The Effect of Firm Entry on Capacity Utilization and Macroeconomic Productivity

Anthony Savagar*
Huw Dixon†

February 9, 2015

Abstract

The paper develops a theory that firm entry causes endogenous fluctuations in macroeconomic productivity through its effect on incumbent firms' capacity utilization. The analysis shows that imperfect competition causes long-run *excess entry*: too many firms each with excess capacity. Because entry occurs slowly, macroeconomic conditions are initially born by the excess capacity incumbents who respond by altering their capacity utilization. Incumbents' efficiency changes because of non-constant returns to scale which aggregates to affect the economy's productivity. In the long run, entry occurs and new firms absorb the change in economic conditions, which alleviates incumbents' alteration in capacity. Therefore the productivity change is ephemeral. Hence, the slow response of firms to economic conditions causes endogenous productivity dynamics. **JEL: E20, E32, D21, D43, L13**

How does industry competition affect firm entry and in turn macroeconomic dynamics? Since Chamberlin 1933 economists have understood that in a monopolistically competitive economy there is *'excess capacity'*–each firm produces below their full capacity, which minimizes costs, because underproduction earns monopoly profits. In terms of firm entry, underproduction means too many firms enter and each produces too little. There is *excess entry*. The focus of this paper is transition toward the excess capacity steady state. During transition the capacity utilization of firms fluctuates which causes endogenous productivity variations because at different capacity levels

*savagaran@cardiff.ac.uk
†dixonh@cardiff.ac.uk

firms have different productivity. The firm entry capacity utilization mechanism is unexplored in macroeconomics, so the aim of the paper is to provide theoretical understanding. It ties with the ongoing conundrum of business cycle productivity in macroeconomics, and has intuitive grounds in light of current productivity puzzles. For example, negative shock to the economy that is not immediately responded to by firms exiting worsens capacity underutilization and exacerbates negative productivity.

The main result is that resources are divided between too many firms in the long run, but a perturbation from this equilibrium causes a more efficient division in the short run as firms can not adjust so incumbents bear the shock. The investigation centres on this endogenous capacity utilization mechanism. In long-run equilibrium firms have excess capacity, but a movement away from equilibrium can cause firms to increase (reduce) capacity so they produce closer to (further from) full capacity for a short period of time whilst other firms enter to arbitrage monopoly profits (exit to avoid negative profits). The key technical results are that entry causes absolutely faster convergence to steady state and reduces the set of complex dynamics, by increasing eigenvalues. Convergence to steady state occurs faster, because firms outside the market are more responsive to monopoly profits, and on entry they decrease profits by more than a firm would under imperfect competition.

The focus on theory and analytical dynamics distinguishes the paper from others that typically focus on quantitative replication of empirical results. They are stochastic and dynamics are simulated, as opposed to our qualitative deterministic study that avoids specifying functional forms or solving particular parameterized, path dependent models. Our analysis of model dynamics provides a mapping between endogenous entry and the dynamics it creates. The model includes important features from the main papers in the literature so far: endogenous labour, and capital accumulation (Bilbiie, Ghironi, and Melitz 2012); imperfect competition (Etro and Colciago 2010) and entry congestion effects that Lewis 2009 finds empirically important and Berentsen and Waller 2009 model structurally. Congestion means entry costs increase with number of entrants. Imperfect competition is the paramount addition since it is ubiquitous in these models but its relation to entry and mapping to dynamics have not been investigated.

To understand the relationship between entry, imperfect competition and macroeconomic dynamics, we build on work by Brito and Dixon 2013. The model is a Cass-Koopmans model with labour-leisure choice and capital. We add Datta and Dixon 2002 entry and imperfect competition via Cournot monopolistic competition d'Aspremont, Dos Santos Ferreira, and Gerard-Varet 1997. Firms control output price (imperfect product market), but do not control input prices (perfect factor market). The role of a firm is as a divisor of resources: too many firms divides resources too much; too few firms divides resources too little. How does imperfect competition affect the division of resources among firms? How does productivity vary as we transition toward this division of capital to firms?

Our theory is that division of output across firms is important for determining efficiency and so the sluggish entry of firms is crucial. In the short run the number firms is fixed, as aggregate output varies, output per firm and capacity utilization vary. Imperfect competition causes firms to produce less than the efficient level, so they operate with locally increasing returns to scale. For example a positive production shock (technological or demand) causes a short-run productivity gain because incumbents raise output as other firms cannot enter in the short run to absorb the positive shock. Therefore firms initially grow and benefit from increasing returns, so productivity improves: fewer inputs produce more output. The output of the incumbents is above the equilibrium level and yields monopoly profits. In the long run, prospective firms slowly enter to arbitrage the profits to zero-profit long-run free entry outcome. Each entrant reduces output per firm: undoing capacity utilization, undoing better returns to scale, undoing productivity gains. Therefore the productivity gain is ephemeral, and excess capacity returns in the long run.

Excess capacity is a standard feature of models with imperfect competition, but its relationship to entry is often overlooked, since three features–imperfect competition, non-constant returns to scale, slow firm entry–are necessary for capacity utilization effects. The contribution of this paper is to understand the relationship between excess capacity and firm entry. Consider the contrapositive of these three assumptions: 1) Without slow entry, there is standard instantaneous free entry[1], short-run capacity utilization by incumbents will not arise because other firms enter instantaneously to meet output, so incumbents do not respond by varying their capacity; their optimal production does not change. 2) If there were perfect competition, firms produce efficiently, at minimum average cost; they do not have excess capacity which if used can improve productivity. 3) If there were constant returns to scale, entry is inert: there is no difference between one large firm producing all output versus many small firms producing all output[2].

Related Literature (UNDER REVISION) Brito and Dixon 2013 study entry implications for macroeconomic dynamics. They show that with

[1]The two extremes entry cases–instantaneous free entry and no entry fixed number of firms–are special cases the model.

[2]Furthermore with a fixed cost, like we have, there would be a one firm natural monopoly.

perfect competition, firm entry causes empirically plausible macroeconomic dynamics in a Ramsey model. The main result is that entry is sufficient to give nonmonotone deviations from equilibrium under fiscal shocks. The research derives sufficient conditions for hump-shaped responses that quantitative DSGE papers observe via simulation.

Our model employs a different entry setup to the popular business cycle with entry paper by Bilbiie, Ghironi, and Melitz 2012. In our model the sunk entry cost is endogenous, and firms technology has decreasing returns to scale.

The employment of non-CRTS is an important consideration in our paper which ties it more closely to IO literature and breaches a common macroeconomic assumption. For example, in the IO literature Luttmer JET 2012 has a fixed cost and decreasing returns, as do Rossi-Hansberg and Wright AER.

A novel aspect of our model is the endogenous sunk cost of entry which is driven by a negative network effect, also called congestion effect. Some recent papers have acknowledged the importance of congestion effects in entry. Lewis 2009 offered initial empirical support for congestion effects of firm entry. Her VAR study showed that congestion effects can account for observed lags in monetary policy.

Roadmap – There are four sections. Section 2 proposes a model of firm entry in the macroeconomy and derives productivity. The analysis follows and consists of section three on stability, section four on comparative statics and section five on comparative dynamics. The final sixth section uses the analytical results to prove that productivity varies endogenously.

1 Endogenous Entry Model with Imperfect Competition

The model follows a Ramsey-Cass-Koopmans setup. Additions are imperfect competition, firm entry, and capital accumulation. The model is deterministic, and labour is endogenous. There are two state variables: capital and number of firms $(K, n) \in \mathbb{M} \subseteq \mathbb{R}^2$, where $\mathbb{M}$ is the state space of the control problem that later forms a subset of the general dynamical system state (or phase) space.

We solve the model as a decentralised equilibrium because imperfect competition distorts the optimising behaviour of the firm which does not coincide with centralised equilibrium of a planner optimising behaviour of the economy. [3]

[3] The conditions we derive could be derived from an optimal control problem with two

1.1 Household

The economy is inhabited by a continuum of identical households whose mass is normalized to one. A household seeks policy functions of consumption $\{C(t)\}_0^\infty \in \mathbb{R}$ and labour supply $\{L(t)\}_0^\infty \in [0,1]$ that maximise lifetime utility $\mathrm{U} : \mathbb{R}^2 \to \mathbb{R}$. We assume $u : \mathbb{R} \times [0,1] \to \mathbb{R}$ is jointly concave and differentiable in both of its arguments. It is strictly increasing in C and strictly decreasing in L. A representative household solves the utility maximisation problem:

$$\mathrm{U} := \int_0^\infty \mathrm{u}(C(t), 1 - L(t)) e^{-\rho t} \mathrm{dt} \tag{1}$$

$$\text{s.t.} \quad \dot{K}(t) = rK(t) + wL(t) + \Pi - C(t) \tag{2}$$

The household owns capital $K \in \mathbb{R}$ and take equilibrium rental rate r and wage rate w as determined by the market. Households own firms and receive firm profits. Using the maximum principle to maximise utility yields six Pontryagin conditions for optimal consumption and labour [4] The conditions simplify to an intertemporal consumption Euler equation, intratemporal labour-consumption trade-off and the resource constraint (2).

$$\dot{C}(r,\rho) = \frac{C}{\sigma(C)}(r - \rho), \quad \text{where } \sigma(C) = -C\frac{u_{CC}(C)}{u_C(C)} \tag{3}$$

$$w = -\frac{u_L(L)}{u_C(C)} \tag{4}$$

The solution of the dynamic optimization problem for household consumption will be one of the solutions of this system of two differential equations that satisfy the initial condition $K(0) = K_0$. To complete the boundary value problem we add two transversality conditions on the upper boundary. This completes the unique solution for the boundary value problem: three variables, three equations, three boundary conditions.

$$\lim_{t\to\infty} K_t e^{-\rho t} \geqslant 0 \tag{5}$$

$$\lim_{t\to\infty} K_t \lambda_t e^{-\rho t} = 0 \tag{6}$$

$$K_0 = K(0) \tag{7}$$

restrictions, but the economic intuition is less clear. However, the equivalence is important for the theory of dynamics we use later.

[4] Appendix A sets up the Hamiltonian and solves it.

Equations (2)-(7) characterize optimal paths of consumption and labour. In equilibrium these equations continue to hold, with factor prices w and r at their market value, which arises from firms profit maximisation as we shall show next.

1.1.1 Intratemporal Labour

The intratemptoral condition (4), given w at equilibrium, defines optimal labour in terms of the other variables (C, K, n), so it can be substituted out of the system, thus reducing the system's dimensionality by one.

Lemma 1 Optimal labour choice.
$\hat{L}(\underset{-}{C}, \underset{+}{K}, \underset{+}{n}; \underset{+}{A}, \underset{-}{\mu})$ [5]

Proof. Follows that L satisfies assumptions for implicit function definition. Implicitly differentiate intratemporal conditions. **See appendix** □

1.2 Firm Production

There is a continuum of one firm industries, so industry-wide there is a countable number of measure zero firms, and within an industry firms have positive measure. Multiproduct firms do not exist, so a firm is a producer, is a product, is an industry. A firm faces a fixed cost and decreasing returns to scale, a scenario that is more common in IO literature for example Rossi-Hansberg and Wright 2007 and Luttmer 2012.

There is imperfect competition in the product market and perfect competition in the factor market. Product market imperfect competition means that firms can control output price; factor market perfect competition means firms cannot control factor prices. Under imperfect competition each firm produces less than it would under perfect competition, so the marginal product of the factors of production, capital and labour, is higher. Under perfect competition factor prices equal their marginal products, but under imperfect competition, output is lower, so marginal product is higher than price, the difference being the so-called mark-up which allows a firm to accrue monopoly profits. Factors are capital and labour and their prices are interest r and wage w. Since all firms face the same factor prices (perfect factor market competition), they employ the same levels of factors. Thus aggregate capital and labour is split evenly among firms.

[5] To obtain the signs differentiate the intratemporal condition (4) with labour defined implicitly by $\hat{L}(C, K, n; A, \mu)$.

$$k(t) = \frac{K(t)}{n(t)}, \quad l(t) = \frac{L(t)}{n(t)} \tag{8}$$

Definition 1 Production.
Firms have the same production technology

$$y(t) = \max\{A\mathrm{F}(k(t), l(t)) - \phi, 0\} \tag{9}$$

where $F : \mathbb{R}^2_+ \supseteq (k, l) \to \mathbb{R}_+$ *is a firm production function with continuous partial derivatives which is homogenous of degree* $\nu < 1$ *(hod-*ν*) on the open cone* $\mathbb{R}^2_+$*, and* $\phi \in \mathbb{R}_+$ *a fixed cost. This firm output function causes a U-shaped average cost curve. The output function* $y : \mathbb{R}^2_+ \supseteq (k, l) \to \mathbb{R}_+ \cup 0$ *is an increasing function of* F*, so is homothetic and inherits the geometric properties of the homogeneous function.*

A firm is inactive if $y(t) = 0$, otherwise it is active. Variable costs are strictly convex (F is homogeneous of $\nu < 1$ in capital and labour; increasing marginal cost); the fixed cost denominated in terms of output is a source of nonconvexity which will cause some firms not to produce. The fixed cost which is paid each period is different to the sunk entry cost, paid one time to enter, which I introduce later[6]. Inada's conditions hold so that marginal products of capital and labour are strictly positive which rules out corner solutions. The average cost curve of (9) is U-shaped because of the fixed cost. Marginal cost is increasing because $\nu < 1$. Hence according to a U-shaped cost curve there are initially IRTS, costs decreasing, then AC=MC then DRTS, costs increasing. Hence there is an optimal scale when average cost is minimized (AC=MC). Despite rising marginal cost and so ultimately decreasing returns to scale in the production function, we shall see that firms operate with increasing returns to scale, known as excess capacity. The trade-off, which drives the U-shpaed cost curved, between an entering firm bringing with it an extra fixed cost, versus incumbents increasing their output but at a higher marginal cost is at the heart of the paper–it is uninteresting in the perfect competition case because firms produce at optimum, whereas we shall see with imperfect competition the negative effect of duplication of resources, too many fixed costs, dominates the returns to scale effect so we are left with locally increasing returns to scale i.e. a more efficient firm would make better use of their fixed cost. The perfect competition, AC=MC, efficient output

[6] As in Jaimovich07 and Rotemberg Woodford the role of this parameter is to reproduce the apparent absence of pure profits despite market power. It allows zero profits in the presence of market power.

is equivalent to the output maximising number of firms $Y_n = 0$ from 102 we have

$$F(k,l)^e = \frac{\phi}{A(1-\nu)} \tag{10}$$

$$y^e = AF(k,l)^e - \phi = \frac{\phi\nu}{1-\nu} \tag{11}$$

$$\mathcal{P}^e = \frac{y^e}{F(k,l)^{\frac{1}{\nu}}} = A^{\frac{1}{\nu}}\nu\left(\frac{\phi}{1-\nu}\right)^{\frac{\nu-1}{\nu}} \tag{12}$$

Technology A does not affect optimal net output, but it reduces optimal gross output $F(k,l)$ and raises productivity. *Measured productivity* is output per unit of production, where production is normalized to be homogeneous of degree 1. From (10) $\left(\frac{1}{n}\right)^{\nu} F(K,L)^e = \frac{\phi}{A(1-\nu)}$ so number of firms is hod-1 in capital and labour. A rise in capital and labour by some proportion will raise number of firms by the same proportion, so capital and labour per firm remain unchanged. This drives the result in (11) that optimal output per firm is a fixed level, independent of model variables.

$$n^e = \left(\frac{A(1-\nu)}{\phi}F(K,L)\right)^{\frac{1}{\nu}}$$

Efficient output is not achieved when we consider the strategic interactions of firms under imperfect competition.

NB REMOVED FACTOR MARKET DERIVATION AS NEEDS UPDATE

So due to imperfect competition, with output price as the numeraire, the firm maximises profits given real wage w and interest rate r by choosing labour and capital such that the following factor market equilibrium holds

$$AF_k(1-\mu) = r \tag{13}$$

$$AF_l(1-\mu) = w \tag{14}$$

The important parameter for the rest of the analysis is μ the fixed mark-up, that indicates market power. An important development that I am currently working on is the case of endogenous μ, under which it is decreasing in number of firms. The imperfectly competitive factor prices r and w increase firm operating profits (**??**) by $\mu\nu AF(k,l)$ relative to the perfect competition case when $\mu = 0$.

$$\pi(L,K,n;A,\mu,\phi) = (1-\nu)AF(k,l) - \phi + \mu\nu AF(k,l) \tag{15}$$

$$= (1-(1-\mu)\nu)AF(k,l) - \phi \tag{16}$$

The next subsection is a brief explanation of the importance this results will later have.

1.2.1 Zero profit outcome

The extra profit $\mu\nu AF(k,l)$ from imperfect competition is crucial to understanding the entry-imperfect-competition relationship. Here we make several assumptions that are later derived.

Profits offer entry incentives, so monopoly profits encourage excess entry. In section (4.1) we show that profits are zero in steady state so there is no incentive to enter the market and entry is zero. Taking zero profits as given, (15) implies

$$F(k^*,l^*) = \frac{\phi}{A(1-(1-\mu)\nu)} \tag{17}$$

$$y^*(\underset{+}{\phi},\underset{+}{\nu},\underset{-}{\mu}) = AF(k^*,l^*) - \phi = \frac{\nu(1-\mu)\phi}{1-(1-\mu)\nu} \tag{18}$$

$$\tag{19}$$

With zero profit $\pi = y - wl - rk$ becomes

$$y^* = wl^* + rk^* = (1-\mu)A\nu F(k^*,l^*) \tag{20}$$

$$\mathcal{P}^* = \frac{y^*}{F(k^*,l^*)^{\frac{1}{\nu}}} = F(k^*,l^*)^{\frac{\nu-1}{\nu}} A\nu(1-\mu) \tag{21}$$

$$= A^{\frac{1}{\nu}}\nu(1-\mu)\left(\frac{\phi}{1-(1-\mu)\nu}\right)^{\frac{\nu-1}{\nu}} \tag{22}$$

Productivity is output per unit of production, production that is normalized to be $hod-1$ since $F(\alpha k,\alpha l)^{\frac{1}{\nu}} = (\alpha^\nu F(k,l))^{\frac{1}{\nu}} = \alpha F(k,l)^{\frac{1}{\nu}}, \alpha \in \mathbb{R}$. Note that this is a general definition of productivity: it is general in the sense that under constant returns to scale $\nu = 1$ then $\mathcal{P}^{\text{CRTS}} = (1-\mu)A$, and therefore with CRTS and perfect competition $\mu = 0$ our definition of measured productivity is A which is TFP. Notice that with imperfect competition and constant returns to scale the mark-up determines measured productivity.

However, less output per firm does not imply less output in aggregate because in general equilibrium there will be a change in the number of firms. Comparing (10) and (11) with (17) and (18) $F(k^*,l^*) < F(k^e,l^e)$, $y^* < y^e$, $\mathcal{P}^* < \mathcal{P}^e$ this is Chamberlain-Robinson excess capacity.

Zero profit equilibrium only exists if the denominator is positive $\nu < \frac{1}{1-\mu}$, $\mu \to 1, \nu \to \infty$ monopoly power allows ever increasing returns to scale. If

$\mu = 0$ returns to scale are necessarily decreasing $\nu < 1$ because no firm is large enough to make use of the fixed cost. With CRTS $\nu = 1$ then $\mu > 0$ because with constant returns and a fixed cost the economy cannot be competitive. Under perfect competition, $\mu = 0$, the long-run zero-profit output is technically efficient, (11), but as market share μ increases long-run output y^* decreases. Hence imperfect competition causes firms to produce less than the efficient level y^e, so AC>MC and AC is decreasing, giving locally increasing returns to scale. Increasing output would improve firm productivity. This result is Chamberlin-Robinson *excess capacity*; a firm could produce more to reach full capacity, but it sits back and underproduces to maximise profits. With entry we shall see that firms can be manipulated to use some of this capacity. So in equilibrium the trade-off between an additional firm bringing an extra fixed cost and reducing overall production possibility frontier outweighs the efficiency gain from smaller firms being more efficient.

1.3 Firm Entry

The number of firms at time t is determined an endogenous sunk cost of entry and an arbitrage condition that equates entry cost with incumbency profits.

Assumption 1 Sunk Entry Cost (congestion effect).
Entry cost $q \in \mathbb{R}$ increases with the number of entrants $\dot{n}$ in t.

$$q(t) = \gamma \dot{n}, \qquad \gamma \in (0, \infty) \tag{23}$$

The process is symmetric, a prospective firm pays q to enter; an incumbent firm receives $-q$ to exit. γ is the marginal cost of entry, and its bounds capture the limiting cases of entry. $\gamma \to 0$ implies instantaneous free entry, and $\gamma \to \infty$ implies fixed number of firms.

Assumption 2 Entry Arbitrage.
Gain from entry equals return from investing the cost of entry at the market rate.

$$\dot{q} + \pi(t) = r(t)q(t) \tag{24}$$

The two assumptions are well supported. The congestion effect assumption is common in industrial organization literature for example Ericson-Pakes91(an alternative theory of firm and industry dynamics) and Das and Das 1997. Justified by increased advertising costs, or competition for a fixed resource, and variously called "negative network effects" or "entry adjustment cost". In terms of the macroeconomy, Lewis 2009 statistically models congestion externalities in entry and concludes that the mechanism improves model fit because it reduces impact responses of entry. citeBerentsenWaller09 also model a congestion effect in a DSGE model.

The arbitrage assumption implies that the return to investing in a firm is equal to the return of that investment at the risk free rate r. And an implication of this is that the value of a firm is equal to present discounted value of future profits as in Bilbiie, Ghironi, and Melitz 2012.

The two assumptions form a dynamical system in number of firms and cost of entry $\{n, q\}$ which reduces to a second-order ODE in number of firms

$$\gamma\ddot{n} - r\gamma\dot{n} + \pi = 0 \tag{25}$$

By defining entry, this second-order ODE is separable into two first-order ODEs

Definition 2 Entry and Exit.
Entry (or exit) is measured by the the change in the number of firms. Negative entry is exit.

$$e(t) = \dot{n} \tag{26}$$

Hence our model of industry dynamics, which determines the number of firms, is defined by two ODEs

$$\dot{n} = e \tag{27}$$

$$\dot{e} = -\frac{\pi(t)}{\gamma} + r(t)e(t) \tag{28}$$

The endogenous entry cost causes a non-instantaneous adjustment path to steady state, which provides an analytical framework to understand short-run dynamics. To observe the importance of the endogenous entry cost (congestion effect), consider the contradiction that entry cost is fixed, $q(t) = \gamma$. In which case the the second-order ODE becomes $\pi = r\gamma$ so there is no dynamic entry.

Remark 1. Stock Market Efficiency

Integrating the entry ODE reveals that the cost of entry equals the stock market value of the firm, which is the value of discounted future dividends (profits)

$$q(t) = \int_t^\infty \pi(s) e^{-\int_t^* r(\tau)d\tau} ds \tag{29}$$

2 Aggregation

The last section discussed firm level production and showed that imperfect competition caused variable production, output and productivity of a firm to be below the perfect competition case in the long run, so-called excess capacity. Importantly these levels are fixed and independent of model variables in the long run because capital per firm and labour per firm always return to the same level–firms enter or exit until the desired per firm levels arise. That each firm ultimately produces the same output, has implications for aggregation: when there is a rise in firms, aggregate output will increase; when there is a fall in firms, aggregate output will decrease. Both productivity and output per firm will always tend to the same level, whereas aggregate output will change, it has constant returns to scale.

The first firm to enter in a period pays 0, whereas the second firm will pay γ and the third firm to enter 2γ and so on. Therefore the economy wide cost of entry, $Z(t)$, is

$$Z(t) = \gamma \int_{0}^{e(t)} i \; di = \gamma \frac{e(t)^2}{2} \tag{30}$$

Hence aggregate profits are aggregate operating profits less entry costs

$$\Pi = n\pi - Z(t) \tag{31}$$

$$\Pi = n[AF(k,l)(1-(1-\mu)\nu) - \phi] - \gamma \frac{e(t)^2}{2} \tag{32}$$

Aggregate output is

$$Y(L,K,n;A,\phi) = \int_{0}^{n} y(i)\mathrm{di} = n\left[\mathrm{AF}\left(\frac{K}{n},\frac{L}{n}\right) - \phi\right] \tag{33}$$

We previously noted at the firm level $F(\frac{K}{n},\frac{L}{n})$ is homogenous of degree $\nu < 1$, so decreasing returns. The aggregate production function Y is homogeneous of degree 1 in (K, L, n)–if you double capital, labour and number of firms, all firms remain as they were, productivity and output at the firm level are unaffected, so aggregate output also doubles because there are twice as many firms each producing the same amount as before–in the long run this can be seen by noting $Y = ny$ from (1.2).

$$Y = n\left[\mathrm{AF}\left(\frac{K}{n},\frac{L}{n}\right) - \phi\right] \tag{34}$$

If we substitute in zero profit output per firm (18) we get long-run output

$$Y^* = n^* \left[AF\left(\frac{K^*}{n^*}, \frac{L^*}{n^*}\right) \nu(1-\mu) \right] \tag{35}$$

$$Y^* = {n^*}^{1-\nu} AF(K,L)\nu(1-\mu) \tag{36}$$

substitute out $n^* = \frac{Y^*}{y^*} = \frac{Y^*}{\frac{\phi\nu(1-\mu)}{1-\nu(1-\mu)}}$

$${Y^*}^{\nu} = \left(\frac{1-\nu(1-\mu)}{\phi\nu(1-\mu)}\right)^{1-\nu} AF(K^*,L^*)\nu(1-\mu) \tag{37}$$

$$Y^* = F(K^*,L^*)^{\frac{1}{\nu}} \underbrace{A^{\frac{1}{\nu}}\nu(1-\mu)\left(\frac{1-\nu(1-\mu)}{\phi}\right)^{\frac{1-\nu}{\nu}}}_{\mathcal{P}} \tag{38}$$

Which is equivalent to $Y^* = n^* y^*$. Since $\nu < 1$, free entry causes long-run increasing returns to scale between $F(K,L)$ and Y.

$$\mathcal{P}^* = \frac{Y^*}{F(K^*,L^*)^{\frac{1}{\nu}}} = A^{\frac{1}{\nu}}\nu(1-\mu)\left(\frac{1-\nu(1-\mu)}{\phi}\right)^{\frac{1-\nu}{\nu}} \tag{39}$$

Long-run aggregate output Y^* is homogeneous of degree 1[7] in $\{K, L, n\}$. $F(K,L)$ is hod-ν so $F^{\frac{1}{\nu}}$ is hod-1. $\mathcal{P}$ is an efficiency parameter, called *measured productivity* since it is a function of true TFP A. Incidentally constant returns to scale $\nu = 1$ imply *underlying productivity* is $\mathcal{P}^* = \frac{Y}{F} = A(1-\mu)$ which is equivalent to TFP if there is perfect competition $\mu = 0$. Measured productivity is decreasing in μ. Measured productivity $\mathcal{P}$ is increasing in technological development A, whereas technology does not affect output per firm y^* (18).

Marginal products with fixed labour are in appendix E.1. The response of output to an extra firm is ambiguous

$$\hat{Y}_n = \pi - \mu\nu AF\left(\frac{K}{n}, \frac{L}{n}\right) + F_l L_n \tag{40}$$

The result is also ambiguous in steady state when $\pi = 0$, whereas it is positive under perfect competition because an extra firm increases labour and there is no negative effect from the mark-up competition, and no endogenous

[7]Long-run aggregate output is $hod - 1$ in its factors. Increasing the factors $\{K, L, n\}$ in Y^* by λ gives $Y^*(\lambda K, \lambda Y, \lambda n) = (\lambda n)^{1-\nu} AF(\lambda K, \lambda L)\nu(1-\mu) = Y^* = \lambda^{1-\nu} n^{1-\nu} \lambda^{\nu} AF(K,L)\nu(1-\mu) = \lambda n^{1-\nu} AF(K,L)\nu(1-\mu) = \lambda Y^*$

labour an additional firm reduces output. In the classic Ramsey case with no endogenous labour, and no imperfect competition $\hat{Y}_n = 0$, so the number of firms maximises output

Regardless of steady state Y_n can be positive or negative. (See appendix). When homogeneity $\nu \to 1$ there are CRTS and it is more likely that an additional firm reduces output. Since returns are near to constant it is less inefficient for a large incumbent to increase output than if returns were strongly decreasing because of a increasing variable costs. Since the incumbent is already going to pay fixed cost, the benefit of it producing albeit slightly less efficiently is better than a new more efficient firm entering but incurring an extra fixed cost. If there are strong decreasing returns to scale it is more likely that the extra efficient of an additional firm relative to an incumbent will outweighs the extra fixed cost incurred because there is one more firm. The size of the fixed cost is also important. Large fixed cost and CRTS encourage fewer firms.

3 General Equilibrium

Let us now consider determination of prices, consumption and labour given the current capital stock and number of firms.

Output from firms Y is split between consumption $C(t)$, investment $I(t) = \dot{K}$ and government spending $G(t)$.

$$\dot{K} = wL + rK + \Pi - C \tag{41}$$

$$= n(1-\mu)\nu AF(k,l) + n[AF(k,l)(1-(1-\mu)\mu) - \phi] - \gamma\frac{e^2}{2} - C \tag{42}$$

$$\dot{K} = n\left[A\mathrm{F}\left(\frac{K}{n}, \frac{L}{n}\right) - \phi\right] - C - \gamma\frac{e^2}{2} \tag{43}$$

There is Ricardian equivalence $T(t) = G(t)$.

Definition 3.
Competitive equilibrium is the 'equilibrium' paths of aggregate quantities and prices $\{C(t), L(t), K(t), n(t), e(t), w(t), r(t)\}_{t=0}^{\infty}$, with prices strictly positive, such that $\{C(t), L(t)\}_{t=0}^{\infty}$ solve the household problem. $\{K(t)\}_{t=0}^{\infty}$ satisfies the law of motion for capital. Labour and capital $\{L(t), K(t)\}_{t=0}^{\infty}$ maximise firm profits given factor prices. The flow of entry causes the arbitrage condition on entry to hold (price of entry equals NPV of incumbent). State variables $\{K(t), n(t)\}_{t=0}^{\infty}$ satisfy transversality. Factor prices are set according to **??** *and ensure goods and factor markets clear.*

4 Dynamical System of Endogenous Entry Economy

Firstly, there is conflicting terminology from optimal control and dynamical systems that needs clarifying. That is, the states, or state space, in the dynamical systems is the $C, e, K, n \in \mathbb{R}^4$, whereas in the optimal control problem it is $K, n \in \mathbb{R}^2$. Actually the system is reducible to the 2-dimensional space as we shall see, but for linearization and studying stability we work in the 4-dimensional space that we shall call the *phase space*.

A system of four autonomous differential equations is sufficient to qualitatively analyse the economy. By a qualitative analysis we mean that we find the set of solutions that satisfy the system given an initial state $x_0 := (K_0, n_0)$ in the phase space $X \in \mathbb{R}^2$ and parameter set $\Omega \in \mathbb{R}^{\mathbb{P}}$. Given our system of differential equations our goal is to describe the qualitative behaviour of the solution set including the invariant sets and limiting behaviour defined by the flow. In order to achieve this qualititative anlysis we employ local theory particulary Hartman-Grobman Theorem and the Stable Manifold Theorem which allow us to analyse the solution set of the nonlinear system of ODEs in a neighbourhood of an equilibrium point (in fact the steady state) through looking at the qualitative behaviour of the solution set of the corresponding linearized system in a neighbourhood of the steady state. So at a point in time $t \in (0, \infty)$ our economy is entirely described by its level of capital and the number of firms. For now we do not specify initial conditions or functional forms that would give specific trajectories of capital and number of firms. The primitives of our boundary value problem are the state of the system defined on an open set $\mathbb{X} := K \times n \in \mathbb{R}^2$, the time $t \in \mathbb{I}$ defined on an open interval of $\mathbb{R}$, the parameterization defined on an open set $\Omega \in \mathbb{R}^{\mathbb{P}}$ and the nonlinear C^1 function $g : \mathbb{X} \times I \times \Omega \supseteq \mathbb{R}^{3+\mathbb{P}} \to \mathbb{R}^2$, or *"time evolution law"*, that maps a given state, time and parameterization into a new state [8]. This rule allows use to determine the state of the system at each moment in time from its state at all previous times. For most of the analysis we work with a system where the state space includes C and e, although these variables can be defined by their *policy functions* which map $K \times n$ into $C \times e$, so the system equations $\dot{K}$ and $\dot{n}$ describe the evolution of the state variables along the stable manifold. Our system is could be reduced to two equations

[8] Without loss of generality, our system is defined across dimensions of the real field $\mathbb{R}$ (Euclidean n-space). The assumption opposes intuition that a variable like number of firms n evolves on the integer ring $\mathbb{Z}$. This eases analysis, for example with the Euclidean norm and associated Euclidean metric, our normed vector space is complete with respect to the metric induced by its norm (a Banach space).

rather than four, so controls are defined in terms of state. This is common to those using dynamic programming techniques. But analysing the stability properties of this system is equivalent to analysing the Hamiltonian system.

Definition 4 Nonlinear system.
The dynamical system is a pair $(\mathbb{X}, g)$*, where* $\mathbb{X} = (\mathbb{R}^4, \psi)$ *is Euclidean space and metric. It defines at a point in time* $t \in \mathbb{R}$ *the state of the system* $x(t) \in \mathbb{X} \subseteq \mathbb{R}^4$ *is described by a* C^1 *vector valued transition map* $g : \mathbb{R}^{5+\mathbb{P}} \supseteq \mathbb{X} \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^4$

$$\dot{K} = Y - \frac{\gamma}{2}e^2 - C - G, \qquad Y = n(F(k,l) - \phi) \tag{44}$$

$$\dot{n} = e \tag{45}$$

$$\dot{C} = \frac{C}{\sigma(C)}(r - \rho), \qquad \sigma(C) = -\frac{Cu_{CC}}{u_C} \tag{46}$$

$$w = \frac{-u_L}{u_C} \tag{47}$$

$$\dot{e} = re - \frac{\pi}{\gamma}, \qquad \pi = AF(k,l)(1 - (1-\mu)\nu) - \phi \tag{48}$$

where factor prices

$$r = (1-\mu)AF_k, \qquad w = (1-\mu)AF_l \tag{49}$$

$$\dot{K} = n(AF(k,l) - \phi) - \frac{\gamma}{2}e^2 - C - G, \tag{50}$$

$$\dot{n} = e \tag{51}$$

$$\dot{C} = -\frac{u_C}{u_{CC}}((1-\mu)AF_k - \rho), \tag{52}$$

$$(1-\mu)AF_l = \frac{-u_L}{u_C} \tag{53}$$

$$\dot{e} = ((1-\mu)AF_k)e - \frac{AF(k,l)(1-(1-\mu)\nu) - \phi}{\gamma}, \tag{54}$$

The *system equations*, (44) and (45), explain how the state of the system evolves. The *optimization conditions*, (46) and (48), restrict the state evolutions. They impose that households maximise utility and potential entrants maximise profits. The economic reiteration is that the system equations determine how capital and number of firms evolve as the economy moves through time, and the optimization conditions ensure that capital and number of firms move so as to maximise consumers' utility and firms' utility.

4.1 Steady-state behaviour

Assume that a solution of the system converges to steady state $(K^*, n^*, C^*, e^*) \rightarrow (K^s, n^s, C^s, e^s)$ as $t \rightarrow +\infty$. [9]In steady state $\dot{K} = \dot{n} = \dot{C} = \dot{e} = 0$.

$$\dot{K} = 0 \Leftrightarrow Y^*(C^*) = C^* + \frac{\gamma}{2}e^* \tag{55}$$

$$\dot{n} = 0 \Leftrightarrow 0 = e^* \tag{56}$$

$$w^* = \frac{-u_L}{u_C} \tag{57}$$

$$\dot{C} = 0 \Leftrightarrow r^*(C^*, K^*, n^*) = \rho \tag{58}$$

$$\dot{e} = 0 \Leftrightarrow r^*(C^*, K^*, n^*)e^* = \frac{\pi^*(C^*, K^*, n^*)}{\gamma} \tag{59}$$

Plug $e^* = 0$ into $\dot{K}$ and $\dot{e}$ and rewrite in terms of variables (C, K, L, n, e)

$$n^* \left[AF\left(\frac{K^*}{n^*}, \frac{L^*}{n^*}\right) - \phi \right] = C^* \tag{60}$$

$$e^* = 0 \tag{61}$$

$$(1-\mu)AF_l\left(\frac{K^*}{n^*}, \frac{L^*}{n^*}\right) = -\frac{u_L(L^*)}{u_C(C^*)} \tag{62}$$

$$(1-\mu)AF_k\left(\frac{K^*}{n^*}, \frac{L^*}{n^*}\right) = \rho \tag{63}$$

$$F\left(\frac{K^*}{n^*}, \frac{L^*}{n^*}\right) = \frac{\phi}{A(1-(1-\mu)\nu)} \tag{64}$$

The system determines $(C^*, L^*, K^*, n^*, e^*)$, where $e^* = 0$ is immediate, and the intratemporal Euler (62) defines $L(C, K, n; A, \mu)$. The results formally confirm what we found in the earlier section on zero profit outcomes. The entry arbitrage condition, which becomes a zero profit condition in steady state, gives steady-state technology $F(\frac{K}{n}, \frac{L}{n})$, which then gives y^*, and a mapping between C^* and n^* via (60). The system is recursive: first find K^*, n^* then the policy rules C^*, L^* as a function of the states. Projecting the optimization conditions (63) and (64) onto the state space determines K^*, n^* for a given L^*, which is fixed via the capital labour ratio. Ceteris paribus, say for the K, L, n at perfect competition steady state levels, (64) implies imperfect competition reduces production per firm and (62) and (63) imply marginal products rise with imperfect competition (because firms underproduce); (60) implies lower consumption. These do not necessarily hold when K, L, n adjust. The intuition is that firms produce less so they benefit from greater increasing returns to scale hence marginal products are higher.

[9]Ignore the trivial steady state that arises when the state vector is the zero vector.

4.1.1 Optimization conditions in k,l and K,n space

The optimization conditions (63) and (64) determine per firm capital k and labour l, and therefore aggregate capital-labour ratio $\frac{k}{l} = \frac{K}{L}$[10]. In k, l space (64) is an isoquant for zero-profit output, and (63) is a locus along which marginal product of capital is at steady state. Figure (1) shows the convex isoquant and linear marginal product of capital in steady state. Later we shall comment on the unamibiguous decrease in k indicated by the dashed lines which is caused by a rise in imperfect competition.

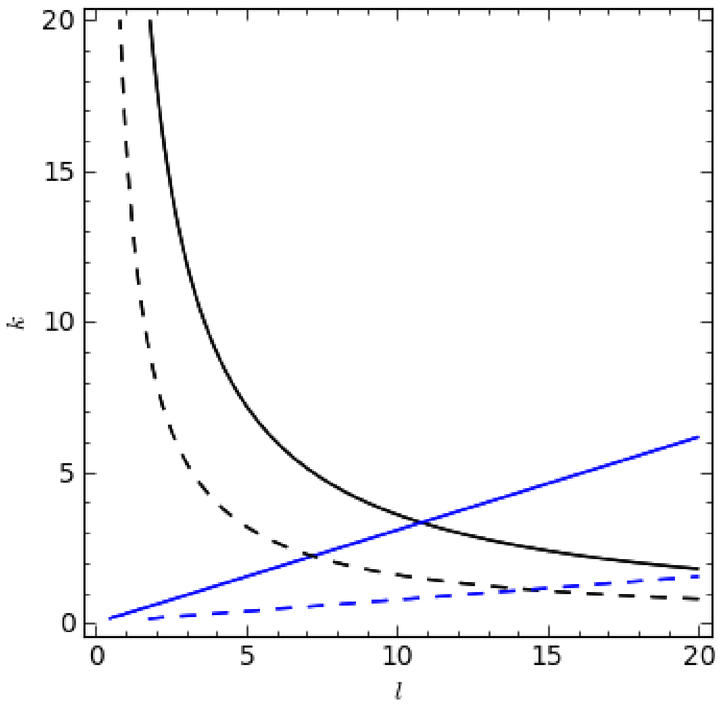


Figure 1: k, l space

Figure (2a) shows K, n combinations that cause $\dot{C} = 0$ and $\dot{e} = 0$[11]. At the intersection both hold. The interpretation of the functions is that consumption does not change when interest rates equal the discount rate $r = \rho$, and entry does not change when profits are zero $\pi = 0$. Incentives cause both results: incentive to consume today is the same as incentive to consume later because discount rate and interest rate are equal, so discount a

[10]In other words, eliminate n in the two equations to give a single relationship between K and L.

[11]Functions (63) and (64) respectively. The loci are equivalent to $\dot{C} = 0$ and $\dot{n} = 0$ only if other vairables are at steady state.

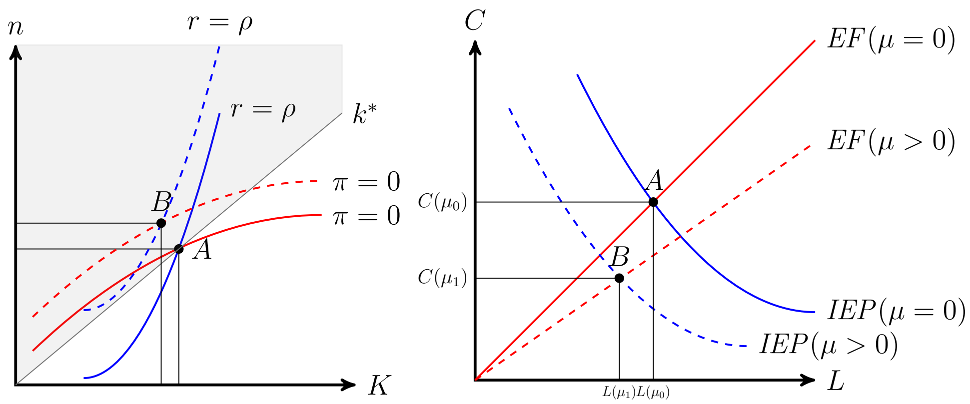
household suffers from waiting to consume is offset by interest earned whilst waiting. There is no incentive for entry when profits are zero. The interest rate condition is upward sloping. A rise in capital per firm decreases the marginal product of capital[12], and therefore the interest rate, but a rise in firms decreases capital per firm back to its steady-state level. The free entry condition ($\pi = 0$) slope is ambiguous when labour varies, and is upward sloping when labour is fixed. Increasing capital increases profits, but capital also raises labour which can reduce profits. If profit falls number of firms decrease until zero profit is restored. Figure (2a) shows the case where an increase in capital raises profits and number of firms increase to syphon the profit causing a positive slope.

Proposition 1.
Interest rates are less sensitive to number of firms than profits. The interest function is steeper than the profit function[13].

$$\left.\frac{dn}{dK}\right|_{r=\rho} - \left.\frac{dn}{dK}\right|_{\pi=0} = \frac{\nu F_{kk}F + (1-\nu)F_k F_k}{-\nu(1-\nu)F_k F} > 0 \tag{65}$$

Proof. Appendix (G) □

4.1.2 Euler Frontier and Income Expansion Path in L, C space



(a) K, n space (b) L, C space

Figure 2: Steady State Condtions

We have k^* and l^*, so we can determine the marginal product of labour, which gives the steady state wage w^*. Steady-state wage is fixed so the

[12]since DRTS F_k is decreasing in its arguments.
[13]In fact, profit function could have negative slope.

intratemporal condition (62) describes L, C combinations that keep the ratio of marginal utilities constant. Thus it represents the *income expansion path*. We call the capital evolution equation (60) the Euler frontier when Y is replaced by zero profit output from (64), equivalent to (18), and number of firms is written in terms of L, so the equations can be plotted in L, C space[14]. $n = \frac{L}{l}$ and l is determined, so when L changes it changes l but n moves to keep l at its steady state level. Then the IEP and EF are

$$\overbrace{(1-\mu)AF_l\left(k^*, l^*\right)}^{w^*} = -\frac{u_L(L)}{u_C(C)} \qquad \text{IEP} \tag{66}$$

$$C = \frac{L}{l^*}\left(\frac{\nu(1-\mu)\phi}{(1-(1-\mu)\nu)}\right) \qquad \text{EF} \tag{67}$$

The IEP is downward sloping because labour is an inferior good and consumption is a normal good. As income increases consumption increases and labour decreases; specifically non-labour income since w^* is fixed. Income and utility expand North West on the IEP. The slope is convex because utility from consumption diminishes and disutility from labor grows. Under quasi-homothetic preferences the IEP is a straight line; if preferences are homomethic IEP intersects (1,0).

4.1.3 Comparative statics: Imperfect competition

Proposition 2.
Market power decreases capital per firm, and ambiguously affects labour per firm.

$$k_\mu < 0, \quad l_\mu \lesseqgtr 0 \tag{68}$$

Proof. Appendix (F.1) □

The result is shown in figure (1) by a downward shift in steady-state output isoquant and downward shift in steady-state marginal product of capital. The result is also shown by the grey region of figure 2a. In the grey region $\frac{K}{n}$ is strictly less, and one can see any shift in the curves caused by a rise in μ will put the new intersection in the grey region, as the example with dotted lines shows.

Imperfect competition reduces the interest rate so it is less than the discount rate. Interest rate will return to parity if the marginal product of capital increases which for given K occurs by increasing n–less capital per

[14]The EF is steeper than the budget constraint see Appendix H.

firm raises MPK as each firm employs capital more efficiently due to DRTS in variable cost. The $r = \rho$ curve shifts up. A rise in imperfect competition increases profits $\pi > 0$, so $\pi = 0$ shifts up too because given K a rise in n will reduce production per firm, and therefore profits, since each firm has less capital and uses it more efficiently (has less excess capacity lower monopoly profits), which will keep profits at zero. Under both steady-state conditions imperfect competition must be offset by a rise in n for a given K, hence capital per firm, k, unambiguously falls. The grey region is $k < k^*$, and any rise in μ will reduce k, to a point in this region[15].

A rise in imperfect competition shifts the IEP down and decreases the EF slope causing a decrease in consumption and an increase or decrease in labour supply, but a fall in consumption for any L. The IEP shifts down because market power μ reduces wage, so for given L, according to the intratemporal condition, consumption must decrease to decrease the utility from consumption and equate the fall in wages[16]. It is a pure substitution effect: leisure $(1 - L)$ is cheaper relative to consumption, therefore the household decrease consumption and increase leisure (for a given level of consumption takes less labour). The EF slope is shallower because each unit of L increases income less, which creates an income effect. The household has less income so decreases intake of both normal goods: consumption and leisure. For a given L the household reduces C. Both substitution and income effects reduce consumption, but the substitution effect reduces labour whereas the income effect increases labour (decreases the normal good leisure). Figure 2b shows the case where substitution effect dominates the income effect–labour falls. Under fixed labour EF would still rotate because return on capital and wage decrease, but the IEP is vertical which removes substitution effect, leaving consumption reducing income effect.

The steady state analysis shows that imperfect competition reduces consumption, but can increase or decrease labour supply. Imperfect competition may increase or decrease capital and number of firms but capital per firm will fall whereas labour per firm is ambiguous. A decrease in capital per firm is important for our thesis because in steady-state firms have excess capacity and employ too few inputs. This result asserts that any more imperfect competition will worsen the situation.

From this snapshot of the economy when capital, number of firms, consumption and entry are constant we find entry is zero, output per firm is

[15]The diagram is schematic; k must be in a region bound above both original curves, which rules out some of the grey, but consider different functions e.g. both are flatter so they tend to the k^* line–this opens up more of the grey region.

[16]The ratio of marginal utilities is negative, so we want the ratio to increase which means a decrease when we consider the negation.

lower, marginal products are higher, consumption is lower, capital, number of firms and labour are higher or lower, but capital per firm is lower and consumption to labour is lower.

4.2 Local Comparative Dynamics IN PROGRESS

Comparative dynamics can explain how imperfect competition affects the economy as it transitions to excess capacity steady state. For example, if the economy is at a more efficient capital-number-of-firms state what mechanisms drive the states to the inefficient steady state? On path to steady state, can the economy pass through more efficient positions in the state space? To answer these questions, we solve the four dimensional system which gives trajectories of the variables over t.

Our system is of nonlinear form $\dot{x} = g(x)$, but we linearize it to $\dot{x} = A\bar{x}$, and analyse the A operator which is a matrix on the space defined. The local behaviour of the nonlinear system is analogous to the behaviour of the linearized system[17]; the systems are *'topologically equivalent'*. Proof in appendix X.

A is the Jacobian matrix where each element is a respective derivative. The derivatives treat $\{K, n, C, e\}$ as independ, as explained earlier labour is a function of these variables through the intratemporal condition $L(K, n, C)$. Later, we derive C and e as functions on K, n.

$$\begin{bmatrix} \dot{C} \\ \dot{e} \\ \dot{K} \\ \dot{n} \end{bmatrix} = \begin{bmatrix} \frac{C}{\sigma} r_C & 0 & \frac{C}{\sigma} r_K & \frac{C}{\sigma} r_n \\ r_C e - \frac{\pi_C}{\gamma} & r & r_K e - \frac{\pi_K}{\gamma} & r_n e - \frac{\pi_n}{\gamma} \\ Y_C - 1 & -\gamma e & Y_K & Y_n \\ 0 & 1 & 0 & 0 \end{bmatrix} \Bigg|_{\substack{\dot{C}=0 \\ \dot{e}=0 \\ \dot{K}=0 \\ \dot{n}=0}} \begin{bmatrix} C \\ e \\ K \\ n \end{bmatrix} \tag{69}$$

Cursory inspection reveals that imperfect competition has an effect on the dynamics. For example consider an economy $\mathcal{E}$ defined by a point in the phase space $\mathcal{E} = \{K', n', C', e'\} \in \mathbb{R}^4$, the system matrix, and therefore the flow of the system, at this point is different for three of the four ODEs when there is imperfect competition. So an economy at this point, will be moving in different directions under perfect competition and imperfect competition. Imperfect competition enters directly through r and π in the $\dot{C}$ and $\dot{e}$ ODEs, and indirectly in $\dot{K}$ through its effect on w and thus L which Y depends on, only $\dot{n}$ will have the same gradient e' in both the economy with and without imperfect competition. Hence even for exactly the same economy imperfect

[17] There is an injective (one-one) mapping between trajectories in a neighbourhood of $\bar{x}$ and an open set containing the origin.

competition will change the trajectory of its movement. Additionally this will mean that the economy will not stop moving at the same place, so steady state is different under each scenario.

Definition 5 Jacobian Matrix and the Jacobian (determinant).
The Jacobian matrix $Df(\bar{x}) : \mathbb{R}^4 \to \mathbb{R}^4$ is the nonlinear system matrix evaluated at steady state $\bar{x}$. It forms the system matrix of the linearized system. The determinant of the Jacobian matrix $|Df(\bar{x})|$ is called the Jacobian.

A reminder that at steady state $e = 0$, $r = \rho$, $Y = C$ and $\pi = 0$.

$$\begin{bmatrix} \dot{C} \\ \dot{e} \\ \dot{K} \\ \dot{n} \end{bmatrix} = \overbrace{\begin{bmatrix} \frac{C^*}{\sigma} r_C^* & 0 & \frac{C^*}{\sigma} r_K^* & \frac{C^*}{\sigma} r_n^* \\ -\frac{\pi_C^*}{\gamma} & \rho & -\frac{\pi_K^*}{\gamma} & -\frac{\pi_n^*}{\gamma} \\ Y_C^* - 1 & 0 & Y_K^* & Y_n^* \\ 0 & 1 & 0 & 0 \end{bmatrix}}^{Df(\bar{x})} \begin{bmatrix} C - C^* \\ e - e^* \\ K - K^* \\ n - n^* \end{bmatrix} \tag{70}$$

In the linearized model, the state vector is deviation from equilibrium point[18]. Labour behaves according to the optimal intratemporal condition $\hat{L} = L(K, n, C)$. The elements of the Jacobian have the same qualitative interpretation as the perfect competition case, for example element $3 times 3$ is the marginal product of capital in both cases. But their magnitudes change for any given point, which means the steady state is also different.

Proposition 3 Effect of entrant on aggregate output.
In steady state an entering firm ambiguously affects aggregate output

$$Y_n^* = -\mu\nu AF\left(\frac{K^*}{n^*}, \frac{L^*}{n^*}\right) + AF_l\left(\frac{K^*}{n^*}, \frac{L^*}{n^*}\right) L_n \tag{71}$$

$$= \frac{-\phi\mu\nu}{1-(1-\mu)\nu} + AF_l\left(\frac{K^*}{n^*}, \frac{L^*}{n^*}\right) L_n \lesseqgtr 0 \tag{72}$$

$$\tag{73}$$

Corollary 1.
Whether an additional firm increases of decreases aggregate output depends on whether the positive labour effect outweighs the negative fixed cost duplication effect

- $\hat{Y}_n^* < \hat{Y}_n^{*\mu=0}$ *firm contributes less than under perfect competition*

[18] In the terminology of the dynamical system all undetermined variables are state variables, whereas in the terminology of the optimization problem K, n are states and C, e are controls. Actually the dynamical system is only 2-dimensional, K, n is a basis for the whole system. C, e will be shown to be functions of K, n

- $\hat{Y}_n^* < 0$ *if* $\mu\nu F > F_l L_n$ *underproduction component exceeds labour boost*
- $\hat{Y}_n^* > 0$ *if* $\mu\nu F < F_l L_n$ *labour boost exceeds capacity underutilization effect*

Firm entry causes a trade-off between the additional firm incurring a fixed cost and the additional firm producing more efficiently because of decreasing returns to scale. The proposition captures the two opposing effects. First, there is a positive labour effect because entry increases labour supply[19]. Second entry has a negative effect that is larger when the fixed cost is higher, there is more market power, or higher returns to scale; so properties akin to a natural monopoly increase the negative effect of a firm entering. We shall call it the "duplication effect", and it creates the possibility that $\hat{Y}_n^* < 0$; whereas, under perfect competition $\hat{Y}_n^* > 0$ [20].

The result alters the direct effect of number of firms on capital accumulation in steady state (element $(3,4)$ in the Jacobian matrix, whereas indirect effects of firms on capital are $(3,1)$,$(3,2)$ and $(3,3)$). Under perfect competition a deviation in firms from equilibrium had an effect of the same sign on capital accumulation–more firms raised capital–but with imperfect competition the sign may be reversed: more firms decumulate capital.

This plays an important role for eigenvalues which I analyse next. It has two effects. Contributes to discouraging complex eigenvalues and ensures faster convergence to steady state.

4.3 Eigenvalues

The four dimensional system has two positive and two negative eigenvalues, so the system is a saddle which is locally asymptotically unstable. By the stable manifold theorem, we show saddlepath stability by setting the constant of integration on the two explosive eigenvalues to zero, thus reducing attention to the stable set. This gives a system of saddle-path conditions that may be solved for C and e. These policy functions for C and e are the stable manifold of the system that ensure the constants on the explosive eigenvalues are zero. Therefore this solution ensures the system is on a subspace of the system without explosive eigenvalues– this is the the stable manifold which asymptotically reaches steady state.

[19] See appendix. The postive effect of a firm on labour supply relies on decreasing returns to the technology $\nu < 1$. Firm entry decreases labour per firm which increases MPL due to decreasing returns. Firms therefore employ more labour. Under under increasing returns the reduction in per firm labour decreases the marginal product of labour. Therefore firms employ less labour

[20] If labour were not endogenous $F_l = 0$ and $\hat{Y}_n^* \leqslant 0$.

First derive the eigenvalues. The 4-dimensional system has symmetry that we exploit to solve the quartic characteristic polynomial analytically. Eigenvalues have the general structure of Feichtinger, Novak, and Wirl 1994, and are qualitatively similar to Brito and Dixon 2013. Despite the same qualitative interpretations, the magnitudes change and this affects the size of the sets of different types of dynamic behaviour.

The characteristic polynomial of the Jacobian takes a standard form. And the solution of the quartic polynomial is symmetric around $\frac{\rho}{2}$, and non-zero since $\frac{\rho}{2} > 0$.

The quartic polynomal takes the following standard form[21]

Proposition 4 Eigenvalues.
There are four eigenvalues

$$\lambda_{1,2}^{s,u} = \frac{\rho}{2} \mp \left[\left(\frac{\rho}{2}\right)^2 - \frac{\mathcal{T}}{2} \mp \left(\left(\frac{\mathcal{T}}{2}\right)^2 - ||J|| \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \tag{74}$$

$\mathcal{T} = M_2 - \rho^2$ where M_2 is the sum of the principal minors of order 2, and $||J|| = M_4$ is the Jacobian determinant which is the sum of principal minors of order 4.

$$M_2 = \rho^2 + \frac{Cr_K}{\sigma} + \frac{\pi_n}{\gamma} \tag{75}$$

$$M_4 = \tag{76}$$

So the eigenvalues are entirely in terms of parameters since at steady state even the variables are in terms of parameters.

Proposition 5 Stable eigenvalues–saddlepath.
There are always two negative eigenvalues, either real or complex, so we can choose state variables freely. There are always two positive and two negative eigenvalues because the root of the outer discriminant (the term in square brackets) is greater than the term it is added/subtracted from $\frac{\rho}{2}$. Because eigenvalues are always positive and negative the matrix is indefinite. We deduce this because $-\frac{\mathcal{T}}{2} \pm \Delta^{\frac{1}{2}} > 0$, note $\mathcal{T} < 0$, so overall the inner discriminant is at least as large as $\frac{\rho}{2}^2$ and therefore its root is larger than $\frac{\rho}{2}$. Formally the least upper bound of the outer discriminate

$$\inf \left\{ \left[\left(\frac{\rho}{2}\right)^2 - \frac{\mathcal{T}}{2} \mp \Delta^{\frac{1}{2}} \right]^{\frac{1}{2}} \right\} = \frac{\rho}{2}$$

[21] See appendix X for my novel proof of this. But the result of the structure of eigenvalues for this type of problem is standard.

This also implies that no eigenvalue has zero real part, so by Grobman-Hartman within a neighbourhood of such a fixed point the linearized system is topologically equivalent to the nonlinear system.

Proposition 6 Nature of eigenvalues.
$\mathcal{T}$ drives stability and is affected by π_n
Δ, more specifically $\mathcal{O}$, determines complex or real roots
The outer discriminant is always positive (and greater than $\frac{\rho}{2}$) because $-\frac{\mathcal{T}}{2} \mp \Delta^{\frac{1}{2}}$ is positive, so only the inner discriminant can cause complex roots.

Proposition 7 Eigenvalues always real.
If $\hat{Y}_n < 0$ then $\mathcal{O} > 0$ so the inner discriminant $\Delta > 0$ and all eigenvalues are real. Imperfect competition encourages nonexistence of complex roots by decreasing $\hat{Y}_n$ which decreases the offsetting negative component $\mathcal{O}$ in the inner discriminant Δ.

The second effect of strictly smaller firm contribution to output Y_n is to increse the absolute values of the eigenvalues

Proposition 8.
The inner discriminant is unambiguously larger (in real case??) so whatever we $\mp$ is larger leading to larger absolute eigenvalues.

$$\frac{\partial \mathcal{O}}{\partial Y_n} < 0, \quad \textit{since} \quad Y_n(\mu) < Y_n(\mu = 0) \tag{77}$$

$$\textit{then} \quad \mathcal{O}(\mu = 0) < \mathcal{O}(\mu) \tag{78}$$

$$\frac{\partial \Delta}{\partial \mathcal{O}} > 0, \textit{therefore} \quad \Delta(\mu]) > \Delta(\mu = 0) \tag{79}$$

That eigenvalues are absolutely larger is driven by having a larger inner discriminant. $\mathcal{T}$ is driven up by π_n component which raises the absolute value of the eigenvalues. Y_n also helps to increase the inner discriminant–both have the ffect of raising the eigenvalues. It means that trajectories converge faster or diverge faster. Why do trajectories move faster, why is the economy more responsive, in the presence of imperfect competition? The intuition is that monopoly profits encourage a faster response by firms so they enter faster to arbitrage profits to zero. Remember the result is driven by π_n being larger, so an additional firm drives profits down more which drives us to zero profit quicker. Note that even though profits are larger they are driven to zero faster than lower normal profits, suggesting a marginal increase in profit causes a more than proportional response in the speed of entry.

We are driven closer to the exogenous labour case for $\mathcal{O}$

5 Main Result: Capacity Utilization

In the short run, the state variables, capital and number of firms, are predetermined, so they do not adjust immediately when there is a shock. This can be seen from the solution to the system, since at $t = 0$ the state variables equal their initial condition x_0, whereas the controls change depending on the eigenvectors. Labour is one of these free variables that will jump instantaneously. The timing difference between immediate labour adjustment and slow capital and firm adjustment exacerbate the capacity utilization effect which causes measured productivity to exceed underlying productivity. Underlying productivity is the steady-state value $\mathcal{P}^*$ defined in (39); it depends on $\{A, \nu, \phi, \mu\}$ and is independent of model variables. Measured productivity $\mathcal{P}$ is productivity observed at any instance.

In steady state firms have excess capacity. Since entry is a slow process, a positive production shock prompts incumbent firms to increase their capacity toward full capacity, making better use of their fixed costs, which improves productivity. A negative production shock causes firms to decrease their capacity, further from full capacity, making less use of their fixed costs and decreasing productivity. In the long run, firms will exit until zero profit returns each firm to producing a long-run level of output which is unchanged. Ignoring labour, with better technology each firm will require less capital to produce the long-run level. Hence capital per firm should fall.

Theorem 5.1 Measured Productivity Overshooting.
A change in technology causes a greater change in measured productivity than underlying productivity.

$$\frac{d\mathcal{P}(0)}{dA} = \frac{1}{(1-\mu)}\frac{d\mathcal{P}^*}{dA} + \mu\mathcal{P}^*\frac{Y_L}{Y^*}L_A, \tag{80}$$

$$\left|\frac{d\mathcal{P}(0)}{dA}\right| > \left|\frac{d\mathcal{P}^*}{dA}\right| \tag{81}$$

Corollary 2.

- *Imperfect compeition $\mu > 0$ is necessary and sufficient for measured productivity overshooting, and overshooting is increasing in μ.*
  PROOF By contrapositive, differentials are equal if and only if $\mu = 0$.
- *Endogenous labour is not necessary for overshooting, but it strengthens the effect.*

A positive technology shock immediately raises output per firm. Firms do not have time to adapt, incumbents absorb the new technology and produce more output given their existing capital. Producing more reduces their excess capacity. Profits increase (π is increasing in A eq (15)). Over time capital and number of firms adjust, returning production to y^* and profits to zero. Capital per firm will falls because better technology the inputs needed to produce the fixed long-run level y^*. Endogenous labour strengthens the impact response, and causes more overshooting.

6 Conclusion

The paper proposes a microfounded endogenous firm entry dynamic general equilibrium model with imperfect competition and endogenous entry costs. The contribution of the paper is to analyse the effect of imperfect competition on entry, and in turn how entry changes capacity utilization which cause endogenous productivity dynamics. Imperfect competition creates monopoly profits which cause "too many" entrants each producing "too little": less than a cost minimizing level of output. Excess capacity is the difference between what they produce, and what they would produce if they minimized costs. Firms underproduce, so underemploy factors, and employing additional factors which expands their output and diminishes excess capacity causes productivity gains. The entry-imperfect-competition interaction has been observed in quantitative DSGE models of firm entry, but the papers only observe simulations, we offer an analytical narrative to the blackbox driving these simulated dynamics. And on the technical side we emphasise that entry reduces the set of complex dynamics, and drives the economy to steady state faster. This can explain why quantitative simulations are able to observe nonmonotone (popularly coined hump-shaped) responses.

A Household Optimization Problem

To obtain the necessary conditions for a solution to the household's utility maximisation problem I use the maximum principle. The current value Hamiltonian is

$$\hat{\mathcal{H}}(t) = u(C(t), L(t)) + \lambda(t)(w(t)L(t) + r(t)K(t) + \Pi(t) - C(t) - G) \quad (82)$$

The costate variable λ_t is the shadow price of wealth in utility units. The Pontryagin necessary conditions are [22]

$$\hat{\mathcal{H}}_C(K, L, C, \lambda) = 0 \implies u_C - \lambda = 0 \tag{83}$$

$$\hat{\mathcal{H}}_L(K, L, C, \lambda) = 0 \implies u_L + \lambda w = 0 \tag{84}$$

$$\hat{\mathcal{H}}_K(K, L, C, \lambda) = \rho\lambda - \dot{\lambda} \implies \lambda r = \rho\lambda - \dot{\lambda} \implies \frac{\dot{\lambda}}{\lambda} = -(r - \rho) \tag{85}$$

$$\hat{\mathcal{H}}_\lambda := \dot{K}_t \implies \dot{K} = rK + wL + \Pi - C - G,\ \Pi = py - \mathrm{TC} \tag{86}$$

Equations (83)-(85) reduce to two equations: a differential equation in consumption, and an injective mapping between labour and consumption.

B Operating Profit

Total costs are

$$\begin{aligned} wl + rk &= AF_l(1-\mu)l + AF_k(1-\mu)k && (87)\\ &= A(1-\mu)[F_l l + F_k k] && (88)\\ &= (1-\mu)\nu AF(k,l) && (89) \end{aligned}$$

Operating profit is output less total costs

$$\begin{aligned} \pi &= y - wl - rk && (90)\\ \pi &= (AF(k,l) - \phi) - ((1-\mu)\nu AF(k,l)) && (91)\\ \pi(L, K, n; A, \mu, \phi) &= AF(k,l)(1 - (1-\mu)\nu) - \phi && (92) \end{aligned}$$

C Output Derivatives

L is fixed.

$$Y(L, K, n; A, \phi) = \int_0^n y(i)\mathrm{di} = n\left[\mathrm{AF}\left(\frac{K}{n}, \frac{L}{n}\right) - \phi\right] \tag{93}$$

[22]

I suppress the function notation e.g. $F_l = F_l\left(\frac{K}{n}, \frac{L}{n}\right)$

$$Y_K = AF_k = \frac{r}{1-\mu} > 0 \tag{94}$$

$$Y_{KK} = AF_{kk}\frac{1}{n} < 0 \tag{95}$$

$$Y_{Kn} = Y_{nK} = A(F_{kk}k_n + F_{kl}l_n) = A\left[F_{kk}\left(-\frac{K}{n^2}\right) + F_{kl}\left(-\frac{L}{n^2}\right)\right] \tag{96}$$

$$= A\frac{-1}{n}(\nu-1)F_k = \frac{A}{n}(1-\nu)F_k > 0 \tag{97}$$

$$Y_L = AF_l = \frac{w}{1-\mu} > 0 \tag{98}$$

$$Y_{LK} = AF_{lk}\frac{1}{n} > 0 \tag{99}$$

$$Y_{LL} = AF_{ll}\frac{1}{n} < 0 \tag{100}$$

$$Y_{Ln} = \frac{A}{n}(1-\nu)F_l > 0 \tag{101}$$

$$Y_n = (AF - \phi) + n\left[AF_k\frac{-K}{n^2} + AF_l\frac{-L}{n^2}\right] = (AF-\phi) + [-\nu AF] \tag{102}$$

$$= (1-\nu)AF\left(\frac{K}{n}, \frac{L}{n}\right) - \phi \lesseqgtr 0$$

Returns to scale ν and fixed cost ϕ both reduce the effectiveness of an additional firm and favour an economy with a low number of firms, which can cause $Y_n < 0$ if there are too many firms. Under constant returns to scale $\nu = 0$ and the contribution of an additional firm is to increase output by per firm output $Y_n = y$.

Line (102) uses Euler's homogeneous function theorem. Since the production function $F : \mathbb{R}^2 \supseteq (k,l) \to \mathbb{R}$ has continuous partial derivatives on the open cone (k,l). Then F is homogeneous of degree ν if and only if

$$F_k k + F_l l = \nu F \quad \forall (k,l) \in \mathbb{R}^2$$

Line (97) uses a lemma, since derivatives of a $hod - \nu$ function are $hod - \nu - 1$ then $xF_{xx} + yF_{xy} = (\nu - 1)F_x$.

D Optimal Labour Derivatives

Partially differentiate the intratemporal Euler with respect to each variable treating labour as an implicit function. Then by implicit function theorem get labour responses $\hat{L}(\underset{-}{C}, \underset{+}{K}, \underset{+}{n}; \underset{+}{A}, \underset{-}{\mu})$

$$u_L + u_C(1-\mu)AF_l\left(\frac{K}{n}, \frac{L}{n}\right) = 0 \tag{103}$$

Assumptions: $F_{ll}, u_{CC}, u_{LL} < 0$ and $u_C, F_L, F_{lk} > 0$, where $l = \frac{L}{n}$ and $k = \frac{K}{n}$

$$u_{LL}L_C + u_{CC}(1-\mu)AF_l + u_C(1-\mu)AF_{ll}\frac{L_C}{n} = 0, \quad L_C = \frac{-u_{CC}(1-\mu)AF_l}{u_{LL} + u_C(1-\mu)A\frac{F_{ll}}{n}} < 0 \tag{104}$$

$$u_{LL}L_K + u_C(1-\mu)AF_{lk}\frac{1}{n} + u_C(1-\mu)AF_{ll}\frac{L_K}{n} = 0, \quad L_K = \frac{-u_C(1-\mu)A\frac{F_{lk}}{n}}{u_{LL} + u_C(1-\mu)A\frac{F_{ll}}{n}} > 0 \tag{105}$$

$$u_{LL}L_n + u_C(1-\mu)A\left[F_{lk}\frac{-K}{n^2} + F_{ll}\left(\frac{-L}{n^2} + \frac{L_n}{n}\right)\right] = 0, \quad L_n = \frac{u_C(1-\mu)A(\nu-1)\frac{F_l}{n}}{u_{LL} + u_C(1-\mu)A\frac{F_{ll}}{n}} > 0 \tag{106}$$

$$u_{LL}L_A + u_C(1-\mu)F_l + u_C(1-\mu)AF_{ll}\frac{L_A}{n} = 0, \quad L_A = \frac{-u_C(1-\mu)F_l}{u_{LL} + u_C(1-\mu)A\frac{F_{ll}}{n}} > 0 \tag{107}$$

$$u_{LL}L_\mu + u_c AF_{ll}\frac{L_\mu}{n} - \left(u_C AF_l + u_C\mu AF_{ll}\frac{L_\mu}{n}\right) = 0, \quad L_\mu = \frac{u_C AF_l}{u_{LL} + u_C(1-\mu)A\frac{F_{ll}}{n}} < 0 \tag{108}$$

The intratemporal Euler is shown graphically as the IEP, it shifts left under a rise in μ reflecting that the choice of L is strictly lower.

Assumptions on the functions, given above, are sufficient to determine the signs in all cases except L_n, which depends on returns to scale of the technology ν. With constant returns $\nu = 0$ an entering firm does not affect labour supply; with decreasing returns $\nu < 1$ labour increases; with increasing returns labour decreases.

The denominator is the same in each case, it is the intratemporal condition differentiated with respect to L and it is negative. The negativity, relects that with more labour utility from labour is less (and therefore when valued in terms of consumption, the second component, it is also less). Therefore the numerator distinguishes signs. The numerator is the derivative of the intratemporal condition with respect to the variable of interest, with labour exogenous. It is the derivative of the value of consumption, so the right hand side of $u_L = -u_C(1-\mu)AF_l\left(\frac{K}{n}, \frac{L}{n}\right)$.

E Optimal interest rate, profit, output

Given optimal labour choice we can evaluate how interest rate, profit and output respond.

E.1 Output

$$Y(L, K, n; A, \phi) = n\left[\mathrm{AF}\left(\frac{K}{n}, \frac{L}{n}\right) - \phi\right] \tag{109}$$

$$\hat{Y}_C = AF_l L_C < 0 \tag{110}$$

$$\hat{Y}_K = A(F_k + F_l L_K) > 0 \tag{111}$$

$$\hat{Y}_n = (1-\nu)AF - \phi + AF_l L_n \lesseqgtr 0 \tag{112}$$

Furthermore in steady state when $F(\frac{K}{n}, \frac{L}{n})^* = \frac{\phi}{A(1-(1-\mu)\nu)}$ then $\hat{Y}_n^* = \frac{-\phi\mu\nu}{1-(1-\mu)\nu} + AF_l L_n$ which is positive or negative depending whether the negative component outweighs the positive labour effect.

E.2 Rents

$$r = (1-\mu)AF_k \tag{113}$$

$$r_C = (1-\mu)Y_{KL}L_C = (1-\mu)\frac{A}{n}F_{kl}L_C < 0 \tag{114}$$

$$r_K = (1-\mu)(Y_{KK} + Y_{KL}L_K) = (1-\mu)\frac{A}{n}[F_{kk} + F_{kl}L_K] \lesseqgtr 0 \tag{115}$$

$$r_n = (1-\mu)(Y_{Kn} + Y_{KL}L_n) = (1-\mu)\frac{A}{n}[(1-\nu)F_k + F_{kl}L_n] > 0 \tag{116}$$

E.3 Profit

$$\pi = AF(k, l)(1 - (1-\mu)\nu) - \phi \tag{117}$$

$$\pi_C = AF_l\frac{L_C}{n}(1 - (1-\mu)\nu) < 0 \tag{118}$$

$$\pi_K = \frac{A}{n}(F_k + F_l L_K)(1 - (1-\mu)\nu) > 0 \tag{119}$$

$$\pi_n = \frac{A}{n}(-\nu F + F_l L_n)(1 - (1-\mu)\nu) \lesseqgtr 0 \tag{120}$$

Notice that for an k, l profit is higher so for any given K, L, n profit is higher, but not necessarily for any given K, n which is why in state space fewer firms can arise–which explains the confusion over why despite higher profits we can have a reduction in firms; the answer is higher profits rely on given L.

F Jacobian

$$\begin{bmatrix} \frac{C^*}{\sigma}(1-\mu)\frac{A}{n}F_{kl}L_C & 0 & \frac{C^*}{\sigma}(1-\mu)\frac{A}{n}[F_{kk}+F_{kl}L_K] & \frac{C^*}{\sigma}(1-\mu)\frac{A}{n}[(1-\nu)F_k+F_{kl}L_n] \\ (1-\mu)\frac{A}{n}F_{kl}L_C e-\frac{AF_lL_C(1-(1-\mu)\nu)}{\gamma n} & (1-\mu)AF_k & (1-\mu)\frac{A}{n}[F_{kk}+F_{kl}L_K]e-\Xi & (1-\mu)\frac{A}{n}[(1-\nu)F_k+F_{kl}L_n]e-\Psi \\ AF_lL_C-1 & -\gamma e & A(F_k+F_lL_K) & (1-\nu)AF-\phi+AF_lL_n \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{121}$$

where $\Xi = \frac{A(F_k+F_lL_K)(1-(1-\mu)\nu)}{n\gamma}$ and $\Psi = \frac{A(-\nu F+F_lL_n)(1-(1-\mu)\nu)}{n\gamma}$. Then in steady state we have $e=0$, $F_k = \frac{\rho}{A(1-\mu)}$, and $F = \frac{\phi}{A(1-(1-\mu)\nu)}$

$$\begin{bmatrix} \frac{-u_C}{u_{CC}}(1-\mu)\frac{A}{n}F_{kl}L_C & 0 & \frac{-u_C}{u_{CC}}(1-\mu)\frac{A}{n}[F_{kk}+F_{kl}L_K] & \frac{-u_C}{u_{CC}}(1-\mu)\frac{A}{n}[(1-\nu)F_k+F_{kl}L_n] \\ -\frac{AF_lL_C(1-(1-\mu)\nu)}{\gamma n} & (1-\mu)AF_k & -\frac{A(F_k+F_lL_K)(1-(1-\mu)\nu)}{n\gamma} & -\frac{A(-\nu F+F_lL_n)(1-(1-\mu)\nu)}{n\gamma} \\ AF_lL_C-1 & 0 & A(F_k+F_lL_K) & (1-\nu)AF-\phi+AF_lL_n \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{122}$$

There is a direct effect of μ which appears as $1-\mu$ in several elements of the Jacobian matrix. There is also an indirect effect which changes value of the functions F and u and their derivatives because they are valuated at a different steady state.

The μ parameter has a direct effect in two rows. In the first row the direct effect which is visible as $(1-\mu)$ decreases the absolute value of $\dot{C}$ responses to changes in the four equilibrium variables, so consumption is less prone to change i.e. consumption volatility dampened. In the second row, $(1-\mu)$ raises the absolute value response of $\dot{e}$ to C, K, n changes, and lowers the response to an e change. The change in entry rates are exacerbated, so a period of lots of entry activity, will be followed by a period of less entry relative to the perfect competition case. The direct effect of entry deviating from equilibrium causes a greater change in the rate of entry than if there were perfect competition.

The indirec effect of μ is to change the steady state, and therefore alter where the functions $F_{kk}, F_k, F_l, u_{CC}, u_C$ are evaluated. The labour derivatives, are defined in terms of the other variables earlier in the appendix. But although these marginal effects may increase or decrease their signs and thus the general dynamics will not change.

F.1 Comparative statics

From (63) and (64)

$$F_k(k,l) = \Upsilon, \quad F(k,l) = \Xi \tag{123}$$

$$\text{where, } \Upsilon = \frac{\rho}{A(1-\mu)} \quad \Xi = \frac{\phi}{A(1-(1-\mu)\nu)} \tag{124}$$

Use Cramer's rule to determine the effect of a change in μ. Differentiate with respect to μ

$$F_{kk}k_\mu + F_{kl}l_\mu = \Upsilon_\mu \tag{125}$$
$$F_k k_\mu + F_l l_\mu = \Xi_\mu \tag{126}$$

$$\overbrace{\begin{bmatrix} F_{kk} & F_{kl} \\ F_k & F_l \end{bmatrix}}^{\mathbf{J}} \begin{bmatrix} k_\mu \\ l_\mu \end{bmatrix} = \begin{bmatrix} \Upsilon_\mu \\ \Xi_\mu \end{bmatrix} \tag{127}$$
$$\begin{bmatrix} k_\mu \\ l_\mu \end{bmatrix} = \underbrace{\frac{1}{det(\mathbf{J})} \begin{bmatrix} F_l & -F_{kl} \\ -F_k & F_{kk} \end{bmatrix}}_{\mathbf{J}^{-1}} \begin{bmatrix} \Upsilon_\mu \\ \Xi_\mu \end{bmatrix} \tag{128}$$

$$det(\mathbf{J}) = F_{kk}F_l - F_{kl}F_k < 0, \quad \Upsilon_\mu = \frac{\rho}{A(1-\mu)^2} > 0 \quad \Xi_\mu = -\frac{\phi}{A(1-(1-\mu)\nu)^2\nu} < 0 \tag{129}$$

Hence,

$$k_\mu = \frac{1}{det(\mathbf{J})}\left(F_l\Upsilon_\mu - F_{kl}\Xi_\mu\right) < 0 \tag{130}$$
$$l_\mu = \frac{1}{det(\mathbf{J})}\left(-F_k\Upsilon_\mu + F_{kk}\Xi_\mu\right) \gtrless 0 \quad \text{if} \quad \left|\frac{F_k}{F_{kk}}\right| \lessgtr \left|-\frac{\phi(1-\mu)^2}{\rho(1-(1-\mu)\nu)^2\nu}\right| \tag{131}$$

G Sensitivity to entry

By analysing the implicit functions n and K within our explicit functions, the isoclines $\dot{C}$ and $\dot{e}$ we have[23]

$$\left.\frac{dn}{dK}\right|_{r=\rho} = -\frac{r_K}{r_n} = \frac{F_{kk}}{-(1-\nu)F_k} > 0 \tag{132}$$
$$\left.\frac{dn}{dK}\right|_{\pi=0} = -\frac{\pi_K}{\pi_n} = \frac{F_k}{\nu F} > 0 \tag{133}$$

[23] Given explicit function $F(x,y,z)$ to analyse the implicit function $z = f(x,y)$ we have $\frac{\partial f}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial f}{\partial y} = -\frac{F_y}{F_z}$.

H Euler Frontier in terms budget constraint

The Euler frontier is the steady state budget constraint. It is the budget constraint with $\frac{K^*}{L^*}$ at constant steady state level, unlike the budget constraint where the ratio changes. The normal budget constraint has positive income when labour is zero because there is nonlabour income from capital, but the EF maintains K/L ratio so zero labour means zero capital. Under the normal budget constraint a rise in labour raises consumption by w, but under the Euler frontier a raise in labour means capital increases to maintain their ratio, so there is a rise in w but also capital income.

I Productivity effect

Denote $F(K, L)$ by F, remembering on impact $t = 0$ the state variables K, n are predetermined so do not adjust; however, L adjusts[24]. By the quotient rule differentiate $\frac{Y}{F^{\frac{1}{\nu}}} = \frac{An^{1-\nu}F+n\phi}{F^{\frac{1}{\nu}}}$

$$\frac{d\mathcal{P}(0)}{dA} = \frac{F^{\frac{1}{\nu}}(n^{1-\nu}F + An^{1-\nu}F_L L_A) - (An^{1-\nu}F + n\phi)\frac{1}{\nu}F^{\frac{1}{\nu}-1}F_L L_A}{F^{\frac{2}{\nu}}} \tag{134}$$

$$= F^{-\frac{1}{\nu}}(n^{1-\nu}F + An^{1-\nu}F_L L_A) - (An^{1-\nu}F + n\phi)\frac{1}{\nu}F^{-\frac{1}{\nu}-1}F_L L_A \tag{135}$$

Evaluate impact effect beginning in steady state $\left.\frac{\partial\mathcal{P}(0)}{\partial A}\right|_{K^*,n^*}$. Furthermore in equilibrium $\mathcal{P}^* = (1-\mu)A\nu {n^*}^{1-\nu}F(K^*, L^*)^{1-\frac{1}{\nu}} = (1-\mu)A^{\frac{1}{\nu}}\nu\left(\frac{\phi}{1-(1-\mu)\nu}\right)^{1-\frac{1}{\nu}}$.

[24]See Caputo 2005 p.426

So $\frac{\partial \mathcal{P}^*}{\partial A} = (1-\mu)A^{\frac{1}{\nu}-1}\left(\frac{\phi}{1-(1-\mu)\nu}\right)^{1-\frac{1}{\nu}} = \frac{\mathcal{P}^*}{A\nu}$ Therefore

$$\frac{d\mathcal{P}(0)}{dA} = n^{1-\nu}F^{1-\frac{1}{\nu}} + \frac{\mathcal{P}^*}{(1-\mu)F}F_L L_A - \mathcal{P}^*\frac{1}{\nu F}F_L L_A \tag{136}$$

$$= \frac{\mathcal{P}^*}{(1-\mu)A\nu} + \frac{\mathcal{P}^* F_L L_A}{\nu F}\left(\frac{\mu}{1-\mu}\right) \tag{137}$$

$$= \frac{1}{(1-\mu)}\frac{d\mathcal{P}^*}{dA} + \frac{\mathcal{P}^* F_L L_A}{\nu F}\left(\frac{\mu}{1-\mu}\right) \tag{138}$$

$$= \left(\frac{\mu}{1-\mu}+1\right)\frac{d\mathcal{P}^*}{dA} + \frac{\mathcal{P}^* F_L L_A}{\nu F}\left(\frac{\mu}{1-\mu}\right) \tag{139}$$

$$\frac{d\mathcal{P}(0)}{dA} - \frac{d\mathcal{P}^*}{dA} = \left(\frac{\mu}{1-\mu}\right)\frac{d\mathcal{P}^*}{dA} + \frac{\mathcal{P}^* F_L L_A}{\nu F}\left(\frac{\mu}{1-\mu}\right) \tag{140}$$

$$= \left(\frac{\mu}{1-\mu}\right)\mathcal{P}^*\left(\frac{1}{A\nu} + \frac{F_L L_A}{\nu F}\right) \tag{141}$$

$$= \left(\frac{\mu}{1-\mu}\right)A^{\frac{1}{\nu}}\nu(1-\mu)\left(\frac{\phi}{1-(1-\mu)\nu}\right)^{\frac{\nu-1}{\nu}}\left(\frac{1}{A\nu} + \frac{F_L L_A}{\nu F}\right) \tag{142}$$

$$= \mu\left(\frac{\phi}{A(1-(1-\mu)\nu)}\right)^{1-\frac{1}{\nu}}\left(1 + \frac{AF_L L_A}{F}\right) \tag{143}$$

[25]Using $\frac{d\mathcal{P}^*}{dA} = \frac{\mathcal{P}^*}{A\nu}$ and $Y^* = (1-\mu)A\nu n^{1-\nu}F$ and $Y_L = An^{1-\nu}F_L$

$$\frac{d\mathcal{P}(0)}{dA} = \frac{1}{(1-\mu)}\frac{d\mathcal{P}^*}{dA} + \mu\mathcal{P}^*\frac{Y_L}{Y^*}L_A \tag{144}$$

J Functional forms

The baseline RBC model assumes isoelastic (constant elasticity) separable subutilties and a Cobb-Douglas production function.

J.1 Utility

$$U(C,L) = \frac{C^{1-\sigma}-1}{1-\sigma} - \xi\frac{L^{1+\eta}}{1+\eta} \tag{145}$$

[25]In section (J) we specify functional forms and we could derive $\frac{AF_L L_A}{F} = \frac{\beta}{1+\eta-\beta}$ so the $\left(1+\frac{AF_L L_A}{F}\right)$ component becomes $\frac{1+\eta}{1+\eta+\beta}$

The derivatives are

$$U_C = C^{-\sigma}, \quad U_{CC} = -\sigma C^{-\sigma-1}, \quad U_L = -\xi L^{\eta} \tag{146}$$

The degree of relative risk aversion is constant $\sigma(C) = -C\frac{U_{CC}}{U_C} = \sigma$. Isoelastic utility implies there is constant elasticity of utility with respect to each good. For example, a percent change in consumption always causes a $(1-\sigma)\%$ change in utility. $\sigma \neq 1$ is the constant coefficient of relative risk aversion. $\sigma \to \infty$ implies infinite risk aversion, so consumption has little effect on utility.

J.2 Production

$$F(k,l) = k^{\alpha}l^{\beta} = K^{\alpha}L^{\beta}n^{-(\alpha+\beta)} = F(K,L)n^{-(\alpha+\beta)} \tag{147}$$

$$F_k = \alpha k^{\alpha-1}l^{\beta} = \alpha K^{\alpha-1}L^{\beta}n^{1-(\alpha+\beta)}, \quad F_l = k^{\alpha}\beta l^{\beta-1} = K^{\alpha}\beta L^{\beta-1}n^{1-(\alpha+\beta)} \tag{148}$$

Cobb-Douglas production conforms to our assumptions on the production function derivatives, and it is homogeneous of degree $\alpha+\beta$, so $\nu = \alpha+\beta$ in our general notation. This implies increasing marginal costs if $\alpha+\beta < 1$.

J.3 Canonical Model

Given these function specifications the canonical model is five equations in five variables; four of the equations are differentional equations in C, e, K, n. The fifth is the intratemporal Euler from which we can define $L(K,n,C) = \left(\frac{\xi C^{\sigma}}{(1-\mu)AK^{\alpha}\beta n^{1-(\alpha+\beta)}}\right)^{\frac{1}{\beta-1-\eta}}$ and thus remove L from the system, leaving the four differential equations in four variables (K, n, C, e) and parameters $(A, \alpha, \beta, \phi, \gamma, \sigma, \mu, \rho, \xi, \eta)$ where $\nu = \alpha + \beta$

$$\dot{K} = n\left[AK^{\alpha}L^{\beta}n^{-(\alpha+\beta)} - \phi\right] - \frac{\gamma}{2}e^2 - C \tag{149}$$

$$\dot{n} = e \tag{150}$$

$$\dot{C} = \frac{C}{\sigma}\left[(1-\mu)A\alpha K^{\alpha-1}L^{\beta}n^{1-(\alpha+\beta)} - \rho\right] \tag{151}$$

$$\dot{e} = (1-\mu)A\alpha K^{\alpha-1}L^{\beta}n^{1-(\alpha+\beta)}e - \frac{1}{\gamma}\left(AK^{\alpha}L^{\beta}n^{-(\alpha+\beta)}(1-(1-\mu)\nu) - \phi\right) \tag{152}$$

We can derive k^*, l^* and $C^*(n^*)$. $\dot{e} = 0$ implies steady-state technology $F(\frac{K^*}{n^*}, \frac{L^*}{n^*}) = \frac{\phi}{A(1-(1-\mu)\nu)}$ which via $\dot{C}$ gives k^* and substitute k^* back into $\dot{C}$ to get l^*

$$k^* = \frac{\phi\alpha(1-\mu)}{(1-(1-\mu)\nu)\rho} \tag{153}$$

$$l^* = \left[\frac{1}{A}\left(\frac{\rho}{\alpha}\right)^{\alpha}\left(\frac{1-(1-\mu)\nu}{\phi(1-\mu)}\right)^{1-\alpha}\right]^{\frac{1}{\beta}} \tag{154}$$

The $C(n)$ realtionship follows from using the steady state technology from $\dot{e}$ in $\dot{K}$, (which also confirms $y^* = \frac{\phi(1-\mu)\nu}{1-(1-\mu)\nu}$

$$n^* = \frac{(1-(1-\mu)\nu)}{\nu(1-\mu)\phi}C^* \tag{155}$$

Thus having determined $e = 0$, k^*, l^*, and $C^*(n^*)$, we are left to find n^*. Substituting k^* and l^* in the intratemporal condition gives another $n^*(C^*)$ condition, which may be solved implicitly using the $C(n)$ condition derived from the $\dot{K}$ goods market condition.

Proposition 9.
With Cobb-Douglas production and isoelastic utility, whether an entering firms explands or contracts output depends $\{\alpha, \beta, \eta, \mu\}$

$$Y_n \lesseqgtr 0 \iff \frac{\beta(1-(\alpha+\beta))}{(\alpha+\beta)(1+\eta-\beta)} \lesseqgtr \mu \tag{156}$$

which comes from $Y_n^* = \frac{-\phi\mu(\alpha+\beta)(1+\eta-\beta)+\phi\beta(1-(\alpha+\beta))}{(1-(1-\mu)(\alpha+\beta))(1+\eta-\beta)}$

Consider a numerical example, $\alpha = 0.3$, $\beta = 0.5$, $\eta = 0$ implies $0.25 < \mu$ is sufficient for an entrant to contract output $Y_n^* < 0$. As intertemporal elasticity of labour η increases less market power μ is necessary. For example, with log utility in labour $\eta = 1$ then $0.125 < \mu$ is sufficient. So for a typical calibration an additional firm contracts output as we normally take μ much higher than either of these two examples. One can also see that if $\mu = 0$ then $Y_n > 0$ and if there are constant returns $\alpha + \beta = 1$ then $Y_n < 0$.

J.4 Cost function

Static optimization problem so drop time subscripts

$$C(r, w, y) = \min_{l,k} wl + rk + \phi \quad \text{s.t.} y \leqslant Ak^{\alpha}l^{\beta} - \phi \tag{157}$$

With Cobb-Douglas production the total cost function from substituting Lagrangean obtained conditional input demands $k(r,w,y) = \left[\left(\frac{w\alpha}{r\beta}\right)^{\beta}\left(\frac{y+\phi}{A}\right)\right]^{\frac{1}{\alpha+\beta}}$ and $l(r,w,y) = \left[\left(\frac{r\beta}{w\alpha}\right)^{\alpha}\left(\frac{y+\phi}{A}\right)\right]^{\frac{1}{\alpha+\beta}}$ into the cost function is

$$C(r,w,y) = (\alpha+\beta)\left(\frac{y+\phi}{A}\right)^{\frac{1}{\alpha+\beta}}\left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{w}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \phi \tag{158}$$

Where the firm takes factor prices as given. The average cost $\frac{C}{y}$ is U-shaped and the marginal cost $\frac{dC}{dy}$ is increasing.

References

Berentsen, Aleksander and Christopher J. Waller (2009). *Optimal stabilization policy with endogenous firm entry*. Working Papers 2009-032. Federal Reserve Bank of St. Louis.

Bilbiie, Florin O., Fabio Ghironi, and Marc J. Melitz (2012). "Endogenous Entry, Product Variety, and Business Cycles". In: *Journal of Political Economy* 120.2, pp. 304 –345.

Brito, Paulo and Huw Dixon (2013). "Fiscal policy, entry and capital accumulation: Hump-shaped responses". In: *Journal of Economic Dynamics and Control* 37.10, pp. 2123 –2155.

Caputo, M.R. (2005). *Foundations of Dynamic Economic Analysis: Optimal Control Theory and Applications*. Cambridge University Press.

Chamberlin, E. (1933). *The theory of monopolistic competition*. Harvard economic studies ... vol: XXXVIII. Harvard University Press.

Das, Sanghamitra and Satya P. Das (1997). "Dynamics of entry and exit of firms in the presence of entry adjustment costs". In: *International Journal of Industrial Organization* 15.2, pp. 217 –241.

d'Aspremont, Claude, Rodolphe Dos Santos Ferreira, and Louis-Andre Gerard-Varet (Mar. 1997). "General Equilibrium Concepts under Imperfect Competition: A Cournotian Approach". In: *Journal of Economic Theory* 73.1, pp. 199–230.

Datta, Bipasa and Huw Dixon (May 2002). "Technological Change, Entry, and Stock-Market Dynamics: An Analysis of Transition in a Monopolistic Industry". In: *American Economic Review* 92.2, pp. 231–235.

Dixit, Avinash K and Joseph E Stiglitz (1977). "Monopolistic competition and optimum product diversity". In: *The American Economic Review* 67.3, pp. 297–308.

Etro, Federico and Andrea Colciago (Dec. 2010). "Endogenous Market Structures and the Business Cycle". In: *Economic Journal* 120.549, pp. 1201–1233.

Feichtinger, Gustav, Andreas Novak, and Franz Wirl (Mar. 1994). "Limit cycles in intertemporal adjustment models : Theory and applications". In: *Journal of Economic Dynamics and Control* 18.2, pp. 353–380.

Lewis, Vivien (Nov. 2009). "Business Cycle Evidence On Firm Entry". In: *Macroeconomic Dynamics* 13.05, pp. 605–624.

Luttmer, Erzo G.J. (2012). "Technology diffusion and growth". In: *Journal of Economic Theory* 147.2, pp. 602–622.

Rossi-Hansberg, Esteban and Mark L. J. Wright (Dec. 2007). "Establishment Size Dynamics in the Aggregate Economy". In: *American Economic Review* 97.5, pp. 1639–1666.