# Cooperative Implementation
# – preliminary and incomplete –

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**Abstract**

Cooperative mechanism design uses cooperative instead of non-cooperative game forms to implement social choice functions. In line with standard implementation theory, a game form implements a social choice function, if the outcome of the social choice function at any given profile of preferences equals the solution of the game induced by the game form and the given profile of preferences. While different solution concepts for cooperative games yield different concepts of implementation, I focus on core implementation. Since core-implementable social choice functions are strategyproof and Pareto optimal, the Gibbard Satterthwaite impossibility result applies. I therefore study core-implementability in four domains that are not afflicted by the Gibbard Satterthwaite result: social choice with single peaked preferences, house matching, house matching with single peaked preferences and division with single peaked preferences.

## 1 Introduction

A game form implements a social choice function if the outcome of the social choice function at any given profile of preferences is the (or a) solution of the game induced by the game form and the given profile of preferences. While different solution concepts then yield different notions of implementation, the common notions of implementation all use non-cooperative game forms to implement social choice functions. Keeping the basic definition of implementation, the present paper proposes to replace non-cooperative with cooperative game forms. In parallel to the different notions of implementation which rely on different solution concepts for non-cooperative games (Nash, Bayes-Nash, in dominant strategies), different solution concepts for cooperative games generate different notions of cooperative implementation.

Fixing a set of agents $N$ and a set of outcomes $X$ a cooperative game form is defined by a material characteristic function. Such a function in turn specifies the outcomes in $X$ enforceable by any given coalition $S \subset N$. In combination with a profile of preferences by all agents over all outcomes, the game form induces a cooperative game. The present paper then focusses on the core as the most popular solution concept for such cooperative games. A social choice function is core-implementable if there exists a material characteristic function such that for each profile of preferences the the outcome of the social choice function is the unique element of the core of the game induced by the material characteristic function and the profile of preferences.

Theorem 1 shows that that any core-implementable social choice function is Pareto optimal and (under generic circumstances) strategyproof. The Gibbard-Satterthwaite impossibility theorem consequently applies to core implementation. Dictatorship is the the only core implementable social choice function on the grand domain. I therefore investigate domains of preferences where the Gibbard-Satterthwaite impossibility result does not apply and show that the set of core-implementable social choice functions is a strict subset of the set of strategyproof and Pareto optimal social choice functions.

Theorem 2 shows that a social choice function on the single peaked domain is core implementable if and only if it is a set-based majority rule. A set based majority rule in turn is defined by a set of decisive coalitions. Each such decisive coalition can implement any policy they choose. If some coalition is decisive, then any larger coalition (containing the decisive one) must also be decisive. Moreover for any coalition either the coalition itself or its complement must be decisive. Any weighed median rule where the agents are assigned weights that sum up to one and where any coalition that weighs more than one half is decisive is a set based median rule.

I next consider social choice problems with private consumption. In house allocation problems (defined by Hylland and Zeckhauser [9]) as well as in the problem of dividing a single resource, agents only care about their own consumption (respectively the house they are matched with and the share of the resource they obtain). In a house allocation problem the set of agents has to be matched with set of indivisible houses so that no agent ends up with more than one house. Each agent has an outside option, which he may rank above any or all houses in the problem. Theorem 3 shows that a social choice function for such a house allocation problem is core implementable if and only if it a hierarchical exchange rule following Papai [13]. Such hierarchical exchange rules generalize Gale's top trading cycles insofar as that each agent may be endowed with multiple houses.

Smaller domains generally permit larger sets of strategyproof and Pareto optimal social choice functions. In that vein Bade [4] proposes the crawler as a novel Pareto optimal and strategyproof social choice rule for house allocation problems with single peaked preferences. I show here that the crawler is not core-implementable. Sprumont [20] has shown that allocation problems of a unidimensional resource permit Pareto optimal and strategyproof social choice rules if all preferences are single peaked. I show here that such allocation problems only permit core-implementable social choice functions (other than serial dictatorship) if there are at least three agents. The uniform rule defined by Sprumont [20] is not core-implementable.

The concept of core implementation is new when agents and groups do not have any pre-defined power. Conversely, in mechanism design problems where some control rights (and thereby effective deviations) are fixed, we often know whether and which social choice functions are core implementable. Take house allocation problems as an example. The seminal first results on such problems pertain to "housing markets" where each agent is endowed with exactly one house. In such housing markets the designer cannot freely vary the material characteristic function: any owner of a house may always opt to keep the house he was endowed with. Shapley and Scarf [19] showed that the core of a housing market is always non-empty. Roth and Postlewaite [17] then showed that each housing market (with linear preferences) has a singleton core. In terms of the present paper their results imply that a Gale's top trading cylces is unique core-implementable social choice function in a housing market. Viewed against the backdrop of house matching, the present proposal can be viewed as a call to go back to the roots.

# 2 Related Literature

To be written.

# 3 Definitions

There is a set of $n$ agents $N$ and a set of outcomes $X$. Each agent $i \in N$ has a complete and transitive preference $\succsim_i$ over the outcomes $X$. If $\succsim_i$ ranks $x \in X$ above all other elements of $X$, I write $x = top(\succsim_i)$. A Profile or preferences and an arbitrary domain of such profiles are denoted $\succsim$ and $\Omega$. A non-empty subset $S \subset N$ of agents is a coalition. The complement of $S$ is denoted $\overline{S}\colon N \setminus S$. The set of all subsets of some set $Y$ is denoted $\mathfrak{P}(Y)$.

A social choice function $scf : \Omega \to X$ maps each profile of preferences $\succsim \in \Omega$ to an outcome $scf(\succsim)$. A cooperative game form is defined by material characteristic function $F : \mathfrak{P}(N) \to \mathfrak{P}(X)$ that maps each coalition $S \subset N$ to a set of social choices $F(S) \subset X$ that are enforceable by the given coalition $S$.[1] Together with a profile of preferences $\succsim \in \Omega$ the material characteristic function $F : \mathfrak{P}(N) \to \mathfrak{P}(X)$ defines a cooperative game $(F, \succsim)$.

Fix a choice $x \in X$. Then a pair $(y, S)$ of a different choice $y \in X$ and a coalition $S$ improves on $x$ at $\succsim$ if if each member $i \in S$ of the coalition $S$ weakly prefers $y$ to $x$ and if at least one member $j \in S$ strictly prefers $y$ to $x$. Such a pair $(y, S)$ is effective if $y \in F(S)$, so that the coalition $S$ can force $y$. An outcome $x$ is in the core $C(F, \succsim)$ of the game $(F, \succsim)$ if no effective pair $(y, S)$ improves on $x$ at $\succsim$.

**Definition 1** *The material characteristic function $F$ core-implements the social choice function $scf$ if $C(F, \succsim) = \{scf(\succsim)\}$ holds for each $\succsim$.*

[1] The function $F$ is also defined on $\emptyset \subset N$ even though $\emptyset$ is not a coalition. This is irrelevant for the results, and makes notation slightly neater.

So $F$ core-implements $scf$ if for each profile of preferences $\succsim$, the outcome of $scf$ at $\succsim$ is the unique element of the core of the game induced by $F$ and $\succsim$. To check whether a a social choice function is core-implementable we, consequently, have to check whether there exists a material characteristic function $F$, such that $\{scf(\succsim)\} = C(F, \succsim)$ for all $\succsim \in \Omega$.

The notion of implementation immediately generalizes to any other solution concept $\mathcal{S}$. The material characteristic function $F$ $\mathcal{S}$-implements the social choice correspondence $scf$ if for each $\succsim \in \Omega$, $scf(\succsim)$ is, according to $\mathfrak{S}$ the solution of the game $(F, \succsim)$.

A social choice $x \in X$ is Pareto optimal at some profile $\succsim \in \Omega$ if there exists no alternative social choice $y \in X$ that is weakly preferred by all agents and strictly by some, so $x$ is Pareto optimal at $\succsim$ if there exists not $y \in X$ such that $y \succsim_i x$ for all $i \in N$ and $y \succ_j x$ for some $j \in N$. A social choice function $scf : \Omega \to X$ is Pareto optimal if for each $\succsim \in \Omega$, $scf(\succsim)$ is Pareto optimal at $\succsim$. The social choice function $scf$ can be viewed as a direct revelation mechanism, where each agent $i$ submits some preference $\succsim_i$ to the designer, who then implements the outcome $scf(\succsim)$. Viewed as such a direct revelation mechanism, the social choice function is strategyproof if $scf(\succsim) \succsim_i scf(\succsim_i', \succsim_{-i})$ holds for all $i \in N$, $\succsim \in \Omega$ and $\succsim_i' \in \Omega_i$. So $scf$ is strategyproof no agent can at any profile of preferences obtain an outcome he prefers by misrepresenting his preferences.

# 4 Basic Facts About Core Mechanisms

The first Theorem establishes a set of basic facts about core-implementable social choice functions. If some $x$ is in the core of a game $(F, \succsim)$ and if $\succsim'$ in all bilateral comparisons of $x$ with some $x'$ regards $x$ at least as favorable as does $x$, then $x$ must also be in the core of $(F, \succsim)$. Now say that the material characteristic function $F$ core-implements some social choice function $scf$: in this case the preceding observation together implies that $C(F, \succsim') = \{scf(\succsim)\}$ if $\succsim'$ views $scf(\succsim)$ at least as favorable as does $\succsim$. Theorem 1 also shows any core-implementable social choice function Pareto optimal and - under mild conditions - strategy proof.

**Theorem 1** *Consider a set of social choice $X$, a set of preference profiles $\Omega$ and material characteristic function $F : \mathfrak{P}(N) \to \mathfrak{P}(X)$. Fix a choice $x \in X$ and two profiles $\succsim, \succsim' \in \Omega$. For parts b) c) and d) assume that $F$ core-implements the social choice function $scf : \Omega \to X$.*

*a) If $x \in C(F, \succsim)$ and if $x' \succ'_i x \Rightarrow x' \succ_i x$ as well as $x' \sim'_i x \Rightarrow x' \succsim_i x$ holds for all $i \in N$ and all $x' \in X$, then $x \in C(F, \succsim')$.*

*b) If $x' \succ'_i scf(\succsim) \Rightarrow x' \succ_i scf(\succsim)$ as well as $x' \sim'_i scf(\succsim) \Rightarrow x' \succsim_i scf(\succsim)$ holds for all $i \in N$ and all $x' \in X$, then $scf(\succsim) = scf(\succsim')$.*

*c) $scf$ is Pareto optimal.*

*d) If for each $i \in N$, $\succsim_i \in \Omega_i$ and $x, x' \in X$ such that $x \succ_i x'$ there exists some $\succsim'_i \in \Omega_i$ such that $\succsim'_i$: $x$ and such that $x'' \succ'_i x' \Rightarrow x'' \succ_i x'$ as well as $x'' \sim'_i x' \Rightarrow x'' \succsim_i x'$ holds for all $x'' \in X$, then $scf$ is strategyproof.*

**Proof**

a) Suppose $x \in X$, $x \in C(F, \succsim)$ and $x \notin C(F, \succsim')$ even though $x' \succ'_i x \Rightarrow x' \succ_i x$ as well as $x' \sim'_i x \Rightarrow x' \succsim_i x$ for all $i \in N$ and all $x' \in X$. So suppose that some effective pair $(x', S)$ improves on $x$ at $\succsim'$. Say $S' \subset S$ is the non-empty set of all agents $j$ with $x' \succ'_j x$ so that $x' \sim_i x$ holds for all $i \in S \setminus S'$. Since $x' \succ'_j x \Rightarrow x' \succ_j x$ for all $j \in S'$ and since $x' \sim'_i x \Rightarrow x_S \succsim_i x$ for all $i \in S \setminus S'$ the pair $(y, S)$ also improves on $x$ at $\succsim$ a contradiction to $x \in C(F, \succsim)$.

b) If $x' \succ'_i scf(\succsim) \Rightarrow x' \succ_i scf(\succsim)$ as well as $x' \sim'_i scf(\succsim) \Rightarrow x' \succsim_i scf(\succsim)$ holds for all $i \in N$ and all $x' \in X$, then part a) implies $scf(\succsim) \in C(F, \succsim')$. Since $C(F, \succsim')$ is a singleton, $scf(\succsim) = scf(\succsim')$.

c) Suppose $scf(\succsim)$ was at $\succsim$ Pareto dominated by some $x$, so that $x \succ_j scf(\succsim)$ holds for some $j \in N$ and $x \succsim_i scf(\succsim)$ for all $i \in N$. Since $x \notin C(F, \succsim)$ some effective pair $(x', S)$ must improve on $x$ at $\succsim$, so that $x' \succ_j x$ holds for some $j \in S$ and $x \succsim_i scf(\succsim)$ for all $i \in S$. Since $x$ Pareto dominates $scf(\succsim)$, we have $x' \succ_j x \succsim_j scf(\succsim)$ as well as $x' \succsim_i x \succsim_i scf(\succsim)$ for all $i \in S$, so that the pair $(x', S)$ also improves on $scf(\succsim)$ at $\succsim$, a contradiction to $scf(\succsim)$ being in the core $C(F, \succsim)$.

d) Suppose $scf$ was not strategyproof, so suppose there exists an agent $i$, a profile $\succsim$ and a deviation $\succsim'_i$ such that $scf(\succsim'_i, \succsim_{-i}) \succ_i scf(\succsim)$. By the assumption on preferences there exists $\succsim''_i \in \Omega_i$ such that $\succsim''_i$: $scf(\succsim'_i, \succsim_{-i})$ and $x \succ''_i scf(\succsim) \Rightarrow x \succ_i scf(\succsim)$ as well as $x \sim''_i scf(\succsim) \Rightarrow x \succsim_i scf(\succsim)$ for all $x \in X$. By part b) of the present Lemma $scf(\succsim'_i, \succsim_{-i}) = scf(\succsim''_i, \succsim_{-i})$ as well as $scf(\succsim) = scf(\succsim''_i, \succsim_{-i})$ a contradiction to the assumption that $scf(\succsim'_i, \succsim_{-i}) \succ_i scf(\succsim)$ so $M$ must be strategyproof.

□

The assumption on $\Omega$ under which any core mechanism must be strategyproof is very weak. Indeed most standard domains of preferences satisfy it. The assumption, in particular, holds on all the domains discussed in the sequel (single peaked preferences, matching, allocation problems with single peaked preferences). To see that core-implementable social choice functions on domains that violate the assumption may be manipulable consider the following example:

**Example 1** Say $N = \{1, 2, 3\}$, $X = \{a, b, x, x'\}$, and $F$ such that $F(\{1, 2\}) = \{a, b\}$, $F(\{3\}) = \{x, x'\}$ and $F(S) = \emptyset$ for all other $S$ with $\mid S \mid < 3$. Define $\Omega$ such that $\Omega_1 = \{\succsim_1, \succsim'_1\}$, $\Omega_2 = \{\succsim_2\}$ $\Omega_3 = \{\succsim_3\}$ with $a \succ_1 x' \succ_1 x \succ_1 b$, $x' \succ'_1 b \succ'_1 x \succ'_1 a$, $b \succ_2 x \succ_2 a \succ_2 x'$ and $x \sim_3 x' \succ_3 a \sim_3 b$.

Since $(x, \{3\})$ as well as $(x', \{3\})$ are both effective and since $x \sim_3 x' \succ_3 a \sim_3 b$ $C(F, \succsim) \subset \{x, x'\}$ as well as $C(F, (\succsim'_1, \succsim_{-1})) \subset \{x, x'\}$. Since $a \succ_1 x'$ as well as $a \succ_2 x'$ and since $(a, \{1, 2\})$ is effective, $x' \notin C(F, \succsim)$. On the other hand $x \in C(F, \succsim)$ since $x \succ_2 a$ but $x \succ_1 b$ so that neither $b$ nor $a$ improves upon $x$ for the coalition $\{1, 2\}$. Mutatis mutandis the same arguments imply $C(F, (\succsim'_1, \succsim_{-1})) = \{x'\}$.

Since $C(F, \succsim) = \{x\}$ and $C(F, (\succsim'_1, \succsim_{-1})) = \{x'\}$ the core $C(F, \cdot)$ is a singleton for all profiles in $\Omega$. So $F$ indeed core-implements a social choice function $scf$. Since $scf((\succsim'_1, \succsim_{-1})) = x' \succ_1 x = scf(\succsim)$, this social choice function is not strategyproof.

Fix a set $X$ with at least three elements and say that $\Omega$ is the domain of all preferences over $X$. Since any core-implementable social choice function $scf : \Omega \to X$ is Pareto optimal and strategy proof, the Gibbard-Satterthwaite impossibility theorem implies that dictatorship is the only core-implementable social choice function on the grand domain.

**Corollary 1** *Fix a domain $X$ containing at least three elements. Say $\Omega$ contains all complete and transitive preferences over $X$. If $scf$ is core implementable then $scf$ must be a dictatorship, so that there exists an agent $i \in N$ such that $scf(\succsim) = top(\succsim_i)$ for each $\succsim \in \Omega$.*

**Proof** Say $scf : \Omega \to X$ is core implementable. Since $\Omega$ contains all complete and transitive preferences over $X$, $scf$ is, by parts c) and d) of Theorem 1, Pareto optimal and strategyproof. By the Gibbard Satterthwaite theorem there exists an agent $i \in N$ such that $scf(\succsim) = top(\succsim_i)$ for each $\succsim \in \Omega$. □

To find core-implementable social choice functions we must restrict ourselves to domains of preferences that are not affected by the Gibbard Satterthwaite impossibility result. Before I consider such domain there is one more generic result for core implementable social choice functions:

At times different sets $D$ and $D'$ define the same core mechanism $M$ so that $\{M(\succsim)\} = C(F, \succsim) = C(F', \succsim)$ holds for all $\succsim \in \Omega$. To simplify notation I chose the unique maximal set of deviations $\overline{D}$ that $M(\succsim) = C(\overline{D}, \succsim)$ for all $\succsim$.

**Lemma 1** *Say $scf$ is core implementable. Then there exists a unique maximal material characteristic function $\overline{F}$ such that $\{scf(\succsim)\} = C(\overline{F}, \succsim)$ for all $\succsim \in \Omega$.*

**Proof** Say $\mathcal{F}$ is the set of all functions $F$ such that $\{scf(\succsim)\} = C(F, \succsim)$ for all $\succsim \in \Omega$. Define $\overline{F} : \mathfrak{P}(N) \to \mathfrak{P}(X)$ such that for each $S \in \mathfrak{P}(N)$: $\overline{F}(S) := \bigcup_{F \in \mathcal{F}} F(S)$ as the - uniquely defined - union over all these sets. To see that $C(\overline{F}, \succsim) = C(F, \succsim)$ holds for each $F \in \mathcal{F}$ and each $\succsim \in \Omega$, fix an arbitrary $F \in \mathcal{F}$ and $\succsim \in \Omega$. If $\{x\} = C(F, \succsim)$ then $\{x\} = C(F', \succsim)$ for each $F' \in \mathcal{F}$. So there exists no $S$ and $x' \in F'(S)$ for any any $F' \in \mathcal{F}$ such that $x'$ dominates $x$ at $\succsim$. □

## 5 The Single Peaked Domain

Suppose $X \subset \mathbb{R}$. A preference $\succsim_i$ is single peaked if there exists a policy $x^* \in X$ such that $x' \succ_i x''$ holds for any $x', x'' \in X$ such that either $x'' < x' \leq x^*$ or $x^* \geq x' > x''$. Say that agent $i$ with ideal point $x^*$ has **distance disutility** if $u_i(x) = - \mid x - x^* \mid$.

Say a coalition $S$ is **decisive** if it may choose any policy and **powerless** if it may not choose any policy. Formally $S$ is decisive (powerless) if $(x, S)$ is effective for each $x \in X$ (for no $x \in X$). A material characteristic function $F : \mathfrak{P}(N) \to \mathfrak{P}(X)$ is a **decisive coalitions function** if each coalition is either decisive or powerless and if the complement of each powerless coalition is decisive and vice versa. So $F$ is a decisive coalitions function if for any $S$ either $F(S) = X$ or $F(\overline{S}) = \emptyset$ and $F(S) = X \Leftrightarrow F(\overline{S}) = \emptyset$.

Majority rule with an odd number of agents, where $F(S) = X$ if $S$ contains at least half of all agents and $F(S) = \emptyset$, is the classic example of a decisive coalitions function. Majority rule core implements the median social choice function $median : \Omega\ toX$ which maps any profile of preferences to the median of its ideal points, so $median(\succsim)$ equals the median of $\{top(\succsim_1), \ldots, top(\succsim_n)\}$. When the number of voters is even the the function $F$ defined above is not a decisive coalitions function, as any set $S$ containing exactly half of all agents as well as its complement are powerless. With an even number of agents the core $C(F, \succsim)$, which consists in all medians of $\{top(\succsim_1), \ldots, top(\succsim_n)\}$ need not be a singleton. Theorem 2 shows that a social choice function for the single peaked domain $[0, 1]$ is core-implementable if and only some decisive coalitions function implements it.

**Theorem 2** *Say $X = [0, 1]$ and $\Omega$ is the domain of single peaked preferences. Then $F : \mathfrak{P}(N) \to \mathfrak{P}(X)$ core-implements some social choice function $scf : \Omega \to X$ if and only if it is a decisive coalitions function.*

Lemma 2 shows that any decisive coalitions function core-implements a social choice function. Lemma 2 applies to any single peaked domain: the assumption that agents have preferences over a policy interval is not needed for this result.

**Lemma 2** *Say $X \subset \mathbb{R}$. If $F : \mathfrak{P}(N) \to \mathfrak{P}(X)$ is a decisive coalitions function, then it core-implements a social choice function $scf : \Omega \to X$.*

**Proof** Fix an arbitrary profile $\succsim$. Define $m$ as the maximal $x \in X$ for which $F(\{i : top(\succsim_i) < x\}) = \emptyset$ and $F(\{i : top(\succsim_i) \geq x\}) = X$. Note that there exists some $i^*$ such that $top(\succsim_{i^*}) = m$, $\{i : top(\succsim_i) > m\} \subset \{i : top(\succsim_i) \geq m\} \setminus \{i^*\}$ and due to the maximality of $m$: $F(\{i : top(\succsim_i) > m\}) = \emptyset$. To see that $m \in C(F, \succsim)$, suppose some effective pair $(y, S)$ did improve on $m$ at $\succsim$. If $y < m$ then $\top(\succsim_i) < m$ for all $i \in S$, so that $S \subset \{i : top(\succsim_i) < m\}$. Since $F(\{i : top(\succsim_i) < m\}) = \emptyset$ $F(S) = \emptyset$. Conversely if $y > m$ then $top(\succsim_i) > m$ for all $i \in S$, so that $S \subset \{i : top(\succsim_i) > m\}$. Since $F(\{i : top(\succsim_i) > m\}) = \emptyset$, $F(S) = \emptyset$. A contradiction arises since $(y, S)$ is not effective whether $y > m$ or $y < m$. Now suppose $C(F, \succsim)$ contained some $y \neq m$. If $m < y$ then the - effective - pair $(m, \{i : top(\succsim_i) \geq m\})$ improves on $y$ at $\succsim$. If $m > y$ then the - effective - pair $(m, \{i : top(\succsim_i) \leq m\})$ improves on $y$ at $\succsim$. □

Weighed median rules construct decisive and powerless coalitions on the basis of voting-power-weights. Say $\pi : N \to \mathbb{R}^*$ with $\sum_{i \in N} \pi(i) = 1$ represents a set of such voting power weights and say that coalition $S$ is decisive if $\sum_{i \in S} \pi(i) > \frac{1}{2}$. Then $\pi$ yields a decisive coalitions function if and only if $\sum_{i \in S} \pi(i) \neq \frac{1}{2}$ for any $S \subset N$. In the median rule with an odd number of agents $n$, each agent $i$ carries equal weight $\pi(i) = \frac{1}{n}$ and the coalition $S$ is decisive if and only if $\sum_{i \in S} \pi(i) > \frac{1}{2}$. With an even number of agents we can decisive coalitions function by assigning one particular agent, say agent 1, tie breaking power. To do so define $\pi(1) = \frac{1}{n} + \epsilon$ and $\pi(i) = \frac{1}{n} - \frac{\epsilon}{n-1}$ for some small $\epsilon > 0$. For any profile $\succsim$ with multiple median ideal points agent 1's the most preferred median is the unique element of the core. For a further deviation from the median rule consider the case of a family with 2 parents and 3 children, where $\pi(i) = \frac{2}{7}$ if $i$ is a parent and $\pi(i) = \frac{1}{7}$ if $i$ is a child, so that any parent is "twice as powerful" than any child. By Lemma 2 any such weighed median rule is core implementable, the converse does not hold:

**Corollary 2** *Any weighed median rule is core-implementable. Not every core-implementable social choice function is a weighed median rule.*

**Proof** Say the decision power weights $\pi$ defines a weighed median rule.

Define a function $F : \mathfrak{P}(N) \to \mathfrak{P}(X)$ such that $F(S) = X$ if $\sum_{i \in S} \pi(i) > \frac{1}{2}$ and $F(S) = \emptyset$ otherwise. Since $\sum_{i \in S} \pi(i) \neq \sum_{i \notin S} \pi(i)$ for all $S \subset N$, $F$ is by Lemma 2 a decisive coalitions function that core-implements a social choice function.

To see that not all core-implementable social choice functions are weighed median rules, consider $N = \{1, 2, 3, 4, 5, 6\}$. Define a decisive coalitions function $F$ such that $F(S) = X$ if $\mid S \mid \geq 4$ or if $S$ is one of the following coalitions with three members.

$$\{1,2,3\},\{1,3,4\},\{1,4,6\},\{1,5,6\},\{1,2,5\}$$
$$\{2,4,6\},\{2,4,5\},\{3,4,5\},\{2,3,6\},\{3,5,6\}.$$

For all other $S$ say $F(S) = \emptyset$. By Lemma 2 $F$ is core-implements a social choice function since it is a decisive coalitions function. To see that this social choice function does not arise out of a weighed majority rule, suppose it did. Suppose the decision power weights $\pi$ did define this weighed majority rule. Since $\sum_{i \in S} \pi(i) > \frac{1}{2}$ holds for each three member coalition $S$ with $F(S) = X$, summing over all such coalitions yields

$$\sum_{S: F(S) = X, |S| = 3} \sum_{i \in S} > 5 \tag{1}$$

.

A contradiction results since the left hand side of the above inequality equals $5(\pi(1) + \pi(2) + \pi(3) + \pi(4) + \pi(5)) = 5$.

□

The converse direction of the proof of Theorem **??** does require the assumption of preferences being defined over an interval. The assumption that this interval is $[0, 1]$ is only made for convenience.

**Proof** Say that $F : \mathfrak{P}(N) \to \mathfrak{P}(X)$ core-implements the social choice function $scf(\succsim)$.

**Claim 1:** For each $x', x'' \in [0, 1]$ with $x' < x''$ there exists some $x^* \in (x', x'')$ such that $x^* \in F(S)$ for some $\emptyset \neq S \subset N$. Suppose not, suppose there exist some $x', x'' \in [0,1]$ with $x' < x''$ and $x \notin F(S)$ for all $x \in (x', x'')$ and $\emptyset \neq S \subset N$. Consider the profile $\succsim$ where each agent has distance disutility with $\frac{x'+x''}{2}$ as the ideal point. To see that $\{\frac{2x'+x''}{3}, \frac{2x'+x''}{3}\} \subset C(F, \succsim)$, suppose some effective pair $(\tilde{x}, S)$ did improve upon $\frac{2x'+x''}{3}$. By the assumption the assumption on $F$, $\tilde{x} \notin (x', x'')$. A contradiction arises since all agents (including the agents in $S$) $\succsim$-prefer $\frac{2x'+x''}{3}$ to any platform outside $(x', x'')$. By Mutatis mutandis, the same arguments imply that $\frac{x'+2x''}{3} \in C(F, \succsim)$, a contradiction to $C(F, \succsim)$ being singleton.

**Claim 2:** For each two effective pairs $(x', S')$ and $(x'', S'')$ with $x' < x''$ there must exist some agent in $S' \cap S''$. Suppose not, so suppose $S' \cap S'' = \emptyset$ for two effective pairs $(x', S')$ and $(x'', S'')$ with $x' < x''$. Define $\succsim$ such that $top(\succsim_i) = x'$ for all $i \in S'$ and $top(\succsim_i)$ for all $i \notin S'$. Since $(x', S')$ is effective and since $top(\succsim_i) = x'$ for all $i \in S'$, $scf(\succsim) = x'$. A contradiction arises since $scf(\succsim) = x''$ must hold since $(x'', S'')$ is effective and since $top(\succsim_i) = x''$ for all $i \in S'$.

**Claim 3:** if $x' \in F(S')$ then $F(S')$ is dense in $[0,1]$. Claim 1 implies that $\bigcup_{\emptyset \neq S \subset N} F(S)$ is dense in $[0,1]$. Since $F(S) \subset F(N)$ for all $S \subset N$, the preceding observation implies that $F(N)$ is dense in $[0,1]$. So consider the case $S' \neq N$. Suppose there exists $x'', x_{S'} \in [0,1]$ such that $x' < x_{S'} < x''$ and $x_{S'} = \sup\{x \leq x'' : x \in F(S')\}$. Define $\succsim$ such that each agent has distance disutility with $top(\succsim_i) = \frac{x_{S'}+x''}{2}$ for each $i \in S'$ and $top(\succsim_i) = x'$ for all $i \notin S'$. To see that $\frac{2x_S+x''}{3} \in C(F, \succsim)$ suppose some effective pair $(y, S'')$ improves on $\frac{2x_S+x''}{3}$ at $\succsim$. Since each agent in $S'$ $\succsim$-prefers only platforms in $(\frac{2x_{S'}+x''}{3}, \frac{x_{S'}+2x''}{3})$ and since no pair $(y, S'')$ with $S'' \subset S'$ is effective if $y \in (\frac{2x_{S'}+x''}{3}, \frac{x_{S'}+2x''}{3})$, there must exist some $i'' \in S'' \setminus S'$. By Claim 1 there also exists some $i' \in S' \cap S''$. For $y \succsim_{i''} \frac{2x_S+x''}{3}$ and $y \succsim_{i'} \frac{2x_S+x''}{3}$ to hold we must have $y \leq \frac{2x_S+x''}{3}$ as well as $y \geq \frac{2x_S+x''}{3}$, implying that no effective pair improves on $\frac{2x_S+x''}{3}$ at $\succsim$. Mutatis mutandis the same arguments imply $\frac{3x_S+x''}{4} \in C(F, \succsim)$, and $C(F, \succsim)$ is not a singleton. The case that there exist platforms $x'', x_{S'} \in [0,1]$ such that $x' > x_{S'} > x''$ and $x_{S'} = \inf\{x \geq x'' : x \in F(S')\}$ is rule out by analogous arguments.

**Claim 4:** Fix some $S \in \mathfrak{P}(N)$ for which $F(S)$ is dense in $[0,1]$. Assume that $x^* \notin F(S)$. Define $F'$ such that $F'(S') = F(S)$ for all $S' \neq S$ and

$F'(S) = F(S) \cup \{x^*\}$. To see that $C(F, \succsim) = C(F', \succsim)$, note firstly that $C(F', \succsim) \subset C(F, \succsim)$ since any pair that is effective under $F$ is also effective under $F'$. For $C(F', \succsim) \neq C(F, \succsim)$ to hold $C(F', \succsim) = \emptyset$ must hold for some $\succsim$. Since $scf(\succsim) \in C(F, \succsim)$, the pair $(x^*, S)$ must improve on $scf(\succsim)$. Since $F(S)$ is dense in $[0, 1]$, there exists a platform $x'' \in F(S)$ and between $x^*$ and $scf(\succsim)$. Since $x^* \succsim_i scf(\succsim)$ for all $i \in S$ and since each $\succsim_i$ is single peaked, $(x'', S)$ improves on $scf(\succsim)$ and $\succsim$. Since $x'' \in F(S)$ we obtain a contradiction to $\{scf(\succsim)\} = C(F, \succsim)$. So $\overline{F}(S) = [0, 1]$.

**Claim 5:** For each $S$ either $\overline{F}(S) = [0, 1]$ or $\overline{F}(\overline{S}) = [0, 1]$, moreover $\overline{F}(S) = [0, 1] \Leftrightarrow \overline{F}(\overline{S}) = \emptyset$. To see this fix any coalition $S$ and two platforms $x', x'' \in [0, 1]$ with $x' < x''$. Fix $\succsim$ such that $top(\succsim_i) = x'$ if $i \in S$ and $\top(\succsim_i) = x''$ if $i \in \overline{S}$. If $\overline{F}(S) = \overline{F}(\overline{S}) = [0, 1]$ then $C(\overline{F}, \succsim)$ is empty. If $\overline{F}(S) = \overline{F}(\overline{S}) = \emptyset$ then $C(\overline{F}, \succsim)$ contains $[x', x'']$.

While Claim 5 establish that $scf$ is core-implementable if it is core-implemented by a decisive coalitions function, Lemma **??** conversely show that any decisive coalitions function implements some social choice funtion.

□

# 6 Private Consumption

Assume that the domain $X$ can be represented as a set $\mathcal{M}$ of allocations $\mu : N \to Y$ for some set $Y$ with the understanding that agent $i$ consumes $\mu(i)$ in the allocation $\mu$. Assume that each agent's preference $\succsim$ over allocations only depends on the object he obtains. So agent $i$ is indifferent between $\mu$ and $\mu'$ if $\mu(i) = \mu'(i)$. Since only agent $i$'s private consumption matters for his preference over allocations $\mu \in \mathcal{M}$, the notation $\succsim_i$ also stands for agent $i$'s preference over his private consumption, so that $y \succsim_i y'$ holds for $y, y' \in Y$ if $\mu \succsim_i \mu'$ holds for two allocations $\mu, \mu' \in \mathcal{M}$ with $\mu(i) = y$ and $\mu'(i) = y'$.

Any coalition $S$ and allocation $\mu$ together define a suballocation $\nu$ as the restriction of $\mu$ to $S$. If $\nu$ is a suballocation then $N_\nu$ is the set of agents for whom $\nu$ is defined. In the context of private consumption such suballocations can be used instead of material characteristic functions $F$ to define game forms. For a fixed set of effective suballocations $V$ and profile of preferences

$\succsim\in\Omega$, the core $C(V,\succsim)$ of the game $(V,\succsim)$ induced by $V$ and $\succsim$, consists in all allocations $\mu$ that are not dominated by some suballocation $\nu$ at $\succsim$, where $\nu$ dominates $\mu$ at $\succsim$ if $\nu(i)\succsim_i \mu(i)$ for all $i\in N_\nu$ and $\nu(j)\succsim_j \mu(j)$ for some $j\in N$.

For any material characteristic function $F$ define a set of effective suballocations $V(F)$ such that $\nu\in V(F)$ if $\mu\in F(N_\nu)$ and $\nu\subset\mu$. Conversely for any set of effective suballocations $V$ define material characteristic function $F(V)$ such that $\mu\in F(N_\nu)$ if $\nu\in V$ and $\nu\subset\mu$. For any core implementable social choice function $scf:\Omega\rightarrow\mathcal{M}$, say that $\overline{V}$ is the maximal set of effective submatchings that core-implements $scf$.

**Lemma 3** *Say $\mathcal{M}$ is a set of allocations and $\Omega$ is a private consumption domain. Say $F$ is a material characteristic function and $V$ a set of effective submatchings.*

*a) Then $C(F(V),\succsim)=C(V,\succsim)$ and $C(F,\succsim)=C(V(F),\succsim)$ hold for all $\succsim$.*

*b) If $scf$ is core implementable then $\overline{V}$ is uniquely defined.*

*c) Say $F$ and $V$ core-implement a social choice function $scf$ and $\overline{F}$ and $\overline{V}$ are the respective maximal sets. Then $F(\overline{V})=\overline{F}$ and $\overline{V}=V(\overline{F})$.*

## Proof

a) To see that $\mu\in C(F,\succsim)\Leftrightarrow\mu\in C(V(F),\succsim)$ note that some $(S,\mu')$ is effective according to $F$ and improves on $\mu$ at $\succsim$ if and only if the restriction of $\mu'$ to $S$ in contained in $V(F)$ and dominates $\mu$ at $\succsim$. So $C(F,\succsim)=C(V(F),\succsim)$. Similarly $\mu\in C(F(V),\succsim)\Leftrightarrow\mu\in C(V,\succsim)$ holds since some $(S,\mu')$ is effective according to $F(V)$ and improves on $\mu$ at $\succsim$ if and only if the restriction of $\mu'$ to $S$ in contained in $V(F)$ and dominates $\mu$ at $\succsim$.

b) Define

□

For any suballocation $\nu'$ define a subproblem $X^{-\nu'}$ $\Omega^{-\nu'}$ for the set of agents $\overline{N}_{-\nu'}$ not matched by $\nu$ and $X^{-\nu'}$ the set of all (sub)allocations $\nu$ for which $\nu' \cup \nu$ is an allocation in the original problem. Say $\Omega^{-\nu'}$ is the set of all preference profiles $\overline{\succsim}$ that are restrictions of some $\succsim \in \Omega$ to $\overline{N}_{\nu'}$ and $X^{-\nu'}$.

**Lemma 4** *Say $\mathcal{M}$ is a set of allocations over $Y$ and $\Omega$ is a private consumption domain. Say the maximal set of effective submatchings $\overline{V}$ core-implements $scf$. Then*

- *a) If $\nu' \in V$, then $V^{-\nu'} = \{\nu : \nu' \cup \nu \in \overline{V}\}$ core-implements $scf^{-\nu'}$ $scf(\succsim) = \nu' \cup scf^{-\nu'}(\overline{\succsim})$ holds for all $\succsim \in \{\succsim \in \Omega\colon \succsim_i\colon \nu'(i)$ for all $i \in N_{\nu'}\}$.*
- *c) If for each $y \in Y$ and $i \in N$ there exists a $\succsim_i \in \Omega_i$ such that $top(\succsim_i) = y$, then $scf$ is non-bossy.*

**Proof**

a) Fix $scf$, $\overline{V}$, and $\nu'$ as set out above. Fix an arbitrary profile of preferences $\overline{\succsim}$ of agents $\overline{N}_{\nu'}$ over allocations $X^{-\nu'}$. Fix $\succsim \in \Omega$ such that $\succsim_i\colon \nu'(i)$ for all $i\imath N_{\nu'}$ and $\overline{\succsim}$ the restriction of $\succsim$ to $\overline{N}_{\nu'}$ and $X^{-\nu'}$. Since $\nu' \in \overline{V}$, $\nu' \subset scf(\succsim)$ holds, for otherwise $scf(\succsim)$ would be dominated by $\nu' \in \overline{V}$. So $scf(\succsim) \setminus \nu'$ is well-defined.

To see that $\{scf(\succsim) \setminus \nu'\} = C(V^{-\nu'}, \overline{\succsim})$ first suppose some $\nu \in V^{-\nu'}$ dominates $scf(\succsim) \setminus \nu'$ at $\overline{\succsim}$. Since $\nu \in V^{-\nu'}$ there exists a submatching $\nu'' \subset \nu'$ such that $\nu'' \cup \nu \in \overline{V}$. Since $\overline{\succsim}$ is the restriction of $\succsim$ to $\overline{N}_{\nu'}$ and $X^{-\nu'}$ $\nu'' \cup \nu$ dominates $\nu' \cup scf(\succsim) \setminus \nu' = scf(\succsim)$, yielding the contradiction $scf(\succsim) \notin C(\overline{V}, \succsim)$.

Now suppose that some $\nu^\circ \neq scf(\succsim) \setminus \nu'$ was an element of $C(V^{-\nu'}, \overline{\succsim})$. Since $\nu' \cup \nu^\circ \neq scf(\succsim)$ we have $\nu' \cup \nu^\circ \notin C(\overline{V}, \succsim)$ and there exists a $\tilde{\nu} \in \overline{V}$ that dominates $\nu' \cup \nu^\circ$ at $\succsim$. Since $\succsim_i\colon \nu'(i)$ for each $i \in N_{\nu'}$, $\tilde{\nu}(i)$ must equal $\nu'(i)$ for each $i \in N_{\tilde{\nu}} \cap N_{\nu'}$ for $\tilde{\nu}$ to dominate $\nu' \cup \nu^\circ$ at $\succsim$. We can therefore represent $\tilde{\nu}$ as $\nu'' \cup \nu$ for some $\nu'' \subset \nu'$ such that $N_\nu \cap N_{\nu'} = \emptyset$. For $\tilde{\nu}$ to dominate $\nu' \cup \nu^\circ$ at $\succsim$, $\nu \neq \emptyset$ must hold. Since $\nu' \cup \nu \in \overline{V}$, we obtain the contradiction that $\nu \in V^{-\nu'}$ dominates $\nu^\circ$ at $\overline{\succsim}$.

b) Say the set of effective submatchings $V$ core-implements $scf : \Omega \to \mathcal{M}$. Say that for each $y \in Y$ and $i \in N$ there exists a $\succsim_i \in \Omega_i$ such that $top(\succsim_i) = y$. Fix a profile $\succsim \in \Omega$ and agent $i$ and a deviation $\succsim'_i \in \Omega_i$. Suppose $scf(\succsim)(i) = scf(\succsim'_i, \succsim_{-i})(i)$. Define $\succsim''_i$ such that $top(\succsim''_i) = scf(\succsim)(i)$. Part b) of Lemma 1 yields $scf(\succsim) = scf(\succsim''_i, \succsim_{-i}) = scf(\succsim'_i, \succsim_{-i})$, so that $scf$ is non-bossy.

□

To see that core-implementable social choice functions can be bossy when the condition on the domain is not satisfied consider the following example.

**Example 2** Let $N = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d\}$. $\Omega$ such that $\Omega_i = \{\succsim_i\}$ with $a \succ_i b \succ_i c \succ_i d$ for $i = 1, 2, 3$ and $\Omega_4 = \{\succsim'_4, \succsim''_4\}$ with $a \succ'_4 x \succ'_4 b \succ'_4 c$ and $b \succ''_4 x \succ''_4 a \succ_4 c$. Say

$$V = \{\{1c\}, \{2c\}, \{3c\}, \{2c, 4a\}, \{3c, 4a\}, \{1c, 2a\}, \{1c, 4b\}, \{3c, 4b\}, \{1a, 2c\}\}$$

. Since $\{ic\} \in \overline{V}$ for $i = 1, 2, 3$ and since $a \succ_i b \succ_i c \succ_i d$ for $i = 1, 2, 3$, $\mu \in C(V, \succsim) \Rightarrow \mu(4) = d$ holds for all $\succsim \in \Omega$. For $\mu' \in C(V, (\succsim'_4, \succsim_{-4}))$ $\mu'(1) = c$ must hold for otherwise $\mu'$ is dominated by either $\{2c, 4a\}$ or $\{3c, 4a\}$. Given that $\mu'(1) = c$ and $\{1c, 2a\} \in V$ we have $\mu' = \{1c, 2a, 3b, 4d\}$. Since none of the remaining suballocations in $V$ dominate $\mu'$, $\{\mu'\} = C(V, (\succsim'_4, \succsim_{-4}))$. For $\mu'' \in C(V, (\succsim''_4, \succsim_{-4}))$ $\mu''(2) = c$ must hold for otherwise $\mu''$ is dominated by either $\{1c, 4b\}$ or $\{3c, 4b\}$. Given that $\mu'(2) = c$ and $\{1a, 2c\} \in V$ we have $\mu' = \{1a, 2c, 3b, 4d\}$. Since none of the remaining suballocations in $V$ dominate $\mu''$, $\{\mu''\} = C(V, (\succsim''_4, \succsim_{-4}))$. So $V$ core-implements a social choice function $scf$.

Since $scf(\succsim'_4, \succsim_{-4}) \neq scf(\succsim''_4, \succsim_{-4})$ even though $scf(\succsim'_4, \succsim_{-4})(4) = scf(\succsim''_4, \succsim_{-4})(4)$ this social choice function is bossy.

# 7 Matching with Outside Options

There is a set of finitely many indivisible houses $H$. Each agent also has access to an outside option, denoted $o$. So $Y$ equals $H \cup \{o\}$ in this context.

For convenience the outside option $o$ is also referred to as a house. When the difference matters the elements of $H$ are referred to as real houses. No agent is ever indifferent between any two different houses, so the set $\Omega$ of preference profiles only contains antisymmetric preferences.[2]

In house matching problems allocations $\mu \in \mathcal{M}$ and subllocations $\nu$ are called a **matchings** and **submatchings**. To qualify as a matching, $\mu$ cannot assign any real house to two different agents, so $\mu(i) = \mu(j)$ can only hold for two different agents $i \neq j$ if $\mu(i) = \emptyset$. For a submatching $\nu$, $H_\nu$ is the set of real houses matched by $\nu$, so $H_\nu = \{h \in H : \nu(i) = h \text{ for some } i \in N_\nu\}$. If $\nu$ and $\nu'$ are two submatchings with $\nu \cup \nu' = \mu$ we consequently must have $\nu(i) = \nu'(i)$ for all $i \in N_\nu \cap N_{\nu'}$, $N_\nu \cup N_{\nu'} = N$ and $H_{\nu\setminus\nu'} \cap H_{\nu'\setminus\nu} \subset \{o\}$.

I first establish necessary conditions for a set $V$ of effective submatchings to define a core mechanism. To do so fix two submatchings. The Lemma, firstly, identifies conditions under which these two submatchings cannot simultaneously belong to $V$. The Lemma, secondly, gives conditions under which a combination of the two submatchings must be in the maximal set of effective submatchings $\overline{V}$ if the two given submatchings are in $V$. The Lemma gives wide reaching conditions for matchings $\mu$ that must be contained in $\overline{V}$. Finally the Lemma shows that any submatching $\nu \cup \{io\}$ must in the maximal set of effective submatchings if $\nu$ itself belongs to this set - and is not a matching.

**Lemma 5** *Say $V$ and $\overline{V}$ respectively are a set and the maximal set of effective submatchings that core-implement the social choice function $scf : \Omega \to \mathcal{M}$. Fix three submatchings $\nu$, $\nu'$, and $\nu''$ with $\nu' \neq \emptyset \neq \nu''$ and $N_{\nu'} \cap N_{\nu''} = \emptyset$.*

*a) If $\nu \cup \nu' \in V$ and $H_{\nu'} = H_{\nu''} \neq \emptyset$ then $\nu \cup \nu'' \notin V$.*

*b) Define $\nu : N_{\nu'} \cup N_{\nu''} \to H_{\nu'} \cup H_{\nu''}$ such that $\nu(i) = \nu'(i)$ if $i \in N_{\nu'} \setminus \{i^*\}$, $\nu(i) = \nu''(i)$ if $i \in N_{\nu''} \setminus \{j^*\}$, $\nu(i^*) = \nu''(j^*)$ and $\nu(j^*) = \nu'(i^*)$ for some fixed $i^* \in N_{\nu'}$ and $j^* \in N_{\nu''}$. If $H_{\nu'} \cap H_{\nu''} = \emptyset$ and $\nu', \nu'' \in V$ then $\nu \in \overline{V}$.*

*c) Say $\nu \in V$ and $i \notin N_\nu$, then $\nu \cup \{io\} \in \overline{V}$.*

[2]This is in addition the the assumption that $\Omega$ is a private consumption domain.

d) *Any matching $\mu$ is contained in $\overline{V}$.*

**Proof**

a) Suppose $\nu \cup \nu', \nu \cup \nu'' \in V$ and $H_{\nu'} = H_{\nu''} \neq \emptyset$. Since $N_{\nu'} \cap N_{\nu''} = \emptyset$ we can define a profile $\succsim$ such that $\succsim_i$: $\nu'(i)$ for all $i \in N_{\nu'}$, $\succsim_i$: $\nu''(i)$ for all $i \in N_{\nu''}$ and $\succsim_i$: $\nu(i)$ for all $i \in N_\nu$. Since $\nu \cup \nu', \nu \cup \nu'' \in V$, we have $\nu' \cup \nu \subset scf(\succsim)$ and $\nu'' \cup \nu \subset scf(\succsim)$. A contradiction results, since $\nu'$ and $\nu''$ match the same houses $H_{\nu'} = H_{\nu''} \neq \emptyset$ to different agents.

b) Fix $\nu$ and $i^*, j^*$ as set out above. Define a profile $\succsim$ such that $\succsim_i$: $\nu(i)$ for all $i \in N_\nu$, $\succsim_{i^*}$: $\nu(i^*), \nu'(i^*)$, and $\succsim_{j^*}$: $\nu(j^*), \nu''(j^*)$. If $i^*$ gets a house worse than his second ranked house $\nu'(i^*) = \nu(j^*)$, then $scf(\succsim)$ is dominated by $\nu'$ - a contradiction. So $scf(\succsim)(i^*) \in \{\nu(j^*), \nu(i^*)\}$. Mutatis mutandis $scf(\succsim)(j^*) \in \{\nu(j^*), \nu(i^*)\}$ must also hold. Since $M$ is by part c) of Theorem 1 Pareto optimal, $\nu \subset scf(\succsim)$ and we have $\nu \in \overline{V}$ as required.[3]

c) Fix any $\nu \in V$, $i \notin N_\nu$, and $\succsim \in \Omega$ with $\succsim_j$: $\nu(j)$ for all $j \in N_\nu$ while $\succsim_i$: $o$. Since $\nu \in V$ and since $scf$ is by part c) of Theorem 1 Pareto optimal $\nu \cup \{io\} \in scf(\succsim)$ holds, so $\nu \cup \{io\} \in \overline{V}$.

d) Fix a matching $\mu$ and profile $\succsim \in \Omega$ such that $\succsim_i$: $\mu(i)$ for all $i \in N$. Then $scf(\succsim) = \mu$, and therefore $\mu \in \overline{V}$, holds since $scf$ is by part c) of Theorem 1 Pareto optimal.

□

## 7.1 Hierarchical Exchange Mechanisms

Theorem 3 characterizes the set of core mechanisms for house allocation problems with outside options as the set of hierarchical exchange mechanisms. A trading algorithm together with a set of control rights $c$ is used to define

[3]note to self: I don't know fi $\nu \in v$ has to hold, has repercussion later in mini effec subm.

a slight extension of Papai's [13] hierarchical exchange mechanisms to cover the domains $\Omega^o$ and $\overline{\Omega}$ with and without outside options. A **control rights function** $c_\nu : \overline{H}_\nu \to \overline{N}_\nu$ at some submatching $\nu$ assigns each unmatched real house $h$ to some unmatched agent $c_\nu(h)$ as its owner. To calculate the outcome of a hierarchical exchange mechanism $c - HEX : \Omega \to \mathcal{M}$ at some fixed profile of preferences $\succsim \in \Omega$ use the following algorithm starting with $\nu_1 = \emptyset$ and assuming that $c_\nu$ is defined wherever needed and say that $c$ is the set of all these control rights.

**Round $r$:** If an agent prefers the outside option to all unmatched houses he points to himself, if not he points to his most preferred unmatched house. Each house points to its owner according to $c_{\nu_r}$. Choose a pointing cycle. If the cycle involves no house, match the only agent in the cycle with the outside option, if not match each agent in the cycle with the house he points to, set $\nu_{r+1}$ to the union of $\nu_r$ and all matches achieved in the current round. If $\nu_{r+1}$ is a matching, stop, otherwise go to Round $r + 1$.

If some round $r$ of the above algorithm for some profile of preferences starts with the submatching $\nu = \nu_r$, then $\nu$ is ***c*-relevant**.[4] If some round $r$ of the above algorithm and a following round $r'$ respectively start with $\nu = \nu_r$ and $\nu' = \nu_{r'}$ for some profile of preferences, then $\nu'$ is a ***c*-successor** to (the $c$-relevant submatching) $\nu$. For the set of control rights functions $c$ to define a hierarchical exchange mechanism, all ownership must continue, in the sense that $c_\nu(h) = c_{\nu'}(h)$ must hold for each $c$-relevant $\nu$, $c$-successor $\nu'$ to $\nu$, and house $h$ not matched by $\nu'$. For the case without outside options ($\Omega = \overline{\Omega}$) the condition that an agent prefers the outside option to all unmatched houses is only met when all real houses have been matched. So in this case the present definition coincides with Papai's [13] original definition of hierarchical exchange mechanisms.

The ***c*-set** consists in all $c$-relevant submatchings other than $\emptyset$, as well as all submatchings that arise out of the union of a $c$-relevant submatching and some matches between agents and the outside option. In the case with outside options ($\Omega = \Omega^o$), the $c$-set can be defined as the set of all $c$-relevant

[4]Since a submatching $\nu$ is used in the calculation of $c - HEX(\cdot)$ if and only if $\nu$ is $c$-relevant, the set $c$ equals the set $\{c_\nu : \nu \text{ is } c- \text{ relevant}\}$.

submatchings other than $\emptyset$, as unions between a $c$-relevant submatchings and some matches between agents and the outside option are themselves $c$-relevant.[5]

We can now formally state Theorem 3.

**Theorem 3** *A social choice function $scf : \Omega \to \mathcal{M}$ for matching problems with outside options is a core-implementable if and only if it is a hierarchical exchange mechanism. The maximal set of effective submatchings $\overline{V}$ that core-implements $scf$ coincides with the c-set derived from the control rights $c$ defining $scf$ as a hierarchical exchange mechanism $c - HEX$.*

Theorem 3 not only claims that the set of core mechanisms coincides with the set of hierarchical exchange mechanisms, it also claims a close connection between the control rights $c$ defining a hierarchical exchange mechanism and the maximal set of effective submatchings defining the same mechanism as a core mechanism. Any hierarchical exchange mechanism defined by the control rights $c$ can alternatively be represented as a core mechanism defined by the $c$-set as the set of effective submatchings. Conversely any core mechanism defined by a maximal set of effective submatchings $\overline{V}$ can be represented as a hierarchical exchange mechanism with control rights $c$ defined such that $c_\nu(h) = i$ holds for any $\nu$, $i$, and $h$ if and only if $\nu \in \overline{V}$ as well as $\nu \cup \{ih\} \in \overline{V}$.

## 7.2 Proof of Theorem 3

I prove Theorem 3 by induction over the number of agents $n$. To do so I define the claim made in Theorem 3 for all problems with $n$ or fewer agents as $A^o - n$:

$A - n$: A social choice function $scf : \Omega \to \mathcal{M}$ for matching problems with outside options and no more than $n$ agents is a core-implementable if and only if it is a hierarchical exchange mechanism. The maximal set of effective

[5] When there are not outside options ($\Omega = \overline{\Omega}$), such unions need not be $c$-relevant. Just think of a serial dictatorship with agent 1 as the first dictator. In this case $\{1o\}$ is in the $c$-set but not $c$-relevant since no preference in $\overline{\Omega}$ ranks the outside options $o$ about the real houses in $H$.

submatchings $\overline{V}$ that core-implements $scf$ coincides with the $c$-set derived from the control rights $c$ defining $scf$ as a hierarchical exchange mechanism $c - HEX$.

For the inductive proof of Theorem 3 it is useful to note that any hierarchical exchange mechanism $c-HEX$ together with any $c$-relevant submatching $\nu$ defines a submechanism $c - HEX^{-\nu}$. This submechanism $c - HEX^{-\nu}$ maps the preferences of $\overline{N}_\nu$ (all agents not matched by $\nu$) over $\overline{H}_\nu \cup \{o\}$ (all houses not matched by $\nu$) to (sub)matchings for $\overline{N}_\nu$ and $\overline{H}_\nu$. The submechanism $c - HEX^{-\nu}$ is itself a hierarchical exchange mechanism, defined by the control rights $c^{-\nu}$, so that $c - HEX^{-\nu} = c^{-\nu} - HEX$. To derive $c^{-\nu}$ from $c$, note that a submatching $\nu'$ is $c^{-\nu}$-relevant if and only if $\nu \cup \nu'$ is $c$-relevant. For each $c^{-\nu}$-relevant $\nu'$, $c^{-\nu}_{\nu'}$ is defined as $c_{\nu\cup\nu'}$. If $\succsim_i$: $\nu(i)$ holds for all $i \in N_\nu$, we have $c - HEX(\succsim) = \nu \cup c - HEX^{-\nu}(\overline{\succsim})$ where $\overline{\succsim}$ is as before the restriction of $\succsim$ to $\overline{N}_\nu$ and $\overline{H}_\nu$. The $c^{-\nu}$-set, consists in all submatchings $\nu'$ such that $\nu \cup \nu'$ is in the $c$-set. In the appendix I show that hierarchical exchange mechanisms are well-defined and that each hierarchical exchange mechanism is defined by a unique set of control rights $c$.

**Lemma 6** *A-2 holds.*

**Proof** **Case 1:** $N = \{1\}$. For $\overline{V}$ to core implement $scf$ $\{1h\} \in \overline{V}$ must for all $h \in H \cup \{o\}$, as $C(\overline{V}, \succsim)$ would contain at least two elements if $\{1h\} \notin \overline{V}$ and $\succsim_i$: $h, h'$ for some $h' \neq h$. Since $\{\{1h\} : h \in H \cup \{o\}\}$ is the set of all submatchings other than $\emptyset$, we have $\{\{1h\} : h \in H \cup \{o\}\} = \overline{V}$. For $c$ to define a hierarchical exchange mechanisms $c_\emptyset(h)$ must be defined for each house. Since there is only one agent we have $c_\emptyset(h) = 1$ for each $h \in H$. So the $c$-set equals $\{\{1h\} : h \in H \cup \{o\}\} = \overline{V}$. For any $\succsim \in \Omega$ the unique matching in $C(\overline{V}, \succsim)$ as well as $c - HEX(\succsim)$ match agent 1 with his favorite house, so $\{c - HEX(\succsim)\} = C(\overline{V}, \succsim)$ holds for all $\succsim \in \Omega$.

**Case 2** $N = \{1, 2\}$. Say $\overline{V}$ core-implements $scf$. Fix $\succsim_i$ such that $\succsim_i$: $h, o$ for $i = 1, 2$ for some house $h \in H$. Since $scf$ is by part c) of Lemma 1 Pareto optimal $scf(\succsim)$ must either equal $\{1o, 2h\}$ or $\{1h, 2o\}$. W.l.o.g say $scf(\succsim) = \{1h, 2o\}$. So $\{1o, 2h\}$ must be dominated by some $\nu \in \overline{V}$ at $\succsim$. Since agents 1 and 2 respectively get the second and top most preferred house in $\{1o, 2h\}$, $\nu(1) = h$ must hold. Since $h$ is the only house agent 2 (weakly) prefers to $h$, $\nu = \{1h\}$ must hold. So $\{1h\} \in \overline{V}$. Since $h \in H$ was chosen arbitrarily there must for each $h$ exist one agent $i \in \{1, 2\}$ such that $\{ih\} \in \overline{V}$. By part a) of Lemma 5 this agent $i$ is unique. By part b) of Lemma 5 any matching $\mu$ is contained in $\overline{V}$.

To define $c$, fix an arbitrary $h \in H$. Say $c_{\emptyset}(h) = i$ if $\{ih\} \in \overline{V}$. If $\{ih\} \in \overline{V}$, say $c_{\{ih\}}(h') = j \neq i$ for any $h' \in H \setminus \{h\}$. Let $c_{\{io\}}(h) = j \neq i$ for all $h \in H$ if $\{io\}$. Conversely if $scf$ is a hierarchical exchange mechanism defined by the control rights $c$ let $\overline{V}$ be the $c$-set.

To see that $\{c{-}HEX(\succsim)\} = C(\overline{V}, \succsim)$ for all $\succsim$, fix any $\succsim$. **Case 2.1:** $\succsim_i$: $h$ for some $\{ih\} \in \overline{V}$ which holds if and only if either $c_{\emptyset}(h) = i$ or $h = o$. For $\mu$ not to be dominated by $\{ih\}$ at $\succsim$, $\mu(i) = h$. Similarly $c{-}HEX(\succsim)(h) = i$ since either $c_{\emptyset}(h) = i$ or $h = o$. Now say $h'$ is such that $h' \succsim_j$: $(H \setminus \{h\}) \cup \{o\}$. Since $\{ih, jh'\} \in \overline{V}$ and $\{jh\} \notin \overline{V}$, $C(\overline{V}, \succsim) = \{\{ih, jh'\}\}$. By the definition of $c$, agent $j$ owns all houses in $H \setminus \{h\}$ at $\{ih\}$, so $c - HEX(\succsim)(j) = h'$ is $\succsim_j$-best house in $(H \setminus \{h\}) \cup \{o\}$ and $c{-}HEX(\succsim)$ equals the unique element of $C(\overline{V}, \succsim)$. **Case 2.2:** $\succsim_i$: $h$ for no $\{ih\} \in \overline{V}$. Then we have $\succsim_1$: $h'$ and $\succsim_2$: $h$ for $\{1h\}, \{2h'\} \in \overline{V}$ which holds if and only if $c_{\emptyset}(h) = 1$ and $c_{\emptyset}(h') = 2$. The first round of trade in $c - HEX(\succsim)$ finds the matching $\{1h', 2h\}$. Since $\{1h', 2h\} \in \overline{V}$, this matching is the unique element of $C(\overline{V}, \succsim)$. □

For the step of the induction assume that $A - n$ holds up to some $n \geq 2$. Consider a social choice function $scf : \Omega \to \mathcal{M}$ for $n+1$ agents. For Lemmas 7 through 10 assume that $scf$ is a core implemented by the maximal set of submdatchings $\overline{V}$. Say $h$ is **owned** in this mechanism if there exists a unique $i$ with $\{ih\} \in V$. If each house is owned, then $scf(\succsim)$ is, by Lemma 7, for each $\succsim \in \Omega$ consistent with the first round of a hierarchical exchange mechanism with $c_{\emptyset}(h) = i$ whenever $\{ih\} \in \overline{V}$. By Lemma 8 the ownership of all houses in $scf$ suffices for $scf$ to be a hierarchical exchange mechanism. Lemmas 9 and 10 show that this sufficient condition holds: all houses in $scf$ are owned. Lemma 11 assumes that $scf$ is a hierarchical exchange mechanism defined by some control rights $c$, and shows that $scf$ is core-implemented by the $c$-set as the set of effective submatchings. To state Lemma 7 define $\overline{V}_{\emptyset}$ as the set of **minimal effective submatchings** in $\overline{V}$, so $\nu \in \overline{V}_{\emptyset}$ implies that $\nu \in \overline{V}$ and $\nu' \subsetneq \nu \Rightarrow \nu' \notin \overline{V}$. Also say that $\nu$ arises out of a cycle of houses and their owners if there exists a pointing cycle where each $i \in N_\nu$ points to $\nu(i)$ and each $\nu(i) \in H_\nu$ points to $j$ if $\{j\nu(i)\} \in \overline{V}$.

**Lemma 7** *Say the maximal set of effective submatchings $\overline{V}$ core implements the social choice function $scf : \Omega \to \mathcal{M}$.*

*a) If house $h \in H$ is owned, then it is owned by a unique agent.*

*b) If $\nu$ arises out of a cycle among houses and their owners, then $\nu \in \overline{V}$.*

*c) If each house $h \in H$ is owned, then $\nu$ is a minimal effective submatching if and only if it arises out of a cycle of houses and their owners or is a match between one agent and the outside option.*

**Proof**

a) If $\{ih\}, \{jh\} \in \overline{V}$ then part a) of Lemma 5 implies $i = j$.

b) Fix a submatching $\nu$ that arises out of a cycle among houses and their owners. If $\mid N_\nu \mid = 1$ the result trivially holds. Suppose now that the result holds for all cycles matching $m$ or fewer agents, for some $m \geq 1$. Say $\mid N_\nu \mid = m+1$. Choose some $i^* \in N_\nu$ and $h^* \in H_\nu$ such that $i^*$ owns $h^*$, so that $\nu' = \{i^*h^*\} \in \overline{V}$. Define $j^*$ such that $\nu(j^*) = h^*$. Define $\nu'' : N_\nu \setminus \{i^*\} \to H_\nu \setminus \{\nu(i^*)\}$ such that $\nu''(j^*) = \nu(i^*)$ and $\nu''(i) = \nu(i)$ for all $i \in N_\nu \setminus \{i^*, j^*\}$. Since $\nu$ arises out of a cycle among houses and their owners and since $i^*$ owns $h^*$, $\nu''$ arises out of a cycle among the houses $H_\nu \setminus \{h^*\}$ and their owners (the set $N_\nu \setminus \{i^*\}$). The inductive hypothesis implies that $\nu'' \in \overline{V}$. Applying part b) of Lemma 5 we see that $\nu \in \overline{V}$.[6]

c) Fix any submatching $\nu$ that arises out of a cycle among houses and their owners, so that $\nu \in \overline{V}$ by part a) of the present Lemma. To see that $\nu \in V_\emptyset$ suppose $\nu' \subsetneq \nu$ held for some $\nu' \in \overline{V}$. Since $\nu$ arises out of a cycle among houses and their owners, there exists a submatching $\{ih\} \in \overline{V}$ such that $h \in H_{\nu'}$ but $i \notin N_{\nu'}$. But then part a) of Lemma 5 implies the contradiction $\nu' \notin \overline{V}$. So $\nu \in \overline{V}_\emptyset$.

[6]The proof of part b) also holds when cycles are replaced with any submatchings, so that $\nu \in \overline{V}$ holds if each agent matched by $\nu$ owns one of the houses matched by $\nu$.

Now fix any $\nu \in \overline{V}_\emptyset$ and any $h \in H_\nu$. The assumption that each house is owned together with part a) of the present Lemma implies that there exists a unique agent $i(h)$ such that $\{i(h)h\} \in \overline{V}$. Part a) of Lemma 5 implies that $i(h) \in N_\nu$. Now let each agent $i(h) \in N_\nu$ point to $\nu(i)$ and let each house $h \in H_\nu$ point to its owner $i(h)$. Since there are finitely many houses and agents there exists a pointing cycle. Say some such cycle yields the submatching $\nu'$. Since each agent points to $\nu(i)$, $\nu' \subset \nu$. Since $\nu \in \overline{V}_\emptyset$, $\nu = \nu'$ and $\nu$ itself must arise out of a pointing cycle among agents and owners.

Note that part c) of Lemma 5 implies $\{io\} \in \overline{V}$ for any $i$. Since $\nu' \subsetneq \nu$ implies $\nu' = \emptyset$ any such $\{io\}$ is in $\overline{V}_\emptyset$. So $\nu \in \overline{V}_\emptyset$ holds whether $\nu$ is a match between one agent and the outside option or arises out of a cycle of houses and their owners.

$\square$

Lemma 7 only contains the statements that are used in the sequel. The proof directly applies to slightly more general statements: the statements in Lemma 7 hold for the case without outside options. Part b) also holds when cycles are replaced with any submatchings, so that $\nu \in \overline{V}$ holds if each agent matched by $\nu$ owns one of the houses matched by $\nu$.

**Lemma 8** *Assume $A - n$ for some $n \geq 2$. Say the maximal set of effective submatchings $\overline{V}$ core-implements a social choice function $scf : \Omega \to \mathcal{M}$ for $n + 1$ agents. If each house in $scf$ is owned, then $scf$ is a hierarchical exchange mechanism defined by the control rights $c$ for which $\overline{V}$ is the c-set.*

**Proof** To define control rights $c$, fix any house $h \in H$. By assumption there exists an agent $i \in N$ with $\{ih\} \in \overline{V}$. By part a) of Lemma 7 this agent $i$ is unique, so $c_\emptyset(h) :\, = i$ if $\{ih\} \in \overline{V}$ is well-defined. By part c) of Lemma 7 the set of minimal effective submatchings $\overline{V}_\emptyset$ consists in all submatchings that either pair one agent with the outside option or arise out of a cycle among pairs $\{ih\} \in \overline{V}$. Now fix an arbitrary $\nu \in \overline{V}$. By the definition of $\overline{V}_\emptyset$ there exists some $\nu' \in \overline{V}_\emptyset$ such that $\nu' \subset \nu$. Part a) of Lemma 7.4 implies that $scf^{-\nu'}$ is core implemented by $V^{-\nu'} :\, = \{\nu'' : \nu' \cup \nu'' \in \overline{V}\}$. Since $scf^{-\nu'}$ is a core-implementable for $n$ or fewer agents it can by $A-n$ be represented as a hierarchical exchange mechanism $c-HEX^{-\nu'}$ defined by the set of control rights $c^{-\nu'}$, such that the $c^{-\nu'}$-set is the maximal set of effective submatchings that defines $c-HEX^{-\nu'}$ as a core mechanism. Since $V^{-\nu'}$ core implements the same social choice function $scf^{-\nu'}$, $V^{-\nu'}$ must be a subset of the $c^{-\nu'}$-set. Since $\nu' \subset \nu \in \overline{V}$, $\nu \setminus \nu' \in V^{-\nu'}$. By $A-n$ we can define $c_\nu$ as $c^{-\nu'}_{\nu\setminus\nu'}$.

To see that $c_\nu$ is well-defined if $\nu \neq \emptyset$, we need to show $c^{-\nu'}_{\nu\setminus\nu'} = c^{-\tilde{\nu}}_{\nu\setminus\tilde{\nu}}$ for any two different $\nu', \tilde{\nu} \in V_\emptyset$ with $\nu' \subset \nu$ and $\tilde{\nu} \subset \nu$. To do so, fix some $h \notin H_\nu$ and say $j' := c^{-\nu'}_{\nu\setminus\nu'}(h)$ and as $j'' := c^{-\tilde{\nu}}_{\nu\setminus\tilde{\nu}}(h)$. Define $\succsim$ such that $\succsim_i: \nu(i)$ for all $i \in N_\nu$, $\succsim_{j'}: h$, $\succsim_{j''}: h$. By part a) of Lemma 7.4 $scf(\succsim) = \nu' \cup scf^{-\nu'}(\overline{\succsim}) = \tilde{\nu} \cup scf^{-\tilde{\nu}}(\overline{\succsim}) = \nu \cup scf^{-\nu}(\overline{\succsim})$.[7] Since $\nu' \cup scf^{-\nu'}(\overline{\succsim}) = \nu \cup scf^{-\nu}(\overline{\succsim})$ we have $\nu \setminus \nu' \subset scf^{-\nu'}(\overline{\succsim})$. Since $j'$ top ranks and controls $h$ at $\nu \setminus \nu'$ in the hierarchical exchange mechanism defined by $c^{-\nu'}$ ($c^{-\nu'}_{\nu\setminus\nu'}(h) = j'$ as well as $\succsim_{j'}: h$), we have $h = scf^{-\nu'}(\overline{\succsim})(j') = scf(\succsim)(j')$. Applying the same arguments to $j''$ and $\tilde{\nu}$ we see that $scf(\succsim)(j'') = h$ also holds. So $j'$ must equal $j''$ and $c_\nu$ is well-defined for each $\nu$.

To see that ownership continues fix a $c$-relevant $\nu$ with a $c$-successor $\nu'$, an agent $i$, and a house $h \in \overline{H}_{\nu'}$ such that $c_\nu(h) = i$. If $\nu \neq \emptyset$ then $\nu'' \subset \nu$ holds for some $\nu'' \in \overline{V}_\emptyset$. In that case $\nu \setminus \nu''$ is $c^{\nu''}$-relevant ($\nu$ may equal $\nu''$). Since $\nu'$ is a $c$-successor to $\nu$, $\nu' \setminus \nu''$ is $c^{\nu''}$-successor to $\nu \setminus \nu''$. The definition of $c$, together with the continuity of ownership in $c^{\nu''} - HEX$ then implies $c_\nu(h) = c^{-\nu''}_{\nu\setminus\nu''}(h) = c^{-\nu''}_{\nu'\setminus\nu''}(h) = c_{\nu'}(h)$. Next consider the case that $\nu = \emptyset$ so that $\nu' \in \overline{V}$. Since $c_\emptyset(h) = i$ we have $\{ih\} \in \overline{V}$. Fix a profile $\succsim$ with $\succsim_i: h$. For $scf(\succsim)$ not to be dominated by $\{ih\}$ it must match $i$ with $h$. Since $i \notin N_\nu$ and $h \notin H_\nu$ and since $\succsim$ was chosen arbitrarily, except for $\succsim_i: h$, $ih \in scf^{-\nu'}(\overline{\succsim})$ holds for any $\overline{\succsim}$ with $\overline{\succsim}_i : h$. Since $scf^{-\nu} = c - HEX^{-\nu'}$, $i$ must own $h$ at the start of $c - HEX^{-\nu}$, so $i = c^{-\nu'}_\emptyset(h) = c_{\nu'}(h)$. Ownership, in sum, alwasy continues in the control rights $c$.

To see that $c - HEX(\succsim) = scf(\succsim)$ fix an arbitrary $\succsim \in \Omega$. Let each agent point to their most preferred house or to himself if he prefers the outside option. Let each house point to its owner according to $c_\emptyset$ - which is

[7] Here $\overline{\succsim}$ here stands for three different restrictions of $\succsim$ (to $\overline{N}_\nu$ and $\overline{H}_\nu$, to $\overline{N}_{\nu'}$ and $\overline{H}_{\nu'}$, and to $\overline{N}_{\tilde{\nu}}$ and $\overline{H}_{\tilde{\nu}}$).

by the definition of $c_\emptyset$ equivalent to $h$ points to $i$ if $\{ih\} \in V$. Say $\nu$ forms as a pointing cycle. The definition of $c-HEX$ implies $c-HEX(\succsim) = \nu \cup c-HEX^{-\nu}(\overline{\succsim})$. Part b) of Lemma 7 implies $\nu \in \overline{V}$. Since each agent $i \in N_\nu$ top ranks $\nu(i)$ part a) of Lemma 7.4 implies $scf(\succsim) = \nu \cup scf^{-\nu}(\overline{\succsim})$. By $A-n$ we have $scf^{-\nu}(\overline{\succsim}) = c-HEX^{-\nu}(\overline{\succsim})$, implying $scf(\succsim) = c-HEX(\succsim)$. So $M$ and $c-HEX$ are identical mechanisms. □

## 7.3 Each House in a Core Mechanisms is Owned

Assume $A-n$ and fix any core mechanism $M$ for $n+1$ agents. Lemma 10 establishes that according to $M$ all houses are owned. The proof of Lemma 10 requires the Lemma 9's result, that houses $e$ and $f$ must be owned by agent 1 or 2 in $M$ if $\{1e, 2f\}$ as well as $\{2f, 1e\}$ are effective.

**Lemma 9** *Assume $A-n$ for some $n \geq 2$. Say the set of effective submatchings $V$ defines a core mechanism $M : \Omega \to \mathcal{M}$ for $n+1$ agents.*

*Fix a core mechanism $M : \Omega \to \mathcal{M}$ for $n+1$ agents defined by the set of effective submatchings $V$ and two houses $e, f \in H$. If $\{\{1e, 2f\}, \{2f, 1e\}\} \subset V$, then agents 1 and 2 together own houses $e$ and $f$.*

**Proof** At any profile where 1 and 2 rank $e$ and $f$ in the top two positions, 1 and 2 must be matched with these two houses. To see this say some matching $\mu$ matches agent $i \in \{1, 2\}$ with a house $h \notin \{1, 2\}$. The submatching in $\{\{1e, 2f\}, \{2f, 1e\}\}$ which matches the other agent $j \in \{1, 2\}$ with his top house and $i$ with the other house in $\{e, f\}$ dominates $\mu$.

**Case 1:** $H = \{e, f\}$. Fix $\succsim$ such that $\succsim_i$: $f, e$ for all $i \in N$. By the above observation we may w.l.o.g. assume $\{1e, 2f\} \subset scf(\succsim)$ so that $\{1f, 2e, 3o, 4o, ...\}$ must at $\succsim$ be dominated by some $\nu \in V$. The only dominating submatchings at $\succsim$ are $\{2f\}$, $\{2f, ie\}$, $\{if\}$, $\{1f, ie\}$, $\{ie\}$, $\{ie, jf\}$ for $i, j \notin \{1, 2\}$. Part a) of Lemma 5, together with the assumption $\{\{1e, 2f\}, \{1f, 2e\}\} \subset V$, implies that only the first of these submatchings can be contained in $V$. So $V$ must contain $\{2f\}$.

**Case 2:** $n = 3$ and $\{e, f, g\} \subset H$. Consider the profile $\succsim$ with $\succsim_i$: $f, e, g$ for $i = 1, 2, 3$. By the introductory observation $scf(\succsim)$ matches agents 1 and 2 with houses $e$ and $f$. Since $M$ is, by part c) of Theorem 1, Pareto optimal, $scf(\succsim)$ must match 3 with $g$. W.l.o.g. say $scf(\succsim) = \{1e, 2f, 3g\}$, so that $\{\{1f, 3e\}, \{2f, 3e\}\} \cap V = \emptyset$ since $\{1f, 3e\}$ and $\{2f, 3e\}$ both dominate $\{1e, 2f, 3g\}$ at $\succsim$. Now consider $scf(\succsim_3^e, \succsim_{-3}^{fg})$. By Pareto optimality $3e \in scf(\succsim_3^e, \succsim_{-3}^{fg})$, so that $C(V, (\succsim_3^e, \succsim_{-3}^{fg})) \subset \{\{1f, 2g, 3e\}, \{1g, 2f, 3e\}\}$. Since $\{1f, 2g, 3e\}$ and $\{1g, 2f, 3e\}$ are respectively only dominated by $\{2f\}, \{2f, 3e\}$ and $\{1f\}, \{1f, 3e\}$, since $\{\{1f, 3e\}, \{2f, 3e\}\} \cap V = \emptyset$ and since $C(V, \succsim)$ is a singleton, $V$ must contain $\{1f\}$ or $\{2f\}$.

**Case 3:** $n > 3$ and $\{e, f, g\} \subset H$. Fix two agents other than agent 1 and 2, say 3 and 4. By part a) Lemma 7.4 there are two well-defined social choice functions $scf^{-3g}$ and $scf^{-4g}$ core implemented by $V^{-3g}$ and $V^{-4g}$. By $A - n$, $scf^{-3g}$ and $scf^{-4g}$ can be represented as hierarchical exchange mechanisms $c - HEX^{-3g}$ and $c - HEX^{-4g}$, defined by the control rights $c^{-3g}$ and $c^{-4g}$. Say $c_{\emptyset}^{-3g}(f) = k$ and $c_{\emptyset}^{-4g}(f) = j$. Part a) of Lemma 7.4 together with $\{\{1e, 2f\}, \{2f, 1e\}\} \subset V$ implies $\{\{1e, 2f\}, \{2f, 1e\}\} \subset V^{-3g}$ as well as $\{\{1e, 2f\}, \{2f, 1e\}\} \subset V^{-4g}$, so that $k$ and $j$ can only be agent 1 or 2. Part a) of Lemma 7.4 then implies implies $\{\{kf\}, \{3g, kf\}\} \cap V \neq \emptyset \neq \{\{jf\}, \{4g, jf\}\} \cap V$. If $\{kf\} \in V$ or $\{jf\} \in V$ we are done. If not $\{\{3g, kf\}, \{4g, jf\}\} \subset V$. Since $k, j \in \{1, 2\}$ we then obtain a contradiction to part a) of Lemma 5.

While the above arguments show that $f$ is owned by agent 1 or 2 (so $\{1f\} \in V$ or $\{2f\} \in V$), symmetric arguments imply that also $e$ is owned by agent 1 or 2. □

**Lemma 10** *Assume $A - n$ for some $n \geq 2$. Say maximal set of effective submatchings $\overline{V}$ defines a core mechanism $M : \Omega \to \mathcal{M}$ for $n+1$ agents. For each $h \in H$ there exists a unique agent $i \in N$ such that $\{ih\} \in \overline{V}$.*

**Proof** Fix a house $f \in H$.

**Case 1:** $H = \{f\}$. Say $\{1f, 2o, 3o, \dots\} = scf(\succsim)$ for $\succsim_i$: $f, o$ for all $i$. By part b) of Theorem 1 $\{1f, 2o, 3o, \dots\}$ equals $scf(\succsim')$ for any $\succsim'$ with $\succsim'_1$: $f$, so $\{1f\} \in \overline{V}$.

**Case 2:** There exists some $e \in H \setminus \{f\}$. Suppose $\{if\} \notin \overline{V}$ for all $i \in N$. By part a) of Lemma 7.4 $scf^{-1e}$ is core implementable and by $A-n$ representable as a hierarchical exchange mechanism $c-HEX^{-1e}$ with control rights $c^{-1e}$. W.l.o.g say $c_{\emptyset}^{-1e}(f) = 2$. By part a) of Lemma 7.4 the social choice function $scf^{-1e}$ is core implemented by the set of effective submatchings $V^{-1e} = \{\nu : \nu'' \cup \nu \in \overline{V} \text{ for some } \nu'' \subset \{1e\}\}$. So $c_{\emptyset}^{-1e}(f) = (2, o)$ implies $\{2f\} \in V^{-1e}$ which in turn implies $\{\{2f\}, \{1e, 2f\}\} \cap \overline{V} \neq \emptyset$. Since $\{2f\} \notin \overline{V}$ we have $\{1e, 2f\} \in \overline{V}$.

Applying the same reasoning to $scf^{-2e}$ and $j$ the initial owner of $f$ according to $c-HEX^{-2e}$, so $c_{\emptyset}^{-2e}(f) = j$ we see that $\{\{jf\}, \{2e, jf\}\} \subset \overline{V}$. Since $\{jf\} \notin \overline{V}$ we have $\{2e, jf\} \in \overline{V}$. If $j = 1$, then Lemma 9 yields $\{\{1f\}, \{2f\}\} \cap \overline{V} \neq \emptyset$ - a contradiction. So $j \notin \{1, 2\}$, w.l.o.g say $j = 3$, so that $\{2e, 3f\} \in \overline{V}$. Now apply the above reasoning to $scf^{-3e}$ and say $c_{\emptyset}^{-3e}(f) = k$, so that $\{\{kf\}, \{3e, kf\}\} \cap \overline{V} \neq \emptyset$ for some $k$. Since $\{kf\} \notin \overline{V}$ we have $\{3e, kf\} \in \overline{V}$. By part a) of Lemma 5 $\{3e, kf\} \in \overline{V}$ and $\{1e, 2f\} \in \overline{V}$ can only hold if $k = 1$.

So we in sum have $\{\{1e, 2f\}, \{2e, 3f\}, \{3e, 1f\}\} \subset \overline{V}$. But that means that we have a "roommate problem" at $\succsim$ with $\succsim_i$: $e, f, g$ for all $i$: It cannot be that $f \succ_2 scf(\succsim)(2)$, as $\{1e, 2f\}$ would then dominate $scf(\succsim)$. Mutatis mutandis, it can also not be that $f \succ_1 scf(\succsim)(1)$ and $f \succ_3 scf(\succsim)(3)$. A contradiction arises since at least one agent in $\{1, 2, 3\}$ must be matched with a house he deems worse than $f$. □

## 7.4 Hierarchical exchange mechanisms are core mechanism

**Lemma 11** *Assume $A - n$ for some $n \geq 2$. Say the control rights $c$ define a hierarchical exchange mechanism $c - HEX : \Omega \to \mathcal{M}$ for $n + 1$ agents. Say $V$ is the c-set. Then $c - HEX$ is a core mechanism $M$ and we have $\{c - HEX(\succsim)\} = C(V, \succsim)$ holds for all $\succsim \in \Omega$.*

**Proof** Fix any $\succsim \in \Omega$. Say a cycle at $\emptyset$ given $c-HEX(\succsim)$ yields $\sigma$, so that $c - HEX(\succsim) = \sigma \cup c - HEX^{-\sigma}(\overline{\succsim})$ and $\succsim_i$: $\sigma(i)$ holds for all $i \in N_\sigma$. By $A - n$, $\{HEX^{-\sigma}(\overline{\succsim})\} = C(V^{-\sigma}, \overline{\succsim})$ where $V^{-\sigma}$ is the $c^{-\sigma}$-set.

To see that $c-HEX(\succsim) \in C(V,\succsim)$, suppose $c-HEX(\succsim)$ was dominated by some $\nu$ in the $c$-set $V$ at $\succsim$. If $i \in N_\sigma \cap N_\nu$, then $\sigma(i)$ must equal $\nu(i)$ since $\nu(i) \succsim_i c - HEX(\succsim)(i) = \sigma(i) \succsim_i H$. If $N_\sigma = \{i\}$, then $\sigma(i)$ either a house owned by $i$ at $\emptyset$ or $\sigma(i) = o$. If $N_\sigma \neq \{i\}$, say $j$ is the owner of $\sigma(i)$ according to $c_\emptyset$. Since $\sigma(i) \in H_\nu$, since $\nu$ is $c$-relevant, and since $c_\emptyset(\sigma(i)) = j$, $\nu$ must match agent $j$. By the same token $\sigma$ must also match agent $j$. Proceeding inductively we see that $i \in N_\sigma \cap N_\nu$ implies $\sigma \subset \nu$. So either $\sigma \subset \nu$ or $N_\sigma \cap N_\nu = \emptyset$ must hold. Since $\succsim_i$: $\sigma(i)$ holds for all $i \in N_\sigma$ and since $\nu$ dominates $c - HEX(\succsim)$, $\nu \setminus \sigma \neq \emptyset$. So $\nu \setminus \sigma \in V^{-\sigma}$ dominates $HEX^{-\sigma}(\overline{\succsim})$ at $\overline{\succsim}$ a contradiction to $A - n$.

To see that $c-HEX(\succsim)$ is the unique element of $C(V,\succsim)$, suppose there exists a $\mu \in C(V,\succsim) \setminus \{c-HEX(\succsim)\}$. Since $\mu \in C(V,\succsim)$, $\sigma$ cannot dominate $\mu$ at $\succsim$. Since $\succsim_i$: $\sigma(i)$ holds for all $i \in N_\sigma$, $\sigma \subset \mu$ must hold. Since $c - HEX(\succsim) \neq \mu$, we then obtain $\mu \setminus \sigma \neq c - HEX^{-\sigma}(\overline{\succsim})$. By $A - n$ some $\nu \in V^{-\sigma}$ dominates $\mu \setminus \sigma$. But then $\sigma \cup \nu \in V$ dominates $\mu$ - a contradiction.

□

With Lemmas 6 through 11 in place, we are ready to prove Theorem 3.

**Proof of Theorem 3** Note that it suffices to show $A - n$ for all $n \in \mathbb{N}$. By Lemma 6 $A - n$ holds for $n = 1, 2$. Now assume that $A - n$ holds for some $n \geq 2$. Fix a mechanism $M : \Omega \to \mathcal{M}$ for $n + 1$ agents.

Say the maximal set of effective submatchings $\overline{V}$ defines $M$ as a core mechanism. By Lemma 10 there exists for each $h \in H$ a unique $i \in N$ such that $\{ih\} \in \overline{V}$. By Lemma 8, the latter observation is sufficient for $M$ to be a hierarchical exchange mechanism defined by the control rights $c$ where the $c$-set $V$ is a subset of $\overline{V}$. To see the converse say the control rights $c$ define $M$ as a hierarchical exchange mechanism $c - HEX$ and say that $V$ is the $c$-set. By Lemma 11 the mechanism $c - HEX$ equals the core mechanism defined by the $c$-set $V$ as the set of effective submatchings.

To see that $A - n$ holds it only remains to be shown that the $c$-set $V$ equals the maximal set of effective submatchings $\overline{V}$ if $c - HEX = M$ holds for $M$ the core mechanism defined by $\overline{V}$. Since the $c$-set $V$ defines the same core mechanism $M$, so $C(V,\succsim) = C(\overline{V},\succsim)$ for all $\succsim \in \Omega$, and since $\overline{V}$ is the *maximal* set of effective submatchings defining $M$, $V$ is a subset of $\overline{V}$. So suppose there exists some $\nu^* \in \overline{V} \setminus V$. Since $\nu^*$ is not in the $c$-set $\nu^*$ is not $c$-relevant. If there exists a $c$-relevant $\nu'$ such that $\nu' \subset \nu^*$, then $nu' \setminus \nu^*$ is not in the $c^{-\nu'}$-set. By Lemma $scf^{-nu'}$ is a core mechanism defined by $V^{-\nu'} = \{\nu : \nu' \cup \nu \in \overline{V}\}$. By $A - n$, $scf^{-\nu'} = c^{-\nu'} - HEX$ and the $c^{-nu'}$-set coincides with the maximal set of effective submatchings that core implements $scf^{-\nu'}$. A contradiction arises since $\nu^* \setminus \nu' \in V^{-\nu'}$ even though $\nu^* \setminus \nu'$ is not $c^{-\nu'}$-relevant.

Now fix any $i_1 \in N_{\nu^*}$. If $\nu^*(i_1)$ is owned by $i_1$ at $\emptyset$, so if $c_\emptyset(\nu^*(i_1)) = i_1$, then $\{i_1\nu^*(i_1)\}$ is $c$-relevant in contradiction to the preceding paragraph. So $c_\emptyset(\nu^*(i_1)) \neq i_1$ must hold, say $c_\emptyset(\nu^*(i_1)) = i_2$, since $\{i_2\nu^*(i_2)\} \in V$, $i_2 \in N_{\nu^*}$ must hold to avoid a contradiction with part a) of Lemma 5. If $\nu^*(i_2)$ is owned by $i_1$ or $i_2$ at $\emptyset$ then we found a $c$-relevant $\nu' \subset \nu^*$ in contradiction to the preceding paragraph. Proceeding inductively we obtain the contradiction that $\nu^*$ must match infinitely many agent. So we can conclude that $A-n+1$ holds. □

# 8 Private consumption with single peaked preferences

## 8.1 The single peaked matching domain

Consider a matching domain where $Y = \{1, \ldots, n\}$ is a set of houses. There is a linear order on all houses: they may for example be ordered by their size, so that house $i$ is smaller than house $j$ if $i < j$. There are equally many agents as there are houses. The set of agents $N$ also equals $\{1, \ldots, n\}$. All agents have single peaked preferences over houses. The domain of preferences $\Omega$ is furthermore restricted so that no agent is ever indifferent between any two houses.

The crawler, defined by [4] is a strategyproof and Pareto optimal mechanism, where each agent is initially endowed with one house. Say w.l.o.g. that agent $i$ initially owns house $i$ for all $i \in N$. The crawler $Crawl : \Omega \to \mathcal{M}$ is defined via a trading algorithm that screens agents and houses in ascending order. The smallest agent who wants to either stay or move to a yet smaller house leaves the mechanism with his most preferred house as his match. All agents who currently occupy houses at least as large as this agent's choice and smaller than the house he vacated "crawl" to the next largest house. The process is repeated until all agents are matched. To differentiate houses from agents, a generic house is now denoted at $h_t$. At Step $k$ the agents in $N^k$ and the houses in $H^k$ remain unmatched.

**Initialize**: $N^1 \leftarrow N$, $H^1 \leftarrow N$.

**Step** $k$:

**Indexation**: Let $m\colon =\mid N^k \mid$. Index agents and houses such that $N^k\colon = \{i_1, \ldots, i_m\}$ and $H^k\colon = \{h_1, \ldots, h_m\}$ with $i_t < i_{t+1}$ and $h_t < h_{t+1}$ for all $t \in \{1, \ldots, m-1\}$.

**Screening**: If $h_t \succ_{i_t} h_{t+1}$ holds for some $t$, let $t^*$ be the minimal such $t$. If not, let $t^*\colon = m$. Say that $h_r$ is the $\succsim_{i_{t^*}}$-best house in $H^k$.

**Matching**: Let $C(\succsim)(i_{t^*}) = h_r$.

**Updating**: Let $N^{k+1}\colon = N^k \setminus \{i_{t^*}\}$ and $H^{k+1}\colon = H^k \setminus \{h_r\}$. If $N^{k+1} = \emptyset$ terminate, otherwise go to Step $k+1$.

Since the crawler is strategyproof and Pareto optimal it makes sense to check wether the crawler is core-implementable. The next result shows that this is not the case.

**Proposition 1** *Let $n \geq 3$. Then the crawler $C$ is not core-implementable.*

**Proof** Suppose *Crawl* was core-implementable, so suppose the maximal effective set of submatchings $\overline{V}$ did core-implement the crawler. Since each agent gets the house he was endowed with if he top ranks this house, so $Crawl(\succsim)(i) = i$ if $\succsim_i\colon i$, the maxiaml set of effective subamtchings $\overline{V}$ contains any submatching $id_S$, mapping a subset $S \subset N$ such that $id(i) = i$, If $n > 3$ consider $id_{\{4,\ldots,n\}} = \nu' \in \overline{V}$. Lemma **??** implies that $\overline{V}^{-\nu'}$ core-implements $Crawl^{-\nu'}$, which is by the definition of the crawler again a crawler. For the crawler with $n$ agents to be core-implementable, the crawler with three agents must be core implementable. So say that $\overline{V}$ core-implements the crawler with three agents.

Call the houses initially owned by agents 1, 2, and 3 $e$, $f$, and $g$, so that $\{1e\}, \{2f\}, \{3g\} \in \overline{V}$. Now consider the profile $\succsim$ with $\succsim_1$: $g, f, e$, $\succsim_2$: $e, f, g$, $\succsim_3$: $g, f, e$. The outcome of the crawler is $\{1g, 2e, 3f\}$. So $\{1g, 2f, 3e\}$ must be dominated by some $\nu \in \overline{V}$ at $\succsim$. Since 1 and 3 get top house in $\{1g, 2f, 3e\}$, $2 \in N_\nu$. Since 2 only prefers $e$ to $f$, $2e \in \nu$. Since $\{1e\} \in \overline{V}$, $1 \in N_\nu$, since $\succsim_1$: $g, f, e$ and since $\nu$ dominates $\{1g, 2f, 3e\}$, $\{1g\} \in \nu$. Since $\{3g\} \in \overline{V}$, $3 \in N_\nu$. Since $\succsim_3$: $e, f, g$ and since $\nu$ dominates $\{1g, 2f, 3e\}$ at $\succsim$ $3e \in \nu$ a contradiction to $2e \in \nu$. □

## 8.2 The division problem with single peaked preferences

In the division problem a set of $n$ agents have to share an infinitely divisible unit of some resource. An allocation $x \in X$ assigns each agent $i$ a share $x_i \geq 0$, so that $\sum_{i \in N} x_i = 1$. Each agent has single peaked preferences over $[0, 1]$.

There are many strategyproof and Pareto optimal social choice functions for the division problem with single peaked preferences. Sprumont [20] characterized the uniform rule as the unique strategyproof, Pareto opitmal and anonymous social choice function $scf : \Omega \to \mathcal{M}$. To define anonymity say that $p : N \to N$ is a permutation on the set of agents $N$. The social choice function is the anonymous if $scf(\succsim_{p(1)}, \dots, \succsim_{p(n)})(i) = scf(succsim)(p(i))$ for all $i$. The uniform rule is then defined as follows:

$$U(\succsim)(i) = \begin{cases} \min\{top(\succsim_i), \overline{\lambda}(\succsim)\}, & \text{if } \sum_{i \in N} top(\succsim_i) \geq 1 \\ \max\{top(\succsim_i), \underline{\lambda}(\succsim)\}, & \text{if } \sum_{i \in N} top(\succsim_i) \leq 1 \end{cases}$$

Where $\overline{\lambda}(\succsim)$ solves the equation $\sum_{i \in N} \min\{top(\succsim_i), \overline{\lambda}(\succsim)\} = 1$ $\underline{\lambda}(\succsim)$ solves the equation $\sum_{i \in N} \max\{top(\succsim_i), \underline{\lambda}(\succsim)\} = 1$.

In division problems with single peaked preferences and two agents, dictatorship is the only core implementable social choice function. Since the uniform rule for any number of agents embeds the division problem with two agents, the uniform rule is not core implementable. There are however some other core implementable mechanisms for division problems with single peaked problems.

**Theorem 4** *Say $X, \Omega$ is a division problem with single peaked preferences and $n$ agents.*

*a) If $n = 2$, dictatorship is the unique core implementable social choice function $scf : \Omega \to \mathcal{M}$.*

*b) The uniform rule $U : \Omega \to \mathcal{M}$ is not core-implementable.*

*c) If $n = 3$ there exists a core-implementable social choice function that differs from serial dictatorship.*

**Proof**

a) Suppose the set $V$ of effective suballocations core-implements $scf$. Suppose $\{1x_1^*\}$ as well as $\{2x_2^*\}$ are contained in $\overline{V}$. Since $scf(\succsim)(i) = x_i^*$ if $top(\succsim_i) = x_i^*$, $\{1x_1^*, 2x_2^*\}$ must be an allocation, so that $x_1^* + x_2^* = 1$ must hold. W.o.l.g assume that $x_1^* \le \frac{1}{2}$ and $x_2^* \ge \frac{1}{2}$. Now consider $\succsim'$ such that each agent has distance preferences with $top(\succsim_1') = .8$ and $top(\succsim_2') = .4$. Say $scf(\succsim') = \{1x_1', 2x_2'\}$. By part c) of Theorem 1 $\{1x_1', 2x_2'\}$ is Pareto optimal at $\succsim'$ such that $x_1' \in [.6, .8]$ and $x_2' \in [.2, .4]$. If $x_1' \ne .8$ then $\{1top(\succsim_1', 2(1 - top(\succsim_1')))\}$ must be dominated at $\succsim'$. Since agent 1 gets his top choice in $\{1top(\succsim_1', 2(1 - top(\succsim_1')))\}$ agent 2 must be effective for some $\tilde{x}_2 \in (.2, .4]$. Now consider the a profile $\succsim''$ with $top(\succsim_1'') = x_1^*$ and $\top(\succsim_2'') = \tilde{x}_2$. Since $\{1x_1^*\}, \{2\tilde{x}_2\} \in V$, $scf(\succsim'') = \{1x^*, 2\tilde{x}_2\}$. A contradiction arises since $\{1x^*, 2\tilde{x}_2\}$ is not an allocation as $x^* + \tilde{x}_2 < 1$. Mutatis mutandis we obtain a similar contradiction if $x_2' \ne .5$, so that $\{1(1 - top(\succsim_2'), 2top(\succsim_2'))\}$ must be dominated at $\succsim'$.

The preceding paragraph implies that only one agent $i = 1$ or $i = 2$ can be effective for any $x_i$. W.l.o.g. say $\{1x_1\} \in V$ implies $i = 1$. If $\{x_1 : \{1x_1\} \in \overline{V}\}$ is not dense in $[0, 1]$ then the arguments given the proof of Theorem 2 imply that $C(\overline{V}, \succsim)$ contains multiple elements for some $\succsim$. So $\{x_1 : \{1x_1\} \in \overline{V}\}$ must be dense in $[0, 1]$. The arguments given the proof of Theorem 2 then imply that $\{1x_1\} \in \overline{V}$ for any $x_1 \in [0,1]$. So $scf(\succsim)(1) = top(\succsim_1)$ for all $\succsim \in \Omega$ and $sc$ is indeed a dictatorship.

b) If there are exactly two agents then the claim follows from part a) of the present result. So suppose the maximal set of effective suballocations $\overline{V}$ did core-implement $U$ for $n > 2$. Fix $\nu'$ such that $N_{\nu'} = N \setminus \{1,2\}$ and $\nu'(i) = \frac{1}{n}$. Since $U(\succsim)(i) = \frac{1}{n}$ for each $\succsim$ with $top(\succsim_i) = \nu'(i) = \frac{1}{n}$, $\nu' \in \overline{V}$. The social choice function $U^{-\nu'} : \Omega^{-\nu'} \to \overline{\mathcal{M}}$ is therefore by Lemma **??** core-implementable. By the definition of $U$, $U^{-\nu'}$ is the uniform rule for two agents. A contradiction then arises since $U^{nu'}$ is by part a) of the present result not core-implementable.

c) Let $N = \{i_1, i_2, i_3\}$. Define a set of effective suballocations.

$$V\colon = \{\{i_1x_1\} : x_1 \in [0, \frac{1}{2}]\} \cup \{\{i_2x_2\} : x_2 \in [0, \frac{1}{2}]\} \cup \{\{i_1x_1, i_2x_2\} : x_1 + x_2 \leq 1\}.$$

To see that $V$ defines a core mechanism fix an arbitrary $\succsim$. Say $top(\succsim_{i_1}) = x_1$, $top(\succsim_{i_2}) = x_2$. If $x_1 > \frac{1}{2}$ and $x_2 > \frac{1}{2}$ then $\{i_1\frac{1}{2}, i_2\frac{1}{2}, i_30\}$ is the unique element of the $C(V,\succsim)$. To see this suppose some suballocation $\nu$ in $V$ did dominate $\{i_1\frac{1}{2}, i_2\frac{1}{2}, i_30\}$ at $\succsim$. Since 3 is not matched by any submatching in $V$, $\nu$ matches either 1 or 2 or both. Say w.l.o.g that $1 \in N_\nu$. For 1 to prefer $\nu(1)$ to $\frac{1}{2}$, $\nu(1) > \frac{1}{2}$ must hold. For $\nu$ to be in $V$, $\nu$ must then also match 2. But since $\nu(1) > \frac{1}{2}$, $\nu(2)$ must be smaller than $\frac{1}{2}$, so that $\frac{1}{2} \succ_2 \nu(2)$, implying that $\nu$ does not improve upon $\{i_1\frac{1}{2}, i_2\frac{1}{2}, i_30\}$. To see that $\{i_1\frac{1}{2}, i_2\frac{1}{2}, i_30\}$ is the unique element of $C(V,\succsim)$, suppose some alternative $\mu$ was contained in $C(V,\succsim)$. For $\mu$ to differ from $\{i_1\frac{1}{2}, i_2\frac{1}{2}, i_30\}$ either $\mu(i_1)$ or $\mu(i_2)$ (or both) must be smaller than $\frac{1}{2}$, say w.l.o.g $\mu(i_1) < frac12$. Since $top(\succsim_{i_1}) = x_1 > \frac{1}{2}$ and since $\succsim_1$ is single peaked $\frac{1}{2} \succ_1 \mu(1)$. Since $\{i_1\frac{1}{2}\} \in V$, there is an effective submatching in $V$, namely $\{i_1\frac{1}{2}\}$ that improves on $\mu$ at $\succsim$. So $C(V,\succsim) = \{\{i_1\frac{1}{2}, i_2\frac{1}{2}, i_30\}\}$.

Now say $x_1 \leq \frac{1}{2}$ (noting that the same arguments apply mutatis mutandis to the case where $x_2 \leq \frac{1}{2}$). In this case $\{1x_1, \tilde{x}_2, 1 - x_1 - x_2\}$

is the unique element of $C(V, \succsim)$, where $\tilde{x}_2 \colon \min\{x_2, 1 - x_1\}$. To see that $\{1x_1, \tilde{x}_2, 1 - x_1 - x_2\}$ is an element of $C(V, \succsim)$, suppose $\nu \in V$ did improve on $\{1x_1, \tilde{x}_2, 1 - x_1 - x_2\}$ at $\succsim$. Since no submatching in $V$ matches agent 3 and since $\{1x_1, \tilde{x}_2, 1 - x_1 - x_2\}$ matches agent 1 with his top choice, $\nu$ must match agent 2 and $\nu(2) \succ_2 \tilde{x}_2$ must hold. If $\tilde{x_2} = x_2$ we obtain a contradiction since $x_2$ is the $\succsim_2$-best choice. If $\tilde{x}_2 \neq x_2$ then $x_2 > \tilde{x}_2 = 1 - x_1$. For $\nu(2) \succ_2 \tilde{x}_2$ to hold $\nu(2) > \tilde{x}_2$ must hold. Since $\tilde{x}_2 = 1 - x_1 \geq \frac{1}{2}$, $\nu(2) > \frac{1}{2}$, implying that $\nu$ must also match 1. Since $\nu(2) > \tilde{x}_2 = 1 - x_1$, $\nu(1) < x_1$. The latter implies that $x_1 \succ_1 \nu(1)$ and $\nu$ cannot improve on $\{1x_1, \tilde{x}_2, 1 - x_1 - x_2\}$ at $\succsim$. Now suppose some $\mu \neq \{1x_1, \tilde{x}_2, 1 - x_1 - x_2\}$ was in $C(V, \succsim)$. Since $\{1x_1\} \in V$ $\mu(1)$ must equal 1 for $\{1x_1\}$ not to improve upon $x_1$. Since any $\{1x_1, 2x_2\}$ with $x_1 + x_2 \leq 1$ is in $V$, $\mu(2)$ must equal $\tilde{x}_2$.

□

# 9 other

The goal of mechanism design is the define game forms that implement social choice functions. A game form in turn implements a social choice function if for each profile of preferences the outcome of the social choice function coincides with the equilibrium outcome of the game induced by the game form and the given profile of preferences. Different notions of equilibrium then correspond to different notions of implementation. Here I propose to use cooperative game theory to define a new concept of implementation. According to this new concept a cooperative game form implements a social choice function if for each profile of preferences the outcome of the social choice function is the unique element of the core of the game induced by the game form at the given profile of preferences.

The crawler on single peaked domain is not a core mechanism.

part b) Lemma 2 which basically shows trading cycles conditioning on initial ownership does not apply: this uses more rich preferences.

Consider crawler where initial ownership is such that $\{1e\}, \{2f\}, \{3g\}$. Assumption of such an initial ownership is w.o.l.g since each initital owner gets the owned house whenever he top ranks it. say line is $e, f, g$

now consider the profile where $\succsim_1$: $g, f, e$, $\succsim_2$: $e, f, g$, $\succsim_3$: $g, f, e$. The outcome of the crawler is $\{1g, 2e, 3f\}$. So $\{1g, 2f, 3e\}$ must be dominated by some $\nu \in V$ at $\succsim$. since 1 and 3 get top house in $\{1g, 2f, 3e\}$, $2 \in N_\nu$. Since 2 only prefers $e$ to $f$, $2e \in \nu$. Since $\{1e\} \in V$, $1 \in N_\nu$, since $\succsim_1$: $g, f, e$ and since $\nu$ dominates $\{1g, 2f, 3e\}$, $\{1g\} \in \nu$. Since $\{3g\} \in V$, $3 \in N_\nu$. Since $\succsim_3$: $e, f, g$ and since $\nu$ dominates $\{1g, 2f, 3e\}$ at $\succsim$ $3e \in \nu$ a contradiction to $2e \in \nu$. So even in the smallest case where the crawler and Gale's top trading cycles differ, the crawler is not a core mechanism on the single peaked domain.

Story: smaller domain more core mechanisms. Here domain without outside options is subdomain of domain with outside options. even smaller: single peaked. But crawler is not. Would have to extend the proof of sp to single peaked domain. bc then have: another PO, sp mechanism (is it nb?) but is not core mechanism.

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