Income Distribution and Macroeconomics

Oded Galor and Joseph Zeira

The Galor-Zeira Model

The Galor-Zeira Model

- Overlapping-Generations economy

The Galor-Zeira Model

- Overlapping-Generations economy
- $t = 0, 1, 2, 3, \ldots$

The Galor-Zeira Model

- Overlapping-Generations economy
- $t = 0, 1, 2, 3, \ldots$
- One good

The Galor-Zeira Model

- Overlapping-Generations economy
- $t = 0, 1, 2, 3, ...$
- One good
- 3 factors:

The Galor-Zeira Model

- Overlapping-Generations economy
- $t = 0, 1, 2, 3, \ldots$
- One good
- 3 factors:
  - $K \equiv$ Physical capital

The Galor-Zeira Model

- Overlapping-Generations economy
- $t = 0, 1, 2, 3, \ldots$
- One good
- 3 factors:
  - $K \equiv$ Physical capital
  - $L^s \equiv$ Skilled Labor

The Galor-Zeira Model

- Overlapping-Generations economy
- $t = 0, 1, 2, 3, \ldots$
- One good
- 3 factors:
  - $K \equiv$ Physical capital
  - $L^s \equiv$ Skilled Labor
  - $L^u \equiv$ Unskilled Labor

Production

Production

Total output produced

$$Y_t = Y_t^s + Y_t^u$$

Production

Total output produced

$$Y_t = Y_t^s + Y_t^u$$

- Production in the skilled-intensive sector:

$$Y_t^s = F(K_t, L_t^s) \equiv L_t^s f(k_t); \qquad k_t \equiv K_t / L_t^s$$

Production

Total output produced

$$Y_t = Y_t^s + Y_t^u$$

- Production in the skilled-intensive sector:

$$Y_t^s = F(K_t, L_t^s) \equiv L_t^s f(k_t); \qquad k_t \equiv K_t / L_t^s$$

- Production in the unskilled-intensive sector:

$$Y_t^u = aL_t^u$$

Inverse Demand for Factors

- Capital:

$$r_t = f'(k_t) \equiv r(k_t)$$

Inverse Demand for Factors

- Capital:

$$r_t = f'(k_t) \equiv r(k_t)$$

- Skilled labor:

$$w_t^s = f(k_t) - f'(k_t)k_t \equiv w^s(k_t)$$

Inverse Demand for Factors

- Capital:

$$r_t = f'(k_t) \equiv r(k_t)$$

- Skilled labor:

$$w_t^s = f(k_t) - f'(k_t)k_t \equiv w^s(k_t)$$

- Unskilled labor:

$$w_t^u = a \equiv w^u$$

Factor Prices

- Small open economy

Factor Prices

- Small open economy
- World interest $= r$

Factor Prices

- Small open economy
- World interest $= r$

$\Longrightarrow$

$$\begin{aligned} r_t &= r \\ k_t &= f'^{-1}(r) \equiv k \\ w_t^s &= w^s(k) \equiv w^s \end{aligned}$$

Factor Prices

- Small open economy
- World interest $= r$

$$\Longrightarrow$$

$$\begin{aligned} r_t &= r \\ k_t &= f'^{-1}(r) \equiv k \\ w_t^s &= w^s(k) \equiv w^s \end{aligned}$$

$$\Longrightarrow$$

$$(r_t, w_t^s, w_t^u) = (r, w^s, w^u) \qquad \forall t$$

Individuals

Individuals

- Continuum of measure 1

Individuals

- Continuum of measure 1
- Each Individual has 1 parent and 1 child

Individuals

- Continuum of measure 1
- Each Individual has 1 parent and 1 child
- Identical in:

  Preferences

  Innate abilities

Individuals

- Continuum of measure 1
- Each Individual has 1 parent and 1 child
- Identical in:

  Preferences

  Innate abilities

- Differ in:

  Parental income $\Rightarrow$ Inv't in HC

Member of Generation t: Period of Life

Member of Generation t: Period of Life

- First period of life (Period t):
  - [invest in HC] or [work as unskilled]

Member of Generation t: Period of Life

- First period of life (Period t):
  - [invest in HC] or [work as unskilled]
- Second period of life (Period $t+1$):
  - [work as unskilled / no inv't in HC] or [work as skilled / inv't in HC in 1st period]

Member of Generation t: Endowment and Preferences

Member of Generation t: Endowment and Preferences

- Time endowment:
  - 1 units of time in each period

Member of Generation t: Endowment and Preferences

- Time endowment:
  - 1 units of time in each period
- Capital endowment:
  - $b_t \equiv$ capital inherited in 1^{st} period

Member of Generation t: Endowment and Preferences

- Time endowment:
  - 1 units of time in each period
- Capital endowment:
  - $b_t \equiv$ capital inherited in 1^{st} period
- Preferences:

$$u^t = \alpha \ln c_{t+1} + (1-\alpha) \ln b_{t+1} \qquad \alpha \in (0,1)$$

Member of Generation t: Budget Constraint

Second period budget constraint:

$$c_{t+1} + b_{t+1} \leq \omega_{t+1}$$

Member of Generation t: Budget Constraint

Second period budget constraint:

$$c_{t+1} + b_{t+1} \leq \omega_{t+1}$$

$c_{t+1} \equiv$ consumption

Member of Generation t: Budget Constraint

Second period budget constraint:

$$c_{t+1} + b_{t+1} \leq \omega_{t+1}$$

$c_{t+1} \equiv$ consumption

$b_{t+1} \equiv$ transfers to offspring

Member of Generation t: Budget Constraint

Second period budget constraint:

$$c_{t+1} + b_{t+1} \leq \omega_{t+1}$$

$c_{t+1} \equiv$ consumption

$b_{t+1} \equiv$ transfers to offspring

$\omega_{t+1} \equiv$ wealth in period $t+1$

Member of Generation t: Optimization

$$\{c_{t+1}, b_{t+1}\} = \arg\max[\alpha \ln c_{t+1} + (1-\alpha)\ln b_{t+1}]$$

$$\text{s.t.} \qquad c_{t+1} + b_{t+1} \leq \omega_{t+1}$$

Member of Generation t: Optimization

$$b_{t+1} = (1-\alpha)\omega_{t+1}$$

$$c_{t+1} = \alpha\omega_{t+1}$$

Member of Generation t: Optimization

$$b_{t+1} = (1-\alpha)\omega_{t+1}$$

$$c_{t+1} = \alpha\omega_{t+1}$$

Indirect Utility: $\Longrightarrow$

$$v^t = \alpha \ln \alpha\omega_{t+1} + (1-\alpha) \ln \omega_{t+1}$$

Member of Generation t: Optimization

$$b_{t+1} = (1-\alpha)\omega_{t+1}$$

$$c_{t+1} = \alpha\omega_{t+1}$$

Indirect Utility: $\Longrightarrow$

$$\begin{aligned} v^t &= \alpha \ln \alpha\omega_{t+1} + (1-\alpha)\ln \omega_{t+1} \\ &= [\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)] + \ln \omega_{t+1} \end{aligned}$$

Member of Generation t: Optimization

$$b_{t+1} = (1-\alpha)\omega_{t+1}$$

$$c_{t+1} = \alpha\omega_{t+1}$$

Indirect Utility: $\Longrightarrow$

$$\begin{aligned} v^t &= \alpha \ln \alpha\omega_{t+1} + (1-\alpha)\ln \omega_{t+1} \\ &= [\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)] + \ln \omega_{t+1} \end{aligned}$$

$\Longrightarrow$ v^t is monotonic increasing in 2nd period wealth, ω_{t+1}

Member of Generation t: Optimization

$$b_{t+1} = (1-\alpha)\omega_{t+1}$$

$$c_{t+1} = \alpha\omega_{t+1}$$

Indirect Utility: $\Longrightarrow$

$$\begin{aligned} v^t &= \alpha \ln \alpha\omega_{t+1} + (1-\alpha) \ln \omega_{t+1} \\ &= [\alpha \ln \alpha + (1-\alpha) \ln(1-\alpha)] + \ln \omega_{t+1} \end{aligned}$$

$\Longrightarrow$ v^t is monotonic increasing in 2nd period wealth, ω_{t+1}
$\Longrightarrow$ maximization of ω_{t+1}, is the basis of occupational choices

Fundamental Assumptions

- Imperfect Capital Markets:

$$r < i \tag{A1}$$

$r \equiv$ interest rate for lender

$i \equiv$ interest rate for borrowers (for inv't in HC)

Fundamental Assumptions

- Imperfect Capital Markets:

$$r < i \tag{A1}$$

$r \equiv$ interest rate for lender

$i \equiv$ interest rate for borrowers (for inv't in HC)

- Fixed cost of education (Indivisibility of inv't in HC) Weighted average of the payments to teachers, administrators, and maintenance workers in the school system (i.e., weighted average of the wages skilled and unskilled workers):

$$C^H = \theta w^s + (1-\theta) w^u \equiv h > 0 \qquad \theta \in [0,1] \tag{A2}$$

Income: Unskilled Individuals

Income: Unskilled Individuals

$$\omega_{t+1}^{u} = (w^{u} + b_t)(1 + r) + w^{u}$$

Income: Unskilled Individuals

$$
\begin{aligned}
\omega_{t+1}^{u} &= (w^{u} + b_t)(1+r) + w^{u} \\
&= w^{u}(2+r) + (1+r)b_t
\end{aligned}
$$

Income: Skilled Individuals

$$\omega_{t+1}^{s} = \begin{cases} w^s - (h - b_t)(1+i) & \text{if} \quad b_t \leq h \\ w^s + (b_t - h)(1+r) & \text{if} \quad b_t \geq h \end{cases}$$

Income: Skilled Individuals

$$\omega_{t+1}^s = \begin{cases} w^s - (h - b_t)(1+i) & \text{if} \quad b_t \leq h \\ w^s + (b_t - h)(1+r) & \text{if} \quad b_t \geq h \end{cases}$$

$\Longrightarrow$

$$\omega_{t+1}^s = \begin{cases} w^s - (1+i)h + (1+i)b_t & \text{if} \quad b_t \leq h \\ w^s - (1+r)h + (1+r)b_t & \text{if} \quad b_t \geq h \end{cases}$$

Assumptions

- Investment in human capital is *not* beneficial for individuals who must finance the entire cost of education via borrowing

$$w^s - (1+i)h < 0 \tag{A3}$$

Assumptions

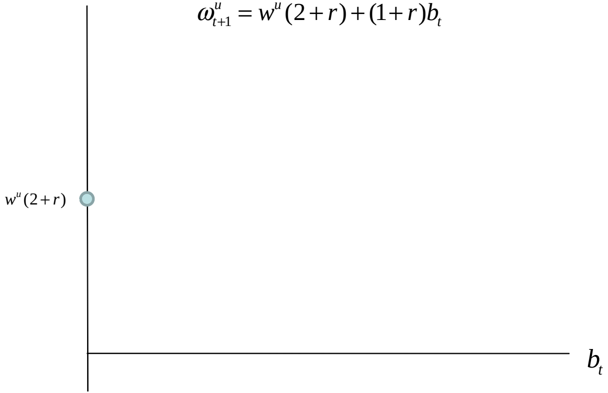
- Investment in human capital is *not* beneficial for individuals who must finance the entire cost of education via borrowing

$$w^s - (1+i)h < 0 \tag{A3}$$

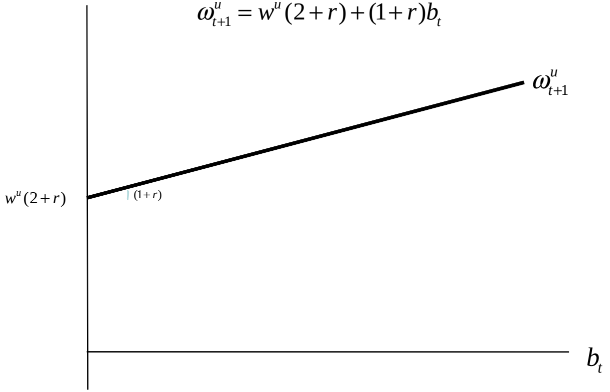
- Investment in human capital is beneficial for individuals who can finance the entire cost of education *without* borrowing

$$w^s - (1+r)h > w^u(2+r) \tag{A4}$$

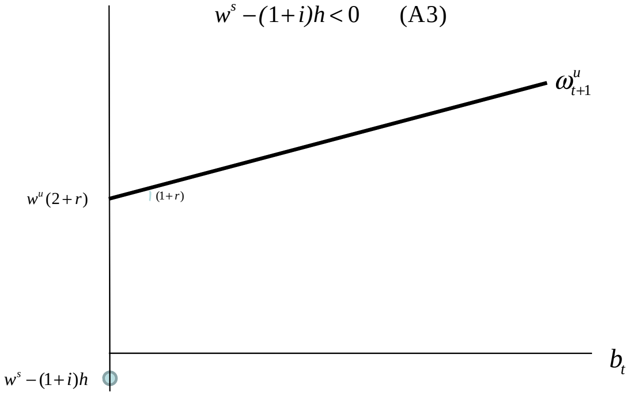
Income from Being Unskilled Worker



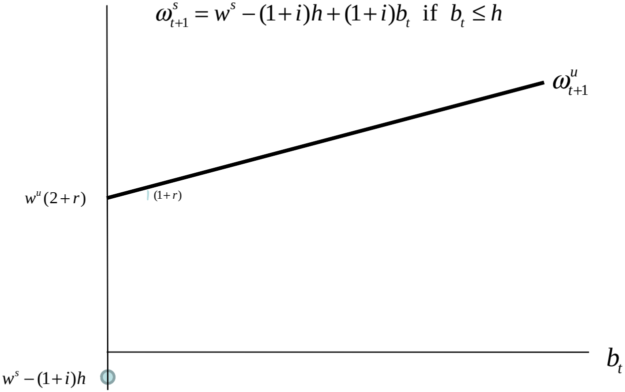
Income from Being Unskilled Worker



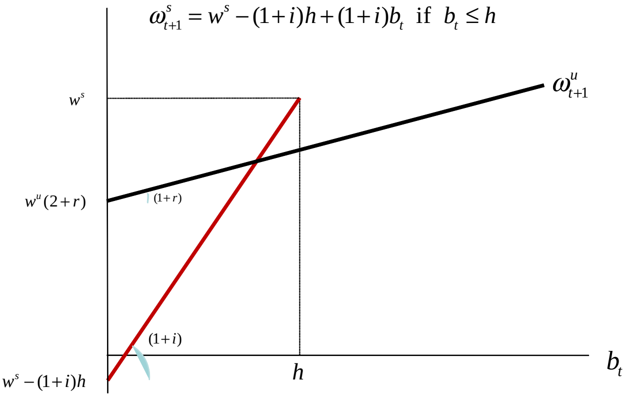
Income from Being Skilled Worker: Borrowers



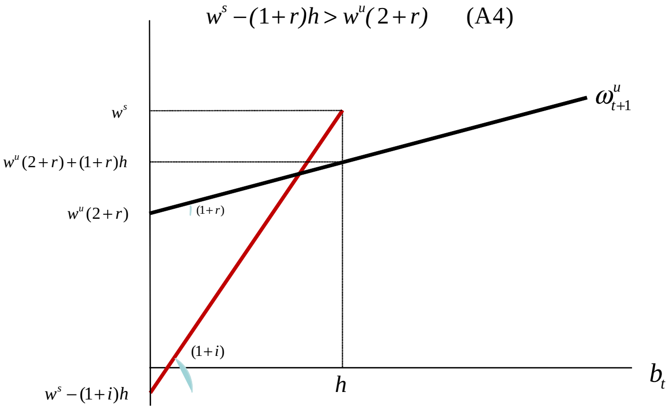
Income from Being Skilled Worker: Borrowers



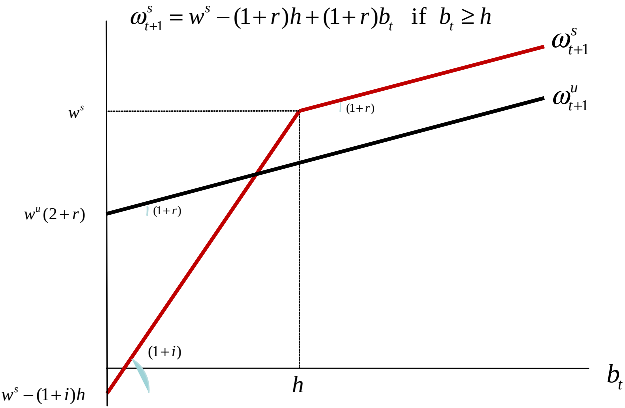
Income from Being Skilled Worker: Borrowers



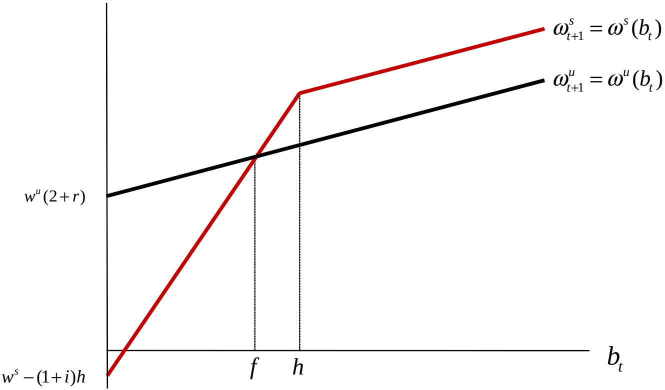
Income from Being Skilled Worker: Borrowers



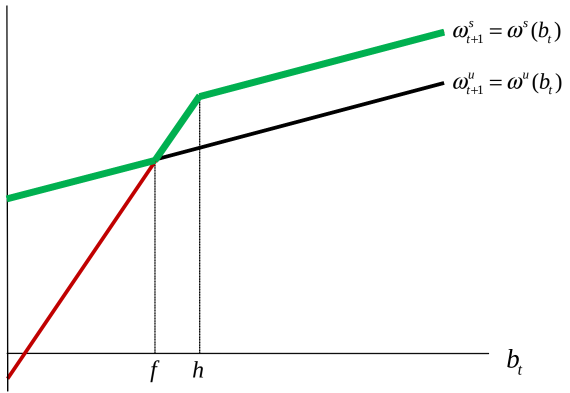
Income from Being Skilled Worker: Lenders



Bequest and Occupational Choice



Bequest and Occupational Choice



Bequest and Occupational Choice

$$b_t \quad \begin{cases} < f & \rightarrow x^u_{t+1} > x^s_{t+1} \text{ (individual } t \text{ becomes unskilled)} \\ > f & \rightarrow x^u_{t+1} < x^s_{t+1} \text{ (individual } t \text{ becomes skilled)} \end{cases}$$

Bequest and Occupational Choice

$$b_t \quad \begin{cases} < f & \rightarrow x^u_{t+1} > x^s_{t+1} \text{ (individual } t \text{ becomes unskilled)} \\ > f & \rightarrow x^u_{t+1} < x^s_{t+1} \text{ (individual } t \text{ becomes skilled)} \end{cases}$$

where

$$f = \frac{w^u(2+r) - [w^s - (1+i)h]}{i-r} > 0$$

Bequest Dynamics

$$b_{t+1} = (1 - \alpha)x_{t+1}$$

Bequest Dynamics

$$b_{t+1} = (1-\alpha)x_{t+1}$$

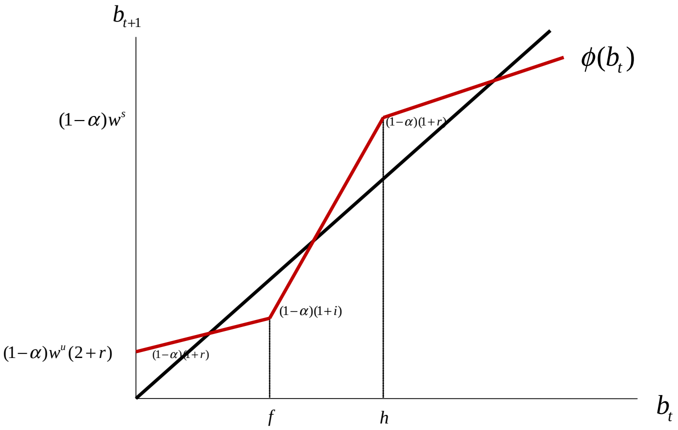
$$b_{t+1} = \begin{cases} (1-\alpha)\left[w^u(2+r) + (1+r)b_t\right] & b_t \in [0, f] \\ (1-\alpha)\left[w^s - (1+i)h + (1+i)b_t\right] & b_t \in [f, h] \\ (1-\alpha)\left[w^s - (1+r)h + (1+r)b_t\right] & b_t \in [h, \infty] \end{cases}$$

Bequest Dynamics: Sufficiet Conditions for Multiplicity of Steady-Sate

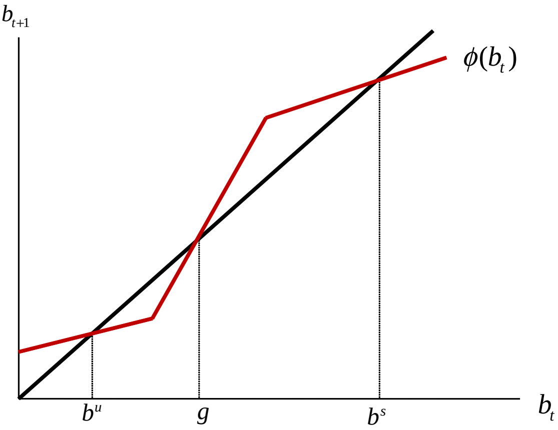
$$\begin{aligned}(1-\alpha)(1+r) &< 1 \\ (1-\alpha)(1+i) &> 1\end{aligned} \tag{A5}$$

$$(1-\alpha)w^s > h \tag{A6}$$

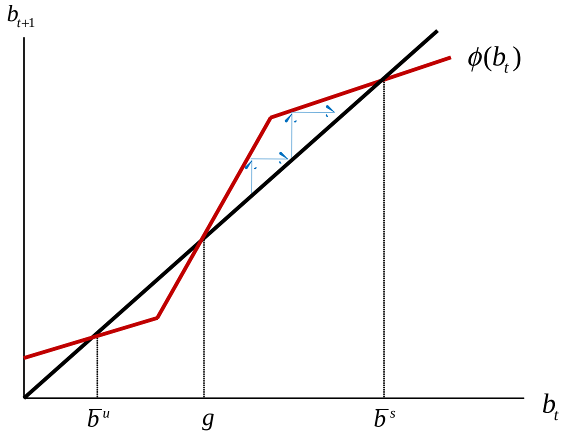
Bequest Dynamics



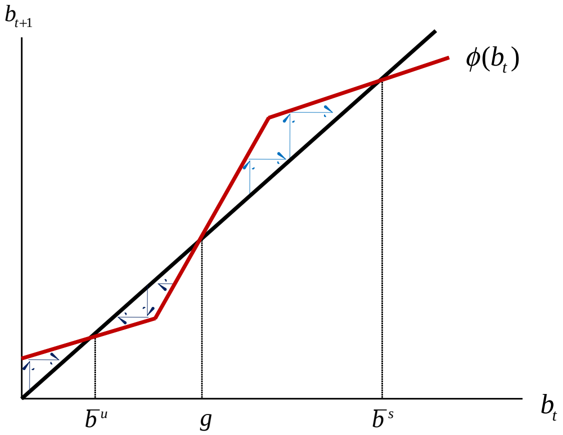
Bequest Dynamics: Multiple Steady-State Equilibrium



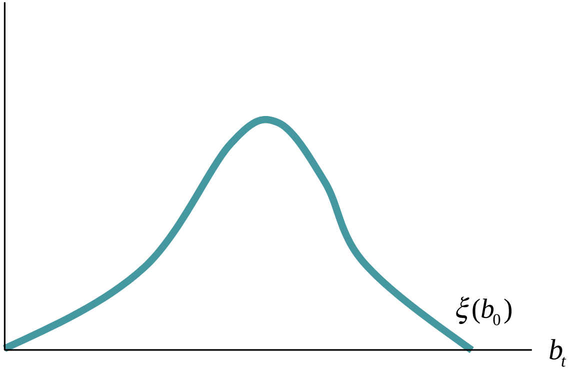
Bequest Dynamics: Stability of High Bequest Equilibrium



Bequest Dynamics: Stability of Steady- State Equilibria



The Distribution of the Inheritance in Period t



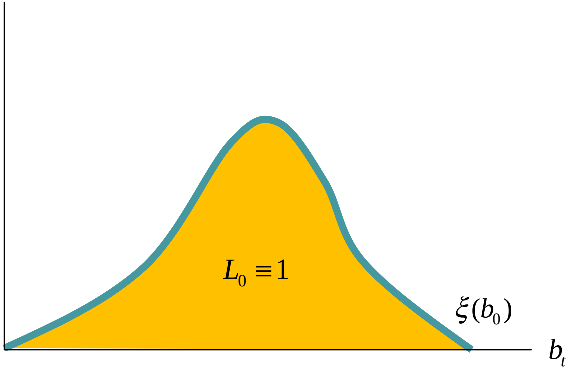
Income Distribution and the Long Run Decomposition of the Labor Force

$\xi_t(b_t) \equiv$ Distribution of inheritance at time t

$\Longrightarrow$

$$L_t = \int_0^\infty \xi(b_t) db_t \equiv 1$$

The Distribution of the Inheritance in Period t



Income Distribution of the Long Run Decomposition of the Labor Force

$$\lim_{t\to\infty} l_t^u = \int_0^g \xi_t(b_t)db_t \equiv \bar{l}^u$$

$$\lim_{t\to\infty} l_t^s = \int_g^\infty \xi_t(b_t)db_t \equiv \bar{l}^s$$

Income Distribution of the Long Run Decomposition of the Labor Force

$$\lim_{t\to\infty} l_t^u = \int_0^g \xi_t(b_t)db_t \equiv \bar{l}^u$$

$$\lim_{t\to\infty} l_t^s = \int_g^\infty \xi_t(b_t)db_t \equiv \bar{l}^s$$

where

$$\partial \bar{l}^s / \partial g < 0$$

Income Distribution of the Long Run Decomposition of the Labor Force

$$\lim_{t\to\infty} l_t^u = \int_0^g \xi_t(b_t)db_t \equiv \bar{l}^u$$

$$\lim_{t\to\infty} l_t^s = \int_g^\infty \xi_t(b_t)db_t \equiv \bar{l}^s$$

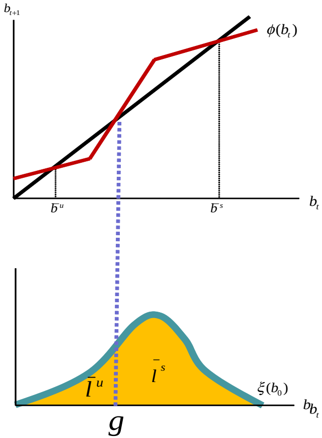
where

$$\partial\bar{l}^s/\partial g < 0$$

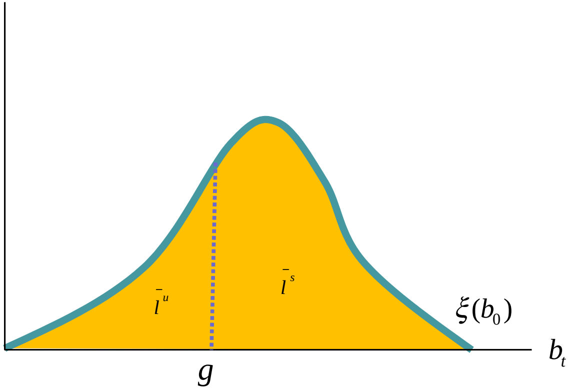
and

$$g = \frac{(1-\alpha)[(1+i)h - w^s]}{(1-\alpha)(1+i) - 1} > 0$$

Income Distribution of Skill Composition



Income Distribution of Skill Composition



Income Per Capita in the Long Run

- Income of a skilled individual in the second period of life (wage and capital income)

$$I_2^s = w^s + (\bar{b}^s - h)r$$

Income Per Capita in the Long Run

- Income of a skilled individual in the second period of life (wage and capital income)

$$I_2^s = w^s + (\bar{b}^s - h)r$$

- Income of an unskilled individual in the second period of life (wage and capital income)

$$I_2^u = w^u + (\bar{b}^u + w^u)r$$

Income Per Capita in the Long Run

- Income of a skilled individual in the second period of life (wage and capital income)

$$I_2^s = w^s + (\bar{b}^s - h)r$$

- Income of an unskilled individual in the second period of life (wage and capital income)

$$I_2^u = w^u + (\bar{b}^u + w^u)r$$

- Income of an unskilled individual in the first period of life (only wage income)

$$I_1^u = w^u$$

Income Per Capita in the Long Run

- Aggregate income in the steady-state

$$\bar{Y} = l_2^s \bar{I}^s + l_2^u \bar{I}^u + l_1^u \bar{I}^u$$

Income Per Capita in the Long Run

- Aggregate income in the steady-state

$$\bar{Y} = I_2^s \bar{l}^s + I_2^u \bar{l}^u + I_1^u \bar{l}^u$$

- Aggregate income (note: $\bar{l}^s + \bar{l}^u = 1$)

$$\begin{aligned} Y &= [w^s - rh + r\bar{b}^s]\bar{l}^s + [w^u(2+r) + r\bar{b}^u](1 - \bar{l}^s) \\ &= w^u(2+r) + r\bar{b}^u + [(w^s - rh) - w^u(2+r) + (\bar{b}^s - \bar{b}^u)]\bar{l}^s \end{aligned}$$

Income Per Capita in the Long Run

- Aggregate income in the steady-state

$$\bar{Y} = I_2^s \bar{l}^s + I_2^u \bar{l}^u + I_1^u \bar{l}^u$$

- Aggregate income (note: $\bar{l}^s + \bar{l}^u = 1$)

$$\begin{aligned} Y &= [w^s - rh + r\bar{b}^s]\bar{l}^s + [w^u(2+r) + r\bar{b}^u](1 - \bar{l}^s) \\ &= w^u(2+r) + r\bar{b}^u + [(w^s - rh) - w^u(2+r) + (\bar{b}^s - \bar{b}^u)]\bar{l}^s \end{aligned}$$

- Income per capita

$$\bar{y} = \bar{Y}/2$$

Skill Composition and Income Per Capita in the Long Run

- An increase in the fraction of skilled workers increases income per capita in the steady-state

$$\frac{\partial \bar{y}}{\partial \bar{l}^s} = [(w^s - rh) - w^u(2+r) + (\bar{b}^s - \bar{b}^u)]/2 > 0$$

Skill Composition and Income Per Capita in the Long Run

- An increase in the fraction of skilled workers increases income per capita in the steady-state

$$\frac{\partial \bar{y}}{\partial \bar{l}^s} = [(w^s - rh) - w^u(2+r) + (\bar{b}^s - \bar{b}^u)]/2 > 0$$

since

$$w^s - (1+r)h > w^u(2+r)$$

$$\bar{b}^s > \bar{b}^u$$

Skill Composition and Income Per Capita in the Long Run

- An increase in the fraction of skilled workers increases income per capita in the steady-state

$$\frac{\partial \bar{y}}{\partial \bar{l}^s} = [(w^s - rh) - w^u(2+r) + (\bar{b}^s - \bar{b}^u)]/2 > 0$$

since

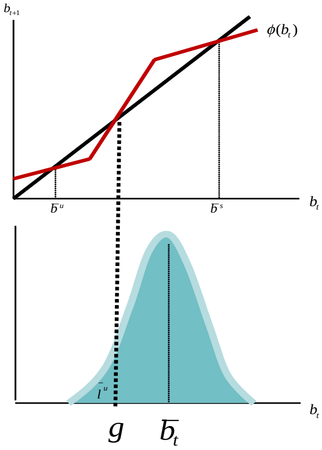
$$w^s - (1+r)h > w^u(2+r)$$

$$\bar{b}^s > \bar{b}^u$$

- An increase in g reduces income per capita in the steady-state

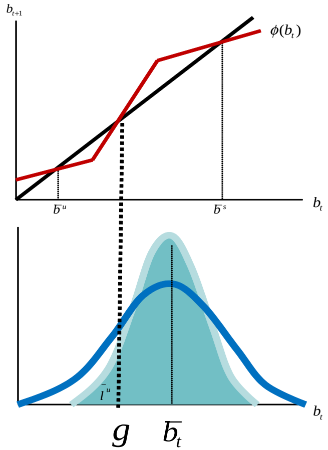
$$\frac{\partial \bar{y}}{\partial g} = \frac{\partial \bar{y}}{\partial \bar{l}^s} \frac{\partial \bar{l}^s}{\partial g} < 0$$

Inequality and Development: Rich Economies

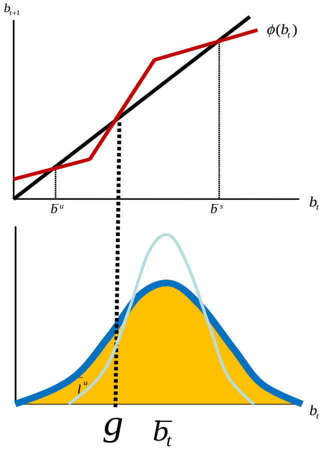


Rich Economies: Inequality is Harmful for Development

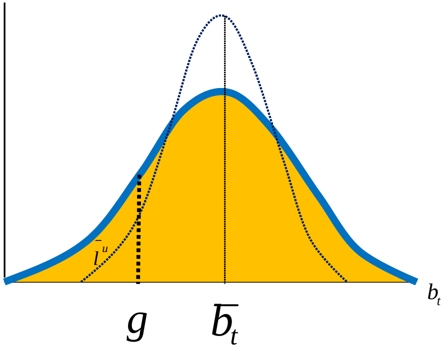
Inequality reduces human capital formation



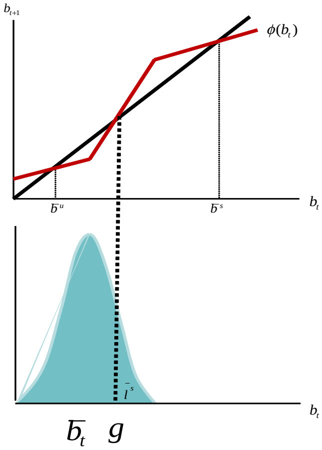
Rich Economies: Inequality is Harmful for Development



Rich Economies: Inequality is Harmful for Development

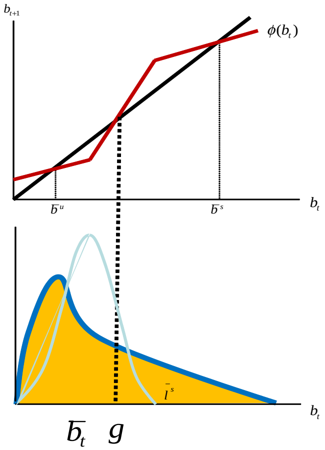


Inequality and Development: Poor Economies

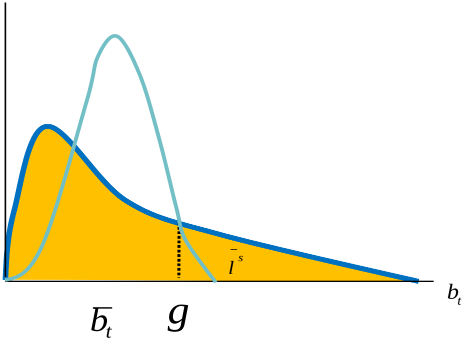


Poor Economies: Inequality may Benefit Development

Inequality stimulates human capital formation



Poor Economies: Inequality may Benefit Development



Robustness

The qualitative results are robust to:

- Education cost that is indexed to wages

Robustness

The qualitative results are robust to:

- Education cost that is indexed to wages
- Labor augmenting technical change

Robustness

The qualitative results are robust to:

- Education cost that is indexed to wages
- Labor augmenting technical change
- Shocks the outcome of investment in human capital, as long as wages are endogenous

Robustness

The qualitative results are robust to:

- Education cost that is indexed to wages
- Labor augmenting technical change
- Shocks the outcome of investment in human capital, as long as wages are endogenous
- Concave production function of human capital (Moav (EL, 2002), Galor-Moav (RES, 2004))

Robustness: Technological Progress and Endogenous Education Cost

Labor Augmenting Technological Progress: increases the productivity of workers in both the skilled-intensive and the unskilled intensive sector.

- Production in the skilled-intensive sector

$$Y_t^s = F(K_t, A_t L_t^s) \equiv A_t L_t^s f(k_t); \qquad k_t \equiv K_t / A_t L_t^s$$

Robustness: Technological Progress and Endogenous Education Cost

Labor Augmenting Technological Progress: increases the productivity of workers in both the skilled-intensive and the unskilled intensive sector.

- Production in the skilled-intensive sector

$$Y_t^s = F(K_t, A_t L_t^s) \equiv A_t L_t^s f(k_t); \qquad k_t \equiv K_t / A_t L_t^s$$

- Production in the unskilled-intensive sector

$$Y_t^u = A_t a L_t^u$$

Robustness: Technological Progress and Endogenous Education Cost

Labor Augmenting Technological Progress: increases the productivity of workers in both the skilled-intensive and the unskilled intensive sector.

- Production in the skilled-intensive sector

$$Y_t^s = F(K_t, A_t L_t^s) \equiv A_t L_t^s f(k_t); \qquad k_t \equiv K_t / A_t L_t^s$$

- Production in the unskilled-intensive sector

$$Y_t^u = A_t a L_t^u$$

- Technological progress

$$A_{t+1} = (1 + \lambda) A_t \qquad \lambda > 0.$$

Robustness: Technological Progress and Endogenous Education Cost

Factor Prices

$$w_t^s = A_t[f(k) - f'(k)k] \equiv A_t w^s$$

$$w_t^u = A_t a \equiv A_t w^u$$

$$r_t = r$$

Cost of Education

- Weighted average of the payments to teachers, administrators, and maintenance workers in the school system

Cost of Education

- Weighted average of the payments to teachers, administrators, and maintenance workers in the school system
- $\Rightarrow$ Weighted average of the wages skilled and unskilled workers

Cost of Education

- Weighted average of the payments to teachers, administrators, and maintenance workers in the school system
- $\Rightarrow$ Weighted average of the wages skilled and unskilled workers

$$C_t^H = \theta A_t w^s + (1-\theta) A_t w^u \equiv A_t h$$

Income: Unskilled Individuals

$$x_{t+1}^{u} = (A_t w^u + b_t)(1+r) + A_{t+1} w^u$$

Income: Unskilled Individuals

$$\begin{aligned} x^u_{t+1} &= (A_t w^u + b_t)(1+r) + A_{t+1} w^u \\ &= A_t w^u (2 + r + \lambda) + (1+r) b_t \end{aligned}$$

Income: Skilled Individuals

$$x_{t+1}^{s} = \begin{cases} A_{t+1} w^{s} - (A_t h - b_t)(1+i) & \textit{if} \quad b_t \leq A_t h \\ A_{t+1} w^{s} + (b_t - A_t h)(1+r) & \textit{if} \quad b_t \geq A_t h \end{cases}$$

Income: Skilled Individuals

$$x_{t+1}^{s} = \begin{cases} A_{t+1}w^{s} - (A_t h - b_t)(1+i) & \textit{if} \quad b_t \leq A_t h \\ A_{t+1}w^{s} + (b_t - A_t h)(1+r) & \textit{if} \quad b_t \geq A_t h \end{cases}$$

$\Longrightarrow$

$$x_{t+1}^{s} = \begin{cases} A_t[w^{s}(1+\lambda) - (1+i)h] + (1+i)b_t & \textit{if} \quad b_t \leq A_t h \\ A_t[w^{s}(1+\lambda) - (1+r)h] + (1+r)b_t & \textit{if} \quad b_t \geq A_t h \end{cases}$$

Threshold level of Bequest for Becoming Skilled Worker in Period t

$$f = \frac{A_t\{w^u(2+r) - [w^s - (1+i)h] - \lambda(w^s - w^u)\}}{(i-r)}$$

Threshold level of Bequest for Becoming Skilled Worker in Period t

$$f = \frac{A_t\{w^u(2+r) - [w^s - (1+i)h] - \lambda(w^s - w^u)\}}{(i-r)}$$

$$\frac{f_t}{A_t} = \frac{A_t\{w^u(2+r) - [w^s - (1+i)h] - \lambda(w^s - w^u)\}}{(i-r)} \equiv \hat{f} > 0$$

Threshold level of Bequest for Becoming Skilled Worker in Period t

$$f = \frac{A_t\{w^u(2+r) - [w^s - (1+i)h] - \lambda(w^s - w^u)\}}{(i-r)}$$

$$\frac{f_t}{A_t} = \frac{A_t\{w^u(2+r) - [w^s - (1+i)h] - \lambda(w^s - w^u)\}}{(i-r)} \equiv \hat{f} > 0$$

for

$$w^u(2+r) > [w^s - (1+i)h] + \lambda(w^s - w^u)$$

Bequest Dynamics

$$b_{t+1} = \begin{cases} (1-\alpha)\{A_t w^u(2+r+\lambda) + (1+r)b_t\} & b_t \in [0, f] \\ (1-\alpha)\{A_t [w^s(1+\lambda) - (1+i)h] + (1+i)b_t\} & b_t \in [f, A_t h] \\ (1-\alpha)\{A_t [w^s(1+\lambda) - (1+r)h] + (1+r)b_t\} & b_t \in [A_t h, \infty] \end{cases}$$

Bequest Dynamics

Let $\hat{b}_{t+1} \equiv b_{t+1} A_{t+1}$

$$\hat{b}_{t+1} = \begin{cases} \left[\frac{1-\alpha}{1+\lambda}\right] \left\{ w^u (2 + r + \lambda) + (1 + r)\hat{b}_t \right\} & \hat{b}_t \in [0, (\hat{f})] \\ \left[\frac{1-\alpha}{1+\lambda}\right] \left\{ [w^s (1 + \lambda) - (1 + i)h] + (1 + i)\hat{b}_t \right\} & \hat{b}_t \in [\hat{f}, h] \\ \left[\frac{1-\alpha}{1+\lambda}\right] \left\{ [w^s (1 + \lambda) - (1 + r)h] + (1 + r)\hat{b}_t \right\} & \hat{b}_t \in [h, \infty] \end{cases}$$

Bequest Dynamics

Let $\hat{b}_{t+1} \equiv b_{t+1} A_{t+1}$

$$\hat{b}_{t+1} = \begin{cases} \left[\frac{1-\alpha}{1+\lambda}\right] \left\{ w^u (2 + r + \lambda) + (1+r)\hat{b}_t \right\} & \hat{b}_t \in [0, (\hat{f})] \\ \left[\frac{1-\alpha}{1+\lambda}\right] \left\{ [w^s (1+\lambda) - (1+i)h] + (1+i)\hat{b}_t \right\} & \hat{b}_t \in [\hat{f}, h] \\ \left[\frac{1-\alpha}{1+\lambda}\right] \left\{ [w^s (1+\lambda) - (1+r)h] + (1+r)\hat{b}_t \right\} & \hat{b}_t \in [h, \infty] \end{cases}$$

$\Rightarrow$ The dynamical system is unaffected qualitatively by labor-augmenting technological progress

Sufficient Conditions for Multiple Steady-States

$$(1-\alpha)(1+r) < (1+\lambda)$$

$$(1-\alpha)(1+i) > (1+\lambda)$$

$$w^s(1+\lambda) - (1+i)h < 0$$

$\Rightarrow$ The system is characterized by multiple steady-state, where the unstable equilibrium

$$\hat{g} = \frac{(1-\alpha)[(1+i)h - w^s(1+\lambda)]}{[(1-\alpha)(i+i) - (1+\lambda)]} > 0$$