# Prices and Inflation when Government Bonds are Net Wealth

Marcus Hagedorn*
University of Oslo and CEPR

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**Abstract**

In this paper I show that models where government bonds are net wealth - their value exceeds the value of tax liabilities (Barro (1974)) - offer a new perspective on various issues in monetary economics.

Foremost, prices and inflation are jointly and uniquely determined by fiscal and monetary policy. In contrast to the conventional view the long-run inflation rate here is, in the absence of output growth and even when monetary policy operates an interest rate rule with a different inflation target, equal to the growth rate of nominal fiscal variables, which are controlled by fiscal policy.

This novel theory also offers a different perspective on a range of important issues. These include the fiscal and monetary transmission mechanism, policies at the zero-lower bound, U.S. inflation history, and recent attempts to stimulate inflation in the Euro area. Several puzzles disappear which arise in New Keyensian models during a liquidity trap. Forward guidance has negligible output effects, the size of the fiscal multiplier decreases if prices are less sticky, and technological regress reduces output.

To derive my findings I first use a reduced form approach where households derive utility from holding bonds to explain my findings. I show how and for which policy rules the price level is globally determinate. I then use a Bewley-Imrohoroglu-Huggett-Aiyagari heterogenous agent incomplete markets model and show that the reduced form results carry over to this model.

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*University of Oslo, Department of Economics, Box 1095 Blindern, 0317 Oslo, Norway. Email: marcus.hagedorn07@gmail.com

# 1 Introduction

The prevailing view on prices and inflation and the conduct of monetary policy (Woodford (2003) ,Galí (2015)) is that monetary policy works through setting nominal interest rates, that monetary policy controls the inflation rate and that prices are determinate if policy responds sufficiently strongly to inflation - the Taylor principle. Money supply is not controlled by monetary policy and money has only a limited role. Often money in circulation is not part of the model and money serves only as a unit of account. Or a money demand equation is added which determines money endogenously as clearing the money market. Fiscal policy is largely irrelevant in this view.

In this paper I propose that one modification of existing models where government bonds are net wealth - the value exceeds the value of tax liabilities (Barro (1974)) - and government bonds are nominal offers a new and different perspective on these topics. While monetary policy operates as in the prevailing view and controls the nominal interest rate fiscal policy is now assigned a significant role. Two results stand out.

First, in contrast to the conventional view the long-run inflation rate here is, in the absence of output growth and even when monetary policy operates an interest rate rule with a different inflation target, equal to the growth rate of nominal fiscal variables, which are controlled by fiscal policy. In the short-run prices and inflation are jointly and uniquely determined by fiscal and monetary policy. In contrast to conventional wisdom, a tough, independent central bank not only is insufficient to guarantee price stability in the long-run, but also has no direct control over long-run inflation even if it follows an interest rate rule which satisfies the Taylor principle. By controlling the nominal anchor, the fiscal policy always wins out when it comes to long-run inflation. What central bank independence ensures however is that the treasury cannot impose fiscal policies on the central bank, e.g. monetizing its debt.

Second, the price level is globally determinate when monetary policy works through setting an arbitrary sequence of nominal interest rates, for example through an interest rate peg or an interest rate rule. Fiscal policy sets sequences of nominal government spending, taxes, and government debt, for example through a fiscal rule, and these sequences satisfy the present value government budget constraint at all times and for all prices. In this environment, I show that the steady-state price level is determinate even if nominal interest rates are constant and I derive conditions on policy rules to ensure global determinacy.

To understand these results it is sufficient to combine a few simple insights and it is instructive to start with the steady state and assume no economic growth.

First, when government bonds are net worth shifts in the stock of real public debt affect real aggregate demand (Barro (1974)). When in addition government bonds are nominal, shifts in the

price level shift the real value of debt and thus affect real aggregate demand. The price level is then determined such that aggregate demand equals aggregate supply.
Second, there is not a unique steady-state real interest rate determined by households' discount factor but instead it depends on the amount of real bonds available. When government bonds are net worth, households are willing to accept a lower return than suggested by their discount factor as bonds provide some extra service. Bonds have a "liquidity premium". Under standard concavity assumptions, the liquidity premium is the lower the more bonds households have. To absorb more government bonds households thus require a higher real interest rate. This shows that depending on the amount of bonds available a continuum of steady-state real interest rates is feasible. Monetary and fiscal policy now jointly choose one out of this continuum of potential steady-state real interest rates. Monetary policy sets the steady-state nominal interest rate. Fiscal policy sets the growth rate of nominal government debt. In a steady state, the value of real government debt is constant, such that the steady-state condition for fiscal policy is that the growth rate of nominal debt is equal to the inflation rate (in the absence of economic growth). The real interest rate is then determined by the Fisher equation as the ratio of the nominal interest rate and the inflation rate. Clearly this logic to determine the long-run inflation rate does not apply if the steady-state real interest rate is pinned down by the discount factor as it is the case in standard New Keynesian models.

Outside steady states I show determinacy for all monetary policy rules, those not responding, weakly responding or strongly responding to prices. Interestingly, if responding too strongly to price increases fiscal policy may induce (not remove) indeterminacy, which in turn requires more active monetary policy to reestablish determinacy. I provide a characterization of these determinacy conditions. I also provide several proofs to rule out hyperinflations and hyperdeflations. Interestingly while price stickiness is largely irrelevant for steady-state and local determinacy it serves to rule out hyperinflations. An equilibrium hyperinflation requires that the price level is eventually infinite. Setting the price to infinity is however not an optimal decision for a firm subject to price adjustment costs and frictions.

This novel theory of price and inflation determination also suggests to rethink various other issues in monetary economics. Applied to recent attempts by the ECB to increase inflation in the Euro area, the findings in this paper suggest that these efforts are unlikely to be successful. Instead higher inflation would require an expansion of nominal fiscal spending by Euro area member states to stimulate nominal demand, assigning an important role to large countries such as Germany. A fiscal stimulus by a small country would have very little impact on inflation, as it has only a negligible effect on area-wide demand, but would lead to a real appreciation (with likely adverse economic consequences) for this small country.

Applied to the growing concerns that the US or the world economy may be stuck in a liquidity trap with zero nominal and real interest rates for an extended time, the findings in this paper suggest an easy solution. Although the ZLB prevents further cuts of the nominal interest rate, fiscal policy can increase the growth rate of nominal spending and therefore the inflation rate, leading to lower real interest rates, provided that this policy is sufficiently persistent and credible. If instead fiscal policy continues its current austerity plan bringing low inflation rates to around zero, then the real interest rate will hover around zero too, even in the long run.

The theory set out in this paper also offers a different perspective on US inflation history. After experiencing high inflation rates in the 1970s, the 1980s saw success in keeping inflation rates low. In the standard interpretation, central banks eventually recognized that keeping inflation low was their primary objective and as a consequence, were successful in doing so. The framework proposed in this paper suggests that it was not the change in the conduct of monetary policy that kept inflation in check, but a shift to a less expansionary fiscal policy during the Reagan administration, perhaps forced on by the prolonged high nominal interest rates set by central banks under chairman Paul Volcker and resulting high deficits.

When government bonds are net worth several puzzles disappear, which are otherwise observed during a liquidity trap in New Keynesian models. There is no forward guidance puzzle, as commitment to future monetary policy has only negligible effects here. Improvements in technology increase output when the price level is determinate, in contrast to New Keynesian models. The size of the fiscal multiplier becomes smaller if prices are less sticky, whereas it gets arbitrarily large in New Keynesian models.

Furthermore, this new view allows for policy analyses not only when the zero lower bound is binding, but also more generally when not using an interest rate rule satisfying the Taylor principle. In standard models the nominal interest rate has to respond to inflation aggressively enough to guarantee a locally determinate inflation rate and to overcome the indeterminacy pointed out in Sargent and Wallace (1975).[1] To avoid indeterminacy, policy analysis is then restricted to this subset of aggressive monetary policies, and hence also during periods when actual policy did not follow the Taylor principle. Indeed, Clarida et al. (2000) find that US monetary policy pre-1979 increased nominal interests rates by less than expected inflation. This case violated the assumptions needed for determinacy and implied that self-fulfilling inflation bursts cannot be ruled out, and that further analysis either is precluded or would have to invoke further assumptions to overcome indeterminacy. The theory proposed here offers an alternative theory of price level determination, where monetary policy can be represented by any arbitrary sequence of nominal

[1] Without determinacy agents long-run price expectations are not anchored and beliefs can coordinate on a high price level, unchanged prices or even lower prices, with potentially quite different short-run implications. Clearly a predetermined price, for example if set in the previous period, does not overcome this indeterminacy issue.

interest rates or an interest rate rule (not) satisfying the Taylor principle. They can do so simply because for each sequence and each rule, determinacy of the equilibrium is ensured. This allows study of fiscal and monetary policy coordination without restrictions on policy rules and for both time periods, pre- and post-1979. For example, fiscal policy could be expansionary to stimulate the economy, while at the same time monetary policy keeps nominal interest rates constant so as not to offset the stimulus. It is also conceivable that the pre-1979 US experience may just reflect a lower level of concern about inflation among both monetary and fiscal policy makers than was the case during the Volcker-Greenspan era, without inducing indeterminacy.[2]

It is important to emphasize, however, that these results do not hold in every model where Ricardian equivalence fails. For example, the price level is not determinate in an economy where a fraction of households are hand-to-mouth consumers while the remaining households act according to the permanent income hypothesis (PIH). The reason for the indeterminacy is that government bonds are not net worth since only permanent income households hold bonds and shifts in the value of public debt have no aggregate demand effects but only shift consumption from one group to the other. Similar arguments apply to the perpetual youth model and variants of it (Yaari (1965), Blanchard (1985), Bénassy (2005, 2008)), where Ricardian equivalence does not hold but the price level is indeterminate, as I explain in the Appendix.

To derive my results formally I start with a reduced form model in Section 2 where government bonds have net worth because they are an argument of the utility function. This simplification allows for a clear and accessible exposition which explains how the steady-state price level is determined, when the economy is locally determinate, how to rule out hyperinflations and hyperdeflations and how the steady-state inflation rate is determined. I also allows to explain the mechanism behind the determinacy result; why models where government bonds are net worth deliver determinacy and why models with zero net worth lead to indeterminacy; why fiscal policy has to be partially nominal; why this is not the Fiscal Theory of the Price Level (FTPL); and why adding capital or cash to the model does not alter these conclusions. In Section 3 I move to a Bewley-Imrohoroglu-Huggett-Aiyagari heterogeneous agents incomplete markets economy and show that the results from the reduced form analysis carry over to this model class. One reason to study incomplete market models is that it is desirable to have a model where government bonds have net worth not because bonds enter the utility function but because net worth arises endogenously as a result of household optimization. In incomplete market models government bonds are net worth since they allow households to better smooth consumption in response to insurable idiosyncratic income shocks. Another reason is that government bonds are (large) net worth in

[2]Price level indeterminacy also implies that adding a small amount of nominal rigidities renders purely real models unstable (Kocherlakota (2016)) implying that "purely real models are too fragile to be of practical value". Price level determinacy overcomes this unpleasant conclusion by removing this instability.

incomplete market models, a questioned that motivated a large literature before and after Barro (1974). A large empirical literature (Campbell and Deaton (1989), Attanasio and Davis (1996), Johnson et al. (2006), Blundell et al. (2008), Attanasio and Pavoni (2011) among many others) has documented significant deviations in households consumption behavior from the complete markets benchmark. There is for example a large heterogeneity in the marginal propensity to consume (MPC) across households, where some households behave according to the permanent income hypothesis and have a small MPC while other have a MPC of an order of magnitude higher. As incomplete market models match these micro-consumption facts, the value of government bonds is large in these models. While it is sufficient for the results in this paper that government bonds have some possibly arbitrarily small net worth, quantitative work is motivated by the observation that the deviation from complete market models, where bonds are not net worth, is large. This brings me to the last reason to consider incomplete market models.

A growing literature has recently emerged which incorporates price rigidities into incomplete market models.[3] One motivation to do so is that the textbook incomplete market model while able to generate a realistic distribution of marginal propensities to consume, neither allows output to be demand-determined as prices are fully flexible, nor does it take the zero lower bound (ZLB) into account. These factors limit its applicability to many questions raised by the Great Recession. Adding a nominal side to the model, allowing for price rigidities and taking the ZLB into account on the other hand forces us to address the same questions we confront in complete market models: How is the price level determinate? What type of monetary and fiscal policies guarantee determinacy? Without answers to these question, for example, one of the most important policy questions, the magnitude of the fiscal multiplier, cannot be addressed satisfactorily. This paper provides these answers and shows that they are quite different from the standard analysis based on complete markets. Section 4 concludes and discusses several contributions which use the insights of this paper to consider forward guidance, the fiscal multiplier and several puzzles which arise in New Keyensian models during a liquidity trap.

[3]Kaplan et al. (2016), Auclert (2016) and Lütticke (2015) study monetary policy in a model with incomplete markets and pricing frictions, however with a different focus. Whereas these authors emphasize and quantify several redistributive channels of the transmission mechanism of monetary policy which are absent in standard complete market models, the price level is endogenously determinate in equilibrium only in my paper. Earlier contributions are Oh and Reis (2012) and Guerrieri and Lorenzoni (2015), who were among the first to add nominal rigidities to a Bewley-Imrohoroglu-Huggett-Aiyagari model and Gornemann et al. (2012) who were the first to study monetary policy in the same environment. More recent contributions include McKay and Reis (2016) (impact of automatic stabilizers), McKay et al. (2015) (forward guidance), Bayer et al. (2015) (impact of time-varying income risk), Ravn and Sterk (2013) (increase in uncertainty causes a recession) and Den Haan et al. (2015) (increase in precautionary savings magnifies deflationary recessions).

# 2 Price Level and Inflation Determinacy when Government Bonds are Net Wealth

In this Section, I use a reduced form approach to explain why and how the price level and inflation are determinate if government bonds are net wealth and government bonds are nominal. For pedagogical reasons I start with an infinite horizon representative agent economy where households derive utility from consumption, leisure and money holdings. Importantly, monetary policy works through setting nominal interest rate and I assume first that the nominal interest rate is constant, the scenario considered by Sargent and Wallace (1975). Money supply is not set by monetary policy but instead adjusts endogenously to satisfy households' money demand at the the nominal interest rate set by the central bank. In this world I show that the steady-state price level is indeterminate as shown by Sargent and Wallace (1975) and that price stickiness does not change this conclusion, echo findings in Nakajima and Polemarchakis (2005). I then assume that households in addition also derive utility from holding real bonds, a reduced form of modeling that government bonds have value (which I use interchangeably with being net worth in what follows). In this extended model I show three layers of determinacy: The steady-state price level is determinate, the economy is locally determinate and I rule out hyperinflations and hyperdeflations. Whereas price stickiness is irrelevant for the first two layers, it becomes important for the latter one. I also characterize the properties of fiscal and monetary policy rules which imply local determinacy. In Section 3 I go beyond the reduced form analysis and present a heterogeneous agent incomplete markets model of the Bewley-Imrohoroglu-Huggett-Aiyagari type and I show that the same arguments and the same results, derived using the reduced form analysis, apply.

## 2.1 Indeterminacy when Government Bonds are not Net Wealth

The economy is populated by a large number of identical household with preferences over consumption $c_t$, hours worked $h_t$, and real money balances $m_t$

$$\sum_{t=0}^{\infty} \beta^t (u(c_t) - v(h_t) + \mu(m_t)), \tag{1}$$

where $u$ and $\chi$ are increasing and concave and $v$ is increasing and convex Households carry nominal money $M_{t-1}$ and nominal bonds $B_{t-1}$ into period $t$ from the previous period and acquire money $M_t$ and nominal bonds $B_t$ in that period. The period $t$ price level is $P_t$ so that nominal consumption expenditures are $P_t c_t$ and real balances $m_t = M_t/P_t$. The real wage is $w_t$, the time endowment is normalized to one and labor income is $w_t h_t$. Households then maximize utility for a budget constraint

$$M_t + B_t = R_t B_{t-1} + M_{t-1} + P_t(1-\tau)w_t h_t - P_t c_t - T_t + P_t d_t, \tag{2}$$

where $R_t = 1 + i_t$ is the nominal interest rate on bonds, $T_t$ are lump-sum taxes, $\tau$ is a linear tax on labor income and $d_t$ are real dividend payments. The real interest rate $1 + r_t = R_t \frac{P_{t-1}}{P_t}$.[4] I now follow the standards in the New Keynesian literature to add price stickiness to the model.

**Final Good Producer** A competitive representative final goods producer aggregates a continuum of intermediate goods indexed by $j \in [0,1]$ and with prices $p_j$:

$$Y_t = \left( \int_0^1 y_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}},$$

where $\epsilon > 1$.Given a level of aggregate demand $Y$, cost minimization for the final goods producer implies that the demand for the intermediate good $j$ is given by

$$y_{jt} = y(p_{jt}; P_t, Y_t) = \left( \frac{p_{jt}}{P_t} \right)^{-\epsilon} Y_t, \tag{3}$$

where $P_t$ is the (equilibrium) price of the final good and can be expressed as

$$P_t = \left( \int_0^1 p_{jt}^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}.$$

**Intermediate good producer** Each intermediate good $j$ is produced by a monopolistically competitive producer using labor input $n_j$. Production technology is linear.

$$y_{jt} = h_{jt}.$$

Intermediate producers hire labor at the nominal wage $P_t w_t$ in a competitive labor market. With this technology, the real marginal cost of a unit of intermediate good is

$$mc_{jt} = w_t.$$

[4]It is important to keep in mind that here as in all of the recent literature in monetary economics the central bank sets the nominal interest rate on short-term bonds and not money supply, which gives rise to price level indeterminacy in the first place (Sargent and Wallace (1975)). In addition to setting the nominal return on bonds, the central bank can also pay interest rates on reserves. But this does not overcome the indeterminacy issue and only changes the opportunity costs of holding money. I thus omit this complication. For determinacy, setting money supply and paying interest rates on reserves is equivalent to not setting the nominal return on bonds but setting money supply instead.

Each firm chooses its price to maximize profits subject to real price adjustment costs as in Rotemberg (1982),

$$\Phi\left(p_{jt}; p_{jt-1}, P_{t-1}\right), \tag{4}$$

which depend on the set price $p_{jt}$, on last period's price $p_{jt-1}$ and/or last periods price $P_{t-1}$, the steady state price level. Costs $\Phi$ are increasing and convex in its first argument and zero in a steady state, $\Phi\left(P_t^*, P_t^*, P_t^*\right) = 0$ and $\lim_{p_{jt}\to\infty} \Phi\left(p_{jt}; p_{jt-1}, P_{t-1}\right) = \infty$. In what follow I consider two specifications. For tractability I always start with a simple specification where deviations of the price $p_{jt}$ from last period's price level $P_{t-1}$ are costly,

$$\Phi\left(p_{jt}, p_{jt-1}; P_{t-1}\right) = \Phi(p_{jt}/P_{t-1}). \tag{5}$$

I also consider a more standard specification,

$$\Phi\left(p_{jt}, p_{jt-1}; P_t^*\right) = \Phi\left(\frac{p_{jt}}{p_{jt-1}} - \pi^{ss}\right), \tag{6}$$

which leads to a standard forward looking New-Keynesian Phillips curve.

Given last period's individual price $p_{jt-1}$ and the aggregate state $(P_t, Y_{t,}, w_t, r_t)$, the firm chooses this period's price $p_{jt}$ to maximize the present discounted value of future profits. The firm satisfies all demand,

$$y(p_{jt}; P_t, Y_t) = n(y(p_{jt}; P_t, Y_t)) \tag{7}$$

by hiring the necessary amount of labor,

$$n_{jt} \quad = \quad n(y(p_{jt}; P_t, Y_t)) = y(p_{jt}; P_t, Y_t) = \left(\frac{p_{jt}}{P_t}\right)^{-\epsilon} Y_t. \tag{8}$$

The firm's pricing problem is

$$V_t\left(p_{jt-1}\right) \equiv \max_{p_{jt}} \frac{p_{jt}}{P_t} y\left(p_{jt}; P_t, Y_t\right) - w_t y(p_{jt}; P_t, Y_t) - \Phi\left(p_{jt}, p_{jt-1}; P_{t-1}\right) + \beta V_{t+1}\left(p_{jt}\right).$$

In equilibrium all firms choose the same price, and thus, aggregate consistency implies $p_{jt} = P_t$ for all $j$ and $t$. Thus, $\frac{p_{jt}}{p_{jt-1}} = \frac{P_t}{P_{t-1}} = \pi_t$ and $\frac{p_{jt+1}}{p_{jt}} = \frac{P_{t+1}}{P_t} = \pi_{t+1}$. The equilibrium real profit of each intermediate goods firm is then

$$d_t = Y_t(1 - w_t).$$

To avoid ruling out some price sequence just because the associated adjustment costs exhaust available resources, I assume that the costs are as-if. They affect firms pricing decisions but neither lower firms profits nor enter the aggregate resource constraint. None of my conclusions are affected by this asumption.

Nominal lump-sum taxes $T_t$ are set to satisfy the government budget constraint at all times

$$T_t := R_t B_{t-1} + P_t g_t + G_t - B_t + M_{t-1} - M_t, \tag{9}$$

also showing that the theory presented here is not the FTPL. The transversality condition

$$\lim_{T\to\infty} \beta^T u'(C_T) \frac{R_T B_T}{P_T} = 0 \tag{10}$$

holds.[5]

### 2.1.1 Equilibrium and Steady State

A *competitive equilibrium* is a sequence of prices $P_t$, taxes $T_t$, bonds $B_t$, money $M_t$, consumption $c_t$, hours $h_t$, prices $R_t$, $r_t$ and $w_t$ such that:

1. Households choose $c_t$, $h_t$ and $M_t$ to maximize utility given prices $P_t, R_t, r_t$ and $w_t$.
2. Prices are set optimally by firms.
3. The government budget constraint is satisfied.
4. The resource constraint: $Y_t = h_t = c_t + g_t + \frac{G_t}{P_t}$.
5. The transversality condition (10) holds.

A *steady state* is a competitive equilibrium where $c, h, Y, r$ and $R$ are constant and

$$\frac{B_t}{B_{t-1}} = \frac{M_t}{M_{t-1}} = \frac{T_t}{T_{t-1}} = \frac{P_t}{P_{t-1}} = 1 + \pi^{ss}. \tag{11}$$

[5]Kocherlakota and Phelan (1999) (KP) impose an additional limiting condition, $\lim_{t\to\infty}(M_t + R_{t+1}B_t)\prod_{s=1}^{t+1}\frac{1}{R_s} = 0$, on the government budget (equation (25) in KP) to rule out the FTPL. In their complete markets model where government bonds are not net wealth this condition is equivalent to the household transversality condition which has to be satisfied in any equilibrium. But the KP condition is not equivalent to the transversality condition when government bonds are net wealth, which for example and in contrast to KP is consistent with a real interest rate below the growth rate of the economy. For example the limiting condition would not be satisfied in a steady state where $R \equiv 1$ (ZLB), constant $M, B > 0$, $g = 0$ and $G = T$ although the government budget constraint is clearly satisfied. Therefore (25) in KP does not apply here and does not indicate whether the FTPL is operating or not.

Adding the optimality conditions,

$$1 + r_t = \frac{R_t}{1 + \pi^{ss}} = \frac{R_t}{\frac{P_t}{P_{t-1}}} = \frac{R_t}{\frac{B_t}{B_{t-1}}} = 1/\beta \qquad \text{[Bond market]} \tag{12}$$

$$\frac{R_t - 1}{R_t} = \frac{\mu'(M_t/P_t)}{u'(c_t)} \qquad \text{[Money market]} \tag{13}$$

$$w_t = \frac{v'(h_t)}{u'(c_t)} \qquad \text{[Labor Supply]} \tag{14}$$

$$w_t = \frac{\epsilon - 1}{\epsilon} \qquad \text{[Pricing]} \tag{15}$$

then characterizes a steady-state.

### 2.1.2 Steady-state price level indeterminacy

The indeterminacy of the steady-state price level is now easy to see. Start with an equilibrium price sequence $\{P_t\}_{t=0}^{\infty}$. Then $\{\lambda P_t\}_{t=0}^{\infty}$ is also an equilibrium price sequence for every $\lambda > 0$. All real variables $c_t, h_t, w_t, r_t$ and $Y_t$ as well as the inflation rate $\frac{P_t}{P_{t-1}} = \frac{\lambda P_t}{\lambda P_{t-1}}$ are unchanged. Since the nominal interest rate $R_t$ set by the central bank is unchanged households demand the same amount of real balances and thus $\lambda M_t$ nominal money. All firms setting a price $\lambda P_t$ is profit-maximizing. By construction these allocations and prices form a new equilibrium for every $\lambda$. Unsurprisingly (see Nakajima and Polemarchakis (2005)), this conclusion holds if prices are flexible or sticky simply because a steady-state requires no price-adjustment.

**Result 1.** *There is a continuum of steady-state price levels. The steady-state price level is indeterminate.*

This confirms the well known result of price level indeterminacy if the nominal interest is fixed. There is no point in discussing local determinacy in this environment if the steady-state price level is already indeterminate. The standard approach is to derive properties of the interest rate rule to ensure determinacy. In this paper I take a different route and argue that a slight modification of the model such that government bonds are net wealth implies global determinacy even if the nominal interest rate is fixed.

## 2.2 Steady-State Price Level Determinacy when Government Bonds are Net Wealth

I just make one (reduced-form) change to the previous model. I add a preference $\chi(B_t/P_t)$ for holding real bonds so that the utility function is

$$\sum_{t=0}^{\infty} \beta^t \Big[ u(c_t) - v(h_t) + \mu(\frac{M_t}{P_t}) + \chi(\frac{B_t}{P_t}) \Big], \tag{16}$$

where $\chi(\cdot)$ is increasing and strictly concave. I now show that this modification leads to price level determinacy even if the preference for holding bonds is quite small. The separability between $\mu$ and $\chi$ is just for convenience and all results are unchanged if I added a non-separable function $\chi(\frac{M_t}{P_t}, \frac{B_t}{P_t})$ instead. The reason is that the first-order condition for bonds is now

$$u'(c_t) = \chi'(\frac{B_t}{P_t}) + \beta \frac{R_{t+1}}{1 + \pi_{t+1}} u'(c_{t+1}). \tag{17}$$

In a steady-state

$$u'(c^*) = \chi'(\frac{B}{P}) + \beta \frac{R}{1 + \pi_{ss}} u'(c^*), \tag{18}$$

where steady-state consumption $c^* = h^*$ and hours $h^*$ solve

$$\frac{v'(h^*)}{u'(h^*)} = \frac{\epsilon - 1}{\epsilon}, \tag{19}$$

$\frac{B_t}{B_{t-1}} = 1 + \pi^{ss}$, $B_t = B(1 + \pi^{ss})^t$ and $P_t = P(1 + \pi^{ss})^t$, where for simplicity but inconsequentially $g = G = 0$.

Households are willing to accept a real return $\frac{R_{ss}}{1+\pi^{ss}}$ lower than $1/\beta$ even in steady-state since holding bonds provides some extra utility other than just intertemporal substitution of consumption. The higher is this (marginal) utility $\chi'(\frac{B}{P})$, the lower is the real interest rate. The magnitude of this extra utility depends on preferences - the function $\chi$ - and fiscal policy, such that the steady state real interest rate also depends on fiscal policy, as this policy determines the amount of outstanding government debt.

This is the key difference to the previous model where government bonds are not net wealth. Since bonds are nominal, the first-order condition now depends on the price level but did not above. Indeed, given a steady-state nominal interest rate $R$ set by monetary policy and a sequence of nominal government bonds growing at a constant rate $\pi^{ss}$, the first-order condition can be used to solve for the steady-state price level. Clearly, the same conclusion is reached if $g$ and $G$ are not

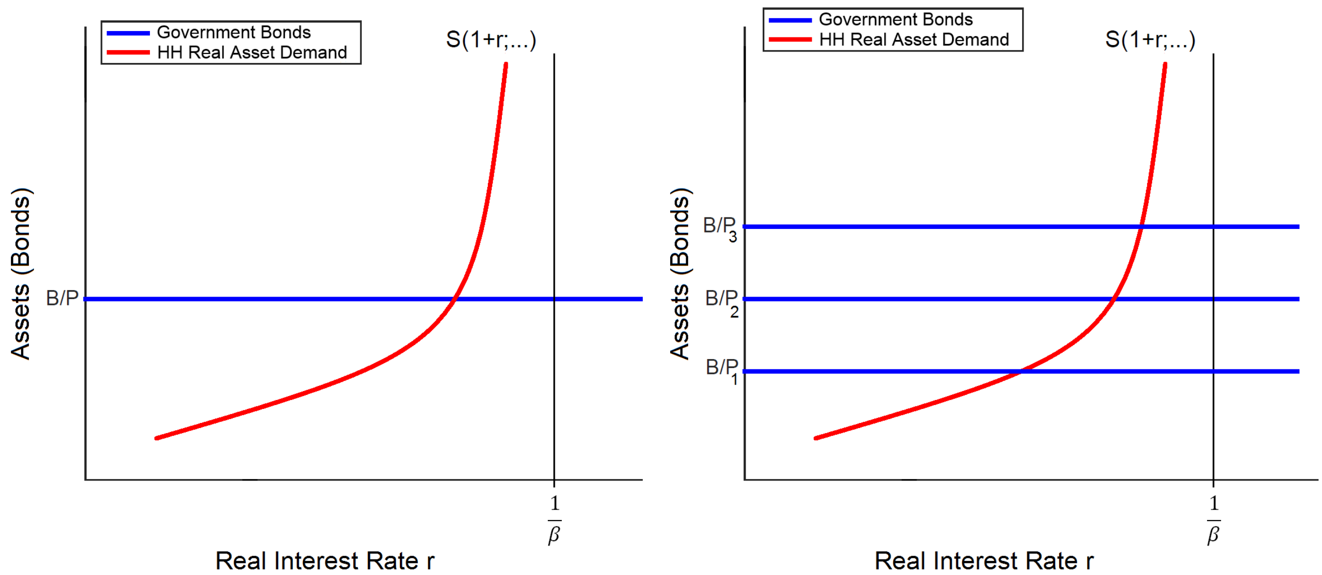



Figure 1: Asset Market in Huggett economy

zero.

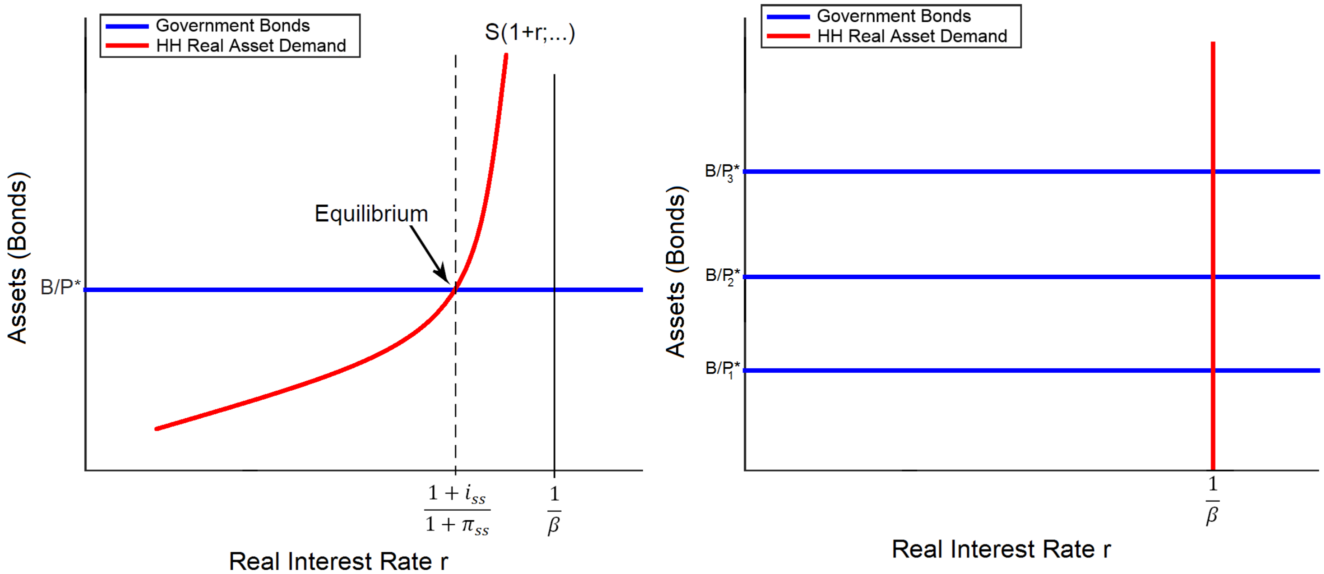
**Result 2.** *The steady-state real interest rate* $\frac{R_{ss}}{1+\pi^{ss}} < 1/\beta$. *The first-order condition for bonds allows to solve for the steady-state price level.*

The infinite horizon assumption ensures that the first-order condition for bonds can be used in every period to determine the price level. To understand this better assume a finite horizon such that in the last period households derive no utility from bonds and the demand for bonds is zero. There is then no first-order condition for bonds that can be used to determine the price level and the logic from Section 2.1 applies and implies price level indeterminacy. This last-period indeterminacy carries over to previous periods such that the price level is indeterminate in all periods as for example in Geanakoplos and Mas-Colell (1989) and Balasko and Cass (1989).

The arguments allow for a graphical analysis which also serves to illustrate the property needed for price level determinacy, a well defined demand for bonds. Solving the first order condition (18),

$$u'(c^*) = \chi'(b) + \beta(1+r)u'(c^*), \tag{20}$$

for real bonds $b$ yields the demand for bonds $S(1+r,\ldots)$ as a function of the real interest rate. The bond market clears if the demand for real bonds $S(1+r,\ldots)$ is equal to the supply of real bonds $B/P$, as represented in Figure 1 (left panel). Households' asset demand $S(1+r,...)$ is an upward sloping function of the real interest rate $1+r$. Preferences, productivity and other exogenous variables shift the asset demand function. Real asset supply by the government equals




(a) Nominal Bonds (b) Price-Indexed Bonds

Figure 2: Asset Market Equilibrium: a) Nominal Bonds; b) Price-Indexed Government Debt $B^{real}$.

$B/P$, where $B$ is nominal bonds and $P$ is the price level.[6] The equilibrium condition is

$$S(1+r,\ldots) = \frac{B}{P}. \tag{21}$$

This is one equation with two unknowns, the real interest rate $1+r$ and the price level $P$. This suggests that a continuum of price levels (associated with a continuum of real interest rates), e.g. $P_1, P_2, P_3$, clears the asset market as illustrated in the right panel of Figure 1. But this argument would overlook that $(1+r_{ss}) = \frac{1+i_{ss}}{1+\pi_{ss}} = \frac{R_t}{\frac{B_t}{B_{t-1}}}$ is set by policy to eliminate the real interest rate from the list of unknowns, equation (21) now has just one unknown, the price level $P^*$:

$$S(\frac{1+i_{ss}}{1+\pi_{ss}},\ldots) = \frac{B}{P^*}, \tag{22}$$

which serves to determine the unique price level, as illustrated in the left panel of Figure 2.

The above reasoning does not extend to the the fist model in Section 2.1 where government bonds are not net wealth. The key implication of this model is that the steady-state real interest rate is determined by the discount factor only, $(1+r_{ss})\beta = 1$, whereas in models where government bonds are net wealth the real interest rate depends on virtually all model primitives. Ricardian equivalence implies that the steady-state demand for real bonds is not a well-defined function but a correspondence and thus the above arguments cannot be used to determine the price level. The right panel of Figure 2 illustrates the indeterminacy, depicting supply and demand in the asset market as before, but with the difference that now the steady-state savings curve is a vertical line at

[6] With positive inflation rate $\pi$, bonds in a steady state at time $t$ equal $B(1+\pi)^t$ and the price level equals $P(1+\pi)^t$ for initial values $B$ and $P$, so that the term $(1+\pi)^t$ term cancels when computing the real value of bonds $B/P$.

the steady-state interest rate $1/\beta$. When government bonds are net worth, it is an upward sloping curve. The vertical asset demand curve when government bonds are not net wealth reflects the result that the real interest rate is independent of the quantity of real bonds such that a continuum of price levels, e.g. $P_1^*, P_2^*, P_3^*$, satisfies all equilibrium conditions.

The graphical representation also suggests another simple way to understand why the price level is indeterminate in the first model and is determinate in the second model where the demand for bonds is well defined. The number of endogenous variables exceeds the number of equilibrium conditions by one in the first model but in the second model where bonds are net wealth provides an extra non-redundant equation, the first-order condition (18) for bond market clearing. At first glance the argument might seem wrong since both models feature a consumption Euler equation. That is correct but in the first model this equation is redundant as it just determines the steady-state real interest rate to be equal to $1/\beta$ with no role for bonds or the price level whatsoever. In contrast, the bond-FOC in the second model defines a trade-off between the real interest rate and the value of bonds. This extra and non-redundant FOC together with $1 + r_{ss} = \frac{R_t}{\frac{B_t}{B_{t-1}}}$ then determines the price level as illustrated analytically and graphically above.[7]

How Monetary and Fiscal Policy Determine the Steady-State Real Interest Rate

A key step in the argument is how monetary and fiscal policy determine the steady-state real interest rate. In both models, a Fisher relation between the steady-state nominal interest $i_{ss}$, real interest rate $r_{ss}$ and inflation $\pi_{ss}$ holds:

$$1 + r_{ss} = \frac{1 + i_{ss}}{1 + \pi_{ss}}. \tag{23}$$

Monetary policy sets the steady-state nominal interest rate $i_{ss}$. Fiscal policy sets the growth rate of nominal debt ($B$) and adjusts nominal tax revenues ($T$) to balance the government budget. In a steady state, real tax revenue and real government debt are constant, such that the steady-state condition for fiscal policy is that the growth rates of nominal tax revenues and nominal debt all are equal to the inflation rate (in the absence of economic growth),[8]

$$1 + \pi_{ss} = \frac{T' - T}{T} = \frac{B' - B}{B}. \tag{24}$$

To be clear about the interpretation of these steady-state conditions: If fiscal policy decides for a

[7] The Fiscal Theory of the Price Level (FTPL) provides an additional equation as it assumes that the government budget constraint is satisfied by only one price level. In this paper I pursue a different approach. Instead of using the government budget constraint as an additional equation, I assume that the government balances the budget for all price levels and show that the asset market clearing condition is the needed additional equation in models where government bonds have real value.

[8] With real economic growth of rate $g$, $(1 + \pi_{ss})(1 + g) = \frac{T'-T}{T} = \frac{B'-B}{B}$.

2% nominal growth rate in nominal debt, $\frac{B'-B}{B}$, then the steady-state condition that steady-state real government debt is constant requires that the steady-state inflation rate equals 2% as well. The steady-state further requires that nominal tax revenues $T$ also grow at 2%. It is important to note that these considerations do not determine the levels of real taxes and real debt except in the sense that these are unchanging over time in a steady state. In particular, the price level is not yet determined.

Equation (24) means that the inflation rate is set by fiscal policy and is equal to the growth rate of nominal government spending, implying that the equilibrium real interest rate is determined jointly by monetary and fiscal policy.[9] These conclusions about the steady-state inflation rate are valid even if monetary policy implements an interest rate rule such as

$$i_t = \max(\bar{i} + \phi(\pi_t - \pi^*), 0), \tag{25}$$

for an inflation target $\pi^*$, an intercept $\bar{i}$ and $\phi > 0$. In this case inflation is still determined by equation (24) and the steady-state nominal interest rate equals[10]

$$i^{ss} = \max(\bar{i} + \phi(\frac{B' - B}{B} - \pi^*), 0). \tag{26}$$

Note that this line of reasoning requires that there is a continuum of potential steady-state real interest rates, and not just one equal to $1/\beta$ as in models where government bonds are not net wealth. Therefore this logic to determine the long-run inflation rate does not apply if government bonds are not net wealth.

Money Demand and Endogenous Money

Note that for these monetary and fiscal policy choices to be consistent with equilibrium, the central bank has to be ready to satisfy the resulting nominal money demand (13) and the fiscal authority has to set taxes as in (9) to balance the steady-state government budget constraint. After the price level is obtained as solving (18), the first-order condition for money is just used to solve for $M_t$. The only purpose of adding a demand for money to the model is thus to determine the quantity of money that the central bank will need to supply in order to implement its nominal interest rate target. Here, the amount of money is endogenous and not a policy instrument since the central bank controls the nominal interest rate. I obtain

[9]Monetary and fiscal policy cannot implement any arbitrary steady-state real interest rate, only one that is consistent with a steady state. In particular $\beta(1 + r_{ss}) < 1$ since otherwise asset demand would become infinite, also a well-known result in incomplete market models.

[10]For example if $\bar{i} = 0.02$, $\phi = 1.5$, debt grows at $\frac{B'-B}{B} = 0.02$ and the inflation target $\pi^* = 0$ then the steady-state inflation is 2% and the nominal interest rate equals $i^{ss} = 0.02 + 1.5 * 0.02 = 0.05$. In the (less realistic) case that the inflation target of monetary policy $\pi^* = 0.04$ exceeds the 2% that follows from fiscal policy, the steady-state nominal interest rate equals $i^{ss} = \max(0.02 + 1.5(0.02 - 0.04), 0) = 0$ and inflation is still 2%.

**Result 3.** *The steady-state price level is determinate when government bonds are net wealth.*

Consider a simple example where $u(c) = \log(c)$, $v(h) = \rho^h \frac{h^{1+\theta}}{1+\theta}$, $\mu(M/P) = \rho^\mu \log(M/P)$ and $\chi(B/P) = \rho^\chi \log(B/P)$ and both the nominal interest rate $R$ and the amount of government bonds $B$ are constant. The steady-state price level $P^*$ then solves

$$\frac{1}{c^*} = \frac{\rho^\chi}{B/P^*} + \frac{\beta R}{c^*}, \tag{27}$$

with an explicit solution

$$P^* = (1 - \beta R)\frac{B}{c^* \rho^\chi} \tag{28}$$

and

$$c^* = h^* = (\frac{\epsilon - 1}{\rho^h \epsilon})^{\frac{1}{\theta - 1}}. \tag{29}$$

An equivalent way to determine the price level, is to first solve for the steady-state demand for real bonds from the FOC,

$$S(1 + r) = \frac{\rho^\chi c^*}{1 - \beta R}, \tag{30}$$

and use the asset market clearing condition

$$S(1 + r) = \frac{\rho^\chi c^*}{1 - \beta R} = \frac{B}{P^*} \tag{31}$$

to solve for $P^*$, the same as in (28).

The steady-state price level $P^*$ depends on policy as expected. Tighter monetary policy - a higher nominal interest rate - lowers the price level whereas a expansionary fiscal policy - a higher debt level - increases it since as explained above $\beta R = \beta R/(1 + \pi_{ss}) < 1$. A larger demand for bonds - a higher $\rho^\chi$ - increases the real value of bonds and thus leads to a lower price level.

If government debt is growing at rate $\frac{B_t}{B_{t-1}} = 1 + \pi^{ss}$ not necessarily one, then

$$P^* = (1 - \frac{\beta R}{1 + \pi^{ss}})\frac{B}{c\rho^\chi}. \tag{32}$$

A positive inflation rate lowers, for a fixed nominal interest rate, the real return on bonds. Households are willing to accept this lower real return only if the utility of holding bonds increases. Concavity implies that that the real value of bonds has to fall, that is the price level has to increase.

Adding Capital

Adding investment $I_t$ and capital $K_t$ to the model does not change the conclusion of this Section. To see this, assume a production function $Y_t = F(K_t, h_t)$ and that capital accumulates as

$$K_{t+1} = F(K_t, h_t) + (1-\delta)K_t - c_t, \tag{33}$$

for a depreciation rate $\delta$. The household budget constraint changes to

$$M_t + B_t + P_t K_{t+1} = R_t B_{t-1} + P_t(1+r_t^k)K_t + M_{t-1} + P_t w_t h_t - P_t(c_t + I_t) - T_t + P_t d_t, \tag{34}$$

for a real return on capital $1 + r_t^k$ and yields the first-order conditions

$$u'(c_t) = \beta(1 + r_{t+1}^k)u'(c_{t+1}), \tag{35}$$

$$F_K(K_t, h_t) + (1-\delta) = 1 + r_t^k, \tag{36}$$

where the first one takes into account that capital does not provide any extra services as bonds do. In a steady state these FOCs simplify to

$$(1 + r^k) = 1/\beta, \tag{37}$$

$$F_K(K^*, h^*) + (1-\delta) = 1/\beta \tag{38}$$

and hours $h^*$ now solve

$$\frac{v'(h^*)}{u'(h^*)} = \frac{\epsilon - 1}{\epsilon} F_h(K^*, h^*), \tag{39}$$

which allows to solve for the steady-state level of capital $K^*$, investment $I^*$ and hours $h^*$ as a function of parameters such as $\beta$ and $\delta$ but independently of the price level. The steady state price level $P^*$ still solves

$$\frac{1}{c^*} = \frac{\rho^\chi}{B/P^*} + \frac{\beta R}{c^*}, \tag{40}$$

but taking into account that consumption $c^* = F(K^*, h^*) - I^*$, which is again independent of the price level.

This completes the first part of showing determinacy, determinacy in a steady state. This is a key but not the only step in establishing global determinacy. A proof also requires to show determinacy when the economy is not in steady state either because policy is not in steady state or because non steady-state beliefs move the economy away from steady-state. In the next step I therefore turn to local determinacy.

## 2.3 Local Determinacy

The steady state is locally unique if there is no other equilibrium in which all variables are within a neighborhood of their steady-state values. To check local determinacy of the steady state it is sufficient to check it for the log-linearized economy (Woodford (2003)). Assessing local determinacy in the linearized model is quite simple as it amounts to checking the properties of the eigenvalues of the transition matrix.

I first assume that fiscal policy follows a stationary exogenous policy and sets a fixed amount of nominal government debt and monetary policy sets a constant nominal interest rate and I allow for feedback policy rules below. For the ease of exposition I first assume that the costs of setting a price $p_{jt}$ is in terms of the deviation from last period's price level $P_{t-1}$, $\Phi(\frac{p_{jt}}{P_{t-1}})$ but I obtain the same conclusion when I consider a standard New Keynesian Phillips curve below. I also assume that government spending $g = G = 0$ and that utility derived from bonds and money holdings is separable and I show in the appendix that a mild parameter restriction delivers the same result.

The (around the steady-state) linearized equations are

$$\kappa \hat{Y}_t = (\hat{p}_t - \hat{p}_{t-1}) \tag{41}$$

$$\hat{Y}_t = \tilde{\beta} E_t \hat{Y}_{t+1} - \psi \hat{p}_t - \sigma(\hat{i}_{t+1} - (E_t \hat{p}_{t+1} - \hat{p}_t)), \tag{42}$$

where $\hat{Y}_t = \log(Y_t/Y^*)$ is the log deviation of output from steady state (also equal to the output gap since the natural rate of output is constant in the absence of real disturbances), $\hat{p}_t = \log(P_t/P^*)$ and $\hat{i}_{t+1} = \log\left(\frac{R_{t+1}}{R_{ss}}\right)$. The parameters $\sigma \equiv \frac{\tilde{\beta} u'(c^*)}{-u''(c^*)c^*} > 0$, $\psi = \frac{-\sigma b^* \chi''(b^*)}{u'(c^*)\tilde{\beta}} > 0$, $\kappa = \frac{\epsilon \varphi}{\Phi''(0)} > 0$ is the slope of the Phillips curve, $\hat{w}_t = \varphi \hat{Y}_t$, $b^* = B/P^*$ is the steady-state real value of bonds and $\tilde{\beta} = \beta(1 + r_{ss}) < 1$ which is less than one when government bonds are net wealth. Substitution of (41) into (42) and using $\hat{i}_t = 0$ yields a second-order difference equation in $\hat{p}_t$,

$$E_t \hat{p}_{t+1} = \underbrace{\frac{1 + \tilde{\beta} + \kappa(\sigma + \psi)}{\tilde{\beta} + \kappa\sigma}}_{=:a_1} \hat{p}_t + \underbrace{\frac{-1}{\tilde{\beta} + \kappa\sigma}}_{=:a_0} \hat{p}_{t-1}, \tag{43}$$

and in matrix form

$$\begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{1+\tilde{\beta}+\kappa(\sigma+\psi)}{\tilde{\beta}+\kappa\sigma} & \frac{-1}{\tilde{\beta}+\kappa\sigma} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix}$$

with one eigenvalue larger and one smaller than one,[11]

$$\lambda_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_0}}{2}. \tag{44}$$

A two-dimensional model with one predetermined endogenous state variable, $\hat{p}_{t-1}$, and one non-predetermined one, $\hat{p}_t$ is determinate if one eigenvalue is outside and the other one is inside the unit circle.[12]

If $g, G > 0$ and the utility derived from money and bonds are not separable, $\epsilon_m := \frac{\sigma \chi_{bm}}{m^* u'(c^*)} \neq 0$, I reach the same conclusion if $1 + \epsilon_m \eta_y > 0$, where log-linearized money-demand is

$$\hat{m}_t = \eta_y \hat{Y}_t - \eta_i \hat{i}_t \tag{45}$$

for a output elasticity $\eta_y$ and an interest-rate elasticity $\eta_i > 0$. I therefore obtain

**Result 4.** *The steady-state is locally determinate when government bonds provide liquidity services ($\psi > 0$).*

I also obtain the same conclusion if I assume that the cost of setting a price $p_{jt}$ is in terms of the deviation from the firm's own last period's price $p_{jt-1}$, $\Phi(\frac{p_{jt}}{p_{jt-1}})$, which leads to the standard linearized New Keynesian Phillips curve,

$$(\hat{p}_t - \hat{p}_{t-1}) \quad = \quad \kappa \hat{Y}_t + \beta(\hat{p}_{t+1} - \hat{p}_t). \tag{46}$$

In matrix form the economy is described through

$$\begin{pmatrix} E_t p_{t+1} \\ p_t \\ E_t Y_{t+1} \end{pmatrix} = \frac{1}{\beta} \begin{pmatrix} 1+\beta & -1 & -\kappa \\ \beta & 0 & 0 \\ \beta\psi - \sigma & \sigma & \beta/\tilde{\beta} + \sigma\kappa \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \\ Y_t \end{pmatrix}.$$

This three-dimensional model with one predetermined endogenous state variable, $\hat{p}_{t-1}$, and two non-predetermined one, $\hat{Y}_t$ and $\hat{p}_t$, is determinate if two eigenvalues are outside and the other one is inside the unit circle. Applying the characterization in Proposition C.2 in Appendix C of Woodford (2003) shows that this is the case. The economy is locally determinate.

[11]If government bonds have no value, $\psi = 0$, then $\lambda_1 = 1$ and the second eigenvalue $\lambda_2 = \frac{1}{1+\kappa\sigma} < 1$ since $\psi = 0$ implies $\tilde{\beta} = 1$ as in the model without value for government bonds.

[12]In the boundary case $\psi = 0$ the eigenvalue is one and the linearized model cannot be used to assess local determinacy. The previous analysis implies however that the steady state is locally indeterminate if $\psi = 0$.

## 2.4 Local Determinacy: Policy Rules

The previous section establishes price level determinacy when monetary and fiscal policy are constant. I now extend the analysis and allow for policies responding to deviation of prices and output from their respective steady-state values and establish conditions on the policy rules which deliver local determinacy. Interestingly, in models where government bonds have a real value policy rules do not overcome indeterminacy but instead may induce it.
I assume an interest rate rule

$$\hat{i}_t = \varphi^i \hat{p}_t \tag{47}$$

where $\varphi^i$ is the response to price deviations. Since prices are the the state-variables it is convenient to specify the rule in terms of prices and not in terms of inflation. Adding in addition a response to output deviations is typically important for quantitative assessments but is less relevant for local determinacy. I report those results below but start with the simpler rule of price deviations only as this allows to convey the intuition better. Similarly I assume a rule for nominal debt

$$\hat{B}_t = \varphi^B \hat{p}_t \tag{48}$$

with a price response $\varphi^B$. Taxes are still set to balance the government budget constraint. The model is now described through

$$\begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{1+\tilde{\beta}+\kappa(\sigma+\tilde{\psi})}{\tilde{\beta}+\kappa\sigma} & \frac{-1}{\tilde{\beta}+\kappa\sigma} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix},$$

which differs from the previous model as

$$\tilde{\psi} = \psi(1-\varphi^B) + \sigma\varphi^i$$

replaces $\psi$. The local determinacy condition with policy response functions follows immediately:

**Result 5.** *The steady-state is locally determinate when government bonds have value ($\psi > 0$) and*

$$\tilde{\psi} = \psi(1-\varphi^B) + \sigma\varphi^i > 0. \tag{49}$$

It is easy to see the conditions for local determinacy for two special cases. When monetary policy is constant ($\varphi^i = 0$) and only fiscal policy is responding or when fiscal policy is constant ($\varphi^B = 0$) and only monetary policy is responding

**Result 6 (Special Cases).** *The economy is locally determinate if government bonds have value ($\psi > 0$) and*

*If fiscal policy is constant ($\varphi^B = 0$),*

$$\varphi^i \geq 0 \tag{50}$$

*If monetary policy is constant ($\varphi^i = 0$),*

$$\varphi^B < 1. \tag{51}$$

The intuition why determinacy can be ensured only if parameter restrictions on fiscal policy are imposed is straightforward. Consider first debt policy and suppose that $P_t > P^*$. If debt policy is not responding to prices, $\varphi^B = 0$, $P_t > P^*$ implies a fall in the real value of debt, so that households require a lower real interest rate or equivalently a higher inflation rate ($P_{t+1} > P_t$) to absorb less real debt. If debt policy is aggressive, $\varphi^B > 1$, this reasoning does not work since in this case $P_t > P^*$ implies a policy-induced increase in the real value of debt. Households then require a higher real interest rate to be willing to absorb more real debt. If the nominal interest rate is not responding to prices this requires a fall in prices, $P_{t+1} < P_t$, that is the eigenvalue is smaller than one.

Note that there are no restrictions on monetary policy. Even a negative response to price hikes, $\varphi^i \geq -1/\sigma$, would still imply determinacy. This is not surprising since already a constant nominal interest rate implies determinacy. A higher price $P_t > P^*$ implies a fall in real debt and requires a decrease in the real interest rate, which is equivalent to $P_{t+1} > P_t$, that is the eigenvalue is larger than one. If the nominal interest rate increases in response to higher prices then an even larger increase in prices is necessary to lower the real interest rate, that is the eigenvalue gets even larger, again implying determinacy.

Note that the intuition when fiscal policy induces local determinacy independently of how monetary policy is conducted did not use much model details and can thus be expected to be valid more generally. As I show in Section 3 this is the case in richer incomplete market models. Result 5 derives the criterion that combines conditions on monetary and fiscal policy. A key number in this endeavour is naturally the intertemporal elasticity of substitution as this allows to compare the demand effects of monetary and fiscal policy. An expansive fiscal policy ($\varphi^B > 1$) now induces determinacy but monetary policy has to be sufficiently contractionary, that is $\varphi^i$ has to be sufficiently high,

$$\varphi^i > \frac{\psi(\varphi^B - 1)}{\sigma}. \tag{52}$$

The intuition for this result builds on the explanations given above for the determinacy of monetary and fiscal policy. Suppose again $P_t > P^*$ and $\varphi^B > 1$. Again real debt increases so that households require a higher real interest rate to be willing to absorb more real debt. If monetary policy was passive this would require a fall in $P_{t+1}$ (relative to $P_t$). But if the nominal interest rate increases more than the required real interest rate, then $P_t = P_{t+1}$ would imply that the real interest rate is too high. As a consequence next period's price $P_{t+1}$ again has to increase to bring the real interest rate down.

For policy rules which respond not only to price but also to output deviations,

$$\hat{i}_t = \varphi_p^i \hat{p}_t + \varphi_y^i \hat{Y}_t, \tag{53}$$

$$\hat{B}_t = \varphi_p^B \hat{p}_t + \varphi_y^B \hat{Y}_t \tag{54}$$

and natural sign restrictions - $\varphi_y^i > 0$ (higher interest rate in a boom) and $\varphi_y^B < 0$(expansionary fiscal policy in a recession) the same condition,

$$\psi(1 - \varphi_p^B) + \sigma\varphi_p^i > 0, \tag{55}$$

ensures local determinacy.[13]

For a cost function $\Phi(p_{jt}/p_{jt-1})$ which induces a New Keynesian Phillips curve and under the same sign restrictions the local determinacy criterion is again unchanged,

$$\psi(1 - \varphi_p^B) + \sigma\varphi_p^i > 0. \tag{56}$$

## 2.5 Hyperinflations and Hyperdeflations

As a last step in establishing global determinacy I have to show that prices are in steady state when monetary and fiscal policy are stationary, which is equivalent to ruling out inflationary and deflationary price spirals. Note that this argument rules out hyperinflations only if nominal variables are growing at a constant rate, but not when policy decides for an explosive path of nominal variables. The theory thus allows for policy induced hyper-inflations but not for belief induced ones. It is important to point out that depending on the fiscal and monetary policy implemented, the economy may experience a hyperinflation, a deflation or price stability, but in each scenario the price level is determinate.

[13]The derivation in the appendix provides a sufficient condition without sign restrictions which rules out that all eigenvalues have modulus larger than one (meaning that the system is not stable) so that the economy is locally determinate.

The starting point is the non-linear pricing equation

$$(\frac{v'(h_t)}{u'(c_t)} - \frac{\epsilon - 1}{\epsilon}) \quad = \quad \frac{P_t}{\epsilon P_{t-1}} \Phi'(\frac{P_t}{P_{t-1}} - \pi_{ss}) \tag{57}$$

defines output as a function $Y_t(\frac{P_t}{P_{t-1}})$ so that consumption

$$c_t = C(\frac{P_t}{P_{t-1}}, P_t) = Y_t(\frac{P_t}{P_{t-1}}) - g - \frac{G_t}{P_t} \tag{58}$$

is a function of inflation $\frac{P_t}{P_{t-1}}$ and the price $P_t$. Substituting this into the FOC for bonds

$$\frac{u'(c_t)}{P_t} = \frac{\chi'(\frac{B_t}{P_t})}{P_t} + \beta R \frac{u'(c_{t+1})}{P_{t+1}} \tag{59}$$

allows to describe the nonlinear equilibrium dynamics of prices through two functions $H$ and $\Gamma$ which relate the price in period $t+1$ to the prices in periods $t$ and $t-1$,

$$H(\tilde{P}_t, \tilde{P}_{t-1}) := \frac{u'(C(\frac{\tilde{P}_t}{\tilde{P}_{t-1}}, \tilde{P}_t))}{\tilde{P}_t} - \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} = \frac{\beta R}{1+\pi_{ss}} \frac{u'(C(\frac{\tilde{P}_{t+1}}{\tilde{P}_t}, \tilde{P}_{t+1}))}{\tilde{P}_{t+1}} =: \Gamma(\tilde{P}_{t+1}, \tilde{P}_t), \tag{60}$$

where I define the detrended price

$$\tilde{P}_t = \frac{P_t}{(1+\pi_{ss})^t}, \tag{61}$$

so that $\frac{B_t}{P_t}$ can be written as $\frac{B}{\tilde{P}_t}$ and $\tilde{P} = P^*$ in steady state.

I now show that (60) is satisfied only if $\tilde{P} = P^*$, that is neither $\tilde{P}_t > P^*$ nor $\tilde{P}_t < P^*$ can be an equilibrium. As a first step I prove that for $\tilde{P}_t > P^*$ to be an equilibrium price, prices have to be monotonically increasing and that for $\tilde{P}_t < P^*$ to be an equilibrium price, prices have to be monotonically decreasing. These two results are not surprising given that the local determinacy analysis implies that the model has sufficiently many eigenvalues with modulus larger than one. In a second step I then show that both cases violate equilibrium conditions.

#### 2.5.1 Ruling out Speculative Hyperinflations

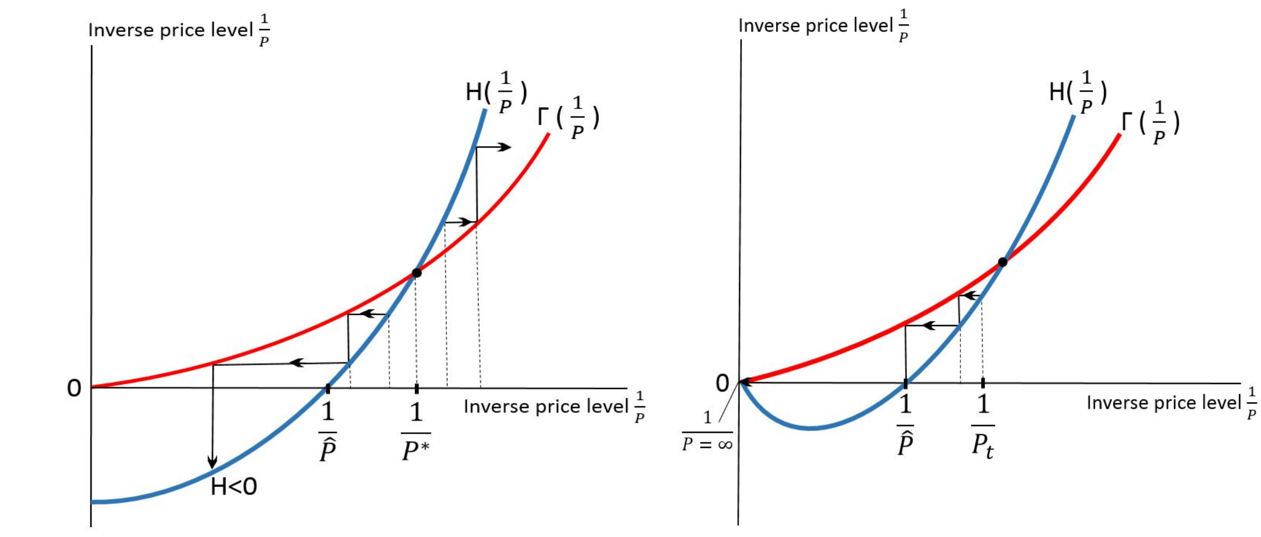
First assume that $\tilde{P}_t > P^*$ is an equilibrium price. Suppose that $\tilde{P}_{t-1} \leq \tilde{P}_t$ (I consider $\tilde{P}_{t-1} > \tilde{P}_t$ below) then using that $\frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} > \frac{\chi'(\frac{B}{P^*})}{\tilde{P}_t} = (1 - \frac{\beta R}{1+\pi_{ss}})\frac{u'(C(1,P^*))}{\tilde{P}_t}$,

$$
\begin{aligned}
H(\tilde{P}_t, \tilde{P}_{t-1}) - \Gamma(\tilde{P}_t, \tilde{P}_t) &\leq H(\tilde{P}_t, \tilde{P}_t) - \Gamma(\tilde{P}_t, \tilde{P}_t) && (62)\\
&= \frac{u'(C(\frac{\tilde{P}_t}{\tilde{P}_t}, \tilde{P}_t))}{\tilde{P}_t}(1 - \frac{\beta R}{1+\pi_{ss}}) - \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} && (63)\\
&< \frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t}(1 - \frac{\beta R}{1+\pi_{ss}}) - \frac{\chi'(\frac{B}{P^*})}{\tilde{P}_t} && (64)\\
&= \frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t}(1 - \frac{\beta R}{1+\pi_{ss}}) - \frac{u'(C(\pi_{ss}, P^*))}{\tilde{P}_t}(1 - \frac{\beta R}{1+\pi_{ss}}) && (65)\\
&= (\frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t} - \frac{u'(C(1, P^*))}{\tilde{P}_t})(1 - \frac{\beta R}{1+\pi_{ss}}) && (66)\\
&\leq 0. && (67)
\end{aligned}
$$

Since $\Gamma(\tilde{P}_{t+1}, \tilde{P}_t)$ is falling in $\tilde{P}_{t+1}$, I obtain $\tilde{P}_{t+1} > \tilde{P}_t$. Iterating this argument delivers a monotonically increasing sequence of prices. Uniqueness of the steady state implies that this sequence is unbounded since a monotone bounded function converges. The intuition is straightforward. A price level $\tilde{P}_t$ higher than $P^*$ implies a lower real value of government bonds and therefore a higher marginal utility $\chi'$. Households are then willing to accept a lower real interest, requiring an even higher price $\tilde{P}_{t+1} > \tilde{P}_t$ since the nominal interest rate is constant here. It is easy to show that this argument generalizes to environments with non-separable utility functions or when the value of bonds is stochastic as long as the trade-off between real interest rates and the amount of bonds is still operative: A higher real value of bonds requires a higher real interest rate.

The literature (Obstfeld and Rogoff (1983, 2017)) provides a very elegant solution to rule out hyper-inflations in monetary models which can be adopted here as well. A slight adaptation is necessary since the central bank controls money supply in Obstfeld and Rogoff (1983, 2017) whereas it sets the nominal interest here, so that their arguments on money demand cannot be applied here. Suppose the government provides some fractional real backing of debt/transfers, that is the government trades bonds/transfers for real consumption at a (very high) price $\bar{P}$ then prices cannot rise above $\bar{P}$ since households always trade with the government and not at a price higher than $\bar{P}$. But then the increasing price sequence must unravel backward, just because the price level cannot converge to infinity as equilibrium would require.[14]

[14] Strictly speaking the arguments use perfect foresight but they are sufficient to rule out sunspots as well. Indeed the same arguments show that $E_t\tilde{P}_{t+1}$ converges to infinity and that again this cannot be an equilibrium.




(a) Condition (71)

(b) P jump to $\infty$

Figure 3: Dynamics of the Price Level

**Result 7.** *(Obstfeld and Rogoff (1983, 2017)) type backing of nominal government obligations at a sufficiently high price $\bar{P}$ rules out hyperinflations.*

Hyperinflations can also be ruled out without such a backing. This is where sticky prices become relevant. To better understand the role of price rigidities, I first assume that prices are flexible which allows for a diagrammatic analysis as in Obstfeld and Rogoff (1983). With flexible prices the functions $H$ and $\Gamma$ simplify to

$$H(\tilde{P}_t) = \frac{u'(C(\tilde{P}_t))}{\tilde{P}_t} - \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} \tag{68}$$

$$\Gamma(\tilde{P}_{t+1}) = \frac{\beta R}{1+\pi_{ss}} \frac{u'(C(\tilde{P}_{t+1}))}{\tilde{P}_{t+1}} \tag{69}$$

Figure 3 illustrates the dynamics of the price level, $\tilde{P}_t$, that is implied by

$$H(\tilde{P}_t) = \Gamma(\tilde{P}_{t+1}), \tag{70}$$

which defines $\tilde{P}_{t+1}$ as a function of $\tilde{P}_t$. To render the comparison with Obstfeld and Rogoff (1983) easier the diagram is presented in terms of the inverse price level, $1/P$. Since both functions $H$ and $\Gamma$ are downward sloping in $P$, they are upward sloping in $1/P$ and from the left panel of the diagram it is apparent that there is a unique steady state at $P^*$ since $H$ is steeper than $\Gamma$. There exists a $\hat{P}$ such that

$$H(\hat{P}) = 0 \text{ and } H(\tilde{P}_t) < 0 \text{ for all } \tilde{P}_t > \hat{P}. \tag{71}$$

A negative value $H < 0$ does not allow to iterate on (70) since $\Gamma(\tilde{P}) > 0$ for all finite $\tilde{P}$. The only remaining possibility is that $\tilde{P}$ jumps to infinity.

The proof to rule out hyperinflations therefore boils down, in the absence of a government backing, to rule out that the price level is infinite, $P = \infty$. If

$$\lim_{\tilde{P}\to\infty} \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} > 0 \tag{72}$$

then $P = \infty$ cannot be an equilibrium since

$$\lim_{\tilde{P}\to\infty} H(\tilde{P}) = -\lim_{\tilde{P}\to\infty} \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} < 0 = \lim_{\tilde{P}\to\infty} \Gamma(\tilde{P}). \tag{73}$$

This is what the left panel of Figure 3 shows. There is no equilibrium at $P = \infty$ ($1/P = 0$) since the curve $H$ is strictly below $\Gamma$ when $1/P \to 0$, $H(P = \infty) < \Gamma(P = \infty) = 0$.

**Result 8.** *Condition (72) rules out hyperinflations.*

In the context of their monetary models, Obstfeld and Rogoff (1983, 2017) consider condition (72) to be implausible. It implies money to be an absolute necessity although money is thought of as just reducing some trading frictions. Here however this condition refers to the value of bonds which arises not because bonds reduce trading frictions but for example because bonds allow to smooth consumption in response to uninsurable idiosyncratic risk. Heterogeneous agent incomplete market economies such as Huggett (1993) is a model class where bonds have value and which I consider in Section 3. In such models condition (72) seems less implausible.

But fortunately this question need not to be resolved here since price rigidities offer an alternative which requires much weaker assumptions. Panel b) of Figure 3 shows the dynamics when condition (72) is not met and $H(P = \infty) = \Gamma(P = \infty) = 0$. The previous argument to rule out $P = \infty$ does not apply anymore and the diagram suggests that prices could now jump to infinity once they reached the level $\hat{P}$.

Whereas all the previous arguments work if prices are flexible or sticky, it is straightforward to show that $P = \infty$ is not possible in models with some price rigidities. The argument is quite simple. In contrast to flexible price economies, prices are set by profit-maximizing firms in models with sticky prices and are not just the magic outcome of a fictitious auctioneer. This simple insight is all what is needed. It is never optimal for a firm to set $P = \infty$. The pricing equation (57) is not satisfied since $\frac{P_t}{\epsilon P_{t-1}}\Phi'(\frac{P_t}{P_{t-1}} - \pi_{ss}) = \infty$ if $\frac{P_t}{P_{t-1}} = \infty$.

Setting prices to infinity would also incur large or even infinite Rotemberg (1982) price adjustment costs which is clearly not an equilibrium. A different way of modeling price rigidities, but with the same conclusion, is to assume Calvo price setting. The fraction of non-adjusting firms would charge a finite price and would absorb all demand whereas the $P = \infty$ firms would face no demand, implying that setting $P = \infty$ is not optimal and that the output weighted price level would be finite. Note that the only assumption needed is that there exists a $\hat{P}$ such that $H(\hat{P}) = 0$ and $H(\tilde{P}_t) < 0$ for all $\tilde{P}_t > \hat{P}$. An assumption that seems quite weak, that is certainly much weaker than condition (72), and that is satisfied and not questioned in Obstfeld and Rogoff (1983, 2017). The only role of price rigidities is to rule out that the price can jump to infinity. This is the case for the two leading models of price stickiness in the literature - price adjustment costs and Calvo price setting - and is a reasonable property of any mechanism that induces price rigidities on firms. I obtain:

**Result 9.** *Price rigidities and condition (71) rule out hyperinflations.*

This completes the proof since $\tilde{P}_{t-1} > \tilde{P}_t > P^*$ can be ruled out as well using the same arguments as above and showing that the price sequence is increasing when moving back in time, $\tilde{P}_{t-1} < \tilde{P}_{t-2} < \tilde{P}_{t-3} < \ldots$. Suppose not and $\tilde{P}_{t-2} < \tilde{P}_{t-1}$. Then the above arguments imply that $\tilde{P}_t > \tilde{P}_{t-1}$, a contradiction. Iterating backwards yields a monotonically increasing function which again by the same arguments cannot be an equilibrium.[15]

### 2.5.2 Ruling out Speculative Hyperdeflations

Now assume that $\tilde{P}_t < P^*$ is an equilibrium price. Suppose that $\tilde{P}_{t-1} \geq \tilde{P}_t$ then using that $\frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} < \frac{\chi'(\frac{B}{P^*})}{\tilde{P}_t} = (1 - \frac{\beta R}{1+\pi_{ss}})\frac{u'(C(1,P^*))}{\tilde{P}_t}$,

$$
\begin{aligned}
H(\tilde{P}_t, \tilde{P}_{t-1}) - \Gamma(\tilde{P}_t, \tilde{P}_t) &\geq H(\tilde{P}_t, \tilde{P}_t) - \Gamma(\tilde{P}_t, \tilde{P}_t) && (74)\\
&= \frac{u'(C(\frac{\tilde{P}_t}{\tilde{P}_t}, \tilde{P}_t))}{\tilde{P}_t}(1 - \frac{\beta R}{1+\pi_{ss}}) - \frac{\chi'(\frac{B}{\tilde{P}_t})}{\tilde{P}_t} && (75)\\
&> \frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t}(1 - \frac{\beta R}{1+\pi_{ss}}) - \frac{\chi'(\frac{B}{P^*})}{\tilde{P}_t} && (76)\\
&= \frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t}(1 - \frac{\beta R}{1+\pi_{ss}}) - \frac{u'(C(\pi_{ss}, P^*))}{\tilde{P}_t}(1 - \frac{\beta R}{1+\pi_{ss}}) && (77)\\
&= (\frac{u'(C(1, \tilde{P}_t))}{\tilde{P}_t} - \frac{u'(C(1, P^*))}{\tilde{P}_t})(1 - \frac{\beta R}{1+\pi_{ss}}) && (78)\\
&\geq 0. && (79)
\end{aligned}
$$

[15]The dynamics of the price level out of steady-state can be characterized to be monotone here in contrast to the analysis of models with money in Woodford (1994). The reason why such a concise characterization is available is that I can use the result that in equilibrium a higher level of real bonds requires a higher real interest. This is the only property used and as long as this is satisfied, the results in this Section hold more generally, for example for non-separable preferences.

Since $\Gamma(\tilde{P}_{t+1}, \tilde{P}_t)$ is falling in $\tilde{P}_{t+1}$, I obtain $\tilde{P}_{t+1} < \tilde{P}_t$. Iterating this argument delivers a monotonically decreasing sequence of prices. Uniqueness of the steady state implies that this sequence is not bounded by a number larger than zero since a monotone bounded function converges. The intuition is agin straightforward. A price level $\tilde{P}_t$ lower than $P^*$ implies a higher real value of government bonds and therefore a lower marginal utility $\chi'$. Households then demand a higher real interest, requiring an even lower price $\tilde{P}_{t+1} < \tilde{P}_t$ since the nominal interest rate is constant here.

I now show that such speculative deflationary spirals are not an equilibrium. A simple proof is to recognize that eventually the price level will be so low that government demand $g + \frac{G}{P}$ exceeds output, which clearly cannot be an equilibrium. An equivalent way to think of this result is that the government sets a nominal anchor $G$ which serves as a lower bound for nominal aggregate demand and thus rules out beliefs of prices converging to zero. But, as noted when discussing local determinacy above, indeterminacy may arise even if government bonds have value if fiscal policy is too accommodative, for example changing spending more than one-for-one with prices. The same applies here. If $G$ is decreased more than prices $\frac{G}{P}$ is not explosive and the simple proof does not rule out hyperdeflations.

**Result 10.** *Hyperdeflations are ruled out if nominal government spending $G > 0$.*

The proof in Aiyagari (1994a) that the steady-state real interest rate is strictly smaller than $1/\beta$ offers another approach. Aiyagari (1994a) considers a heterogeneous agent incomplete markets model but his arguments can be used in the reduced form model as well subject to one small caveat which I discuss below. In the incomplete market model in Section 3 the arguments apply one-for-one without such a caveat.

I now show that the arguments in Aiyagari (1994a) can be adopted here under the reasonable assumption that

$$\lim_{P \to 0} \chi'(\frac{B}{P}) = 0. \tag{80}$$

The FOC for bonds (82) is equivalent to

$$1 = \frac{\chi'(\frac{B_t}{P_t})}{u'(c_t)} + \beta R \frac{P_t}{P_{t+1}} \frac{u'(c_{t+1})}{u'(c_t)}, \tag{81}$$

so that

$$\lim_{t \to \infty} \beta R \frac{P_t}{P_{t+1}} \frac{u'(c_{t+1})}{u'(c_t)} = 1. \tag{82}$$

Assumption (80) thus delivers the result, which is also holds in all incomplete market models, that asset demand is infinite if the real interest rate is equal to $1/\beta$. For large enough T

$$\bar{\gamma}_t := \prod_{j=T}^{T+t-1} \frac{1}{1+r_{j+1}} \approx \beta^t \frac{u'(c_{T+t})}{u'(c_T)}, \tag{83}$$

so that the transversality condition implies, since the capital stock $K_t$ is bounded,

$$\lim_{t\to\infty} \bar{\gamma}_t (\frac{B_{T+t}}{P_{T+t}} + K_t) = \lim_{t\to\infty} \bar{\gamma}_t \frac{B_{T+t}}{P_{T+t}} = 0. \tag{84}$$

The government budget constraint (for $G = 0$) implies

$$\frac{B_t}{P_t} \leq \frac{\frac{B_{t+1}}{P_{t+1}}}{1+r_{t+1}} + \frac{Y_t - g}{R_t}. \tag{85}$$

which uses that period $t$ tax revenues are bounded from above by $Y_t$. This is a result in Aiyagari (1994a) based on a standard Laffer curve logic and that lump-sum taxes are zero or have to be very small (less than the lowest labor income). This is where the caveat shows up. In a representative agent economy lump-sum taxes could be large enough to fully tax all interest rate payments. Thus in this Section it is an assumption and only a result in Section 3 that tax revenues are bounded. Combining the government budget constraint (85) with (84) shows that $K_t + \frac{B_t}{P_t}$ is bounded, contradicting that $\frac{B_t}{P_t} \to \infty$ in a hyperdeflation.

**Result 11.** *Hyperdeflations are ruled out if government tax revenues are bounded.*

A standard proof in the literature shows that deflationary price paths violate the transversality condition. The same arguments apply here if $\frac{R}{1+\pi_{ss}} \leq 1$. It follows from (82) that $\frac{u'(c_t)}{P_t}$ is eventually growing at rate $1/(\beta R)$ so that $u'(c_t)\frac{B_t}{P_t}$ is growing at rate $\frac{1+\pi_{ss}}{\beta R} \geq \frac{1}{\beta}$, implying that the transversality condition

$$\lim_{t\to\infty} \beta^t u'(c_t) \frac{B_t}{P_t} > 0 \tag{86}$$

is violated. Mechanically it is clear that the transversality condition is not violated if $\frac{1+\pi_{ss}}{\beta R} < \frac{1}{\beta}$. Economically it is related to a point for example made by Obstfeld and Rogoff (1986). A hyperdeflation allows for a utility gain from selling a bond worth 1$ and rolling it over forever. [16] As Obstfeld and Rogoff (1986) point out selling a bond worth 1$ has to be feasible. Here

[16] Iterating on the first-order condition for bonds yields

$$\frac{u'(c_t)}{P_t} = \sum_{s=t}^{\infty} (\beta R)^{s-t} \frac{\chi'(\tilde{P}_s)}{P_s} + \lim_{T\to\infty} (\beta R)^T \frac{u'(c_{t+T})}{P_{t+T}},$$

where the latter limit term is positive as shown above, so that the LHS is large than the sum on the RHS.

this is the case only if rolling over this 1$ forever is feasible, requiring that the asset position stays nonnegative. Selling one 1$ lowers the asset position by $R^t$ within $t$ periods which keeps assets positive since $B$ is growing at rate $1 + \pi_{ss} \geq R$. If one the other hand $1 + \pi_{ss} < R$ real debt would eventually become negative and then converge to infinity at rate $1/\beta$, violating the transversality condition or equivalently the No-Ponzi condition. Therefore selling 1$ is not feasible if $1 + \pi_{ss} < R$.[17]

**Result 12.** *If* $\frac{R}{1+\pi_{ss}} \leq 1$*, hyperdeflations can be ruled out using a transversality condition argument.*

Together the results on steady-state determinacy, local determinacy and hyperinflations and hyperdeflations imply that the only price sequence which forms an equilibrium is the one where the price is constant and equal to the steady-state price level $P^*$, that is, the price at time $t$ equals $P^*(1 + \pi_{ss})^t$:

**Proposition 1 (Global Determinacy).** *The price level is globally determinate and unique:*

$$\begin{aligned} \tilde{P}_t &= P^* \\ P_t &= P^*(1 + \pi_{ss})^t. \end{aligned}$$

# 3 A Heterogeneous Agent Incomplete Market Model

In this Section I develop a model where markets are incomplete as in Aiyagari (1994b, 1995) and which in reduced form resembles the model of Section 2.2, so that the determinacy arguments carry over from the reduced form model to my incomplete market model. A key difference is that now the value of bonds and of money arise endogenously and are not assumed to be an argument of the utility function. I follow Lucas and Stockey (1987) to generate a values for money, which requires assumptions on the timing of events and on payment arrangements. The main challenge is to integrate those with the trading restrictions of incomplete market models such that the properties of both model classes are preserved. The main changes to the previous model then concern the household sector which I describe in detail first. The labor market and firms' price setting are unaffected and I therefore do not repeat those model features here.

[17] The condition $1 + \pi_{ss} \geq 1$ in the monetary model in Obstfeld and Rogoff (1986) is a special case since money has a nominal return $R = 1$.

## 3.1 The Model

To obtain a reduced form that resembles the model of Section 2.2 I follow Lucas and Stockey (1987) and assume that households derive utility from three different consumption goods and leisure:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \Big[ u^k(c_t^k) + u^b(c_t^b) + u^m(c_t^m) - v(h_t) \Big] \tag{87}$$

where $c_t^k$ are "capital goods", $c_t^b$ are "bond goods" and $c_t^b$ are "money goods". The only difference between these goods besides different utility functions is in terms of which period $t$ assets can be used to purchase them. All assets including capital can be used to acquire capital goods, bonds and money can be used to buy bond goods and only money can be used to purchase money goods.

Agents' labor productivity $\{e_t\}_{t=0}^{\infty}$ is stochastic and is characterized by an $N$-state Markov chain that can take on values $e_t \in \mathcal{E} = \{e_1, \cdots, e_N\}$ with transition probability characterized by $p(e'|e)$ and $\int e = 1$. Agents rent their labor services, $eh$, to firms for a real wage $w_t$. The resource constraint is

$$c_t^k + c_t^b + c_t^m + g + \frac{G}{P_t} = F(K_t, h_t) + (1 - \delta) K_t. \tag{88}$$

As in Lucas and Stockey (1987) sellers receive payment at the beginning of the following period implying that all three goods sell at the the same nominal price.

Household $i$'s beginning of period $t$ capital is $k_{it}$, real bond holdings $b_{it}$ and real money $m_{it}$ with nominal value $P_{t-1} m_{it}$. Before period $t$ productivity is known asset markets open and the household purchases capital goods $c_t^k \geq 0$, capital $k_{it+1}$, bonds $b_{it+1}$ and money $m_{it+1}$, such that the credit constraint for a credit limit $\bar{b}$

$$k_{it+1} + b_{it+1} + m_{it+1} > -\bar{b} \tag{89}$$

and the budget constraint

$$\begin{aligned} & P_t k_{it+1} + P_t b_{it+1} + P_t m_{it+1} R_{it+1} + P_t c_t^k && (90) \\ = \; & P_t(1 + r_t^k) k_{it} + R_t P_{t-1} b_{it} + P_{t-1}(m_{it} - c_{it-1}^m) - P_{t-1} c_{it-1}^b + P_t d_{it-1} + P_{t-1} e_{t-1} h_{t-1} (1 - \tau) w_{t-1} - T_t \end{aligned}$$

are satisfied. Instead of engaging in open market operations, the central bank here rents money $m_{it+1}$ to households at the nominal interest rate $R_{it+1}$, so that the household has cost $m_{it+1} R_{it+1}$ to have $m_{it+1}$ units of real money available.[18] As I show below this leads to the same money demand

[18]Equivalently one can assume that the households acquires $m_{it+1}$ but has only $m_{it+1}/R_{it+1}$ available for spending.

function as in the reduced form model. The first two terms in the left hand side (first line) are the nominal value of capital and bonds and the fourth term is the payment for capital good. The first two terms on the right side are income of assets bought in the preceding period, the third term is unspent cash, the fourth term is payments for bond goods, the fifth term is dividends and the sixth term is after tax receipts from labor income and the last term are lump-sum taxes.

After this trading each household splits into a money good shopper and a Huggett consumer. The shopper must use money $m_{it+1}$ to buy money goods satisfying a cash-in-advance constraint

$$c_{it}^m \leq m_{it+1}. \tag{91}$$

and the Huggett consumer learns the income shock $e_{it}$. While the labor income $P_t e_{it} h_t (1-\tau) w_t$ is paid at the beginning of next period when sellers receive the payment for good sales, consumption of $c_t^b$ by the Huggett consumer results in invoices which are settled next period. It is the possibility to use invoices and not cash to pay for bond goods what distinguishes those goods from money goods, but in Lucas and Stockey (1987) households are not restricted in their use of invoices to pay for non-cash goods. Such an assumption would circumvent the credit-constraint which are essential in incomplete market models. To reconcile both models I allow for payment by invoices but impose an upper bound which ensures that households can pay back fore sure using labor income and bonds,

$$P_t c_t^b \leq P_t e_{it} h_t (1-\tau) w_t + P_t (b_{it+1} - \bar{b}) R_{it+1}). \tag{92}$$

There are various ways to allow households a more generous credit line which would be certainly quantitatively appealing but less tractable. None of the conclusions would change if I added capital $k_{it+1}$ to the invoice constraint. Capital and bonds would then be perfect substitutes and as I explain below the determinacy results are unaffected.

The transversality condition is

$$\lim_{T\to\infty} E_0 [R_{T-1} \frac{b_{iT-1}}{P_{T-1}} \beta^T u^k (c_T^k)] = 0. \tag{93}$$

To formulate the household choice problem, it is convenient to follow the timing assumptions just described and write the household's problem recursively both for the beginning of the period before the income shock has realized,

$$V(k,b,m;\Omega,P,P_{-1}) = \max_{c^k \geq 0, k', b', m', 0 \leq l \leq \hat{l}} u^k(c^k) + u^m(c^m) - v(h) + \sum_{e \in \mathcal{E}} p(e|e_{-1}) V^1(k^1, b^1, m^1, d; \Omega) \tag{94}$$

$$
\begin{aligned}
\text{subj. to} \qquad & Pk' + Pb^1 + Pm^1 + Pc^k \\
= \ & P(1+r^k)k + P_{-1}Rb + P_{-1}(m - c^m) - P_{-1}c^b + Pd + Pe_{-1}h_{-1}(1-\tau)w_{-1} - T \\
c^m \ \leq \ & m' \\
k' + b' + m' \ \geq \ & -\bar{b}
\end{aligned}
$$

and for the decision after the income shock has realized

$$
V^1(k^1, b^1, m^1, d; \Omega) = \max_{c^m, c^b \geq 0, b' \geq 0, M^2 \geq 0} u^b(c^b) + \beta \sum_{e \in \mathcal{E}} p(e'|e) V(k', b', m'; \Omega') \tag{95}
$$

$$
\begin{aligned}
\text{subj. to} \qquad & Pc^b \leq Peh(1-\tau)w + Pb'R \\
& \Omega' = \mathcal{T}(\Omega),
\end{aligned}
$$

where $\Omega(k, b, m, e) \in \mathcal{M}$ is the distribution on the space $X = \mathcal{K} \times \mathcal{B} \times \mathcal{M} \times \mathcal{E}$, agents capital holdings $k \in \mathcal{K}$, bond holdings $b \in \mathcal{B}$, money holdings $m \in \mathcal{M}$ and labor productivity $e \in \mathcal{E}$, across the population, which will together with the policy variables determine the equilibrium prices. $\mathcal{T}$ is an equilibrium object that specifies the evolution of the wealth distribution.

Market clearing conditions for capital and bonds are

$$
K_t = \int k_{it} di \qquad \text{[Capital]} \tag{96}
$$

$$
\frac{B_t}{P_t} = \int b_{it} di \qquad \text{[Bonds]} \tag{97}
$$

and the central bank provides money

$$
M_t = \int P_t m_{it} di. \tag{98}
$$

As in the reduced form model above the central bank has to provide just enough money to implement the interest rate target.[19] This may require to issue more money or to reduce the money stock. The government budget constraint is as before

$$
T_t := R_t B_{t-1} + P_t g + G_t - B_t + M_{t-1} - M_t R_{t+1} - \tau w_t h_t. \tag{99}
$$

In an incomplete markets model lump sum taxes are bounded by $e_1 w_t h_t$ requiring $\tau$ to be large enough so that $T_t \leq 0$ is a transfer and not a tax.

[19] If I considered a cashless economy (Woodford (2003)) instead, I could dispense with cash goods, money entering the utility function and cash-in-advance constraints altogether. I do not choose this route since adding (a small amount of) money makes it easier to understand how the central bank can set the nominal interest rate.

An equilibrium is then a sequence of prices $P_t$, $R_t$, $r_t$ and $w_t$, taxes $T_t$, bonds $B_t$, money $M_t$, and value functions $V$ and $V^1$ with policy functions $c^k$, $c^b$, $c^m$, $k$, $b$, $m$ such that:

1. Households maximize utility taking prices and policies as given.

2. Prices are set optimally by firms.

3. The government budget constraint is satisfied.

4. The resource constraint is satisfied.

5. The transversality condition (93) holds.

The household decision problem yields an optimal consumption choices $c^b(b', e)$ and $c^m(m') = m'$ so that utility can be rewritten as

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \Big[ u^k(c_t^k) - v(h_t) + \underbrace{u^m(m')}_{=:\mu(m')} + \underbrace{E_{t-1} u^b(c^b(b', e))}_{=:\chi(b')} \Big], \tag{100}$$

resembling utility in the reduced form model.

The Euler equation for acquiring bonds can then be written as

$$u_c^k(c_{it}^k) \;\; \geq \;\; \frac{\partial \chi(b', e)}{\partial b'} + E_{t-1} \frac{\beta R_{t+1}}{P_{t+1}/P_t} u_c^k(c_{it+1}^k), \tag{101}$$

with equality if $k' + b' + m' > -\bar{b}$. Again resembling the reduced form model but now this is the FOC of a household who in addition faces income risk.

The capital decision is described through the FOC

$$u_c^k(c_{it}^k) \;\; \geq \;\; E_{t-1} \beta (1 + r_{t+1}^k) u_c^k(c_{it+1}^k) \tag{102}$$

with equality if $k' + b' + m' > -\bar{b}$. Money demand is determined through

$$\frac{u^m(c_{it}^m)}{u^k(c_{it}^k)} = R_{t+1} \tag{103}$$

and money goods $c^m = m'$. Bond goods satisfy

$$u_c^b(c_{it}^b(b_{it+1}, e_{it})) \;\; \leq \;\; \frac{\beta P_t}{P_{t+1}} u_c^k(c_{it+1}^k) \tag{104}$$

with equality if the invoice constraint is not binding.

The similarity of the FOC for bonds suggests that the determinacy results carry over to the incomplete markets model. Building on the previous analysis I now turn to discussing determinacy in the same order as before, starting with the steady state.

## 3.2 Steady State

The arguments in the reduced form and in the incomplete markets model are basically identical. The steady-state demand for bonds is now the aggregation of all individual asset demands $b_i$

$$S(1+r) = \int b_i di, \tag{105}$$

the Fisher relation (23) between the steady-state nominal interest $i_{ss}$, the real interest rate $r_{ss}$ and inflation $\pi_{ss}$ holds:

$$1 + r_{ss} = \frac{1+i_{ss}}{1+\pi_{ss}}, \tag{106}$$

and the steady-state inflation rate satisfies (24)

$$1 + \pi_{ss} = \frac{T' - T}{T} = \frac{B' - B}{B} = \frac{G' - G}{G}. \tag{107}$$

The steady-state price level $P^*$ is again determined through equation (22),

$$S(\frac{1+i_{ss}}{1+\pi_{ss}}) = \frac{B}{P^*}. \tag{108}$$

The central bank has to provide nominal money

$$M_t = P^* \int m_i di \tag{109}$$

to implement the nominal interest rate $R = 1 + i_{ss}$ where $\int m_i di$ is the steady-state aggregate real money demand. Steady-state capital $K_{ss}$ is determined by combining market clearing and the FOC for firms such that $1 + r^k$ is the market clearing price and solves

$$F_K(\int k_i, h) + (1-\delta) \;\; = \;\; 1 + r^k, \tag{110}$$

where individual demand $k_i$ depends on $1 + r^k$ and aggregates to $K_{ss}$,

$$K_{ss} = \int k_i di, \tag{111}$$

so that both $r^k$ and $K_{ss}$ are determined. If capital and bonds were perfect substitutes, for example if both enter symmetrically into the invoice constraint (92), then total household demand for $k+b$ is

$$S(1+r) = \int (b_i + k_i) di, \tag{112}$$

and the price level is determined through

$$S(1+r) = K_{ss} + \frac{B}{P^*}, \tag{113}$$

where $K_{ss}$ satisfies $F_K(K_{ss}, h) + (1-\delta) = 1 + r^k = 1 + r_{ss}$.

Here the assumption is that households exchange consumption goods for money. If one assumes instead that households obtain money through open market operations, then $P^*$ and $M$ solve

$$\frac{M}{P^*} = \int m_i \tag{114}$$

$$K_{ss} + \frac{B-M}{P^*} = \int b_i + k_i, \tag{115}$$

again two equations in two unknowns. Clearly, equation (114) alone does not determine the price level since the central bank sets $i$ and not $M$, which adjusts endogenously to satisfy the quantity equation. It is the asset market clearing condition that determines the price level, which depends on fiscal variables $G$, $T$ and $B$ *and* on $i$.

As in the reduced form model incomplete market model deliver an aggregate demand for bonds which depend on the real interest rate. And as in the reduced form model the associated asset market clearing condition delivers a non-redundant additional equation which then determines the price level. And both models allow for the same graphical representation of price level determination through panel a) of Figure 2.[20]

Another way to see that the asset market clearing condition adds a non-redundant equation is to assume that fiscal policy fixes the real value of bonds $b^{real}$. In this case there is generically no equilibrium since the real bond demand is not equal to real bond supply, $b^{real} = \int b_i$, simply because now there is one more equation than unknowns. The assumption that bonds are nominal adds an unknown, the price level, so that now the number of equations is equal to the number of unknowns.

[20]Incomplete markets models may have multiple *real* equilibria, which is orthogonal to the nominal multiplicity issues here. The results in this paper show that that each real equilibrium is associated with a unique price, that is adding a nominal element to the model does not add any multiplicities. Of course, adding nominal elements cannot remove any real multiplicities.

## 3.3 Local Determinacy

While the analysis of the steady state in the incomplete market model is straightforward to analyze, dynamics outside steady states is quite intractable and I therefore consider an economy without capital and assume a natural borrowing limit and log utility so that the FOC for bonds holds with equality for all households (see Bhandari et al. (2015, 2016a,b, 2017) for the same simplification),

$$u_c^k(c_{it}^k) = \frac{\partial \chi(b', e)}{\partial b'} + E_{t-1}\frac{\beta R_{t+1}}{P_{t+1}/P_t} u_c^k(c_{it+1}^k), \tag{116}$$

which allows for a linear approximation around the median household's values as

$$\sigma^{-1}\hat{c}_{it}^k = \chi\hat{b}' - \tilde{\beta}(\hat{\imath}_t - (E_{t-1}\hat{p}_{t+1} - \hat{p}_t)) + \sigma^{-1}\tilde{\beta}E_{t-1}\hat{c}_{it+1}^k, \tag{117}$$

where $\chi = \frac{\chi_{bb}\bar{b}}{u_c^k} < 0$ and $\tilde{\beta} = \beta(1 + r_{ss})$. Aggregating across households and noting that capital good consumption changes are proportional to output changes yields

$$\hat{Y}_t = \tilde{\beta}E_t\hat{Y}_{t+1} \underbrace{-\sigma\chi(\hat{B} - \hat{p}_t)}_{=:-\psi\hat{p}_t} - \sigma\tilde{\beta}(\hat{\imath}_t - (E_t\hat{p}_{t+1} - \hat{p}_t)),, \tag{118}$$

which is identical to (42) in the reduced form model. Combining this again with the Phillips curve (41) and using that $\hat{\imath} = 0$ yields the same second-order difference equation for prices as in (43) in the reduced form model,

$$E_t\hat{p}_{t+1} = \underbrace{\frac{1 + \tilde{\beta} + \kappa(\sigma + \psi)}{\tilde{\beta} + \kappa\sigma}}_{=:a_1}\hat{p}_t + \underbrace{\frac{-1}{\tilde{\beta} + \kappa\sigma}}_{=:a_0}\hat{p}_{t-1}. \tag{119}$$

I therefore obtain here the same conclusion that the steady-state is locally determinate when government bonds provide liquidity services ($\psi > 0$). Allowing for policy reaction function also yields the same conclusion as before that the economy is locally determinate if

$$\tilde{\psi} = \psi(1 - \varphi^B) + \sigma\varphi^i > 0. \tag{120}$$

## 3.4 Ruling out Hyperinflations and Hyperdeflations

As a last step in establishing global determinacy I ruled out hyperinflations and hyperdeflations in the reduced form model. I now show that the same arguments apply here subject to some caveat. In the reduced from model I showed first that for $\tilde{P}_t > P^*$ to be an equilibrium price, prices have to be monotonically increasing and that for $\tilde{P}_t < P^*$ to be an equilibrium price, prices have to be monotonically decreasing. Prices moving away from the steady-state is a consequence of local determinacy since the eigenvalue larger than one eventually dominates the dynamics. As the incomplete markets model is also locally determinate, the same result can be expected to hold here. But the intractability of outside steady-state dynamics in incomplete markets model does not allow me provide conditions on primitives to rule out cycles, that is prices moving away from the steady-state level but neither converging to infinity nor to zero but instead to form a cycle.

However, inspecting the derivations in the reduced form model shows that the only property needed is that a lower real value of bonds requires a higher real interest rate. A price $P_t > P^*$ leads to a lower real value of bonds and thus requires a lower real interest rate which in turn requires prices to increase even more, $P_{t+1} > P_t$. Similarly a price $P_t < P^*$ leads to a higher real value of bonds and thus requires a higher real interest rate which in turn requires prices to decrease even more, $P_{t+1} < P_t$. In numerical applications the aggregate savings curve is typically increasing in the real interest rate, as in the left panel of Figure 2, but this cannot be ensured theoretically. The remaining main arguments to rule out hyperinflations and hyperdeflations still hold as I will argue now.

### 3.4.1 Hyperinflations

In the reduced form model I provided three different proofs to rule out hyperinflation - Obstfeld and Rogoff (1983, 2017)) type backing of nominal government, Condition (72) holds and sticky prices - which I show to carry over to the incomplete markets model as well.

The (Obstfeld and Rogoff (1983, 2017)) argument holds obviously since trading bonds/transfers for real consumption at a (very high) price $\bar{P}$ prevents prices from rising above $\bar{P}$ since households always trade with the government and not at a price higher than $\bar{P}$, eliminating any hyperinflation.

The second proof used condition (72),

$$\lim_{P\to\infty} \frac{\chi'(\frac{B}{P})}{P} > 0. \tag{121}$$

It is easy to provide a sufficient condition that this condition holds here as well. Assume that a state $e^0 = 0$ exists where households have no labor income, so that all consumption $c^b$ needs to be financed through savings, and that the utility of zero consumption is minus infinity (satisfied by the standard assumption of log utility and CRRA with risk aversion large than one). In state $e^0 = 0$,

$$\lim_{P\to\infty} \frac{u_c^b(0+\frac{B}{P})}{P} > 0, \tag{122}$$

which holds for example if $u^b(c) = log(c)$. Furthermore $\frac{\partial c^b}{\partial b} = R$ so that $\lim_{P \to \infty} \frac{u^b_c(0+\frac{B}{P})}{P} \frac{\partial c^b}{\partial b} > 0$, implying Condition (72) since $\chi'$ is an expectation of nonnegative terms with at least one strictly positive. In Obstfeld and Rogoff (1983, 2017) condition (72) implies that money is an absolute necessity for everyone. The corresponding assumption in the incomplete market model means that bonds are an absolute necessity in some state of the world for someone. One may argue that households could hold capital instead to avoid zero consumption in state $e^0$. Adding a fixed cost of capital adjustment (e.g. as in Kaplan et al. (2016)) would eliminate this possibility and make bonds without adjustment costs a necessity.

A third possibility with much weaker assumptions on income, adjustment costs etc. arises if prices are sticky. Again the arguments from the reduced form model hold here as well. If condition (71) holds that is for large enough $P$ for some household in at least one state of the world,

$$\frac{u^k_c(c^k)}{P} < \frac{\chi'(\frac{B}{P})}{P}, \tag{123}$$

then hyperinflations can be ruled out. The reason is simple. A hyperinflation requires the price to jump to infinity at some point but no firm subject to pricing frictions would set a price $P = \infty$. Clearly, condition (72) is sufficient as it implies that the RHS is bounded from below whereas the LHS converges to zero. But it is way too strong as (123) does not require a state $e^0 = 0$. Condition (71) just requires slower convergence of the RHS than the LHS which is likely to be case for some households since the real value of bonds converges to zero depriving households of any consumption smoothing abilities. If the real value of bonds is low enough, consumption smoothing is so poor that for some households the marginal utility from bond goods exceeds the marginal utility of capital goods, which they would like to correct by accumulating more bonds but they cannot. I therefore obtain

**Result 13.** *Hyperinflations can be ruled out if either*

1. *A Obstfeld &Rogoff backing of nominal government obligations is available,* or
2. *Condition (121) holds,* or
3. *Price are rigid and condition (123) holds.*

### 3.4.2 Hyperdeflations

I also provided three different proofs in the reduced form model to rule out hyperinflation - nominal government spending $G > 0$, bounded tax revenues and transversality condition - which I again argue carry over to the incomplete markets model as well.

If nominal government spending $G > 0$, a price converging to 0 implies that finally $\frac{G}{P}$ exceeds output, which is neither an equilibrium in the reduced form model nor in the incomplete markets model here.

As a second proof I showed that hyperdeflations are ruled out if government tax revenues are bounded. The boundedness of tax revenues is as I argued above a is a result in Aiyagari (1994a) based on a standard Laffer curve logic and that lump-sum taxes are zero or have to be very small (less than the lowest labor income). In the reduced form model the proof required also the (weak) assumption (80), $\lim_{P\to 0}\chi'(\frac{B}{P}) = 0$ to show that the real interest rates converges to $1/\beta$. Such an additional assumption is not necessary now since it is a result in Aiyagari (1994a) that asset demand converges to infinity if the real interest rate approaches $1/\beta$, so that the net worth of bonds converges to zero. This shows that hyperdeflations can be ruled out here without any further assumptions.

Finally the result that the real interest rate converges to $1/\beta$ means that I can also use the same transversality condition argument as before. The same arguments apply here if $\frac{R}{1+\pi_{ss}} \leq 1$.

Prices are eventually growing at rate $1/(\beta R)$ so that $\frac{B_t}{P_t}$ is growing at rate $\frac{1+\pi_{ss}}{\beta R} \geq \frac{1}{\beta}$, implying that the transversality condition,

$$\lim_{T\to\infty} E_0[R_{T-1}\frac{b_{iT-1}}{P_{T-1}}\beta^T u^k(c_T^k)] > 0, \tag{124}$$

is violated. I therefore obtain

**Result 14.** *Hyperdeflations can be ruled out since government tax revenues are bounded or*

1. *If nominal government spending $G > 0$ or*
2. *If $\frac{R}{1+\pi_{ss}} \leq 1$ using a transversality condition argument.*

# 4 Conclusion

This paper shows that the price level is globally determinate in models where government bonds are net wealth, including Bewley-Imrohoroglu-Huggett-Aiyagari heterogenous agent incomplete markets models. A key finding is that the price level is determined jointly by monetary and fiscal policy, with long-run inflation determined by the growth rate of nominal government debt even if monetary policy is operating an interest rate rule with a different inflation target. The nominal anchor - nominal fiscal variables - is controlled by fiscal policy, which therefore has the power to set the long-run inflation rate.

Building on these findings, Hagedorn (2016), Hagedorn et al. (2016, 2017b) and Hagedorn et al. (2017a) use incomplete market models where government bonds are net wealth to investigate several heatedly debated topics in monetary economics. Not surprisingly, the novel way of thinking proposed in this paper leads to different answers from what the conventional view suggests.

The literature has documented several puzzles in New Keynesian models during a liquidity trap: the forward guidance puzzle, technological regress is expansionary (output increases) and the fiscal multiplier becomes larger if prices are less sticky. Since government bonds are net wealth in the incomplete market models we use, we obtain price level determinacy even if the nominal interest is constant at zero. As a result these puzzles disappear.

Forward guidance refers to the idea that commitment of the central bank to keep the nominal interest rate low during a period after the liquidity trap leads to large output gains already during the liquidity trap. We show that the large output gains from forward guidance do not occur in incomplete market models. Independently of whether the nominal interest rate is raised earlier or later than in the benchmark, the output path is almost unchanged. The same conclusion is reached if instead of raising the nominal interest rate according to some exogenous rule to its steady-state level an interest rate rule is applied.

In New Keynesian models fiscal multipliers increase as price stickiness is reduced. In contrast we show that output gets smaller and the fiscal multiplier decreases if price stickiness is reduced in incomplete market models.

Furthermore New Keynesian models also imply that technological regress is expansionary in a liquidity trap. Again in contrast, we show that this is not the case in incomplete market models. Technological progress is expansionary and technological regress is contractionary.

We conclude that in models where government bonds are net wealth such as incomplete market models, the forward guidance puzzle disappears as commitment to future monetary policy has only negligible effects, technological regress decreases output and the size of the fiscal multiplier becomes smaller if prices are less sticky.

We also numerically compute impulse responses to monetary and fiscal policy shocks as well as to technology and discount factor shocks. We find that all impulse responses are in line with their empirical counterparts, a finding which is easily explained by the supply/demand logic at the foundation of the determinacy results in this paper. An increase in nominal interest rates stimulates saving and therefore lowers consumption demand, implying a drop in prices. An increase in government spending stimulates aggregate demand, implying a rise in prices. An increase in technology raises supply and households' incentives to save, implying a drop in prices. And finally an increase in the discount factor stimulates savings since households are more patient, implying a drop in prices. Quite remarkably, our models deliver these results not only for sticky prices but also when prices are flexible. In particular, price rigidities are not needed for monetary policy to have effects.[21] However, for these effects to be quantitatively consistent with empirical findings, nominal rigidities are likely needed (Hagedorn et al. (2017b)).

A main result of this paper is that fiscal and monetary policy jointly determine the price level. Understanding how prices move thus requires a good grasp of the policy coordination by the treasury and the central bank. Since here price level determination is quite different, policy coordination problems are quite different from those in Sargent and Wallace (1981)'s classic "monetarist arithmetic" or in Leeper (1991)'s active and passive monetary and fiscal policies. In particular the question is not which combination of fiscal and monetary policy leads to local determinacy of the price level and which combination does not. Instead prices are globally determinate for a much larger set of policies, including an interest rate peg. Policy coordination is therefore very different: I have shown that fiscal policy can control the long-run inflation rate which might give the impression that monetary policy is quite powerless and that fiscal policy can always get its way, not only determining fiscal policy but also setting the inflation rate, which in every textbook model is under the control of monetary policy exclusively. This impression would underestimate the power of a central bank as an increase in nominal interest rates is very unpleasant for the treasury since increases in nominal interest rates raise the interest payments on government debt, leading to higher debt and eventually higher taxes. In practice this could be the most effective way for the central bank to impose its will on fiscal policy. It offers the possibility for central banks to tame inflation in the medium and long run, which requires to constrain fiscal policy from running an inflationary spending plan. Although the central bank does not set the spending itself, and therefore the treasury can control medium and long-run inflation if it wants to, control of the nominal interest rate is an effective tool for making an inflationary policy quite costly. Raising the nominal interest rate raises the interest payments on government debt, which can sharply constrain government spending. High nominal interest rates may force the treasury into a less expansionary fiscal policy, and thus indirectly lead to a lower inflation rate. Since there is no upper bound on nominal interest rates, there is no limit on the cost the central bank can inflict on the treasury. But the central bank can also support an expansionary fiscal policy through lowering (or not raising) nominal interest rates, if it considers inflation to be too low. However, the ZLB puts a limit on the budgetary support the central bank can provide. The independence of central banks guarantees that those interest rate decisions are taken through monetary and not fiscal policy. As central banks arguably put more weight on price stability than treasuries do (treasuries being more interested in taming deficits), independence leads to a more active interest rate policy to curb inflation than if the treasury were to control the interest rate.

[21] Garriga et al. (2013), Sterk and Tenreyro (2015) and Buera and Nicolini (2016) among others also find that monetary policy has real effects in an incomplete market models with flexible prices.

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# APPENDIX

## A.I Proofs and Derivations

### A.I.1 Derivations of Section 2.3

Local Determinacy

As shown in the main text, the economy is described through

$$M := \begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{1+\tilde{\beta}+\kappa(\sigma+\psi)}{\tilde{\beta}+\kappa\sigma} & \frac{-1}{\tilde{\beta}+\kappa\sigma} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix}.$$

Since

$$trace(M) - det(M) = \frac{\tilde{\beta}+\kappa(\sigma+\psi)}{\tilde{\beta}+\kappa\sigma} = 1 + \frac{\kappa\psi}{\tilde{\beta}+\kappa\sigma} > 1, \tag{A1}$$

the characteristic polynomial $q(x)$ is strictly negative evaluated at 1, $q(1) < 0$, if $\psi > 0$. Since $q$ is eventually increasing to infinity in $x$ one eigenvalue, $\lambda_1$, is real and larger than one. If $det(M) < 1$ the other eigenvalue, $\lambda_2$, is smaller than one since $det(M) = \lambda_1\lambda_2$. If $det(M) > 1$, $q(0) = det(M) > 0$ so that $0 < \lambda_2 < 1$ by the intermediate value theorem.

Local Determinacy with Non-separable utility and $G > 0$.

Non-separability of $\chi(B/P, M/P)$ and nominal government spending $G > 0$ requires two changes. First, I have to log-linearize money-demand as in (45),

$$\hat{m}_t = \eta_y \hat{Y}_t - \eta_i \hat{i}_t \tag{A2}$$

for an output elasticity $\eta_y$ and an interest-rate elasticity $\eta_i > 0$, and consumption as

$$\hat{C}_t = s_Y \hat{Y}_t - s_G(\hat{G}_t - \hat{p}_t), \tag{A3}$$

where $s_Y = \frac{Y^*}{c^*}$ and $s_G = \frac{g^*}{c^*}$ and $Y^*$ and $g^*$ are steady-state real output real government expenditures respectively. Plugging this into the linearized Euler Equation yields

$$s_Y \hat{Y}_t - s_G(\hat{G}_t - \hat{p}_t) = s_Y \tilde{\beta} E_t \hat{Y}_{t+1} - s_G E_t(\hat{G}_{t+1} - \hat{p}_{t+1}) - \psi \hat{p}_t - \epsilon_m(\eta_y \hat{Y}_t - \eta_i \hat{i}_t) - \sigma\tilde{\beta}(\hat{i}_t - (E_t \hat{p}_{t+1} - \hat{p}_t)),$$

where $\epsilon_m = \frac{\sigma \chi_{mb} m^*}{u'(c^*)}$. Using that $\hat{G}_t = \hat{i}_t = 0$ and collecting terms yields

$$s_Y(1 + \epsilon_m \eta_y)\hat{Y}_t = s_Y \tilde{\beta} E_t \hat{Y}_{t+1} - s_G(\hat{p}_t - E_t \hat{p}_{t+1}) - \psi \hat{p}_t - \sigma(\hat{i}_t - (E_t \hat{p}_{t+1} - \hat{p}_t)). \tag{A4}$$

Finally using (41), $\kappa\hat{Y}_t = (\hat{p}_t - \hat{p}_{t-1})$, and rearranging terms,

$$(s_Y\tilde{\beta} + \kappa s_G + \kappa\sigma)E_t\hat{p}_{t+1} \quad = \quad (s_Y(1+\tilde{\beta}+\epsilon_m\eta_y) + \kappa(\sigma+\psi+s_G))\hat{p}_t - s_Y(1+\epsilon_m\eta_y)\hat{p}_{t-1}. \text{(A5)}$$

Equivalently in matrix form

$$M := \begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{s_Y(1+\tilde{\beta}+\epsilon_m\eta_y)+\kappa(\sigma+\psi+s_G)}{s_Y\tilde{\beta}+\kappa(\sigma+s_G)} & \frac{-s_Y(1+\epsilon_m\eta_y)}{s_Y\tilde{\beta}+\kappa(\sigma+s_G)} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix}$$

Since

$$trace(M) - det(M) = \frac{s_Y\tilde{\beta} + \kappa(\sigma+\psi+s_G)}{s_Y\tilde{\beta}+\kappa(\sigma+s_G)} = 1 + \frac{\kappa\psi}{s_Y\tilde{\beta}+\kappa(\sigma+s_G)} > 1, \tag{A6}$$

the characteristic polynomial $q(x)$ is strictly negative evaluated at 1, $q(1) < 0$, if $\psi > 0$. Since $q$ in eventually increasing to infinity in $x$ one eigenvalue, $\lambda_1$, is real and larger than one. If $|det(M)| < 1$ the other eigenvalue, $\lambda_2$, is smaller than one since $det(M) = \lambda_1\lambda_2$. If $det(M) > 1$, $q(0) = det(M) > 0$ so that $0 < \lambda_2 < 1$ by the intermediate value theorem.

The parameter restriction $\frac{s_Y(1+\epsilon_m\eta_y)}{s_Y\tilde{\beta}+\kappa(\sigma+s_G)} > -1$ for which $\epsilon_m > -1/\eta_y$ is sufficient means that $det(M) > -1$. A violation of the parameter restriction would deliver two eigenvalues with modulus larger than one. The economy is locally determinate.

Derivation of Phillips Curve

The firm's pricing problem is

$$V_t(p_{jt-1}) \equiv \max_{p_{jt}} \frac{p_{jt}}{P_t} y(p_{jt}; P_t, Y_t) - w_t y(p_{jt}; P_t, Y_t) - \Phi\left(\frac{p_{jt}}{p_{jt-1}} - \pi_{ss}\right) Y_t + \beta V_{t+1}(p_{jt}),$$

subject to the constraints $n_{jt} = n(y(p_{jt}; P_t, Y_t)) = y(p_{jt}; P_t, Y_t) = \left(\frac{p_{jt}}{P_t}\right)^{-\epsilon} Y_t$.
Equivalently

$$V_t(p_{jt-1}) \equiv \max_{p_{jt}} \frac{p_{jt}}{P_t}\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon} Y_t - w_t\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon} Y_t - \Phi\left(\frac{p_{jt}}{p_{jt-1}} - \pi_{ss}\right) Y_t + \beta V_{t+1}(p_{jt})$$

The FOC w.r.t $p_{jt}$

$$(1-\epsilon)\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon}\frac{Y_t}{P_t} + \epsilon w_t\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon-1}\frac{Y_t}{P_t} - \Phi'\left(\frac{p_{jt}}{p_{jt-1}} - \pi_{ss}\right)\frac{Y_t}{p_{jt-1}} + \beta V'_{t+1}(p_{jt}) = 0 \tag{A7}$$

and the envelope condition

$$V'_{t+1}(p_{jt+1}) \quad = \quad \Phi'\left(\frac{p_{jt+1}}{p_{jt}} - \pi_{ss}\right)\frac{p_{jt+1}}{p_{jt}}\frac{Y_{t+1}}{p_{jt}}. \tag{A8}$$

Combining the FOC and and the envelope condition

$$(1-\epsilon)\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon}\frac{Y_t}{P_t} + \epsilon w_t\left(\frac{p_{jt}}{P_t}\right)^{-\epsilon-1}\frac{Y_t}{P_t} - \Phi'\left(\frac{p_{jt}}{p_{jt-1}} - \pi_{ss}\right)\frac{Y_t}{p_{jt-1}} + \beta\Phi'\left(\frac{p_{jt+1}}{p_{jt}} - \pi_{ss}\right)\frac{p_{jt+1}}{p_{jt}}\frac{Y_{t+1}}{p_{jt}}$$

And using that all firms choose the same price in equilibrium

$$(1-\epsilon) + \epsilon w_t - \Phi'\left(\pi_t - \pi_{ss}\right)\pi_t + \beta\Phi'\left(\pi_{t+1} - \pi_{ss}\right)\pi_{t+1}\frac{Y_{t+1}}{Y_t} = 0 \tag{A9}$$

<u>Linearization of the Phillips Curve</u>

Linearization yields

$$\epsilon\varphi\hat{Y}_t - \theta\pi_t + \beta\theta\pi_{t+1} = 0 \tag{A10}$$

where $\theta = \Phi''(0), \Phi'(0) = 0$ and $\hat{w}_t = \varphi\hat{Y}_t$. Equivalently for $\kappa = \frac{\epsilon\varphi}{\theta}$

$$\pi_t = \kappa\hat{Y}_t + \beta\pi_{t+1} \tag{A11}$$

or in terms of prices

$$\hat{p}_t - \hat{p}_{t-1} = \kappa\hat{Y}_t + \beta(\hat{p}_{t+1} - \hat{p}_t). \tag{A12}$$

<u>Local Determinacy with New Keynesian Phillips Curve</u>

The characteristic equation of the matrix

$$\begin{pmatrix} E_t Y_{t+1} \\ E_t p_{t+1} \\ p_t \end{pmatrix} = \frac{1}{\beta}\begin{pmatrix} 1+\beta & -1 & -\kappa \\ \beta & 0 & 0 \\ \beta\psi - \sigma & \sigma & \beta/\tilde{\beta} + \sigma\kappa \end{pmatrix}\begin{pmatrix} Y_t \\ p_t \\ p_{t-1} \end{pmatrix}.$$

is

$$x^3 + \underbrace{\left(\frac{-\kappa\sigma}{\beta} - 2 - \frac{1}{\beta}\right)}_{=:a_2} x^2 + \underbrace{\left(\frac{(\psi+\sigma)\kappa}{\beta} + \frac{1+\beta}{\beta\tilde{\beta}} + \frac{1}{\beta}\right)}_{=:a_1} x + \underbrace{\frac{-1}{\beta\tilde{\beta}}}_{=:a_0}. \tag{A13}$$

Checking the conditions in Proposition C.2 in Appendix C of Woodford (2003) requires to compute

$$1 + a_2 + a_1 + a_0 = \frac{\kappa\psi}{\beta} > 0 \tag{A14}$$

$$-1 + a_2 - a_1 + a_0 = \frac{((-\psi - 2\sigma)\kappa)}{\beta} - \frac{4\beta + 4}{\beta\tilde{\beta}} < 0 \tag{A15}$$

since $\psi > 0$. The conditions of (Case III) are satisfied since $\tilde{\beta} < 1$ and thus $\mid a_2 \mid > \frac{1}{\tilde{\beta}} + \frac{1}{\beta} + 1 > 3$ and the equilibrium is locally determinate. Note that the condition is $a_0^2 - a_0 a_2 + a_1 - 1 < 0$ is not needed for (Case III).

Local Determinacy with Policy Rules responding to prices and output

The economy is now described through the matrix

$$A \equiv \begin{pmatrix} E_t p_{t+1} \\ p_t \end{pmatrix} = \begin{pmatrix} \frac{1+\tilde{\beta}+\kappa(\hat{\psi}+\hat{\sigma})}{\tilde{\beta}+\kappa\sigma} & \frac{-(1+\sigma\varphi_y^i-\psi\varphi_y^B)}{\tilde{\beta}+\kappa\sigma} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix}$$

where $\hat{\psi} = \psi(1 - \varphi_p^B - \frac{\varphi_y^B}{\kappa})$ and $\hat{\sigma} = \sigma(1 + \varphi_p^i + \frac{\varphi_y^i}{\kappa})$. It follows that

$$detA = \frac{1 + \sigma\varphi_y^i - \psi\varphi_y^B}{\tilde{\beta} + \kappa\sigma} \tag{A16}$$

$$detA - traceA = \frac{\tilde{\beta} + \kappa\psi(1 - \varphi_p^B) + \kappa\sigma(1 + \varphi_p^i)}{\tilde{\beta} + \kappa\sigma} = 1 + \frac{\kappa\tilde{\psi}}{\tilde{\beta} + \kappa\sigma}. \tag{A17}$$

The latter is smaller than $-1$ if $\tilde{\psi} = \psi(1 - \varphi_p^B) + \sigma\varphi_p^i > 0$ (the condition in the text). The two eigenvalues $\lambda_{1,2}$ therefore satisfy

$$(\lambda_1 - 1)(\lambda_2 - 1) < 0, \tag{A18}$$

implying that both eigenvalues are real (otherwise $\lambda_1 - 1$ and $\lambda_2 - 1$ would be a complex pair with positive norm) and that one eigenvalue is larger than one. The second value eigenvalue can then be either smaller or larger than one in modulus.

Proposition C.1 in Appendix C of Woodford (2003) shows that if the condition

$$detA + traceA = \frac{2 + \tilde{\beta} + \psi(\kappa - \kappa\varphi_p^B - 2\varphi_y^B) + \sigma(\kappa + \kappa\varphi_p^i + 2\varphi_y^i)}{\tilde{\beta} + \kappa\sigma} < -1 \tag{A19}$$

is satisfied then the modulus of both eigenvalues is larger than one and if the condition is not satisfied the second eigenvalue is smaller then one in modulus. Thus the latter case is satisfied

($|\lambda_2| < 1$) if

$$\frac{2+\tilde{\beta}-2\psi\varphi_y^B+2\sigma+\varphi_y^i}{\tilde{\beta}+\kappa\sigma} > -2 \tag{A20}$$

and is thus sufficient for local determinacy. Thus under natural sign restrictions

$$\frac{2+\tilde{\beta}-2\psi\varphi_y^B+2\sigma+\varphi_y^i}{\tilde{\beta}+\kappa\sigma} > \frac{2+\tilde{\beta}+2\sigma}{\tilde{\beta}+\kappa\sigma} > 0 \tag{A21}$$

and the economy is locally determinate.

Local Determinacy with New Keynesian Phillips Curve and Policy Responses

The characteristic equation of the matrix

$$A \equiv \begin{pmatrix} E_t p_{t+1} \\ p_t \\ E_t Y_{t+1} \end{pmatrix} = \frac{1}{\beta} \begin{pmatrix} 1+\beta & -1 & -\kappa \\ \beta & 0 & 0 \\ \beta\hat{\psi}-\sigma & \sigma & \beta/\tilde{\beta}+\sigma\kappa+\hat{\epsilon} \end{pmatrix} \begin{pmatrix} p_t \\ p_{t-1} \\ Y_t \end{pmatrix}.$$

where $\hat{\psi} = \psi(1-\varphi_p^B)+\sigma\varphi_p^i$ and $\hat{\epsilon} = \sigma\varphi_y^i - \psi\varphi_y^B \geq 0$.

The characteristic equation is

$$x^3 + \underbrace{\frac{(-\hat{\epsilon}-1-1/\tilde{\beta})\beta-\kappa\sigma-1)}{\beta}}_{=:a_2} x^2 + \underbrace{\frac{\hat{\epsilon}(1+\beta)+\hat{\epsilon}+1+(\psi+\sigma)\kappa)+(1+\beta)/\tilde{\beta}}{\beta}}_{=:a_1} x + \underbrace{\frac{-1-\hat{\epsilon}\tilde{\beta}}{\beta\tilde{\beta}}}_{=:a_0} \tag{A22}$$

Checking the conditions in Proposition C.2 in Appendix C of Woodford (2003) requires to compute

$$1+a_2+a_1+a_0 = \frac{\kappa\hat{\psi}}{\beta} > 0 \tag{A23}$$

$$-1+a_2-a_1+a_0 = \frac{(-2\hat{\epsilon}-2)(1+\beta)+(-\hat{\psi}-2\sigma)\kappa-(2\beta+2)/\tilde{\beta}}{\beta} \tag{A24}$$

The first term (A23) is positive since $\hat{\psi} > 0$. The second term (A24) is negative since $\hat{\epsilon} > 0$. Furthermore $\mid a_2 \mid > 3$ since $\hat{\epsilon} > 0$ implies that $a_2 < \frac{(-1-1/\tilde{\beta})\beta-1)}{\beta} < -3$. The proof shows that the natural sign restrictions can be replaced with $\hat{\epsilon} \geq 0$.

Price Level Indeterminacy: Hand-to-Mouth Consumers

The same basic arguments for the economy where government bonds are not net wealth apply to models where a fraction of households is always hand-to-mouth and the remaining ones behave according to the permanent income hypothesis (PIH). Since hand-to-mouth consumers do not participate in the asset market, the real interest rate is determined by the discount factor of PIH households only, $(1+r_{ss})\beta = 1$, and equilibrium in the asset market is again characterized through

$$\frac{1+i_{ss}}{1+\pi_{ss}} = 1 + r_{ss} = 1/\beta, \tag{A25}$$

which does not depend on the price level, implying that the price level is indeterminate. This model shows that it is not any form heterogeneity by itself that delivers the result. Rather it is the combination of heterogeneity and market incompleteness that makes government bonds net wealth which leads to a well-defined aggregate savings function and implies price level determinacy. By the same argument, permanent heterogeneity in productivity but otherwise complete markets will not lead to price level determinacy either, since again $(1 + r_{ss})\beta = 1$ in a steady state.

Price Level Indeterminacy: Perpetual youth model

Similar arguments hold in "perpetual youth" models (Yaari (1965), Blanchard (1985)) since the steady-state interest rate is again equal to the discount rate, but now adjusted for the probability of death or retirement, so that $(1 + r_{ss})\tilde{\beta} = 1$ in a steady state for the adjusted discount rate $\tilde{\beta}$. Again, the steady-state real interest rate is independent of the price level and the price level is not determinate.

In this class of models, this is however not the only equilibrium if the Samuelson dynamic inefficiency condition is satisfied. In this case both a bubbleless as well as a continuum of bubbly equilibria exist, a scenario explored in recent work by Galí (2017). Whereas most papers assume that the bubble is a real asset affecting the stock market or housing, a monetary bubble may coexist so that money has value as in Samuelson's work. As a result there is a continuum of equilibria each associated with a different value of money (= different size of the monetary bubble) and each associated with a different price level. As an example, suppose that a bubble exists that has a real value of one. In one equilibrium nominal money has a value of one, the price level is one and thus there are no real bubbles. In another equilibrium the price level is two and a real bubble with value one half exists. Or the price level is three and the real bubble has a value of two thirds. Or the real bubble has a value of one and money has no value.

Bénassy (2005, 2008) make a particular choice on the size of the monetary bubble through ruling out real bubbles (the first case in the previous example) and conditional on this choice find a unique bubbly price level. This approach however does not overcome the indeterminacy problem in the "bubbleless" equilibrium and it rules out other bubbly equilibria with different price levels by assumption.[22] Bénassy (2005, 2008) need to make this equilibrium selection to obtain a well-defined demand for money (or more generally for nominal government liabilities) since the

[22]Bénassy (2005, 2008) implicitly assume a particular strong dynamic inefficiency condition - the population growth rate exceeds the real interest rate (which exceeds $1/\beta$) - since consumption of the initial generation would eventually exceed GDP otherwise.

Samuelson logic only delivers existence of a monetary equilibrium but not uniqueness. This shows again, as in the Hand-to-Mouth economy, that the failure of Ricardian equivalence is a necessary but not a sufficient condition for price level determinacy.

Price Level Indeterminacy: Representative agent and aggregate risk

Price level indeterminacy also arises in representative agent economies with aggregate risk. Suppose there are $n$ aggregate shocks $s_1, \dots, s_n$ with associated consumption levels of the representative household $c_1, \dots, c_n$ and marginal utilities of consumption $u_1, \dots, u_n$. The FOC for nominal bonds are therefore

$$v_i := \frac{u_i}{\tilde{P}_i} = \beta \frac{1+i_{ss}}{1+\gamma} \sum_{j=1}^{n} q_{ij} \frac{u_j}{\tilde{P}_j},$$

where $q_{ij} = Prob(s_j \mid s_i)$, $\tilde{P}_i$ is the price level in state $i$ as a deviation from the trend inflation rate $\gamma = \pi_{ss}$, which is equal to the constant growth rate of nominal debt. In matrix from the FOCs read

$$\underbrace{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}_{v} = \beta \frac{1+i_{ss}}{1+\gamma} \underbrace{\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ \vdots & \ddots & & \vdots \\ q_{n1} & \cdots & & q_{nn} \end{pmatrix}}_{Q} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

The vector $v$ is an eigenvector with eigenvalue one of the matrix $\beta \frac{1+i_{ss}}{1+\gamma} Q$ and the largest eigenvalue of $Q$ is one with eigenvector $(1, 1, \dots, 1)^{tr}$. Since $\beta \frac{1+i_{ss}}{1+\gamma} \leq 1$ it follows that $\beta \frac{1+i_{ss}}{1+\gamma} = 1$. Therefore for each $\kappa > 0$ the vector

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \kappa \\ \vdots \\ \kappa \end{pmatrix}$$

is a solution to the FOCs. This continuum of solutions corresponds to different price levels $\tilde{P}_i > 0$, establishing the indeterminacy. Not surprisingly since government bonds do not become net wealth just by adding aggregate uncertainty. Adding aggregate risk to an economy with PIH-households and hand-to-mouth households also does not overcome the indeterminacy problem. The same arguments for the representative agent now apply to the PIH households.