candidate model of Osborne and Slivinski (1996), there are only two types of three-candidate equilibria in runoff elections: either all three candidates share an ideal policy with the median voter or they all have particular distinct positions and each receives one third of the vote. While I show that aggregate uncertainty eliminates centrist equilibria, many additional three-candidate equilibria arise. In addition to each candidate receiving one third of the vote in the first round, which may occur with more diverse ideal policies in my model, equilibria also exist where one candidate obtains a plurality in the first round. The set of two-candidate equilibria also features more differentiated equilibria than in previous work, but two-candidate equilibria exist for fewer parameter values. In that sense, the effect of aggregate uncertainty on the size of the set of two-candidate equilibria is ambiguous. I interpret this result as a strong form of support for Duverger's (1954) Hypothesis: *"simple majority with a second ballot* [the runoff rule] *favors multipartyism."*

My results indicate that aggregate uncertainty is an important component of modeling runoff elections.[3] Models of candidate entry or positioning in runoff elections with sincere

[3]It appears unlikely that strategic voting is capable of generating my results. Bouton and Gratton (2013) show that the "push-over effect" (voting strategically for an inferior candidate in the first round to increase the probability that a voter's preferred candidate has an easier second-round matchup) cannot occur in strictly perfect equilibria in a Poisson voting game. Voters choosing not to vote for a candidate in the second round, having already voted for her in the first round, therefore is not likely explained by strategic voting.

voting and perfect foresight generate a much smaller and more precise set of equilibria, but a set which excludes many empirically relevant outcomes. In addition to improving the predictive power of our models, my results indicate that runoff systems may have some undesirable characteristics. Traditionally, the runoff rule has been perceived to encourage preference revelation while also minimizing the chances of a minority candidate obtaining office. I show that even with a high threshold for first-round victory, a Condorcet losing candidate may obtain office in equilibrium. I also show that equilibria where all candidates enter with a platform of the median voter's ideal policy, which may be normatively desirable, do not exist in runoff elections with aggregate uncertainty.

My model augments the citizen-candidate model of Osborne and Slivinski (1996) with aggregate uncertainty over voter preferences in the second round of the election. I assume that the set of voters who exercise their vote have single-peaked preferences which differ across the two rounds of the election in a way that candidates do not foresee. Specifically, I assume that voters have horizontally differentiated policy preferences, and that the location of the median voter's ideal policy differs according to which state is realized between rounds. My results are, in general, robust to allowing the possibility that the distribution of preferences is constant across the two rounds, but I do not present that version of the model here for concision. The form of aggregate uncertainty I assume does not encompass all uncertainty candidates face at the entry stage, but does generate a very tractable model with novel incentives for candidate behavior. Moreover, the repeated voting structure of runoff elections suggests that events occurring between rounds may provide differing incentives for candidates relative to elections where voters only act once. The set of voters who choose to exercise their vote may vary widely over the two rounds of the election (see Wright 1989, Bullock III and Johnson 1992, Morton and Rietz 2006 for empirical evidence) and events which occur between rounds may reveal more information to voters than they had in the first round of the election. My focus, therefore, is on the unique feature which separates the runoff rule from many other electoral systems.

I adopt the citizen-candidate model rather than a Downsian model for several reasons. First, combining free entry and free choice of policy platform in a model (e.g. a Hotelling-Downs model with free entry) generally results in each candidate winning with positive, and often equal, probability (see, e.g., Brusco, Dziubinski and Roy 2010, Haan and Volkerink 2001). Since one goal of my model is to generate a more realistic set of equilibria, including equilibria where candidates may have incentives to enter strategically and lose with certainty, this Hotelling-Downs result is a substantial restriction. Second, the structure of runoff elections forces an arbitrary choice of the degree of policy commitment allowed to potential candidates. Many candidates have preexisting policy positions (e.g. from prior elections for

Candidate	Round 1 votes	Round 1 share	Round 2 votes	Round 2 share
David Dewhurst	624,170	44.6	480,165	43.2
Ted Cruz	479,079	34.2	631,316	56.8
Tom Leppert	186,675	13.3	–	–
Craig James	50,211	3.6	–	–
Glenn Addison	22,888	1.6	–	–
Lela Pittenger	18,028	1.3	–	–
Ben Gambini	7,193	0.5	–	–
Curt Cleaver	6,649	0.5	–	–
Joe Argis	4,558	0.3	–	–
Total	1,399,451	100	1,111,481	100

Table 1: Vote totals, Republican US Senate Primary, Texas, 2012

incumbents, or positions taken in a primary election). In a standard Hotelling-Downs model, these previous commitments do not affect a candidate's ability to commit to an alternative policy in the general election. Two possible extensions to runoffs suggest themselves: full commitment in the first round of the election, or repositioning between rounds. A priori, I see no particular reason to prefer either extension. In the citizen-candidate model, however, no commitment devices are available to potential candidates; all policy preferences are common knowledge, and therefore candidates who choose to enter must do so at their preferred policy platform. Thus, the commitment technology available to candidates remains consistent across electoral systems, and facilitates easier comparative statics than a Hotelling-Downs framework.

Two examples of electoral results which are equilibria in my model, but not in standard models, are depicted in Table 1, which summarizes the outcome of the Texas Republican party primary election for US Senate in 2012, and Table 2, which summarizes the results from the Brazilian presidential election of 2006. Both elections feature a candidate receiving fewer votes in the second round of the election than in the first, as well as a substantial number of candidates who may be considered to be sure losers. In the case of Table 1, Tom Leppert obtained sufficiently many votes that had he chosen not to run, it is possible that David Dewhurst could have obtained a majority in the first round of the election.[4] The constituents who voted for Leppert and Dewhurst constitute a majority, although a divided one. Nevertheless, after Leppert forced a runoff pitting Dewhurst against Ted Cruz, Dewhurst's vote share dropped dramatically and Cruz won the nomination. I show that

[4]The Washington Post, in an article about Cruz's upset victory, suggests that many of Leppert's supporters would likely have been Dewhurst voters if Leppert was not in the race. http://www.washingtonpost.com/blogs/the-fix/wp/2012/11/28/the-biggest-upset-of-2012/

Candidate	Round 1 votes	Round 1 share	Round 2 votes	Round 2 share
Luiz Inacio Lula Da Silva	46,662,365	48.61	58,295,042	60.83
Geraldo Alckmin	39,968,369	41.64	37,543,178	39.17
Heloisa Helena	6,575,393	6.85	–	–
Cristovam Buarque	2,538,844	2.64	–	–
Ana Maria Rangel	126,404	0.13	–	–
Jose Maria Eymael	63,294	0.07	–	–
Luciano Bivar	62,064	0.06	–	–
Total	95,996,733	100	95,838,220	100

Table 2: Vote totals, Brazilian Presidential Election, 2006

aggregate uncertainty is not only a potential explanation for the results of this election, but that this pattern of results is a possible equilibrium outcome.[5]

While Dewhurst's declining vote total may be explained by a decrease in turnout (although it seems implausible), Geraldo Alckmin's declining vote total in the 2006 Brazilian Presidential election cannot be due to turnout. In 2006, Lula Da Silva won reelection in the second round of a runoff against his main challenger Geraldo Alckmin. Alckmin had obtained 41.64 percent of the votes in the first round, approximately 40 million votes, whereas Lula Da Silva received 48.61 percent of the votes, approximately 46.6 million. The third placed candidate, Heloisa Helena, obtained 6.85 percent of the votes; had she chosen not to run, it is possible that Lula Da Silva would have won in the first round of the election. While in Table 1 the number of votes declined by one fifth between rounds, voting in Brazilian federal elections is mandatory (with minor exceptions). As a result, the total number of votes differed by less than 160,000 votes in an election with nearly 96 million votes cast. Alckmin's vote total declined by over 2.4 million votes.

2 Related Literature

There are three principal strands of literature relating to my paper. First, my model is a contribution to the large literature on candidate entry incentives. Feddersen, et al (1990) is one of the first papers to incorporate both endogenous entry and positioning by candidates in the Hotelling-Downs framework with a plurality election. My result differ substantially; in their model candidates are purely office-motivated, and therefore all candidates must win with positive probability in equilibrium. As a result, all voters are pivotal between every pair of candidates. Their model cannot explain the presence of "sure losers," nor the

[5] Bouton (2012) shows a similar result in a model with strategic voters, although his result requires a threshold for first-round victory less than 50%, which is not the case in this election.

existence of lopsided elections, both of which are regularly observed outcomes.[6] Osborne and Slivinski (1996) and Besley and Coate (1997) independently developed the citizen-candidate model. My results are an extension of Osborne and Slivinski which substantially improves the predictive power of the set of equilibria. Besley and Coate, instead, consider a version of the model with a finite number of voters and do not examine entry incentives in runoff elections. In addition to theoretical work, several papers have looked at empirical evidence regarding Duverger's Law and Hypothesis. The two most convincingly identified papers are Fujiwara (2011) and Bordignon and Tabellini (2009). Fujiwara (2011) exploits a discontinuity in Brazilian electoral rules at the municipality level to identify the causal effect of a change from plurality rule to a runoff rule. He finds that the vote share attributed to third and lower ranked candidates increases by approximately 8.8 percentage points, or approximately 56%. Bordignon and Tabellini (2009) exploit a similar discontinuity in Italian municipal elections and find that a switch from plurality rule to runoff rule increases the number of mayoral candidates by approximately one candidate. My results are complementary; I provide theoretical evidence suggesting that aggregate uncertainty generates dramatically more three-candidate equilibria in runoff elections without a corresponding increase in the amount of two-candidate equilibria.

My model contributes to a growing literature on the effects of uncertainty in elections. Brusco and Roy (2011) utilize the same form of aggregate uncertainty as I do in a citizen-candidate model to study plurality elections. Brusco and Roy show that aggregate uncertainty generates extremist parties in the sense that in all two candidate equilibria, each candidate lies to the outside of the interval formed by the two potential locations of the median voter. My model also generates more extreme two-candidate equilibria than in a model without aggregate uncertainty, but not all equilibria are extremist. My results also complement Brusco and Roy's in that I generate substantially more diverse three-candidate equilibria, including equilibria with extremist candidates. Agranov (2012) considers a model of two-stage elections where voters must infer candidates' ideologies from signals during the campaign and voter preferences differ across the two electoral stages. In contrast to my model, Agranov allows candidates to choose positions freely and change their positions across stages of the election. My results differ in that I focus on entry and number of candidates, whereas Agranov studies positioning in a model with an exogenous number of candidates. Some of our results are similar, though, in that we both generate equilibria where the candidates for office do not share the position of the (expected) median voter. Eguia (2007)

[6] Some parties have such low probabilities of obtaining office that they do not even self-identify the prospect of winning the election as a reason they continue to run, e.g. http://www.il.lp.org/campaigns/reasons2run.php.

considers a citizen-candidate model of plurality elections under uncertainty about whether all votes will be counted. My model differs from Eguia's in both the assumptions I use and the electoral rule considered. Eguia's model has a finite number of strategic voters and focuses on the question of existence of two-candidate equilibria in plurality elections, whereas my model has a continuum of sincere voters and I characterize the set of equilibria for different numbers of candidates. Eguia's results can be read in part as evidence for Duverger's Law, whereas my results provide theoretical evidence for Duverger's Hypothesis. Riviere (2000) considers a citizen-candidate model of pluralities which, in the author's words, relies on very restrictive assumptions. My model, in contrast, focuses on runoff elections and is considerably more general with respect to the distribution of voter preferences.

My model also contributes to the literature concerned with the differing incentives generated by runoff elections. Bouton (2012) examines runoff elections with low (less than 50%) thresholds, although from the perspective of strategic voting rather than strategic candidate behavior. Bouton shows that runoff elections with strategic voters may admit two-candidate equilibria, which provides evidence against Duverger's Law, and that a Condorcet losing candidate can win office in some equilibria. My results are complementary to Bouton's. I confirm that with sincere voting, strategic candidates, and uncertainty over voter preferences, two-candidate equilibria still exist in runoff elections. I also show that a Condorcet losing candidate, defined with respect to the first-round distribution of preferences, wins office with positive probability in some equilibria. Moreover, I show that a Condorcet winning candidate, similarly defined, may enter and win office with probability zero in equilibrium. In my paper, none of these results require a runoff threshold less than 50%. Callander (2005) considers candidate incentives to enter runoff elections in a Downsian model with multiple first-mover candidates. Candidates are purely office-motivated in Callander's model, however, while evidence suggests that candidates are also policy-motivated (see, e.g., Levitt 1996). As a result, my model is capable of generating equilibria which feature candidates entering strategically despite being sure losers in addition to equilibria where all candidates have positive probability of victory.

3 Model

I start with the form of the citizen-candidate model used by Osborne and Slivinski (1996). I assume that the electorate is comprised of a unit mass of citizens $\mathcal{I}$ with single peaked preferences over policy. Policies are represented by points on the real line, $\mathbb{R}$. The ideal point for a citizen i is denoted τ_i, and the ideal points of the electorate are distributed along $\mathbb{R}$ according to an arbitrary distribution F with associated probability density function f. I assume that F is continuous, strictly increasing, and, in the first round of the election, has a unique median, m. The action set for a player i is denoted $\mathcal{A}_i = \{E, N\}$ where E represents entering the race, and N represents not entering. Each time the populace is called to vote, all citizens do so sincerely and myopically.[7] A strategy, therefore, is a mapping $\sigma_i : \tau_i \longrightarrow \mathcal{A}_i$. If a citizen chooses to enter the race, I call her a candidate.

Citizens are policy-motivated and have no preference over the identity of any candidate. Following the notation in OS, citizens who choose to stand for office incur a utility cost c, and obtain office-related benefits b (e.g. "ego rents" as in Rogoff 1990) if she is victorious. If a citizen who chooses N has ideal point a and the winner has ideal point w, the citizen's payoff is

$$\pi_i(N, \sigma_{-i}; \tau_i = a) = -|w - a|.$$

A citizen who chooses N and whose ideal point is the same as the ideal point of the winner obtains a payoff of zero, the maximal possible payoff for a non-candidate. If a citizen chooses E, however, her payoff is dependent on whether she wins office and can be written

$$\pi_i(E, \sigma_{-i}; \tau_i = a) = \begin{cases} b - c & \text{if wins outright} \\ -|w - a| - c & \text{if loses outright.} \end{cases}$$

Each citizen obtains a payoff of $-\infty$ if no citizen chooses to enter. The return to winning an election, b, is the payoff above any policy preferences. The magnitude of b, therefore, represents the relative weight of the incentives to run from holding office as compared to the incentives given by the ability to affect policy. The vote share of candidate i is denoted v_i. I denote the candidate who receives the most votes in the first round of the election by i^* and the candidate who receives the second-most votes in the first round by i^{**}.

The electoral rule considered is as follows. Let $\mathcal{K}$ denote the set of candidates: $\mathcal{K} = \{i \in \mathcal{I} : \sigma_i(\tau_i) = E\}$. If, at any point in the election $\#\mathcal{K} = 2$, the election is decided by plurality rule: $i^* = \arg\max_{i \in \mathcal{K}} v_i$. If $\#\mathcal{K} > 2$ and $\max_{i \in \mathcal{K}} v_i > \frac{1}{2}$, the election ends and the winner is the candidate who received the most votes: $i^* = \arg\max_{i \in \mathcal{K}} v_i$. If, however, $\#\mathcal{K} > 2$ and $\max_{i \in \mathcal{K}} v_i \leq \frac{1}{2}$, the set of candidates is reduced to the two candidates with the largest vote shares $\{i^*, i^{**} : v_{i^*} + v_{i^{**}} \geq v_j + v_k \;\forall\; j, k \in \mathcal{K}\}$. All ties are broken with equal probability.

I model aggregate uncertainty over preferences by introducing two possible states of the world in the event of a second round of the election. Denote the state space $\mathcal{S} = \{L, R\}$. In state L, the distribution of voter preferences is F_L satisfying the same conditions as F, and

[7] I assume sincere, myopic voting for two reasons. First, the assumption that F is a continuous cumulative distribution function makes the model extremely tractable, and also implies that each vote has a pivot probability of zero. Second, the purpose of this paper is to investigate how aggregate uncertainty over voter preferences will affect incentives for candidates' behavior. Combining strategic entry and strategic voting in the citizen-candidate model is left for further work.

with a unique median m_L, but with the additional requirement $m_L < m$. In state R, voters' preferences are distributed according to F_R with unique median $m_R > m$. I assume that state L is realized with probability θ, whereas state R is realized with probability $1-\theta$. Denote the shift in the median voter's ideal policy over states by $m - m_L = s_L$ and $m_R - m = s_R$.

While the game has a dynamic structure necessary to capture the sequential nature of runoff elections, all of the non-trivial actions are taken simultaneously at the start of the game. Therefore, the appropriate solution concept for this game is Nash equilibrium. In particular, a strategy profile σ constitutes a Nash equilibrium for voters with a profile of types τ if $\pi_i(\sigma_i, \sigma_{-i}; \tau) \geq \pi_i(\sigma_{i'}, \sigma_{-i}; \tau)\ \forall\ i \in \mathcal{I}$.

4 Results

I begin by proving a useful lemma. The first result is an adaptation of a result in OS (Lemma 1) to this particular setting with aggregate uncertainty. Lemma 1 is useful because it eliminates many possible configurations of candidates, which reduces the set of potential equilibria that must be checked. Intuitively, the lemma states that candidates who are extreme among the set of entrants cannot lose with certainty because they could obtain a weakly better policy outcome by exiting, helping their most preferred candidate among the other entrants, and saving the cost of running for office. Additionally, and for similar reasons, no candidate can lose with certainty if they share a position with another entrant. This holds for any equilibrium with $k \neq 3$ candidates.[8]

Lemma 1. *In equilibrium, a candidate does not lose with certainty if* $\#\mathcal{K} \neq 3$ *and either:*
(i) There are other candidates with the same ideal position or
(ii) The ideal position of all other candidates are on the same side of her ideal position.

Proof: Suppose $\#\mathcal{K} \neq 3$, a candidate is losing with certainty, and she shares an ideal position with at least one other candidate. By exiting, she obtains a weakly better electoral outcome, as all her voters now vote for a candidate with the same ideal point, and no other voters change their votes. She also saves cost c, and is therefore strictly better off by exiting. A similar argument holds for fringe candidates, except that their voters are now transferred to the candidate closest to her ideal point. This is a weakly better electoral outcome, since she is losing with certainty before exiting, and also saves cost c. If $\#\mathcal{K} = 3$, however, a

[8] The restriction that $k \neq 3$ is a vestige of the particular specification of the runoff rule used here. Runoff rules are typically specified in constitutions as requiring a strict majority (or strictly greater than the threshold), whereas I do not have a second round if only two candidates enter and each obtain exactly half the vote. Two candidates exactly tying is an event never observed in actual runoff elections with finitely many voters, and obtaining exactly the threshold is also extremely unlikely. The particular form of the runoff rule here is necessary only for Proposition 4, which can be easily generated by a slightly more complex two-dimensional citizen-candidate model.

fringe candidate may choose to enter and lose with certainty because by doing so she forces a second round that would not have otherwise happened. ∎

The proof of Lemma 1 provides some insight into the strategic motivations for "sure losers" to run that will reappear in three candidate equilibria. Suppose a potential leftist fringe candidate expects two other candidates to exactly split the vote in a two candidate election (thus evaluated by plurality rule) and therefore faces an even-odds lottery over those two candidates. If she believes that state L is more likely to be realized than state R, i.e. $\theta > \frac{1}{2}$, and running for office has sufficiently low cost, she may also choose to enter the race despite being a sure loser. In essence, she can improve the odds of her favored candidate winning office from an even-odds lottery to a θ to $1-\theta$ lottery. These equilibria don't exist in models without aggregate uncertainty because the sure loser pays cost c, but after failing to make the second round still faces an even-odds lottery over the two remaining candidates.

4.1 Three Candidate Equilibria

Three candidate equilibria are of particular interest in the study of runoff elections. Duverger's Hypothesis predicts that runoff rules encourage multipartyism relative to plurality systems. Duverger's Hypothesis is intuitively appealing due to a perceived reward of finishing second in the first round of a runoff election. In plurality elections, a second place finish may suggest to voters that you (or your party) is a serious challenger, but a second place finish is still a lost election. In a runoff election, however, a second place finish may allow a candidate to continue competing for office. Uncertainty with respect to the distribution of voter preferences in the second round captures this benefit in a way that models with fixed preferences cannot. In a model where the second round distribution of voters preferences is known ex-ante, for any configuration of candidates the outcome of the election is perfectly anticipated (up to a tiebreaking rule). Thus, in order to generate an equilibrium with multiple candidates, the candidates' vote shares must tie in at least one round of the election. Aggregate uncertainty relaxes this constraint substantially. The remainder of Section 4.1 is devoted to illustrating in exactly which ways this constraint relaxes.

In a citizen-candidate model where the spatial distribution of citizens' preferences is constant over time, equilibria with three candidates must feature all candidates sharing the vote equally. This precise form can be generated in two ways, either by a cluster of candidates with identical policy preferences to the median voter or in the form characterized shortly in Proposition 1. Intuitively, if any individual candidate obtains less than a third of the vote in the first round, the consistency of preferences implies that there must be a candidate who is a sure loser regardless of whether she reaches the second round. Since the sure loser can save herself cost c and shift the expected policy to be (weakly) closer to her ideal point by exiting, this cannot possibly be an equilibrium. Equilibria where three candidates all tie for first still exist in my model, although not with three candidates clustered at the median voter's ideal policy (Proposition 7). The possibility of the spatial distribution of preferences differing between rounds, however, opens up a much more diverse set of equilibria. Lemma 2 partially characterizes the set of first-round outcomes generated by a three candidate equilibrium.

Lemma 2. *In any three candidate equilibrium, in the first round the candidates must be configured such that one of the follownig holds:*
(i) all tie for first place and have locations $a_1 \leq a_2 \leq a_3$ *where* $a_1 = F^{-1}(\frac{1}{3}) - \varepsilon_1$*,* $a_2 = F^{-1}(\frac{1}{3}) + \varepsilon_1 = F^{-1}(\frac{2}{3}) - \varepsilon_2$*, and* $a_3 = F^{-1}(\frac{2}{3}) + \varepsilon_2$ *where* $\varepsilon_1, \varepsilon_2 \geq 0$*, or*
(ii) one candidate is alone in first place, but obtains fewer than half the votes, or
(iii) two candidates tie for first place and the third place candidate is a sure loser.

Lemma 2 allows for substantially more types of equilibrium than in OS, but also admits the differentiated three candidate equilibrium from OS. Furthermore, as Proposition 1 summarizes, the necessary condition for such an equilibrium is less restrictive than in OS.

Proposition 1. *In any three candidate equilibrium where each candidate ties for first place in the first round of the election, the interior candidate never has the uniquely smallest probability of winning or expected payoff. Moreover, the candidates have ideal policies* $a_1 = F^{-1}(\frac{1}{3}) - \varepsilon_1$*,* $a_2 = F^{-1}(\frac{1}{3}) + \varepsilon_1 = F^{-1}(\frac{2}{3}) - \varepsilon_2$*, and* $a_3 = F^{-1}(\frac{2}{3}) + \varepsilon_2$ *where* $\varepsilon_1, \varepsilon_2 \geq 0$*. If such an equilibrium exists it requires*

$$b \geq \min\left\{\frac{3}{\theta}c - 2(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})); \frac{3}{1-\theta}c - 2(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))\right\}.$$

The equilibria characterized by Proposition 1 include the differentiated three candidate equilibria in OS, but also include other equilibrium configurations. If the spatial distribution of preferences is fixed over the two rounds of the runoff election, an additional restriction on the values of ε_1 and ε_2 is required so that each fringe candidate has positive probability of winning. This requirement, $m - a_1 = a_3 - m$, pins down the exact ideal policies of each candidate in equilibrium. Uncertainty over the distribution of preferences has two effects, both of which appear in the weaker necessary condition here. Whereas the equilibrium in OS has a necessary condition of $b \geq 6c$, the potentially different probabilities of realizing state R and state L modify the right hand side to include $\frac{3}{1-\theta}c$ or $\frac{3}{\theta}c$. Additionally, since aggregate uncertainty no longer allows us to pin down the exact location of the candidates, but rather restricts their locations relative to each other, the right hand side also includes the subtraction of $F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})$. Together, these changes substantially relax the necessary condition. If the equilibria characterized by Proposition 1 were the only three candidate equilibria, aggregate uncertainty would still have increased the diversity of three candidate equilibria and parameter values for which a three candidate equilibrium could exist.

A similar set of equilibria exist for configurations where a single candidate obtains a plurality, but not a majority, of the votes in the first round and the other two candidates tie for second place. Notably, this has almost exactly the same formulation as Proposition 1; the principal difference is that there are fewer restrictions on the ideal policies of the candidates.

Proposition 2. *In any three candidate equilibrium where a candidate obtains the uniquely largest vote share and the other two candidates tie for second place, the interior candidate never has the uniquely smallest probability of winning or expected payoff. Moreover, the ideal policies of the candidates, denoted* $a_1 \leq a_2 \leq a_3$*, must satisfy* $F(\frac{1}{2}(a_1 + a_2)) \leq \frac{1}{2}$, $F(\frac{1}{2}(a_2 + a_3)) - F(\frac{1}{2}(a_1 + a_2)) \leq \frac{1}{2}$ *and* $1 - F(\frac{1}{2}(a_2 + a_3)) \leq \frac{1}{2}$. *If such an equilibrium exists, it requires*

$$\frac{3}{4}b \geq \frac{3}{\theta}c - 2(a_2 - a_1) \quad \text{and}$$
$$b \geq \frac{3}{1-\theta}c - 2(a_3 - a_2)$$

or

$$b \geq \frac{3}{\theta}c - 2(a_2 - a_1) \quad \text{and}$$
$$\frac{3}{4}b \geq \frac{3}{1-\theta}c - 2(a_3 - a_2)$$

or

$$\frac{3}{4}b \geq \frac{3}{\theta}c - 2(a_2 - a_1) \quad \text{and}$$
$$\frac{3}{4}b \geq \frac{3}{1-\theta}c - 2(a_3 - a_2)$$

.

The qualitative properties of some equilibria in Proposition 1 and Proposition 2 predict some across-round behavior of vote shares that is observed in electoral data. Suppose that an equilibrium of the type characterized in Proposition 2 exists where the candidate with ideal policy a_3 obtains 48 percent of the vote and the candidates with ideal policies a_1 and a_2 each obtain 26 percent of the vote. If state R is realized in the second round, and it happens to be the case that the candidate with ideal policy a_1 is selected via the tiebreaking rule, it is not unreasonable to suspect that her vote share will decrease across the two rounds.

Without imposing structure on the distributions F_R and F_L, and on the location of the candidates' ideal policies in equilibrium, it's difficult to generate conditions under which this is the case. Nevertheless, consider the example sketched above and where citizens' ideal policies are distributed uniformly over the $[0,1]$ interval in the first round. One possible configuration of candidates would be $a_1 = 0.05$, $a_2 = 0.515$, and $a_3 = 0.525$. If F_L is a uniform distribution over the interval $[-1,1]$ and F_R is a uniform distribution over the interval $[0,2]$, the second round of the election will feature a candidate who loses in the second round and does so having received a smaller vote share than she did in the first round regardless of which state is realized. Moreover, if state F_L is realized, not only will the candidate located at a_3 lose votes, she will lose the election in a reversal of the first round results. Such a result does not require the support of the distribution to change; if F_R features a truncated normal distribution over the interval $[0,1]$ with a mean close to $\frac{1}{2}$, for a sufficiently small variance, the result will be replicated. The next two propositions characterize equilibria which predict another as yet unexplained feature of the data, the existence of "sure loser" candidates.

Proposition 3 characterizes a set of three candidate equilibria which also appear in plurality elections even without the addition of aggregate uncertainty. In plurality elections, for some distributions of preferences, OS find three candidate equilibria in which a centrist candidate may choose to enter and lose with certainty, but by doing so induces an even-odds lottery over the other two candidates. These equilibria disappear when a runoff system is implemented, as there is no longer any incentive for the centrist candidate to enter. If the centrist candidate induces a first round tie, then either there will be a tie in the second round as well or one candidate will win with certainty in the second round. Regardless, the "sure loser" has paid c to enter, but has no effect on the distribution of outcomes. This no longer holds if the distribution of preferences in the second round may be different than in the first round of the election. Nevertheless, we often observe candidates who are not serious contenders to reach the second round have substantial effects on the resolution of the election. Let $v(a_i, \mathcal{K}; F)$ denote candidate i's vote share as a function of their ideal policy and the set of candidates, where $\mathcal{K}$ is the set of candidates in a given equilibrium.

Proposition 3. *A three-candidate equilibrium with a centrist sure loser exists for any configuration of candidates with ideal policies a_1, a_l, and a_2 if and only if:*
(i) $c \leq \max\{a_2 - a_l, a_l - a_1\} - \theta(|a_1 - a_l|) - (1-\theta)(|a_2 - a_l|)$,
(ii) $b \geq \max\left\{\frac{1-\theta}{\theta}(a_2 - a_l) - (a_l - a_1) + c, \frac{\theta}{1-\theta}(a_l - a_1) - (a_2 - a_l) + c\right\}$,
(iii) there does not exist a citizen with ideal policy $a_e \in (a_1 - 2s_L, a_2 + 2s_R)/\{a_1, a_2\}$ who can reach the second round with certainty, and
(iv) Distributional Conditions:

1. *If $\exists\ a_e \in (a_1, a_2)$ such that $v(a_1, \mathcal{K}\cup\{a_e\}; F) = v(a_e, \mathcal{K}\cup\{a_e\}; F)$, then $c \geq \frac{\theta}{2}(b + a_e - a_1)$.*

2. *If* $\exists\, a_e \in (a_1, a_2)$ *such that* $v(a_2, \mathcal{K} \cup \{a_e\}; F) = v(a_e, \mathcal{K} \cup \{a_e\}; F)$*, then* $c \geq \frac{1-\theta}{2}(b + a_2 - a_e)$*.*

3. *If* $v(a_1, \mathcal{K}; F) > 2v(a_l, \mathcal{K}; F)$*, then* $\frac{\theta}{2} b \leq c$*.*

4. *If* $v(a_2, \mathcal{K}; F) > 2v(a_l, \mathcal{K}; F)$*, then* $\frac{1-\theta}{2} b \leq c$*.*

5. *If* $\exists\, a_e \in (a_1, a_l)$ *such that* $v(a_1, \mathcal{K} \cup \{a_e\}; F) < v(a_l, \mathcal{K} \cup \{a_e\}; F)$*, then* $c \geq \theta(a_l - a_1)$*.*

6. *If* $\exists\, a_e \in (a_l, a_2)$ *such that* $v(a_2, \mathcal{K} \cup \{a_e\}; F) < v(a_l, \mathcal{K} \cup \{a_e\}; F)$*, then* $c \geq (1-\theta)(a_2 - a_l)$*.*

An equilibrium of this type exists for any $a_1 < m < a_2$ *and* a_l *that satisfy* $|m_L - a_1| < |m_L - a_2|$, $|m_R - a_2| < |m_R - a_1|$ *and* $\min\{|a_1 - m|, |a_2 - m|\} > |a_l - m|$*. Moreover, if* $a_2 - m > m - a_1$*, then* $a_l - a_1 > a_2 - a_l$ *and if* $m - a_1 > m - a_2$*, then* $a_2 - a_l > a_l - a_1$*.*

In this setting, a centrist citizen may choose to enter in order to transform a certain victory into a θ, $1-\theta$ lottery over the other two candidates. Moreover, if $\theta \neq \frac{1}{2}$, she may desire to enter in order to improve the odds that her preferred candidate has of winning the election regardless of whether a candidate was winning with certainty before her entry. Moreover, the configuration of candidates is also more diverse than in a plurality election without aggregate uncertainty. In OS, to have equilibria with sure losers F cannot be symmetric around the median. If F is symmetric, a centrist entrant would reduce the vote share of her preferred candidate by a greater amount than she reduces the vote share of her less preferred candidate. Therefore, if she induces a tie, she must have done so by preventing her preferred candidate from winning with certainty. No such restriction holds here; if F is symmetric and the other two candidates are equidistant from m, a center-left entrant may still choose to enter and take more votes from her preferred candidate if $\theta > \frac{1}{2}$.

Unlike in plurality elections, the structure of the runoff rule combined with aggregate uncertainty also allows for fringe candidates to enter strategically as sure losers in equilibrium. If two candidates have ideal policies that are equidistant from the median voter's ideal policy, a fringe candidate may choose to enter and transform the lottery over those two candidates from an even-odds lottery to a θ, $1-\theta$ lottery. In this case, though, the two candidates who win office with positive probability must be equidistant from the median voter. Suppose, without loss of generality, that the sure loser is a fringe candidate with an ideal policy to the left of the median. If the potential winning candidate to the left of the median is closer to the median voter than the candidate to the right, she would win with certainty if the sure loser exits. The sure loser's payoff must therefore strictly increase by exiting since she obtains a more preferable policy outcome with certainty and saves the cost of running for office. If, on the other hand, the candidate who is closer to the median voter is on the right fringe, the sure loser has no effect on the electoral outcome since the election is decided in the first round, and therefore her payoff strictly increases by exiting and saving the cost of running.

Proposition 4. *A three-candidate equilibrium with a fringe sure loser and two candidates who win with positive probability located at distinct points* a_1 *and* a_2 *at a distance* $\varepsilon \in (0, e_r(F))$ *from the median exists if and only if* $\theta \neq \frac{1}{2}$, $c \leq (\frac{1}{2} - \theta)(|a_1 - a_l| - |a_2 - a_l|)$, $b \geq \max\left\{\frac{c}{1-\theta} - 2\varepsilon; \frac{c}{\theta} - 2\varepsilon\right\}$, *and there does not exist a citizen with ideal policy* $a_e \in (a_1 - 2s_L, a_1) \cup (a_2, a_2 + 2s_R)$ *who can reach the second round. Moreover, the sure loser is located at a distinct point* a_l *where* $a_l < m$ *if* $\theta > \frac{1}{2}$ *and* $a_l > m$ *if* $\theta > \frac{1}{2}$.

Equilibria with fringe candidates running as sure losers do not exist in plurality elections in OS. The exact structure of the runoff rule here is crucial to existence, but the importance of the structure of the runoff rule is a vestige of simplifying assumptions on the domain of preferences. A slightly richer model with two-dimensional preferences (e.g. payoffs are decreasing with Euclidean distance from the ultimate winner) would easily admit the analogue to these equilibria. A configuration with two candidates who are equidistant from the center of mass of the distribution of citizens and a third candidate who is located further from the center of mass enters and surely loses due to an improvement in the probability of her preferred candidate winning in the second round could be an equilibrium even if the election is resolved by plurality rule when a candidate's vote share in the first round ties the threshold. The addition of the second dimension would allow the sure loser to reduce both candidates' vote shares strictly below the threshold and thus cause a second round. In this sense, the equilibria characterized in Proposition 4 are less of a "knife edge" case than may be apparent at first glance, since a multidimensional model may be a more realistic depiction of political competition, although certainly a less tractable one.

4.2 Two Candidate Equilibria

The potential for two candidate equilibria in my model is also of interest. If two candidate equilibria are particularly prevalent, it would be difficult to interpret the existence of three candidate equilibria as a strong theoretical basis for Duverger's Hypothesis. While two candidate equilibria do exist in my model, aggregate uncertainty has an ambiguous effect on the quantity of two candidate equilibria we should expect. Two countervailing forces are at play in my model. Aggregate uncertainty generates two candidate equilibria which are more differentiated than in previous models because centrist entrants may be deterred from entering. This expands the potential set of two candidate equilibria. On the other hand, entry is now more attractive to fringe candidates because they have positive probability of obtaining office due to a favorable realization of uncertainty. This reduces the set of parameter values for which a two candidate equilibria can be sustained. Due to my model generating substantially more three candidate equilibria than OS and having an ambiguous effect on two candidate equilibria, I interpret the result to strengthen the argument OS makes for Duverger's Hypothesis.

From Lemma 1, we know that in any two-candidate equilibrium, the two candidates must be equidistant from the median voter at a distance $\varepsilon \geq 0$; otherwise, one of the candidates would be a sure loser. Moreover, as argued in Lemma 2, multiple candidates located at the median also does not constitute an equilibrium[9]. Clearly, these two candidates must also have distinct ideal policies. Were the two equilibrium candidates to share the same ideal policy, at a location $a \neq m$, a median candidate could enter and win with certainty in the first round of the election. Nevertheless, diverging from the results obtained in OS, the possibility of a centrist entrant reaching the second round will not necessarily disrupt a two-candidate equilibrium. Therefore, I separately characterize the set of two-candidate equilibria according to whether or not such a potential entrant exists. To do so, it's useful to define a critical value for the equilibrium candidates' differentiation. Let $\bar{\varepsilon}(F)$ denote the supremal distance the two equilibrium candidates' ideal policies can be from the median voter such that any potential entrant with a more centrist ideal policy receives a strictly smaller vote share than either of the two equilibrium candidates

$$\begin{aligned}\bar{\varepsilon}(F) = \sup\{\varepsilon : F(\tfrac{1}{2}(a+m-\varepsilon)) > F(\tfrac{1}{2}(a+m+\varepsilon)) - F(\tfrac{1}{2}(a+m-\varepsilon)) \text{ and} \\ 1 - F(\tfrac{1}{2}(a+m+\varepsilon)) > F(\tfrac{1}{2}(a+m+\varepsilon)) - F(\tfrac{1}{2}(a+m-\varepsilon)),\ \forall\ a \in (m-\varepsilon, m+\varepsilon)\}.\end{aligned}$$

Two other potential entrants are particularly useful to highlight in order to characterize the set of two-candidate equilibria. Since the median voter's ideal policy will differ across the two rounds of the election, it may be the case that a fringe candidate desires to enter, knowing that she will reach the second round, and hope for a favorable state to be realized. I use $a_+(\varepsilon, F)$ and $a_-(\varepsilon, F)$ to denote the supremal distance that such a potential candidate can be from the median voter. Thus, $a_+(\varepsilon, F)$ and $a_-(\varepsilon, F)$ denote the most extreme candidates who may enter, reach the second round with certainty, and win office in the event of a

[9] This holds for every configuration with only $k \geq 2$ candidates located at the median, and is proved in Proposition 7.

favorable realization of uncertainty.

$$a_-(\varepsilon, F) = \arg\sup_{a\in(0,2s_L)} a \times \mathbf{1}_A \left(F(m-\varepsilon-\frac{a}{2}) > F(m) - F(m-\varepsilon) \right) \text{ and}$$
$$a_+(\varepsilon, F) = \arg\sup_{a\in(0,2s_R)} a \times \mathbf{1}_A \left(1 - F(m+\varepsilon+\frac{a}{2}) > F(m+\varepsilon) - F(m) \right).$$

where $\mathbf{1}_A$ denotes the indicator function. If $a_-(\varepsilon, F)$ or $a_+(\varepsilon, F)$ are single-valued, then there exists a fringe citizen who can enter and reach the second round of the election with certainty. If, however, $a_-(\varepsilon, F)$ and $a_+(\varepsilon, F)$ are not unique, then no such citizen exists. Note that restricting the possible values of a considered to $(0, 2s_i)$ in state i is without loss of generality; if a citizen can enter and win in a state, she has strictly greater incentive to do so than a citizen who would enter and surely lose. If no fringe citizen can enter and win with positive probability, then all fringe citizens have the same marginal change in payoff from entry. A final piece of useful notation is to define $s(\varepsilon, F)$ as the citizen who, if they chose to enter in a hypothetical two-candidate equilibrium, would take equal vote share from each of the two equilibrium candidates (i.e. the two equilibrium candidates would still tie).[10] In the event that $a_-(\varepsilon, F)$ and/or $a_+(\varepsilon, F)$ is nonempty, additional payoff restrictions are necessary to deter their entrance.

Condition 1. *If $a_-(\varepsilon, F)$ is a singleton,*

$$c \geq \theta(b + a_-(\varepsilon, F)) + (2\theta - 1)\varepsilon.$$

If $a_+(\varepsilon, F)$ is a singleton,

$$c \geq (1-\theta)(b + a_+(\varepsilon, F)) + (1 - 2\theta)\varepsilon.$$

Additionally, while $\bar{\varepsilon}(F)$ denotes the maximum differentiation possible without a centrist entrant being able to reach the second round, it does not provide any information about the location of the potential entrant. Thus, a second condition is required.

Condition 2. *If $d \in (m - \varepsilon, s(\varepsilon, F))$ is the entrant who makes $\bar{\varepsilon}(F)$ bind,*

$$c \geq \frac{\theta}{2}(b + \varepsilon) + (\frac{3}{2}\theta - 1)(m - s(\varepsilon, F)).$$

[10] If f is symmetric about the median, then $s(\varepsilon, F) = m$.

If $d \in (s(\varepsilon, F), m+\varepsilon)$ is the entrant who makes $\bar{\varepsilon}(F)$ bind,

$$c \geq \frac{1-\theta}{2}(b+\varepsilon) + \frac{3\theta-1}{2}(m - s(\varepsilon, F)).$$

If $d = s(\varepsilon, F)$ is the entrant who makes $\bar{\varepsilon}(F)$ bind,

$$c \geq \frac{1}{3}(b + (1+4\theta)\varepsilon) - \frac{2}{3}(1-2\theta)(m - s(\varepsilon, F)).$$

I begin by considering the set of two-candidate equilibria where no potential centrist entrant may reach the second round of the runoff. For the purposes of characterizing some two-candidate equilibria, I impose one regularity condition on the distribution of preferences in each state. The regularity condition guarantees that in the event state R is realized, a centrist entrant would lose to the right candidate, and in the event state L is realized, the centrist entrant would lose to the left candidate. In the proof of Proposition 5 I assume that $\theta \in [\frac{1}{3}, \frac{2}{3}]$ and impose the following regularity condition. Suppose two candidates have ideal policies a_1 and a_2 with $a_1 = m - \bar{\varepsilon}(F)$ and $a_2 = m + \bar{\varepsilon}(F)$.

$$F_L(\frac{1}{2}(a_1 + s(\varepsilon, F))) > 1 - F_L(\frac{1}{2}(a_1 + s(\varepsilon, F))) \quad \text{and}$$
$$1 - F_R(\frac{1}{2}(a_2 + s(\varepsilon, F))) > F_R(\frac{1}{2}(a_2 + s(\varepsilon, F)))$$

This condition is not particularly restrictive, and is only used in the proof of Proposition 5. Moreover, a similar result holds without this condition (proof available upon request).

Proposition 5. *1. In any two-candidate equilibrium where a centrist entrant cannot guarantee reaching the second round, the candidates' ideal policies are $m - \varepsilon$ and $m + \varepsilon$ for some $\varepsilon \in (0, \bar{\varepsilon}(F)]$.*

2. An equilibrium of this type in which the candidates' ideal policies are $m - \varepsilon$ and $m + \varepsilon$ exists if and only if $\varepsilon > 0$, $\varepsilon \geq c - \frac{b}{2}$, and one of $\varepsilon < \bar{\varepsilon}(F)$ and Condition 1 or $\varepsilon = \bar{\varepsilon}(F)$, $c \geq \max\{(1-2\theta)\varepsilon, (2\theta-1)\varepsilon\}$ and Conditions 1 and 2.

When compared to the baseline case of perfectly consistent preferences across the two rounds, characterized in OS, there are clearly fewer two-candidate equilibria of this type when candidates are ex-ante unsure about the distribution of preferences in the second round. Notably, all of the parameter restrictions in OS for two-candidate equilibria of this form reappear here; aggregate uncertainty, however, forces the addition of several additional parameter restrictions. Specifically, if candidates are unsure about the location of the median voter in the second round, they may choose to enter for several possible strategic motivations.

If they could reach the second round (i.e. one of $a_+(\varepsilon, F)$ or $a_-(\varepsilon, F)$ is a singleton), then they may choose to enter and hope for a favorable state to be realized if the cost of running is not sufficiently large. If, on the other hand, they cannot reach the second round (both $a_+(\varepsilon, F)$ and $a_-(\varepsilon, F)$ are multi-valued) potential entrants may still choose to enter knowing that they will force a second round which could be beneficial to their preferred equilibrium candidate. In OS, there is no similar an incentive to enter, since there would be no change in the probability of victory for any candidate.

The possibility of forcing a second round may also deter some candidates from entering the race who would enter if the distribution of preferences was consistent across both rounds of the election. Call the ideal policies of the two equilibrium candidates a_1 and a_2 with $a_1 < a_2$. Let $\tilde{\varepsilon}(F)$ represent the maximal distance from the median voter of two candidates such that a centrist entrant with ideal policy a_e cannot obtain a vote share strictly greater than $\frac{1}{2}$ for all $a_e \in (a_1, a_2)$,

$$\tilde{\varepsilon}(F) = \sup\left\{\varepsilon : F(\frac{1}{2}(m+\varepsilon+a_e)) - F(\frac{1}{2}(m-\varepsilon+a_e) \leq \frac{1}{2}\ \forall\ a_e \in (m-\varepsilon, m+\varepsilon)\right\}.$$

While $\tilde{\varepsilon}(F)$ provides an upper bound on how differentiated candidates can be in a two-candidate equilibrium, it is not a particularly tight upper bound. For example, if citizens are distributed uniformly over the unit interval, bounding candidates' differentiation by $\tilde{\varepsilon}(F)$ does not rule out any configuration. This is because, even if candidates are maximally differentiated and located at $a_1 = 0$ and $a_2 = 1$, any potential entrant will receive a vote share of $\frac{1}{2}$ if they enter at a distinct position $a_e \neq a_1$, a_2, and a vote share of $\frac{1}{4}$ if they enter at either a_1 or a_2. Finally, define $\mathcal{E}$ as the set of ideal policies of all citizens who could enter and reach the second round with certainty

$$\mathcal{E} = \{a_e : F(\frac{1}{2}(a_2+a_e)) - F(\frac{1}{2}(a_1+a_e)) > F(\frac{1}{2}(a_1+a_e))\} \cup \{a_e : F(\frac{1}{2}(a_2+a_e)) - F(\frac{1}{2}(a_1+a_e)) > 1 - F(\frac{1}{2}(a_2+a_e))\}$$

Let $\mathcal{E}_l$ be defined by $\mathcal{E}_l = \mathcal{E} \cap (m-\varepsilon, s(\varepsilon, F))$ and $\mathcal{E}_r$ be defined by $\mathcal{E}_r = \mathcal{E} \cap (s(\varepsilon, F), m+\varepsilon)$. The set $\mathcal{E}_l$ represents all citizens who, by choosing to enter the race, would reach the second round of the election instead of the candidate with ideal policy a_1, and $\mathcal{E}_r$ is the analogous group of citizens who could replace the candidate with ideal policy a_2. If $\mathcal{E}$ is nonempty, but $\mathcal{E}_l$ and $\mathcal{E}_r$ are both empty, the location of the citizen who can reach the second round with certainty is $s(\varepsilon, F)$.

Proposition 6. *1. In any two-candidate equilibrium where a potential centrist citizen could enter and reach the second round with certainty, the candidates' ideal policies are*

$m-\varepsilon$ and $m+\varepsilon$ for some $\varepsilon \in (\bar{\varepsilon}(F), \tilde{\varepsilon}(F)]$.

2. *An equilibrium with two candidates whose ideal policies are $m-\varepsilon$ and $m+\varepsilon$ for $\varepsilon \in (\bar{\varepsilon}(F), \tilde{\varepsilon}(F)]$ exists if and only if $\varepsilon \geq c - \frac{b}{2}$, Conditions 1 and 2, and*
   (a) *If $\mathcal{E}_l$ is nonempty, $c \geq (1-\theta)(b+\varepsilon) - \theta(a_e - m)$ and $F_L(\frac{1}{2}(a_e+a_1)) \geq 1 - F_L(\frac{1}{2}(a_e+a_1))$ for all $a_e \in \mathcal{E}_l$.*
   (b) *If $\mathcal{E}_r$ is nonempty, $c \geq \theta(b+\varepsilon) - (1-\theta)(m-a_e)$ and $1 - F_R(\frac{1}{2}(a_e+a_2)) \geq F_R(\frac{1}{2}(a_e+a_2))$ for all $a_e \in \mathcal{E}_r$.*
   (c) *If $s(\varepsilon, F) \in \mathcal{E}$, $c \geq \frac{1}{2}(b + (1-2\theta)m - s(\varepsilon, F) + 3\varepsilon)$ and both $F_L(\frac{1}{2}(a_1 + s(\varepsilon, F))) \geq 1 - F_L(\frac{1}{2}(a_1 + s(\varepsilon, F)))$ and $1 - F_R(\frac{1}{2}(a_2 + s(\varepsilon, F))) \geq F_R(\frac{1}{2}(a_2 + s(\varepsilon, F)))$.*

The equilibria characterized in Proposition 6 do not exist in a model without aggregate uncertainty. If the spatial distribution of preferences is constant across rounds of the election, no centrist citizen can be deterred from entering if she reaches the second round with certainty. In such a model, such an entrant would be a sure winner, as, regardless of which other candidate reaches the second round, she would have an ideal policy more similar to the median voter's, and therefore win with certainty in the second round. In this sense, aggregate uncertainty has a similar effect in runoff elections as it does in plurality elections. Brusco and Roy (2011) show that in plurality elections with a similar form of aggregate uncertainty over voter preferences, candidates enter with ideal policies more extreme than the ideal policies of the two potential median voters.

4.3 Equilibrium Clusters

Traditionally, Hotelling-Downs models of elections have illustrated a strong incentive to choose a policy platform similar to the median voter's ideal policy. While candidates cannot choose their policy platform in citizen-candidate models, a natural question is to what extent this incentive remains. In OS there exist "centrist" equilibria, where all candidates share the median voter's ideal policy, for every possible number of candidates, conditional on appropriate parameter values. While Lemma 1 restricts the possible equilibrium configurations of candidates it does not, however, provide many restrictions on the potential policy platforms we may expect to observe in equilibrium. Moreover, centrist equilibria may be normatively desirable; depending on the social welfare function and the exact shape of the distribution of voters, the median voter's ideal policy may maximize social welfare. It may also be socially desirable to choose an electoral rule that generates centrist and likeminded policymakers. Unfortunately, Proposition 7 rules out these equilibria as a potential outcome in runoff elections with aggregate uncertainty.

Proposition 7. *There does not exist an equilibrium with all $k \geq 2$ candidates sharing ideal policy m.*

If k identical candidates are seeking office, they must necessarily receive equal vote shares in equilibrium. If $k = 2$, then the election is resolved by plurality rule with an even-odds lottery. If, however, $k > 2$, the election reaches a second round; each candidate is chosen for the second round with probability $\frac{2}{k}$, then ties in the second round. Thus, such an equilibrium requires $b \geq kc$. If the ego-rents of office are too large, however, another median candidate would seek office; thus a centrist equilibrium also requires $b \leq (k+1)c$. Supposing those parameter values are satisfied, though, does not make a centrist equilibrium particularly likely. In a runoff election where the identify of the voters is known with certainty, and their preferences are static over time, such an equilibrium may be sustained because any potential entrant would lose with certainty in the second round. If, however, the median voter's ideal policy in the second round of the election will be $m_l < m$, where $m - m_l = \epsilon$, with probability $\theta > \frac{1}{k}$, there exists a citizen at $m - \frac{\epsilon}{2}$ who can enter and reach the second round of the election with certainty. This citizen, therefore, faces a θ to $1 - \theta$ lottery of obtaining office. Since $\max\{\theta, 1-\theta\} \geq \frac{1}{k}$, this is a strictly preferable lottery than the centrist candidates are facing in the hypothetical equilibrium (due to a slight shift in the expected policy towards this citizen's ideal policy), and this citizen will prefer to enter. This holds for any k and any value of θ. While this is a direct consequence of the assumption of a continuous distribution of voters, it is readily apparent that the probability of such a citizen existing for a large, finite electorate is substantial.[11] Centrist equilibria may exist in IRV systems, however, since IRV is a "repeated counting" rather than "repeated voting" procedure. Since all citizens vote only once there is no possibility of a shift in the distribution of preferences and it may be possible to support a centrist equilibrium.

4.4 Comparative Statics

In addition to being used in a majority of direct presidential elections, runoff rules are also used in many lower stakes elections, including relatively small town mayoral elections or primary elections where the winner is not expected to be competitive (e.g. Democratic party primaries in Republican-dominated districts). The wide variety of contexts for application of the runoff rule and the extensive multiplicity of equilibria in my model suggest that performing comparative statics exercises are particularly useful for generating more precise predictions for candidate behavior. Since I do not characterize plurality results, my focus

[11] A similar argument establishes that centrist equilibria cannot exist in runoff elections with thresholds less than $\frac{1}{2}$, even without uncertainty over second round preferences.

is on how the properties of the runoff election and electorate will change the likelihood of observing a particular type of equilibrium characterized above. Nevertheless, it's useful to first note that in Osborne and Slivisnki, a citizen-candidate model with consistent preferences across rounds predicts that the set of two candidate equilibria in a runoff is a subset of the set of two candidate equilibria in a plurality election. My model generates a set of two candidate equilibria that is more differentiated, but exists for a smaller set of parameters (b, c) than in Osborne and Slivinski, but generates many more three candidate equilibria. In that sense, my model suggests that Duverger's Law is likely to hold also in a setting with aggregate uncertainty. Differing from Osborne and Slivinski, my model can support a three candidate equilibrium for any distribution F satisfying the conditions in Section 3 given appropriate parameters (b, c).

In general, it is hard to distinguish between situations in which the relative likelihood of equilibria characterized in Proposition 1 and Proposition 2 existing differs. The set of parameters necessary to generate equilibria in which three candidates all tie for first is not a subset of the parameters which are necessary for equilibria with a leading candidate and two candidates tied for second, nor the reverse. This is due to the similarity between the two types of equilibrium; in each case, each candidate has positive (and substantial) probability of obtaining office. While in Proposition 2 two candidates obtain office with a smaller probability than their analogues in Propositon 1, they can nevertheless be more differentiated and obtain a larger payoff from a shift in the expected policy outcome due to their presence. As a result, my model suggests that as the cost of running for office and the benefit to obtaining office increase, for a fixed level of differentiation in voter preferences, the necessary condition for equilibria in Proposition 1 is relatively more likely to be satisfied than that in Proposition 2. Nevertheless, my results provide no definitive statement on this question.

The characteristics of the electorate, the cost of pursuing office, and the personal benefits obtained by elected officials do, however, significantly affect the predicted existence of a sure loser and where she lies on the policy spectrum. As opposed to the equilibria in Propositions 1 and 2, equilibria with a sure loser (Propositions 3 and 4) require an upper bound on how costly it is to run for office independent of the benefits extracted by the officeholder. This restriction has an appealing intuitive basis: sure losers will never obtain office in equilibrium, and are motivated to enter the election to shift the expected policy outcome closer to their ideal policy. As a result, for a fixed degree of voter heterogeneity, as the costs of running for office and benefits to holding office grow, sure loser equilibria become less likely. The converse also holds; for a fixed level of costs and benefits, as voters become more heterogeneous, we should expect to see more candidates seeking office despite losing with certainty in equilibrium. These results to suggest that runoff elections in districts with poor political isntitutions (e.g. one where politicians extract large rents in office and running for office is potentially dangerous), should generate fewer sure losers.

The characteristics of the election also predict differences regarding the type of sure loser we should observe in equilibrium. As noted above, no equilibria with a fringe sure loser (Proposition 4) exist if the shock is unbiased ($\theta = \frac{1}{2}$). This is a simple consequence of the necessary positions for the candidates who win with positive probability in this type of equilibrium; if either of these candidates had an ideal policy closer to that of the median voter, there would either be a sure winner, which cannot occur in equilibrium, or the sure loser would have an incentive to exit and allow her preferred candidate to win with certainty. As a result, in equilibrium, the two candidates who win with positive probability are equidistant from the median voter and would win with equal probability if the sure loser had not entered the race. Thus, unbiased shocks cannot support equilibria with fringe sure losers. For shocks that are relatively unbiased, equilibria with fringe sure losers are still rare; the sure loser faces a tradeoff between saving c or having a small effect on the probability of her preferred candidate entering.

Equilibria with centrist sure losers (Proposition 3) may still be common, however, for unbiased shocks. Centrist sure losers may still choose to run even when the candidates who win with positive probability are not equidistant from the median voter. This lack of symmetry no longer implies a sure winner or a favorable policy outcome if the sure loser exits, and thus even if the shock is unbiased, may still provide a strong strategic motivation for the sure loser to enter. If, on the other hand, the shock is very biased (θ near 0 or 1), the benefit to holding office must be very large to incentivize the strong candidates to remain in the election. This is also a reasonably intuitive result. Reaching the second round of the election has no separate payoff; doing so and then losing with high probability is thus not particularly appealing to a candidate. This result would relax somewhat if there was a distinct payoff associated with reaching the second round (e.g. a reduced form for future electoral possibilities generated by appearing as a “serious contender”).

5 Conclusion

In this paper I extend the citizen-candidate model of Osborne and Slivinski to a setting where candidates face some aggregate uncertainty over the distribution of voter preferences in the second round of a runoff election. I show that the addition of aggregate uncertainty generates a substantially more diverse set of three-candidate equilibria which more closely matches observed electoral outcomes than the set of equilibria in models with perfect foresight. I characterize equilibria where candidates choose to enter strategically despite losing the election with certainty in order to induce a second round which improves the odds of victory of their preferred contender. I also show that equilibria exist where the candidate who obtains a plurality in the first round loses in the second round; empirical evidence in Bullock III and Johnson (1992) indicates that reversals occur in approximately 30% of runoff elections in the US.

A principal limitation of my model is the assumption of a continuous distribution of citizens. While this assumption buys some tractability for the model, it also costs me the ability to consider strategic voting in my framework. While estimates of the amount of misaligned voting tend to be small, the same papers estimate that the proportion of strategic voters is quite large (Spenkuch 2013, Kawai and Watanabe 2011). Nevertheless, it appears unlikely that the addition of strategic voting is sufficient to replicate the additional equilibria featured here. Strategic voting, however, may plausibly expand or refine the equilibrium set generated here. The relatively small fraction of misaligned voters suggests that much of the effects of strategic voting are on the set of candidates who choose to enter, thereby making a setting with a finite number of voters who act strategically a natural extension.

An additional limitation of my model is the degree of aggregate uncertainty I assume. While the formulation is relatively general in that I do not impose many assumptions on the distributions in either state, I do assume that the ideal policy of the median voter will differ from the first round to the second round with certainty. Nevertheless, many of my results are robust to a model where the median voter's ideal policy may remain the same with some probability. Intuitively, the general incentive to enter and force lotteries remains unchanged even with the possibility of the median voter's ideal policy remaining constant. The main difference in such a model would be slightly more complicated payoff restrictions stemming from the fact that there are now three possible states that may be realized. As an example, Proposition 7 would be restated as "There does not exist an equilibrium with all $k \geq 2$ candidates sharing ideal policy m if the probability of realizing either state L or state R is at least $\frac{1}{k}$."

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A Proofs of Theorems

A.1 Three Candidate Equilibria

Proof of Lemma 2: If an equilibrium configuration of candidates yields a sure winner with ideal policy a, then each sure loser with ideal policy w obtains $-|w-a|-c$ in equilibrium, and would obtain at least $-|w-a|$ by exiting. Since $c>0$, sure losers cannot be best responding if there is a sure winner. This rules out any configuration where a single candidate obtains more than half the votes. Three candidates who tie for first with the ideal policy of the median voter is also not an equilibrium. If $\max\{\theta, 1-\theta\} \geq \frac{1}{k}$ and $b \geq kc$, there exists a potential entrant with ideal policy $d \neq m$ who can reach the second round with certainty and win with at least probability $\max\{\theta, 1-\theta\}$. Without loss of generality, let $\theta \geq 1-\theta$. One such entrant obtains a payoff $\theta b-(1-\theta)|m-d|-c > -|m-d|$ whenever $\theta \geq \frac{1}{k}$ and $b \in [kc, (k+1)c]$. Therefore if all three candidates tie for first, they must be at the positions above in order to each obtain $\frac{1}{3}$ of the vote. ∎

Proof of Proposition 1: I begin by enumerating all possible configurations of candidates that satisfy the requirement that each candidate ties for first.[12]

Config.	1v2 state L	1v2 state R	2v3 L	2v3 R	1v3 L	1v3 R
1	1 wins	2 wins	2 wins	2 wins	1 wins	3 wins
2	1 wins	2 wins	2 wins	3 wins	1 wins	3 wins
3	2 wins	2 wins	2 wins	3 wins	1 wins	3 wins
4	2 wins	2 wins	2 wins	2 wins	1 wins	3 wins
5	1 wins	2 wins	2 wins	2 wins	3 wins	3 wins
6	2 wins	2 wins	2 wins	3 wins	1 wins	1 wins

Table 3: Potential Eq. Configurations, 3 candidates

Config. Number	Cand. 1	Cand. 2	Cand. 3
1	$\frac{2}{3}\theta$	$\frac{2}{3}(1-\theta)+\frac{1}{3}\theta$	$\frac{1}{3}(1-\theta)$
2	$\frac{2}{3}\theta$	$\frac{1}{3}$	$\frac{2}{3}(1-\theta)$
3	$\frac{1}{3}\theta$	$\frac{2}{3}\theta+\frac{1}{3}(1-\theta)$	$\frac{2}{3}(1-\theta)$
4	$\frac{1}{3}\theta$	$\frac{2}{3}$	$\frac{1}{3}(1-\theta)$
5	$\frac{1}{3}\theta$	$\frac{2}{3}(1-\theta)+\frac{1}{3}\theta$	$\frac{1}{3}$
6	$\frac{1}{3}$	$\frac{2}{3}\theta+\frac{1}{3}(1-\theta)$	$\frac{1}{3}(1-\theta)$

Table 4: Candidates' probabilities of victory in potential eq. configurations

If $\theta \geq \frac{1}{2}$, configuration 5 has the smallest equilibrium payoff for candidate 1; it combines her minimal chance of obtaining office with the maximal probability of candidate 3 obtaining office. If $\theta \geq \frac{1}{2}$, configuration 1 yields the smallest equilibrium payoff for candidate 3. If $\theta \leq \frac{1}{2}$, configuration 3 yields the smallest possible equilibrium payoff for candidate 1. Similarly, configuration 6 gives candidate 3 the lowest possible equilibrium payoff if $\theta \leq \frac{1}{2}$. What remains is to determine the worst configuration for candidate 2. Denote by Pr_i the probability that candidate i obtains office. Candidate 2's payoff to running is $Pr_2 b - 2\varepsilon_1 Pr_1 - 2\varepsilon_2 Pr_3 - c$, and her payoff from not running is either $-2\varepsilon_1$ or $-2\varepsilon_2$. Since ε_1 and ε_2 are free parameters in finding a potential equilibrium, configuration 2 must be the worst possible configuration for candidate 2: if θ is very large, then we choose an equilibrium configuration where $\varepsilon_2 = 0$ and $\varepsilon_1 = F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})$. If θ is very small, then we choose the opposite: $\varepsilon_1 = 0$ and $\varepsilon_2 = F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})$.

Necessary conditions:

[12] Given the lack of information about the size of the shift and the distribution of preferences, an equilibrium configuration of candidates is described by which candidate would win in a second round matchup against each of the other candidates, conditional on a given shift in the distribution of preferences.

1. Candidate 1's participation, $\theta \geq \frac{1}{2}$:

$$\theta b \geq 3c - \theta 2\varepsilon_1 + 2\varepsilon_2$$

   Candidate 1's participation, $\theta < \frac{1}{2}$:

$$\theta b \geq 3c - \theta 2\varepsilon_1 + 4(1-\theta)\varepsilon_2$$

2. Candidate 2's participation:

$$b \geq 3c + 4\theta\varepsilon_1 + 4(1-\theta)\varepsilon_2 - 6\varepsilon^*$$

   where ε^* refers to whichever of ε_1 or ε_2 is the relevant distance from the winner if 2 exits. If $\varepsilon^* = \varepsilon_1$, this condition becomes

$$b \geq 3c - (6-4\theta)\varepsilon_1 + 4(1-\theta)\varepsilon_2,$$

   whereas if $\varepsilon^* = \varepsilon_2$, the condition is

$$b \geq 3c - (6-4(1-\theta))\varepsilon_2 + 4\theta\varepsilon_1.$$

3. Candidate 3's participation, $\theta \geq \frac{1}{2}$:

$$(1-\theta)b \geq 3c - (1-\theta)2\varepsilon_2 + 4\theta\varepsilon_1$$

   Candidate 3's participation, $\theta < \frac{1}{2}$:

$$(1-\theta)b \geq 3c - (1-\theta)2\varepsilon_2 + 2\varepsilon_1$$

After choosing ε_1 and ε_2 subject to $\varepsilon_1 + \varepsilon_2 = F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})$ to relax these constraints as much as possible, these conditions reduce to:

1. Candidate 1's constraint, $\theta \geq \frac{1}{2}$:

$$\theta b \geq 3c - 2\theta(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

   Candidate 1's constraint, $\theta < \frac{1}{2}$:

$$\theta b \geq 3c - 2\theta(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

2. Candidate 2's entry constraint turns out not to vary with θ, but rather only with ε^*; regardless of the value of θ, candidate 2's constraint relaxes as ε^* increases. If $\varepsilon^* = \varepsilon_1$, this becomes

$$b \geq 3c + 4\theta(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})) - 6(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

Candidate 2's constraint, $\varepsilon^* = \varepsilon_2$:

$$b \geq 3c + 4(1-\theta)(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})) - 6(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

3. Candidate 3's constraint, $\theta \geq \frac{1}{2}$:

$$(1-\theta)b \geq 3c - 2(1-\theta)(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

Candidate 3's constraint, $\theta < \frac{1}{2}$:

$$(1-\theta)b \geq 3c - 2(1-\theta)(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

Reducing even further to eliminate redundant conditions, we obtain

1. $$b \geq \frac{3}{\theta}c - 2(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

2. $$b \geq \frac{3}{(1-\theta)}c - 2(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

3. $$b \geq 3c + 4\theta(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})) - 6(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

4. $$b \geq 3c + 4(1-\theta)(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})) - 6(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))$$

To find a lower bound on equilibrium payoffs, I first find a lower bound for candidates 1 and 3. Since there is no overlap between the payoff minimizing equilibria for candidate 3 and the payoff minimizing equilibria for candidate 1, it's apparent that candidate 1 must be earning a larger payoff in candidate 3's worst equilibrium than in her own worst equilibrium. Symmetrically, in candidate 1's worst equilibrium, candidate 3 must be earning a larger payoff than in her own worst equilibrium. Denote an equilibrium payoff for candidate i by

π_i^*, and let Π_i^* be the set of equilibrium payoffs for candidate i in this type of equilibrium. Suppose θ, b and c are such that $\min \pi_3^* \le \min \pi_1^*$. Then, we know the relevant lower bound on payoffs must be derived from candidate 3's minimal equilibrium payoff, and symmetrically so for candidate 1 if $\min \pi_3^* \ge \min \pi_1^*$. Therefore, if $\min_{\{\pi_1,\pi_3\}\in\Pi_1^*\times\Pi_3^*}\{\pi_1,\pi_3\}$ serves as a lower bound for candidate 2's payoffs, the lower bound on payoffs is also satisfied by candidate 2.

In order to find a single necessary condition on b and c for varying values of θ, it remains to be shown that candidate 2's constraints are never the relevant necessary condition. Showing that candidate 2's payoff is bounded below by candidates' 1 and 3 is straightforward. In every potential equilibrium configuration of this type, candidate 2 never has the uniquely smallest probability of winning. Furthermore, for any possible outcome where candidate 2 does not win or reach the second round, she earns a higher payoff than the other losing candidate since her ideal policy differs from that imposed by the winner by less than does the ideal policy of the losing external candidate. Candidate 2's ex-ante expected payoff, therefore, is always bounded below by whichever candidate has the (possibly jointly) minimal probability of eventual victory. The relevant necessary condition, however, depends on candidates' marginal value of entry. If $\theta \ge \frac{1}{2}$, candidate 3's entry condition binds more tightly than candidate 1's due to her low probability of obtaining victory conditional on entry. Simple algebra shows that the right hand side of candidate 3's constraint $(3c/(1-\theta) - 2(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})))$ is always strictly larger than the right hand side of candidate 2's constraint. Thus, if $\theta \ge \frac{1}{2}$, any set of parameters satisfying candidate 3's constraint also satisfy candidate 2's constraint. The same argument holds for $\theta \le \frac{1}{2}$ and a comparison between candidate 1's constraint and candidate 2's constraint. Therefore, the relevant necessary condition for existence of an equilibrium where all three candidates tie for first place in the first round is

$$b \ge \min\left\{\frac{3}{\theta}c - 2(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})); \frac{3}{1-\theta}c - 2(F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3}))\right\}.$$

■

Proof of Proposition 2: The argument here follows that of Proposition 1 with only slight deviation. Note that each candidate has a positive probability of reaching the second round. Moreover, without more structure on the equilibrium or the distributions of voter preferences, we cannot characterize equilibrium configurations in ways other than Table 3. In order to be an equilibrium that does not feature a sure loser, the possible equilibrium probabilities of victory are almost identical; the lone difference in the probability of holding office is that one candidate reaches the second round with certainty, and the other two candidates reach the second round with probability $\frac{1}{2}$ instead of probability $\frac{2}{3}$. Since at least one of the external candidates will tie for second in the first round, the centrist candidate still does not have the uniquely lowest probability of winning or expected payoff. Therefore, the conditions implied by equilibrium existence are identical to that in Proposition 1 with but two differences. First, instead of $F^{-1}(\frac{2}{3}) - F^{-1}(\frac{1}{3})$ in the condition, we must substitute $a_3 - a_2$ and $a_2 - a_1$ where relevant, i.e. $a_2 - a_1$ to generate necessary conditions on candidate 1's participation and $a_3 - a_2$ to generate conditions on candidate 3's participation. Second, we require three pairs of conditions, only one pair of which need be satisfied in a given equilibrium:

$$\frac{3}{4}b \geq \frac{3}{\theta}c - 2(a_2 - a_1) \quad \text{and}$$
$$b \geq \frac{3}{1-\theta}c - 2(a_3 - a_2)$$

or

$$b \geq \frac{3}{\theta}c - 2(a_2 - a_1) \quad \text{and}$$
$$\frac{3}{4}b \geq \frac{3}{1-\theta}c - 2(a_3 - a_2)$$

or

$$\frac{3}{4}b \geq \frac{3}{\theta}c - 2(a_2 - a_1) \quad \text{and}$$
$$\frac{3}{4}b \geq \frac{3}{1-\theta}c - 2(a_3 - a_2)$$

■

Proof of Proposition 3: I begin by noting that $a_1$and a_2 must satisfy the conditions $|m_L - a_1| < |m_L - a_2|$ and $|m_R - a_2| < |m_R - a_1|$. If these conditions are not satisfied, at least one of the two candidates who reaches the second round does not win in the second round after realizing a favorable distribution of preferences, and therefore would prefer to exit saving cost c and causing the sure loser to win in the first round, thus obtaining a preferable policy outcome. The candidate located at a_1 obtains a payoff of $\theta b - (1-\theta)(a_2 - a_l) - c$ by running, and a payoff of $-(a_1 - a_l)$ if she exits. Therefore, to keep her from exiting we require $b \geq \frac{1-\theta}{\theta}(a_2 - a_l) - (a_l - a_1) + c$. Symmetrically for the candidate with ideal policy a_2, we require $b \geq \frac{\theta}{1-\theta}(a_l - a_1) - (a_2 - a_l) + c$. In equilibrium, the sure loser obtains a payoff of $-\theta(a_l - a_1) - (1-\theta)(a_2 - a_l) - c$, whereas she obtains a payoff of $-\max\{a_2 - a_l, a_l - a_1\}$ if she exits and causes her least preferred candidate to win. If by exiting she would cause her more preferred candidate to win, she would prefer to exit, thereby necessitating the condition that a_l be determined by satisfying: (1) if $a_2 - m > m - a_1$, then $a_l - a_1 > a_2 - a_l$ and (2) if $m - a_1 > m - a_2$, then $a_2 - a_l > a_l - a_1$. Thus, for the sure loser to run, we also require $\max\{a_2 - a_l, a_l - a_1\} - \theta(a_l - a_1) - (1-\theta)(a_2 - a_l) \geq c$.

If a citizen with ideal policy $a_e \in (a_1 - 2s_L, a_2 + 2s_R)/\{a_1, a_2\}$ could enter and reach the second round with certainty, they surely would: any potential entrant's payoff to winning office is equal to that of the equilibrium candidate they displace, and by entering they obtain a better expected policy outcome. If an entrant would tie with the candidate at a_1 for second place, they would have a $\frac{\theta}{2}$ probability of winning office instead of the candidate at a_1 and therefore increasing their payoff by $\frac{\theta}{2}(b + a_e - a_1)$ but paying cost c. To deter their entry, it must be the case that $c \geq \frac{\theta}{2}(b + a_e - a_1)$. Symmetrically, if the entrant would tie with the candidate located at a_2, then we require $c \geq \frac{1-\theta}{2}(b + a_2 - a_e)$ for them not to seek office. An identical entrant at a_1 or a_2 will choose not to enter if she is a sure loser since she does not improve her expected policy outcome and also pays cost c. If, however, $F(\frac{a_1+a_l}{2}) > 2[F(\frac{a_2+a_l}{2}) - F(\frac{a_1+a_l}{2})]$, then we require that $c \geq \frac{\theta}{2}b$ for an equilibrium. Symmetrically, if $1 - F(\frac{a_2+a_l}{2}) > 2[F(\frac{a_2+a_l}{2}) - F(\frac{a_1+a_l}{2})]$, then we require $\frac{1-\theta}{2}b \leq c$ for an equilibrium. No other citizens with fringe ideal policies would choose to enter as they would be sure losers and could only have adverse effects on the expected policy. ∎

Proof of Proposition 4: First note that in any three-candidate equilibrium with a fringe sure loser, the candidates who win with positive probability must be symmetrically equidistant from the median voter. Suppose this is not the case for a contradiction, and suppose without loss of generality that the fringe sure loser has an ideal policy to the left of the median voter. If the candidate located at a_1 is closer to the median voter, then the fringe sure loser can exit and her preferred candidate will win with certainty in the first round. This saves the sure loser cost c, and also results in an increased probability of her preferred policy being implemented. If, instead, the candidate located at a_2 is closer to the median voter, then she wins with certainty in the first round and both other candidates would prefer to exit.

A fringe sure loser with position a_l would gain a payoff of $-\theta(|a_1 - a_l|) - (1-\theta)(|a_2 - a_l|) - c$ by running, and a payoff of $-\frac{1}{2}(|a_1 - a_l|) - \frac{1}{2}(|a_2 - a_l|)$ by not running. Thus, we require $c \leq (\frac{1}{2} - \theta)(|a_1 - a_l| - |a_2 - a_l|)$. If $\theta > \frac{1}{2}$, then $\frac{1}{2} - \theta < 0$ and therefore the sure loser must be on the left fringe so that $(|a_1 - a_l| - |a_2 - a_l|) < 0$, and symmetrically on the right fringe if $\theta < \frac{1}{2}$. The payoff to the candidate located at a_1 in equilibrium is $\theta b - 2(1-\theta)\varepsilon - c$, whereas if she chooses not to run, her payoff is -2ε. Thus, these equilibria require $b \geq \frac{c}{\theta} - 2\varepsilon$. Symmetrically for the candidate at a_2, it must be the case that $b \geq \frac{c}{1-\theta} - 2\varepsilon$, and thus we require $b \geq \max\left\{\frac{c}{1-\theta} - 2\varepsilon;\ \frac{c}{\theta} - 2\varepsilon\right\}$. Note that no other sure losers have incentive to enter. Their payoffs strictly decrease by c since the election is already guaranteed to go to a second round.

It remains to be shown that no other citizen has an incentive to enter. By restricting $\varepsilon \in (0, e_r(F))$, there is no internal citizen who could reach the second round with positive probability. Thus, all internal citizens are sure losers if they choose to run, and therefore prefer not to as they have no effect on the outcome of the election. We also require that there does not exist a fringe candidate with an ideal policy within $2s$ of the equilibrium contenders' policies who can reach the second round with positive probability. This is because any such fringe candidate has the same probability of victory as the equilibrium contender they would replace in the second round, yet a greater incentive to enter due to their increased disutility from their least preferred contender winning. Thus, there does not exist any set of parameter values which would incentivize the equilibrium contenders to enter but not fringe candidates who can reach the second round. Therefore, in order for an equilibrium of this type to exist, all fringe citizens must be sure losers if they choose to run and thus would prefer not to enter. ■

A.2 Two Candidate Equilibria

Proof of Proposition 5: Lemma 1 applies for $k = 2$ candidates, and therefore each candidate must tie for first place in a two-candidate equilibrium (thus they must be symmetric and equidistant from m). Moreover, $\varepsilon > 0$ follows directly from the Proposition 7. Each equilibrium candidate obtains a payoff of $\frac{b}{2} - c - \varepsilon$ in equilibrium, and would obtain a payoff of -2ε by exiting. Therefore, we require $\frac{b}{2} - c - \varepsilon \geq -2\varepsilon$, which reduces to $\varepsilon \geq c - \frac{b}{2}$, to keep equilibrium candidates in the race. Conditions (a) i-iv and (b) i-iii are required to deter entry by candidates in a variety of cases.

Case 1: $\varepsilon < e_r(F)$. There is no internal entrant who can win with positive probability, so we restrict attention to potential fringe entrants (any fringe entrant has a weakly greater payoff to running conditional on an internal entrant being a sure loser). Suppose $a_-(\varepsilon, F)$ and $a_+(\varepsilon, F)$ are not singletons; then, there does not exist a fringe entrant who can win with positive probability. A fringe entrant with ideal policy $m - \varepsilon - a$ obtains a payoff of $-\varepsilon - a$ if she chooses not to run, or a payoff of $-\theta a - (1-\theta)(a + 2\varepsilon) - c$. In order to keep her from running, we require the condition $-\varepsilon - a \geq -\theta a - (1 - \theta(a + 2\varepsilon) - c \Leftrightarrow c \geq (2\theta - 1)\varepsilon$. Symmetrically, for a fringe entrant with ideal policy $m + \varepsilon + a$ we require $c \geq (1 - 2\theta)\varepsilon$. If $a_-(\varepsilon, F)$ is a singleton, then there exists a potential fringe entrant with ideal policy $m - \varepsilon - a > m - \varepsilon - 2s$ who can reach the runoff and win given a favorable shift in the distribution of voters. Her payoff to not entering is $-\varepsilon - a$ and her payoff to entering is $\theta b - (1-\theta)(2\varepsilon + a) - c$. Therefore, to ensure that she does not enter, we require $-\varepsilon - a \geq \theta b - (1-\theta)(2\varepsilon + a) - c \Leftrightarrow c \geq \theta(b + a) + (2\theta - 1)\varepsilon$. The binding entrant is located at $a_-(\varepsilon, F)$, making the condition $c \geq \theta(b + a_-(\varepsilon, F)) + (2\theta - 1)\varepsilon$. We also require $c \geq (1 - 2\theta)\varepsilon$ to exclude a sure loser on the right. Symmetrically, if $a_+(\varepsilon, F)$ is a singleton and $a_-(\varepsilon, F)$ is not, we require $c \geq (1 - \theta)(b + a_+(\varepsilon, F)) + (1 - 2\theta)\varepsilon$ to keep out the potential entrant at $a_+(\varepsilon, F)$, and $c \geq (2\theta - 1)\varepsilon$ to keep out a sure loser on the left. If both $a_+(\varepsilon, F)$ and $a_-(\varepsilon, F)$ are singletons, we require $c \geq (1 - \theta)(b + a_+(\varepsilon, F)) + (1 - 2\theta)\varepsilon$ to keep out the entrant at $a_+(\varepsilon, F)$, and $c \geq \theta(b + a_-(\varepsilon, F)) + (2\theta - 1)\varepsilon$ to keep out the entrant at $a_-(\varepsilon, F)$.

Case 2: $\varepsilon = e_r(F)$, and the internal entrant who can tie for second place is located at $d \in (m - \varepsilon, s(\varepsilon, F))$. Note that all of the Case 1 conditions must still hold with respect to external entrants. The internal entrant obtains a payoff of $-\varepsilon$ if she chooses not to enter, and a payoff of $\frac{\theta}{2}(b - d + m - \varepsilon) + (1-\theta)(d - m - \varepsilon) - c$ if she chooses to enter. We therefore require $-\varepsilon \geq \frac{\theta}{2}(b - d + m - \varepsilon) + (1-\theta)(d - m - \varepsilon) - c$. Note that, since we assume $\theta \in [\frac{1}{3}, \frac{2}{3}]$, this restriction binds tighter as d approaches $s(\varepsilon, F)$. Therefore, we require $c \geq \frac{\theta}{2}(b + \varepsilon) + (\frac{3\theta}{2} - 1)(m - s(\varepsilon, F))$.

Case 3: $\varepsilon = e_r(F)$, and the internal entrant who can tie for second place is located at $d \in (s(\varepsilon, F), m + \varepsilon)$. Note that all of the Case 1 conditions must still hold with respect to external entrants. The internal entrant obtains a payoff of $-\varepsilon$ if she chooses not to enter, and a payoff of $\frac{1-\theta}{2}(b+d-m-\varepsilon)+\theta(m-d-\varepsilon)-c$ if she chooses to enter. Thus, we require $-\varepsilon \geq \frac{1-\theta}{2}(b+d-m-\varepsilon)+\theta(m-d-\varepsilon)-c$. Similarly to Case 2, since $\theta \in [\frac{1}{3}, \frac{2}{3}]$, this requirement binds tightest as d approaches $s(\varepsilon, F)$. The condition is, therefore $c \geq \frac{1-\theta}{2}(b+\varepsilon)+\frac{3\theta-1}{2}(m-s(\varepsilon,F))$.

Case 4: $\varepsilon = e_r(F)$, and the internal entrant who can tie for second place is located at $s(\varepsilon, F)$. Note that all of the Case 1 conditions must still hold with respect to external entrants. The entrant at $s(\varepsilon, F)$ obtains a payoff of $-\varepsilon$ if she chooses not to run, and a payoff of $\frac{1}{3}b-\frac{2}{3}((1-2\theta)(m-s(\varepsilon,F)+\varepsilon)-c$ if she chooses to run. The condition to keep her out, therefore, is $-\varepsilon \geq \frac{1}{3}b-\frac{2}{3}((1-2\theta)(m-s(\varepsilon,F)+\varepsilon)-c \Leftrightarrow c \geq \frac{1}{3}(b+(1+4\theta)\varepsilon)-\frac{2}{3}(1-2\theta)(m-s(\varepsilon,F))$.

■

Proof of Proposition 6: This proof follows the argument for Proposition 5 closely. Note that the incentives for the equilibrium candidates are exactly the same in both types of equilibrium, and furthermore, so are the incentives for a potential fringe candidate. The only players for whom the payoffs change are the potential centrist entrants who could reach the second round of the election. Thus, all conditions of Proposition 5 are relevant here except conditions (b). Let a_1 and a_2, $a_1 < a_2$, represent the ideal policies of the equilibrium candidates, and let a_e represent the ideal policy of a potential centrist entrant, $a_e \in (a_1, a_2)$. Define $\mathcal{E}$ as the set of ideal policies of all citizens who could enter and reach the second round with certainty

$$\mathcal{E} = \{a_e : F(\frac{1}{2}(a_2 + a_e)) - F(\frac{1}{2}(a_1 + a_e)) > F(\frac{1}{2}(a_1 + a_e)) \text{ and}$$
$$F(\frac{1}{2}(a_2 + a_e)) - F(\frac{1}{2}(a_1 + a_e)) > 1 - F(\frac{1}{2}(a_2 + a_e))\}$$

Let $\mathcal{E}_l$ be defined by $\mathcal{E}_l = \mathcal{E} \cap (m - \varepsilon, s(\varepsilon, F))$ and $\mathcal{E}_r$ be defined by $\mathcal{E}_r = \mathcal{E} \cap (s(\varepsilon, F), m + \varepsilon)$. Given the distributional restriction $1 - F_R(\frac{1}{2}(a_e + a_2)) \geq F_R(\frac{1}{2}(a_e + a_2))$ for all $a_e \in \mathcal{E}_r$, if $\mathcal{E}_r$ is nonempty, a citizen with ideal policy $a_e \in \mathcal{E}_r$ obtains a payoff of $-\theta(a_e - (m - \varepsilon)) + (1 - \theta)b - c$ by entering, and a payoff of $-\varepsilon$ if she does not enter. Thus, to deter such a citizen from entering, we require $c \geq (1 - \theta)(b + \varepsilon) - \theta(a_e - m)$ for all $a_e \in \mathcal{E}_r$. Similarly, if $\mathcal{E}_l$ is nonempty, a citizen with ideal policy $a_e \in \mathcal{E}_l$ obtains a payoff of $\theta b - (1 - \theta)(m + \varepsilon - a_e) - c$ by entering, and $-\varepsilon$ by choosing not to enter given the distributional restriction $F_L(\frac{1}{2}(a_e + a_1)) \geq 1 - F_L(\frac{1}{2}(a_e + a_1))$ for all $a_e \in \mathcal{E}_l$. Thus, if $\mathcal{E}_l$ is nonempty, we require $c \geq \theta(b + \varepsilon) - (1 - \theta)(m - a_e)$ for all $a_e \in \mathcal{E}_l$. If a citizen with ideal policy $s(\varepsilon, F)$ can enter and reach the second round with certainty, she obtains a payoff of $\frac{1}{2}(b + (1 - 2\theta)m + \varepsilon - s(\varepsilon, F)) - c$ by entering and a payoff of $-\varepsilon$ by choosing not to enter. Therefore, we also require $c \geq \frac{1}{2}(b + (1 - 2\theta)m - s(\varepsilon, F) + 3\varepsilon)$. Finally, it must be the case that $\varepsilon > \bar{\varepsilon}(F)$ in order for a centrist citizen to reach the second round with certainty, and it must be the case that $\varepsilon \leq \tilde{\varepsilon}(F)$ in order for a potential centrist entrant to not win with certainty in the first round. ■

A.3 Equilibrium Clusters

Proof of Proposition 7: Suppose $b \in [kc, (k+1)c]$ and the median voter will be the same in both rounds (no uncertainty and no shift). Then a k-candidate configuration with all candidates located at m yields an expected payoff of $\frac{b}{k} - c \geq 0$. Another entrant at m would obtain a payoff of $\frac{b}{k+1} - c \leq 0$, and therefore choose not to enter. An entrant at $d \neq m$ could reach the second stage with certainty, but then face an opponent who is the median voter, and therefore lose with certainty in the second round and obtain a payoff bounded above by $-c$.

Now, suppose there exists a k-candidate equilibrium where all k candidates share ideal policy m. If $b < kc$, candidates obtain a payoff of $\frac{b}{k} - c < 0$ in equilibrium, and obtain a zero payoff by exiting. If $b > (k+1)c$, another citizen with ideal policy m could enter and obtain a payoff of $\frac{b}{k+1} - c > 0$. Therefore, we require $b \in [kc, (k+1)c]$. Suppose that there will be a shift in the distribution of voters between rounds of size s, but that shift is known with perfect foresight. Without loss of generality, assume that the shift is to the left. Since F is continuous and non-atomistic, there exists and ideal policy $d \neq m$ such that a citizen with ideal policy d could choose to enter and obtain a vote share of $F(\frac{1}{2}(d+m)) = \frac{1}{2} - \frac{\epsilon}{2}$. Moreover, for any $s > 0$, there exists a $d \in (m-s, m)$ such that this citizen reaches the second round with certainty. That citizen wins in the second round with certainty, since she will be located between any potential opponent and the median voter. This is a best response for the citizen with ideal policy d since $b \geq 2c$. Finally, suppose the distribution shifts by a distance s, but that shift may happen in either direction. Let the probability of the distribution shifting left be θ and the probability of the distribution shifting right be $1-\theta$. If $\max\{\theta, 1-\theta\} \geq \frac{1}{k}$ and $b \geq kc$, there exists a potential entrant with ideal policy $d \neq m$ who can reach the second round with certainty and win with at least probability $\max\{\theta, 1-\theta\}$ and obtain a payoff $\Pi > \frac{b}{k} - c \geq 0$. ■

Proposition 8. *There does not exist an equilibrium with an even number $k \geq 4$ candidates where $\frac{k}{2}$ candidates share an ideal policy $m-a$ and the remaining $\frac{k}{2}$ candidates share an ideal policy $m+a$ for any $a>0$.*

Proof of Proposition 8: Suppose for a contradiction that there exists an equilibrium with an even number $k \geq 4$ candidates arranged in symmetric clusters of $\frac{k}{2}$ candidates. Breaking ties with equal probability implies each candidate reaches the second round with probability $\frac{2}{k}$. In the second round, a candidate will face another member of their cluster with probability $(\frac{k}{2}-1)/(k-1)$ or a member of the other cluster with probability $(\frac{k}{2})/(k-1)$. Each candidate in the left cluster therefore wins office with probability $(\frac{k}{2}-1)/(2k-2)+\theta(\frac{k}{2})/(k-1)$ and each candidate in the right cluster wins office with probability $(\frac{k}{2}-1)/(2k-2)+(1-\theta)(\frac{k}{2})/(k-1)$. Since F is continuous, for any location of each cluster, there exists a potential entrant who can obtain arbitrarily close to $\frac{1}{4}$ of the vote. If such a citizen entered, the cluster candidates would split $\frac{1}{4}+\epsilon$ of the vote, and therefore there exists a citizen on either side of the median who can reach the second round of the election with certainty. Moreover, these potential entrants are within s_L and s_R distance from the clusters. If f is symmetric and the clusters are located at $F^{-1}(\frac{1}{4})$ and $F^{-1}(\frac{3}{4})$, then the relevant citizen is arbitrarily close on either side of the cluster. If f is not symmetric, or if f is symmetric and the clusters are located at points other than $F^{-1}(\frac{1}{4})$ and $F^{-1}(\frac{3}{4})$, then there exists a citizen who is arbitrarily close on the side with more voters who can reach the second round of the election with certainty. In the second round, because a right entrant would certainly face a candidate from the left cluster and vice versa, such a right citizen would win with probability $1-\theta$, and a left citizen would win with probability θ. For any value of θ, at least one of the right or left potential entrants would win with a higher probability than some cluster candidates. This citizen would obtain a strictly larger payoff than the cluster candidates who win with smaller probability. This is because by exiting, the cluster candidates would guarantee that another member of their cluster wins with certainty, and obtain a payoff of 0, so their payoffs must be weakly positive in equilibrium. But the potential entrant obtains b with greater probability than the cluster candidates while paying the same c and influences the expected policy in a manner which is preferable to her. Therefore, this citizen is not best responding, which is a contradiction to the supposition of this configuration being a Nash equilibrium. Therefore there cannot exist an equilibrium with $k \geq 4$ candidates arranged in symmetric clusters. ■

■