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RESEARCH ARTICLE

General equilibrium and the new neoclassical synthesis

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Abstract We present a general equilibrium model of the new neoclassical synthesis that has the same level of generality as the Arrow–Debreu model. This involves a stochastic multi-period economy with a monetary sector and sticky commodity prices. We formulate the notion of a sticky price equilibrium where all agents form rational expectations on prices for commodities and assets, interest rates, and rationing. We present a general result showing that monetary policy imposes no restrictions whatsoever on nominal equilibrium price levels and that the set of sticky price equilibria has a dimension equal to the number of terminal date-events. Stickiness of prices implies that this indeterminacy is real.

Keywords General equilibrium · Monetary policy · Sticky prices · New neoclassical synthesis · Indeterminacy

JEL Classification D50 · D90 · E40 · E50

1 Introduction

The new neoclassical synthesis is a term used by Goodfriend and King (1997) to refer to a class of models that incorporate elements of apparently irreconcilable traditions of macroeconomic thought. On the one hand, there are the flexible price models of the

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new classical macroeconomists and real-business-cycle analysis, in which monetary policy is unimportant for real economic activity, and on the other hand, there are the sticky price models of the New Keynesian economics, in which monetary policy is central to understanding real economic activity. The integration of these two streams in the literature leads to a class of models with four central elements: intertemporal optimization, rational expectations, optimal price-setting, and costly price adjustment.

The workhorse of the new neoclassical synthesis is a simple general equilibrium model involving inflation, output, and nominal interest rates at various date-events as its main variables. Crucial in these models is the existence of imperfections in setting commodity prices, which causes them to respond with some lag to changes in market conditions. Price stickiness is then the main channel through which monetary policy affects real variables. The main objective is to study how aggregate variables like inflation and output are affected by the adoption of various alternative policies.

The aim of the current paper is to provide a formulation of the new neoclassical synthesis that has the same level of generality as the Arrow–Debreu general equilibrium model. This involves an extension of the new neoclassical workhorse to a setting with multiple commodities, multiple heterogeneous agents having general preferences, general monetary transaction technologies, and a general approach toward price stickiness. In such an extension, households optimize intertemporally given rationally anticipated prices of commodities and assets, interest rates, and the monetary transactions technology, and whenever prices are not sticky, they do not adjust mechanically to some measure of disequilibrium, but are derived endogenously.

To achieve these objectives, we extend the Arrow–Debreu model in three ways. First, we follow Arrow (1953) and specify a setting with sequentially opening markets for commodities as opposed to the Arrow–Debreu framework with a complete set of markets for contingent commodities. Second, we follow Clower (1967) and require that all purchases and sales of commodities in spot markets are made against money. We specify a general monetary transaction technology in the spirit of Drèze and Polemarchakis (2001). Third, we introduce a general model of price stickiness where at each date-event, the price of a commodity is either sticky or is flexible and can be adjusted. If the price of a commodity is flexible, its value will be determined by the forces of supply and demand to achieve market clearing. If the price of a commodity is sticky, it is equal to the price in the previous date-event. Thereby, its value is endogenously determined by the first predecessor date-event where the price of this commodity is flexible. Our modeling of price stickiness incorporates frequently used specifications by Taylor (1980) and Calvo (1983) as special cases.

A particular period in the model starts with transactions on the asset markets with households trading Arrow securities and collecting dividends on their previous period's asset portfolios. Next, during the period, transactions on spot markets for commodities take place against money, and transactions with the bank occur exchanging bank loans against money. The bank loan corresponds to an obligation of the household to the bank, and money to an obligation of the bank toward the household. Contrary to the bank loan, money balances do not earn interest.

Monetary needs are determined by the monetary transactions technology and depend on commodity prices and the consumption bundles that are chosen. The bank charges interest on nominal debts, which households pay at the end of the period.

Since households hold nominal debts in the aggregate, the bank creates seignorage, which is returned to the bank's shareholders at the end of the period in the form of dividends. This modeling choice implies that we implicitly assume a Ricardian fiscal policy.

The policy of the bank involves the specification of interest rates that are set conditional on the date-event. This is in accordance with the observation in Woodford (2003) that monetary policy decision making by central banks almost everywhere means a decision about the operating target for an overnight interest rate. The reason for making the interest rate the operating target is that the alternative where the bank tries to directly control monetary aggregates has become less effective as a consequence of increases in the sophistication of the financial system.

A central feature of the new neoclassical synthesis is that monetary policy has nontrivial consequences. The key reason is the assumption that commodity prices are not continually adjusted, but remain fixed for at least short periods. Indeed, price stickiness is a well-documented empirical phenomenon. Nakamura and Steinsson (2010), for instance, report that the median duration of a price across sectors is around one year. Price stickiness can be caused by a variety of reasons, but the typical explanation follows Sheshinksi and Weiss (1977) and is based on the existence of menu costs, caused both by the real costs associated with the transmission of prices to the consumers as well as with the decision process itself.

The standard approach in the macroeconomic literature is to specify constant elasticity of substitution in utility and production functions à la Dixit and Stiglitz (1977). This makes it possible to derive closed-form expressions for prices that are set by producers in an environment of imperfect competition. Such an approach does not generalize to a setting with general preferences as has been argued by Roberts and Sonnenschein (1977), who show that equilibrium price and quantity choices may fail to exist even in extremely simple cases. To keep the feature that prices are determined endogenously in equilibrium, we stick to the standard general equilibrium paradigm of competitive markets. At all date-events where the price of a commodity can be adjusted, its level is determined by the forces of supply and demand, where supply and demand are derived from fully rational intertemporally optimizing agents.

If a commodity price is sticky, its value is inherited from the predecessor date-event and is ultimately determined by the most recent predecessor date-event where its price is flexible. If a commodity price is sticky, it is typically not market clearing, resulting in excess supply or excess demand. If the price of a commodity is sticky, its market is equilibrated by quantity adjustments, where the long side of the market is rationed by the amount of trade desired by the short side. Here, we follow the modeling approach developed by Drèze (1975) in the context of a general equilibrium model with upper and lower bounds on prices.

We formulate the concept of a sticky price equilibrium. At a sticky price equilibrium, all households optimize given rational expectations. More precisely, at a sticky price equilibrium, all households hold common and correct point expectations of all prices, rationing, interest rates, and dividends conditional on all possible date-events. Prices and rationing schemes are determined endogenously by the requirement of market clearing on commodity markets and asset markets, where sticky prices are set equal to the previous period's value. Allocations of commodities, assets, and money follow from optimizing behavior by the households, subject to the constraints imposed by the monetary transaction technology.

Our goal is to demonstrate the existence of a sticky price equilibrium under general assumptions on initial endowments, preferences, transaction technologies, and price stickiness. In an ad hoc macroeconomic model, Sargent and Wallace (1975) developed the insight that the price level is indeterminate under an interest rate rule. In a fully articulated macroeconomic model, similar indeterminacy results have been derived by Woodford (1994). These insights have spurred an extensive literature debating the indeterminacy of equilibrium in macroeconomic models with Ricardian fiscal policy, see Woodford (2003) for a detailed treatment of this literature and Cochrane (2011) for a recent discussion. Beyond equilibrium existence, we are therefore interested in obtaining an indeterminacy of equilibrium result in our general setting.

For each terminal date-event, we select one commodity with a flexible price and set this price equal to an arbitrary value. Next, we prove that each such specification of prices is consistent with some sticky price equilibrium, which demonstrates that sticky price equilibria exist and that the set of sticky price equilibria has dimension at least equal to the number of terminal date-events. The equilibrium nominal price level at terminal date-events is arbitrary, irrespective of the interest rate policy by the bank, and it is not even possible to control expected inflation by nominal interest rate policy. We argue that price stickiness implies that this indeterminacy is real.

Our model contains several widely studied general equilibrium models as special cases. This brings up the issue as to how the indeterminacy result is related to equilibrium existence results that have appeared previously in the literature. The standard Arrow–Debreu model corresponds to the case with one time period, one terminal date-event, no price stickiness, and zero interest rates charged by the bank on nominal debt. Such a model has a one-dimensional multiplicity of equilibrium indeed, which is usually suppressed by making use of zero-homogeneity of demand functions to normalize prices. Indeterminacy of equilibrium is entirely nominal.

The standard model of price rigidities as presented in Drèze (1975) corresponds to the case with one time period, one terminal date-event, and zero interest rates charged by the bank on nominal debt. For this model, one-dimensional multiplicity of equilibrium is shown in Herings (1996a), extending such a result for supply-constrained equilibria in van der Laan (1982). For a one-period model where some prices are flexible and some are downward rigid, Citanna et al. (2001) find a one-dimensional multiplicity of equilibria that is real.

In a multi-period model of a monetary economy without price stickiness, a model that contains Arrow (1953) as a special case, Drèze and Polemarchakis (2001) find that the dimension of the set of equilibria is equal to the number of terminal date-events, where the multiplicity is entirely nominal. Nakajima and Polemarchakis (2005) extend these ideas to a simple fully articulated two-period macroeconomic model, where the multiplicity is real when producers set prices one period in advance.

From a technical point of view, the main complication in our model is a result of the fact that nominal prices are sticky, which implies that nominal commodity prices have to enter the fixed point argument, and that it is not convenient to restrict attention to commodity prices in present-value terms. Since there are no a priori upper bounds on nominal price variables, we consider appropriate limits of economies with compactified price variables. A particular difficulty that has to be addressed is to make sure that the well-known cheaper-point assumption is satisfied in terms of present-value prices, both for compactified as for limit economies. Cases where exploding nominal commodity prices are offset by Arrow security prices that converge to zero have to be dealt with carefully.

The indeterminacy result can be understood as a simple consequence of counting equations and unknowns. There are as many markets for commodities and assets as there are instruments to clear them, which would suggest that equilibria are determinate. However, there is a budget constraint at the beginning of each date-event, and there is a budget constraints at the end of each terminal date-event, where each constraint serves as a Walras' law and leads to one additional degree of freedom for equilibrium. Since the policy of the bank consists of setting as many interest rates as there are date-events, each interest rate reducing the degrees of freedom by one due to a no-arbitrage condition on asset prices, we are left with the number of terminal date-events as the dimension of the set of equilibria.

One channel by which indeterminacy of equilibrium could be reduced is suggested in Magill and Quinzii (2014a, b) and consists of introducing additional instruments for the central bank. Another channel would consist of specifying non-Ricardian fiscal policies, where solvency of the fiscal authority implicitly specifies additional equilibrium restrictions and thereby reduces the degrees of freedom. It has been shown in Sims (1994) how this can lead to uniquely determined price levels in a fully articulated macroeconomic model. The macroeconomic literature has emphasized the role of active monetary policy (Leeper 1991) for instance in the form of Taylor rules to obtain determinacy of price levels. At the same time, Benhabib et al. (2001) show that indeterminacy can even result under active interest rate feedback rules. The analysis of equilibrium indeterminacy in a full-fledged general equilibrium model under various assumptions regarding the behavior of the monetary and the fiscal authority features therefore high on the research agenda. As we will argue later, to establish just equilibrium existence under such assumptions in a general setup is far from trivial.

The paper is organized as follows. Section 2 describes the main ingredients of our model—an intertemporal stochastic economy with a general monetary transaction technology and sticky prices—and the concept of sticky price equilibrium. Section 3 explains how the sequence of budget constraints can be replaced by a single budget constraint. Section 4 specifies a fully articulated example and makes a case for the desirability of strictly positive interest rates in an economy with price stickiness and impatient households. It establishes a relation between the rate of impatience of households and the level of welfare optimizing interest rates. Section 5 explicitly lists all assumptions that are needed for a proof of the general indeterminacy result. To study sticky price equilibria, it is helpful to represent the price and rationing in the market of a single commodity by a single parameter, and to define the concept of a parametrized sticky price equilibrium, which is equivalent to but more tractable than the notion of a sticky price equilibrium. This is the topic of Sect. 6. In Sect. 7, we present the main result of the paper about indeterminacy of sticky price equilibria. Section 8 discusses this result and presents the second main result, being that all indeterminacy is real. Section 9 considers potential extensions. Section 10 concludes. The appendix is devoted to a study of the continuity properties of the budget correspondence and the proof of the main theorem in Sect. 7.

2 The model

We provide a formulation of the new neoclassical synthesis that has the same level of generality as the Arrow–Debreu general equilibrium model. There is an event tree $\mathcal{T}$ with the set of date-events S as nodes. The set S is partitioned into subsets $S_0, \ldots, S_T$, where S_t consists of the date-events s_t in period t. We distinguish between dates and periods, where date t represents the starting point of period t and date $t+1$ its end point. We will also refer to date-events s_t and periods s_t to distinguish between points and intervals of time. There is a unique date-event s_0 at $t = 0$, the current date-event.

The set of successors of date-event s_t is denoted by s_t^+, a subset of S_{t+1}. For notational convenience, we introduce a set of date-events S_{T+1} with the same cardinality as S_T. There is a one-one relationship between date-events in S_T and those in S_{T+1}. The related date-event in S_{T+1} corresponds to the end point of period s_T and is the unique element of s_T^+. We denote $S^+ = \cup_{s_t \in S} s_t^+$, so $S^+ = (S \cup S_{T+1}) \backslash \{s_0\}$. The unique predecessor of $s_t \in S^+$ is denoted by s_t^-, an element of S_{t-1}.

In each period $s_t \in S$ there is trade in a finite set L of commodities by households in a finite set H. The price of commodity ℓ at date-event s_t equals $p_{\ell s_t}$. Among other things, the event tree is used to describe when price adjustments take place. Based on a detailed analysis on the distribution of the frequency of price changes in Nakamura and Steinsson (2008), Nakamura and Steinsson (2010) report that the median frequency of monthly price change across sections in the US economy is 8.7 %, implying that the median duration of a particular price across sectors is around one year. These authors also report considerable heterogeneity in this frequency across sectors. For most commodities, therefore, price adjustments do not take place across periods. As in Debreu (1959), the event tree is sufficiently refined for all prices to be constant within periods.

At each date-event $s_t \in S$, the price of commodity $\ell \in L$ is either sticky or can be adjusted. For each $\ell \in L$, this leads to a partition of S consisting of the sets N_ℓ^{s} and N_ℓ^{a}. The price of commodity ℓ can be adjusted at a date-event $s_t \in N_\ell^{\mathrm{a}}$. If the price of a commodity is flexible, its value will be determined by the forces of supply and demand to achieve market clearing. The set N_ℓ^{s} consists of those date-events, where the price of commodity ℓ is sticky and therefore equals the price in the previous date-event, $p_{\ell s_t^-}$. Notice that, contrary to earlier work in the general equilibrium literature with price rigidities, the value of a commodity with a sticky price is endogenously determined by the first predecessor date-event where the price of this commodity is flexible. When the price of commodity ℓ is sticky at date-event s_0, it is inherited from the price $p_{\ell s_{-1}}$ set at date-event s_{-1}, a price that is exogenously given at s_0. We allow for the case where the price of commodity ℓ is flexible at all date-events, in which case $N_\ell^{\mathrm{a}} = S$, the case where the price of commodity ℓ is fixed at all date-events, $N_\ell^{\mathrm{a}} = \emptyset$, as well as all the intermediate cases. The specification in Taylor (1980), where prices are sticky for a fixed number of periods, and the specification of Calvo (1983), where it is determined by chance whether a price can be adjusted, are both obtained as special cases.

Since the price of commodity ℓ at date-events in N_ℓ^{s} is sticky, the markets of these commodities are cleared by quantity adjustments. Violations of voluntary trading are not allowed for. This deviates from part of the macroeconomic literature where quantity adjustments are made by forcing the short side of the market to accommodate the trades desired by the long side. Such an approach is not feasible in our general setup.

In our model, trading on a particular commodity market is not only influenced by the price, but also by the maximal amount a household is able to supply of every commodity, called the *rationing scheme on supply*, and by the maximal amount a household is able to demand for every commodity, called the *rationing scheme on demand*. These constraints are imposed on the long side of the market and are determined by the short side. Rationing schemes serve as the matching technology between supply and demand. Since markets are assumed to be fully transparent, rationing affects the long side of the market only.

Rationing can take many forms. For the sake of simplicity, we consider uniform rationing, implying that the rationing scheme on supply is described by $\underline{z} \in -\mathbb{R}_+^{*LS}$ and the rationing scheme on demand by $\bar{z} \in \mathbb{R}_+^{*LS}$, where $\mathbb{R}^* = \mathbb{R} \cup \{+\infty\}$ denotes the set of extended real numbers. We model the absence of constraints on a particular market ℓs_t by setting $\underline{z}_{\ell s_t} = -\infty$ and $\bar{z}_{\ell s_t} = +\infty$, for instance at date-events in N_ℓ^{a}, where rationing does not take place at equilibrium. The way rationing is modeled is taken from the approach used by Drèze (1975) to study general equilibrium models with real price rigidities, see Herings 1996b for an overview of this literature. In the macroeconomic literature, such an approach is taken for instance in Svensson (1986). The values of the variables $\underline{z}$ and $\bar{z}$ are determined endogenously in an equilibrium.

Price stickiness involves *nominal prices*. For nominal prices to be meaningful, we need to extend the model by a monetary sector. To a large extent, we follow the monetary sector model of Drèze and Polemarchakis (2001), a model that is compatible with Chapter 2 of Woodford (2003), and that can be viewed as its general equilibrium extension. Households hold money for transaction purposes and have a bank loan that is adjusted whenever withdrawels or deposits of money are made.

On top of the real and the monetary part, we assume sequentially complete asset markets, where households trade Arrow securities. Contrary to what is common in the macroeconomic literature, we will not resort to specific functional forms or log-linearizations, but rather consider the actual supply and demand correspondences of commodities, money, and assets under assumptions similar as in the Arrow–Debreu model.

The timing of our model is as follows. A period $s_t \in S$ starts at date t with transactions on the asset markets and with the bank. Asset market transactions involve buying and selling Arrow securities contingent on future date-events and collecting dividends from the asset portfolio held in the previous period. Households use the proceeds from asset market transactions to adjust holdings of the bank loan and money balances. At dates in the interval $(t, t+1)$, transactions with other households on the spot markets for commodities take place against money, and transactions with the bank occur exchanging bank loans against money. At date $t+1$, period s_t terminates with the payment of interest due to the bank and the collection of the bank seignorage by the bank's shareholders.

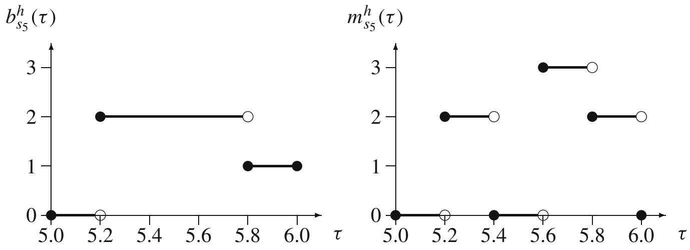

Fig. 1 The development of the bank loan and of money balances in Example 2.1

A more detailed account of the monetary part of the model is as follows. At each period $s_t \in S$, the bank offers loan facilities to households against a non-negative nominal interest rate r_{s_t}, which is allowed to be an arbitrary function of the state s_t. The bank supplies money balances as demanded by the households. Money balances do not earn any interest. At $\tau \in [t, t+1]$, household h therefore has a bank loan $b^h_{s_t}(\tau)$ and holds money balances $m^h_{s_t}(\tau)$. A withdrawal of money balances from the bank by the household increases the bank loan by the same amount, whereas a deposit of money at the bank leads to a decrease in the bank loan by the same magnitude. Transactions with other households increase money balances by an amount equal to the value of net sales, but do not affect the bank loan. At the end of period s_t, the average bank loan of household h in period s_t, $b^h_{s_t} = \int_{\tau=t}^{t+1} b^h_{s_t}(\tau)d\tau$, gives rise to interest payments by households to the bank equal to $r_{s_t} b^h_{s_t}$. We do allow for the special case where all interest rates are equal to zero. When interest rates are all zero and all prices can be adjusted at all date-events, the monetary part of our model is inessential and our model reduces to Arrow (1953) if, moreover, all prices are flexible at all date-events.

Example 2.1 Consider a household who starts a period s_5 with a bank loan $b^h_{s_5}(5) = 0$ and money balances $m^h_{s_5}(5) = 0$. At date 5.2, the household withdraws 2 monetary units from the bank and uses the money to make a purchase with value 2 at date 5.4. The household makes a sale with value equal to 3 monetary units at date 5.6 and brings 1 monetary unit back to the bank at date 5.8. The household makes a purchase with value 2 at date 6. Figure 1 illustrates the development of the bank loan and money balances for this example. The value of the bank loan changes at dates 5.2 and 5.8 when the household transacts with the bank. The amount of money balances held changes at dates 5.2, 5.4, 5.6, 5.8, and 6. The average bank loan of the household in period s_5 is equal to 1.4 units, so gives rise to interest payments equal to $1.4r_{s_5}$. To keep the example simple, it has been assumed that $r_{s_5} = 0$. Otherwise, the value of the bank loan would have been increased by this amount at date 6. Moreover, if the household is a shareholder of the bank, the bank loan is decreased by the household's share in the bank seignorage.

A more detailed account of the asset market part of the model follows next. At each date-event $s_t \in S$ there are $|s_t^+|$ Arrow securities, one for each date-event in s_t^+. An Arrow security for date-event s_{t+1} is traded at date-event s_{t+1}^- against a price $q_{s_{t+1}}$ and pays one nominal unit if and only if date-event s_{t+1} occurs. Because of the availability of Arrow securities, markets are sequentially complete. A standard no-arbitrage argument implies that at equilibrium the sum of the prices of the Arrow securities traded at date-event s_t must be equal to $1/(1 + r_{s_t})$. At no-arbitrage prices, asset demand is indeterminate as any household is indifferent between holding one unit less of the bank loan and one unit more of every Arrow security. To lift this indeterminacy, we set beginning-of-period bank loans equal to money balances for every household, $b^h_{s_t}(t) = m^h_{s_t}(t)$. Notice that we distinguish between end-of-period bank loans $b^h_{s_t^-}(t)$ at s_t^-, where a discrepancy between $b^h_{s_t^-}(t)$ and $m^h_{s_t^-}(t)$ is possible, and beginning-of-period bank loans $b^h_{s_t}(t)$ at s_t, which are equal to $m^h_{s_t}(t)$ by definition.

Aggregate money balances issued by the bank at τ are $m^{\mathrm{c}}_{s_t}(\tau) = \sum_{h\in H} m^h_{s_t}(\tau)$, a non-negative quantity. Average aggregate money balances issued in period s_t equal $m^{\mathrm{c}}_{s_t} = \int_{\tau=t}^{t+1} m^{\mathrm{c}}_{s_t}(\tau)d\tau$. Since beginning-of-period bank loans are set equal to money balances for every household, it holds that $b^{\mathrm{c}}_{s_t}(\tau) = \sum_{h\in H} b^h_{s_t}(\tau) = \sum_{h\in H} m^h_{s_t}(\tau)$ and $b^{\mathrm{c}}_{s_t} = \sum_{h\in H} b^h_{s_t} = \sum_{h\in H} m^h_{s_t}$. Notice that the equality between $b^{\mathrm{c}}_{s_t}$ and $\sum_{h\in H} m^h_{s_t}$ only holds at the aggregate level. Example 2.1 illustrates that at the household level discrepancies between $b^h_{s_t}$ and $m^h_{s_t}$ are likely to occur. At the end of period s_t, the bank collects an amount $v^{\mathrm{c}}_{s_t} = r_{s_t} b^{\mathrm{c}}_{s_t}$ of interest payments as seignorage. The bank issues the entire seignorage as dividends to its shareholders at the end of the period. Household h receives $v^h_{s_t} = \theta^h v^{\mathrm{c}}_{s_t}$ at the end of period s_t with θ^h the shareholdings of household h.

At date-event $s_t \in S$, i.e., at the beginning of period s_t, household h has wealth given by the returns from investments in Arrow securities in the previous period $a^h_{s_t}$, plus monetary holdings at the end of the previous period $m^h_{s_t^-}(t)$, minus the bank loan at the end of the previous period $b^h_{s_t^-}(t)$. Making use of the property that $b^h_{s_t^-}(t-1) = m^h_{s_t^-}(t-1)$, it follows that the monetary holdings minus the bank loan at the end of the previous period equal net sales of commodities in the previous period plus dividends received minus interest payments, $m^h_{s_t^-}(t) - b^h_{s_t^-}(t) = p_{s_t^-}(e^h_{s_t^-} - x^h_{s_t^-}) + v^h_{s_t^-} - r_{s_t^-} b^h_{s_t^-}$.

Household h invests his wealth at date-event s_t in Arrow securities $a^h_{s_{t+1}}$, where $s_{t+1} \in s_t^+$. It follows that household h faces the following sequence of budget constraints,

$$\begin{aligned}
&\textstyle\sum_{s_1\in s_0^+} q_{s_1} a^h_{s_1} + m^h_{s_0}(0) - b^h_{s_0}(0) = 0,\\
&\textstyle\sum_{s_{t+1}\in s_t^+} q_{s_{t+1}} a^h_{s_{t+1}} + m^h_{s_t}(t) - b^h_{s_t}(t) = a^h_{s_t} + m^h_{s_t^-}(t) - b^h_{s_t^-}(t), \quad s_t \in S\backslash\{s_0\},\\
&a^h_{s_{T+1}} + m^h_{s_{T+1}^-}(T+1) - b^h_{s_{T+1}^-}(T+1) = 0, \quad s_{T+1} \in S_{T+1},
\end{aligned}$$

the lifting-of-indeterminacy identities

$$b^h_{s_t}(t) = m^h_{s_t}(t), \quad s_t \in S,$$

and the accounting identities

$$m^h_{s_t}(t+1) - b^h_{s_t}(t+1) = p_{s_t}(e^h_{s_t} - x^h_{s_t}) + v^h_{s_t} - r_{s_t} b^h_{s_t}, \quad s_t \in S.$$

Substitution of the accounting identities and the lifting-of-indeterminacy identities in the budget constraints eliminates the monetary balances $m^h_{s_t}(t)$ and bank loan holdings $b^h_{s_t}(t)$ and leads to the following reduced form of the budget constraints for household h:

$$\begin{aligned}\textstyle\sum_{s_1 \in s_0^+} q_{s_1} a^h_{s_1} &= 0,\\ \textstyle\sum_{s_{t+1} \in s_t^+} q_{s_{t+1}} a^h_{s_{t+1}} &= a^h_{s_t} - p_{s_t^-}\left(x^h_{s_t^-} - e^h_{s_t^-}\right) - r_{s_t^-} b^h_{s_t^-} + v^h_{s_t^-}, \quad s_t \in S\backslash\{s_0\},\\ 0 &= a^h_{s_{T+1}} - p_{s_{T+1}^-}\left(x^h_{s_{T+1}^-} - e^h_{s_{T+1}^-}\right) - r_{s_{T+1}^-} b^h_{s_{T+1}^-} + v^h_{s_{T+1}^-}, \quad s_{T+1} \in S_{T+1}.\end{aligned} \tag{2.1}$$

It follows from the reduced form of the budget constraints that the only relevant aspects of the monetary transactions are the interest payments $r_{s_t} b^h_{s_t}$ made by household h and the bank seignorage $v^h_{s_t}$ received by h at the end of period s_t.

The value of b^h is determined by the *transaction technology correspondence* $\beta^h : \mathbb{R}^{LS}_+ \times X^h \to \mathbb{R}^S$ of household h. It assigns to each $p \in \mathbb{R}^{LS}_+$ and consumption bundle $x^h \in X^h$ a subset $\beta^h(p, x^h)$ of $\mathbb{R}^S$. An element b^h of $\beta^h(p, x^h)$ specifies the bank loans that are sufficient to carry out purchases and sales involved in consumption of x^h when prices are p. A typical example concerns a Clower (1967) type cash-in-advance technology, where $b^h \in \beta^h(p, x^h)$ if and only if $b^h_{s_t} \geq p_{s_t} x^h_{s_t}$, $s_t \in S$. Alternatively, when cash is only needed for net purchases, $b^h \in \beta^h(p, x^h)$ if and only if $b^h_{s_t} \geq p_{s_t}(x^h_{s_t} - e^h_{s_t})^+$, $s_t \in S$, where for a real-valued vector z we use the notation $z^+ = \max\{0, z\}$.[1] Both specifications make the implicit assumption that cash needed for purchases is needed in advance, whereas cash resulting from sales is only available at the end of the period. If, instead, cash resulting from sales is immediately available for purchases, then the natural specification becomes $b^h \in \beta^h(p, x^h)$ if and only if $b^h_{s_t} \geq (p_{s_t}(x^h_{s_t} - e^h_{s_t}))^+$, $s_t \in S$.

As in Lucas and Stokey (1987), we can incorporate the distinction between "cash goods," which are subject to a cash-in-advance constraint, and "credit goods," which do not need to be paid for in cash as would be the case for instance for leisure, and it is entirely possible to make the cash requirements good specific. The transaction technology can be made state-dependent, which enables us to model that the monetary transactions technology is subject to velocity shocks.

The modeling of the transaction technology incorporates a rich variety of other specifications and allows for cases where households have interest elastic money demand as in the Baumol–Tobin model developed independently in Baumol (1952) and Tobin (1956). All that is needed is to have one of the commodities ℓ representing cash-withdrawal services, the consumption of which diminishes the need for cash balances.

The transaction technology correspondence approach avoids specifications where prices enter the utility function. Utility is derived from the consumption of goods, and the only way money holdings and prices affect utilities is via the commodities that can be purchased.

[1] The maximum of two vectors is defined by taking the componentwise maximum. Similarly, the minimum of two vectors is defined by taking the componentwise minimum.

The description of an economy $\mathcal{E} = (\mathcal{T}, (X^h, \preceq^h, e^h, \theta^h, \beta^h)_{h\in H}, (N^{\mathrm{a}}_\ell, N^{\mathrm{s}}_\ell)_{\ell\in L}, p_{s_{-1}}, r)$ is completed by a specification of $(\preceq_h)_{h\in H}$, with $\preceq^h$ the preference relation of household h defined on X^h, and a specification of prices $p_{s_{-1}}$ at date-event s_{-1}, where only the prices $p_{\ell s_{-1}}$ for ℓ such that $s_0 \in N^{\mathrm{s}}_\ell$ matter.

The monetary part of the model deviates from the treatment in Drèze and Polemarchakis (2001) in two, relatively minor, aspects. The transaction technology there is a correspondence that assigns a set of feasible consumption bundles and bank loans (x^h, b^h) to each price system p. In this paper, we assign a set of feasible bank loans b^h to each price system and consumption bundle (p, x^h). A second, more substantial, difference is that we allow bank loans to be negative, which would naturally occur when a household makes many sales in a particular period, resulting in a bank deposit rather than a bank loan.

A household takes prices (p, q), interest rates r, rationing schemes $(-\underline{z}, \bar{z})$, and dividends v^h as given, and chooses a maximal element (x^h, a^h, b^h) for $\preceq^h$ subject to the constraints $x^h \in X^h, b^h \in \beta^h(p, x^h), \underline{z} \leq x^h - e^h \leq \bar{z}$, and the sequence of budget constraints (2.1). The budget set $\gamma^h(p, q, r, \underline{z}, \bar{z}, v^h)$ consists of all tuples (x^h, a^h, b^h) satisfying all these constraints.

We use the notational convention that x is indexed by $h \in H$, $\ell \in L$, and $s_t \in S$, b and v are indexed by $h \in H$ and $s_t \in S$, a is indexed by $h \in H$ and $s_t \in S^+$, p, $\underline{z}$ and $\bar{z}$ by $\ell \in L$ and $s_t \in S$, and q by $s_t \in S^+$.

Definition 2.2 A *sticky price equilibrium* for the economy $\mathcal{E}$ is $(p^*, q^*, \underline{z}^*, \bar{z}^*, v^*, x^*, a^*, b^*)$ in $\mathbb{R}^{LS} \times \mathbb{R}^{S^+} \times -\mathbb{R}^{*LS}_+ \times \mathbb{R}^{*LS}_+ \times \mathbb{R}^{HS} \times \mathbb{R}^{HLS} \times \mathbb{R}^{HS^+} \times \mathbb{R}^{HS}$ such that

(a) for $h \in H$, (x^{*h}, a^{*h}, b^{*h}) is $\preceq^h$-maximal on $\gamma^h(p^*, q^*, r, \underline{z}^*, \bar{z}^*, v^{*h})$,
(b) commodity markets clear, $\sum_{h\in H} x^{*h} = \sum_{h\in H} e^h$,
(c) Arrow security markets clear, $\sum_{h\in H} a^{*h} = 0$,
(d) the no-arbitrage conditions hold, for $s_t \in S$, $\sum_{s_{t+1}\in s_t^+} q^*_{s_{t+1}} = 1/(1 + r_{s_t})$,
(e) for $h \in H$, $s_t \in S$, dividends satisfy $v^{*h}_{s_t} = \theta^h r_{s_t} \sum_{h\in H} b^{*h}_{s_t}$,
(f) for $s_t \in N^{\mathrm{s}}_\ell$, prices equal the previous period's value, $p^*_{\ell s_t} = p^*_{\ell s_t^-}$,
(g) no rationing when the price is flexible, for $\ell \in L$, $s_t \in N^{\mathrm{a}}_\ell$, $\underline{z}^*_{\ell s_t} = -\infty$ and $\bar{z}^*_{\ell s_t} = +\infty$,
(h) rationing is one-sided, for $\ell \in L$, $s_t \in N^{\mathrm{s}}_\ell$,

$$\underline{z}^*_{\ell s_t} > -\infty \text{ implies } \bar{z}^*_{\ell s_t} = +\infty,$$
$$\bar{z}^*_{\ell s_t} < +\infty \text{ implies } \underline{z}^*_{\ell s_t} = -\infty.$$

In Definition 2.2, there is no reference to the variables $b^{\mathrm{c}}_{s_t}$ and $m^{\mathrm{c}}_{s_t}$. They follow from the accounting identities $b^{\mathrm{c}}_{s_t} = \sum_{h\in H} b^h_{s_t}$ and $m^{\mathrm{c}}_{s_t} = b^{\mathrm{c}}_{s_t}$.

A household is not necessarily influenced by the constraints on his choices caused by the rationing scheme $(\underline{z}, \bar{z})$. A household h is *constrained* on its supply in the market for contingent commodity ℓs_t at $(p, q, r, \underline{z}, \bar{z}, v^h)$ if there exists $(\hat{x}^h, \hat{a}^h, \hat{b}^h) \in \gamma^h(p, q, r, \hat{\underline{z}}, \bar{z}, v^h)$, where $\hat{\underline{z}}$ equals $\underline{z}$, except that $\hat{\underline{z}}_{l s_t} = -\infty$, such that for all $(x^h, a^h, b^h) \in \gamma^h(p, q, r, \underline{z}, \bar{z}, v^h)$, $\hat{x}^h$ is strictly preferred to x^h. The definition for a household to be constrained on its demand in the market for contingent commodity ℓs_t is analogous. There is supply (demand) rationing in the market for contingent commodity ℓs_t at $(p, q, r, \underline{z}, \bar{z}, v^h)$ if at least one household is constrained on its supply (demand) in the market for commodity ℓs_t at $(p, q, r, \underline{z}, \bar{z}, v^h)$. There is rationing in the market for contingent commodity ℓs_t at $(p, q, r, \underline{z}, \bar{z}, v^h)$ if there is supply rationing or demand rationing in the market for commodity ℓs_t at $(p, q, r, \underline{z}, \bar{z}, v^h)$.

For the sake of simplicity, we have presented a model of a pure exchange economy. It is a routine exercise to extend the model and the sticky price equilibrium concept to a production economy.

3 Present-value prices

The sequence of budget constraints can be rewritten in a more convenient way. We denote the present-value price at s_0 of one unit of income at date-event $s_t \in \{s_0\} \cup S^+$ by $q^0_{s_t}$. With $s_t(s_{t'})$ denoting the predecessor of $s_{t'}$ at date t, we have

$$q^0_{s_0} = 1,$$
$$q^0_{s_t} = q_{s_1(s_t)} q_{s_2(s_t)} \cdots q_{s_{t-1}(s_t)} q_{s_t}, \quad s_t \in S^+.$$

When we multiply the budget constraint at date-event s_t by $q^0_{s_t}$ and add up we get

$$\sum_{s_t \in S} q^0_{s_t} \sum_{s_{t+1} \in s_t^+} q_{s_{t+1}} a^h_{s_{t+1}} = \sum_{s_t \in S^+} q^0_{s_t} \left(a^h_{s_t} - p_{s_t^-} \left(x^h_{s_t^-} - e^h_{s_t^-} \right) - r_{s_t^-} b^h_{s_t^-} + v^h_{s_t^-} \right).$$

After canceling the a^h-terms which appear on both sides with identical multiplicands, and rearranging terms, we obtain

$$\sum_{s_t \in S^+} q^0_{s_t} \left(p_{s_t^-} x^h_{s_t^-} + r_{s_t^-} b^h_{s_t^-} \right) = \sum_{s_t \in S^+} q^0_{s_t} \left(p_{s_t^-} e^h_{s_t^-} + v^h_{s_t^-} \right). \tag{3.1}$$

Since $\sum_{s_{t+1} \in s_t^+} q_{s_{t+1}} = 1/(1 + r_{s_t})$, we find

$$\sum_{s_t \in S} \left(\frac{q^0_{s_t}}{1 + r_{s_t}} p_{s_t} x^h_{s_t} + \frac{q^0_{s_t}}{1 + r_{s_t}} r_{s_t} b^h_{s_t} \right) = \sum_{s_t \in S} \left(\frac{q^0_{s_t}}{1 + r_{s_t}} p_{s_t} e^h_{s_t} + \frac{q^0_{s_t}}{1 + r_{s_t}} v^h_{s_t} \right). \tag{3.2}$$

It is now straightforward to verify that the original sequence of budget constraints is equivalent to (3.2) plus the recursive system of equations

$$\begin{aligned} a^h_{s_{T+1}} &= p_{s^-_{T+1}} \left(x^h_{s^-_{T+1}} - e^h_{s^-_{T+1}} \right) + r_{s^-_{T+1}} b^h_{s^-_{T+1}} - v^h_{s^-_{T+1}}, \quad s_{T+1} \in S_{T+1}, \\ a^h_{s_t} &= p_{s_t^-} \left(x^h_{s_t^-} - e^h_{s_t^-} \right) + r_{s_t^-} b^h_{s_t^-} - v^h_{s_t^-} + \textstyle\sum_{s_{t+1} \in s_t^+} q_{s_{t+1}} a^h_{s_{t+1}}, \quad s_t \in S \backslash \{s_0\}. \end{aligned} \tag{3.3}$$

Since the a^h-terms are neither part of the preferences nor of any of the other constraints, the consumer choice problem has been reduced to the choice of (x^h, b^h) with $x^h \in X^h$

and $b^h \in \beta^h(p, x^h)$ subject to the single budget constraint (3.2) and the quantity constraints $\underline{z} \leq x^h - e^h \leq \bar{z}$. Moreover, by substituting $\tilde{q}_{s_t} = q^0_{s_t}/(1 + r_{s_t})$, $\tilde{p}_{s_t} = \tilde{q}_{s_t} p_{s_t}$, $\tilde{b}^h_{s_t} = \tilde{q}_{s_t} b^h_{s_t}$, and $\tilde{v}^h_{s_t} = \tilde{q}_{s_t} v^h_{s_t}$, the budget constraint (3.2) can be rewritten as

$$\sum_{s_t \in S} (\tilde{p}_{s_t} x^h_{s_t} + r_{s_t} \tilde{b}^h_{s_t}) = \sum_{s_t \in S} (\tilde{p}_{s_t} e^h_{s_t} + \tilde{v}^h_{s_t})$$

or, even shorter,

$$\tilde{p} x^h + r \tilde{b}^h = \tilde{p} e^h + \tilde{w}^h,$$

where $\tilde{w}^h = \sum_{s_t \in S} \tilde{v}^h_{s_t}$.

We refer to prices $\tilde{p}$ as present-value prices and, more generally, refer to variables with a tilde as present-value variables. Although our definition for $\tilde{p}$ is the most convenient one, it deviates from what is also referred to as a present-value price, the variable p^0 defined by $p^0_{s_t} = q^0_{s_t} p_{s_t}$ for $s_t \in S$. To see that p^0 is also an appropriate present-value price in our model, consider the case of a cash-in-advance constraint with $b^h_{s_t} = p_{s_t} x^h_{s_t}$. We see that the budget constraint (3.2) reduces to

$$\sum_{s_t \in S} p^0_{s_t} x^h_{s_t} = \sum_{s_t \in S} \left(\frac{1}{1 + r_{s_t}} p^0_{s_t} e^h_{s_t} + \tilde{v}^h_{s_t} \right).$$

The right-hand side of this constraint consists entirely of variables which are taken as given by the household. The prices relevant for the purchase of consumption goods are given by p^0.

When we consider the cash-in-advance constraint on net trades, $b^h_{s_t} = p_{s_t}(x^h_{s_t} - e^h_{s_t})^+$, we can rewrite the budget constraint (3.2) to

$$\sum_{s_t \in S} p^0_{s_t} (x^h_{s_t} - e^h_{s_t})^+ = \sum_{s_t \in S} \left(\frac{1}{1 + r_{s_t}} p^0_{s_t} (e^h_{s_t} - x^h_{s_t})^+ + \tilde{v}^h_{s_t} \right).$$

In this case, we obtain a wedge between buying prices $p^0_{s_t}$ and selling prices $(1/(1 + r_{s_t}))p^0_{s_t}$. Positive nominal interest rates create distortions and cause equilibrium marginal rates of substitution of households to differ from one another, thereby leading to absence of Pareto optimality even in the absence of price stickiness.

4 An example

To get a feel for the model and the equilibrium concept, and to illustrate how the model can be used to analyze the effects of monetary policy on inflation and output, a simple, but fully articulated, example is presented next.

Consider an economy with two households, $H = \{1, 2\}$, two periods with no uncertainty, $S = \{0, 1\}$, and one commodity per period. The end of period 1 is denoted by 2, so $S^+ = \{1, 2\}$. The price in period 0 is assumed to be flexible and the price in period 1 sticky, $N^{\mathrm{a}} = \{0\}$ and $N^{\mathrm{s}} = \{1\}$, and consequently, it holds that $p_1 = p_0$ with p_0 flexible. Initial endowments are $e^1 = (1, 0)$ and $e^2 = (0, 1)$, so household 1 can be thought of as an old household that produces today in exchange for consumption in the future and household 2 as a young household that produces in the future in exchange for consumption today.

The preferences of the households are represented by utility functions u^1 and u^2 defined by

$$u^1(x^1) = \ln(x_0^1) + \delta \ln(x_1^1), \quad x^1 \gg 0,$$
$$u^2(x^2) = \ln(x_0^2) + \delta \ln(x_1^2), \quad x^2 \gg 0,$$

where $\delta \in (0, 1]$ denotes the common discount factor.

It is assumed that cash is needed for net purchases,

$$\beta^h(p, x^h) = \left\{ b^h \in \mathbb{R}^S \mid \forall s \in S,\ b_s^h \geq p_s(x_s^h - e_s^h)^+ \right\}, \quad h = 1, 2.$$

The initial endowments are chosen such that household 1 is supplying the good in period 0 and demanding it in period 1, and vice versa for household 2. We can therefore without loss of generality assume that $b_0^1 = 0$, $b_1^1 = p_1 x_1^1$, $b_0^2 = p_0 x_0^2$, and $b_1^2 = 0$.

The bank sets nominal interest rates r_0 and r_1 and issues its entire seignorage as dividends to household 2 at the end of each period, so $\theta^1 = 0$ and $\theta^2 = 1$.

As we have derived in Eq. (3.1), the sequence of budget constraints can be replaced by the single budget constraint

$$q_1(p_0 x_0^h + r_0 b_0^h) + q_1 q_2(p_1 x_1^h + r_1 b_1^h) = q_1(p_0 e_0^h + v_0^h) + q_1 q_2(p_1 e_1^h + v_1^h). \quad (4.1)$$

We argue that whenever $(p^*, q^*, \underline{z}^*, \bar{z}^*, v^*, x^*, a^*, b^*)$ is a sticky price equilibrium, it holds that $(\lambda p^*, q^*, \underline{z}^*, \bar{z}^*, v^*, x^*, a^*, b^*)$ is a sticky price equilibrium for every positive value of the scalar λ. Indeed, since b_0^h and b_1^h are linear in p_0 and p_1, respectively, it follows that the budget constraints are homogeneous of degree 1 in prices (p_0, p_1). The same holds for the transaction technology correspondences. In this example, price stickiness is equivalent to the requirement that $p_0 = p_1$, a condition that remains satisfied when all prices are multiplied with a positive scalar. Prices do not enter the utility function. In what follows, we can therefore without loss of generality restrict our attention to sticky price equilibria with $(p_0^*, p_1^*) = (1, 1)$.

Since there is no uncertainty, the Arrow securities traded on the asset markets reduce to nominal bonds with equilibrium prices pinned down by the no-arbitrage conditions, $q_1 = 1/(1 + r_0)$ and $q_2 = 1/(1 + r_1)$. We substitute $e^1 = (1, 0)$, $e^2 = (0, 1)$, $p = (1, 1)$, $q_1 = 1/(1 + r_0)$, $q_2 = 1/(1 + r_1)$, $b_0^1 = 0$, $b_1^1 = p_1 x_1^1$, $b_0^2 = p_0 x_0^2$, $b_1^2 = 0$, and $v_0^1 = v_1^1 = 0$ in (4.1) and obtain the budgets constraints for households 1 and 2, respectively, given by

$$\tfrac{1}{1+r_0} x_0^1 + \tfrac{1}{1+r_0} x_1^1 = \tfrac{1}{1+r_0} 1,$$
$$x_0^2 + \tfrac{1}{(1+r_0)(1+r_1)} x_1^2 = \tfrac{1}{1+r_0} v_0^2 + \tfrac{1}{(1+r_0)(1+r_1)} (1 + v_1^2),$$

or, equivalently,

$$x_0^1 + x_1^1 = 1,$$
$$(1 + r_0)x_0^2 + \tfrac{1}{1+r_1}x_1^2 = v_0^2 + \tfrac{1}{1+r_1}(1 + v_1^2).$$

If interest rates are positive, the cash-in-advance transaction technology creates a wedge between the prices against which purchases and sales can be made. This becomes apparent from the budget constraints above, where the real relative price of commodity 0 over commodity 1 is equal to 1 for household 1 and equal to $(1 + r_0)(1 + r_1)$ for household 2.

Let $(p^*, q^*, \underline{z}^*, \bar{z}^*, v^*, x^*, a^*, b^*)$ be a sticky price equilibrium. Since the price of commodity 1 is sticky, its market is in general cleared by quantity adjustments. Either household 1 is constrained on the demand of commodity 1, or household 1 is constrained on the supply of this commodity, or we have the degenerate case where no household is constrained.

We first consider sticky price equilibria where household 1 is constrained on the demand of commodity 1 by the rationing scheme $\bar{z}_1^*$, so $x_1^{*1} = \bar{z}_1^*$. By market clearing it holds that $x_1^{*2} = 1 - \bar{z}_1^*$. Since rationing is one-sided in equilibrium by Condition (h) of Definition 2.2, it holds that household 2 is unconstrained, so optimization by household 2 implies that

$$x_0^{*2} = \frac{1}{\delta(1 + r_0)(1 + r_1)} x_1^{*2} = \frac{1 - \bar{z}_1^*}{\delta(1 + r_0)(1 + r_1)}.$$

The budget constraint of household 1 implies that $x_0^{*1} = 1 - \bar{z}_1^*$. Market clearing of the commodity in period 0 yields

$$1 = x_0^{*1} + x_0^{*2} = 1 - \bar{z}_1^* + \frac{1 - \bar{z}_1^*}{\delta(1 + r_0)(1 + r_1)},$$

so

$$\bar{z}_1^* = \frac{1}{1 + \delta(1 + r_0)(1 + r_1)}.$$

Since $\bar{z}_1^*$ is assumed to be binding, it holds that $\bar{z}_1^*$ should be less than the unconstrained demand for commodity 1 by household 1, which equals $\delta/(1 + \delta)$. A sticky price equilibrium where household 1 is constrained on the demand of commodity 1 therefore results if and only if $1/\delta^2 < (1 + r_0)(1 + r_1)$.

It is now straightforward to compute the equilibrium values of all the endogenous variables. The sticky price equilibrium allocation is given by

$$x^{*1} = \left(\tfrac{\delta(1+r_0)(1+r_1)}{1+\delta(1+r_0)(1+r_1)}, \tfrac{1}{1+\delta(1+r_0)(1+r_1)} \right),$$
$$x^{*2} = \left(\tfrac{1}{1+\delta(1+r_0)(1+r_1)}, \tfrac{\delta(1+r_0)(1+r_1)}{1+\delta(1+r_0)(1+r_1)} \right).$$

At equilibrium, household 1 supplies an amount $1/(1 + \delta(1 + r_0)(1 + r_1))$ of the commodity in period 0, invests the money coming from these sales in Arrow securities $a_2^{*1} = (1+r_1)/(1+\delta(1+r_0)(1+r_1))$ at the beginning of period 1, withdraws money $m_1^{*1} = 1/(1+\delta(1+r_0)(1+r_1))$ from the bank in order to make purchases in period 1, and in doing so creates a bank loan $b_1^{*1} = 1/(1 + \delta(1 + r_0)(1 + r_1))$. Household 1 uses m_1^{*1} to purchase $1/(1+\delta(1+r_0)(1+r_1))$ units of the commodity in period 1. At the end of period 1, household 1 has to redeem the bank loan plus interest $(1+r_1)b_1^{*1}$, which is exactly equal to the proceeds from the Arrow securities.

We next consider sticky price equilibria where household 2 is constrained on his supply of commodity 1 by the rationing scheme $\underline{z}_1^*$. It holds that $x_1^{*2} = 1 + \underline{z}_1^*$. By market clearing we find that $x_1^{*1} = -\underline{z}_1^*$. Since rationing is one-sided in equilibrium by Condition (h) of Definition 2.2, it holds that household 1 is unconstrained, so optimization by household 1 implies that $x_0^{*1} = x_1^{*1}/\delta = -\underline{z}_1^*/\delta$. Market clearing of the commodity in period 0 yields $x_0^{*2} = 1 - x_0^{*1} = (\delta + \underline{z}_1^*)/\delta$. The budget constraint of household 1 implies that $1 = x_0^{*1} + x_1^{*1} = -\underline{z}_1^*/\delta - \underline{z}_1^*$, so $\underline{z}_1^* = -\delta/(1+\delta)$. Since $\underline{z}_1^*$ is assumed to be binding, it holds that $1 + \underline{z}_1^*$ should exceed the unconstrained demand for commodity 1 by household 2, which equals

$$\frac{\delta(1+r_1)}{1+\delta}\left(v_0^{*2} + \frac{1+v_1^{*2}}{1+r_1}\right).$$

Since $v_0^{*2} = r_0 x_0^{*2} = r_0(\delta + \underline{z}_1^*)/\delta$ and $v_1^{*2} = r_1 x_1^{*1} = -r_1\underline{z}_1^*$, we can simplify the resulting inequality and find that a sticky price equilibrium where household 2 is constrained on his supply of commodity 1 results if and only if $1/\delta^2 > (1+r_0)(1+r_1)$.

Finally, in case $1/\delta^2 = (1+r_0)(1+r_1)$, the sticky price equilibrium does not involve rationing and the equilibrium allocation is given by $x^{*1} = (1/(1+\delta), \delta/(1+\delta))$ and $x^{*2} = (\delta/(1+\delta), 1/(1+\delta))$.

Observe that in the sticky price equilibria with rationing, the unconstrained demand of the short side of the market determines the amount by which the long side of the market is rationed. When impatience as measured by $1/\delta$ is less than the geometric average of $1+r_0$ and $1+r_1$, there is too much demand for the commodity in the future period, and the long side of the market, household 1, is constrained in the demand for the commodity in the future period by the supply of household 2. The reverse holds if impatience $1/\delta$ exceeds the geometric average of $1 + r_0$ and $1 + r_1$ and sticky price equilibria are then characterized by rationing of the supply of household 2 of the future commodity by the demand of household 1.

In case the bank chooses an interest rate policy such that the geometric average of $1 + r_0$ and $1 + r_1$ is equal to $1/\delta$, a sticky price equilibrium without rationing results. It is easily verified that this equilibrium weakly Pareto dominates the sticky price equilibria resulting from all other interest rate policies. Intuitively, under sticky nominal prices, interest rate policy can be used to equate real commodity price ratios with the discount factor, which calls for strictly positive interest rates when agents are impatient. On the other hand, choosing higher interest rates increases the wedge between buying and selling prices, which leads to inefficiencies. In the example, as long as $(1+r_0)(1+r_1) \leq 1/\delta^2$, both effects are exactly equal to each other, and the sticky price equilibria lead to identical utilities for the households. When $(1 + r_0)(1 + r_1)$ exceeds $1/\delta^2$, increases in $(1+r_0)(1+r_1)$ lead to lower values of $\bar{z}_1^*$, so more stringent demand rationing of household 1, and lower utilities for both households 1 and 2. In the limit, when the product of $1 + r_0$ and $1 + r_1$ tends to infinity, there is no supply of commodity 1 by household 2 and all demand of commodity 1 by household 1 is rationed, resulting in the no-trade outcome.

The example has a number of special features. For instance, interest rate policy is (trivially) completely ineffective to control inflation. More surprisingly, it is also completely ineffective to control price levels as it has been shown that all equilibrium prices can be multiplied by a positive scalar λ without affecting the equilibrium. There is a 1-dimensional set of equilibria, where all indeterminacy is nominal. The question is how such conclusions extend to the general model specification with many periods, uncertainty, multiple commodities, all kinds of price stickiness, and general transaction technologies. We will demonstrate that interest rate policy is not effective to control expected inflation and price levels, the degree of indeterminacy is equal to the number of terminal date-events and, under reasonable assumptions, all indeterminacy is real.

5 Assumptions

The assumptions below are made throughout the paper without further mentioning.

A1. For $h \in H$, X^h is closed, convex, has a lower bound, and $X^h + \mathbb{R}_+^{LS} \subset X^h$. There exists $x^h \in X^h$ such that $x^h \ll e^h$.

A2. For $h \in H$, $\preceq^h$ is transitive, complete, continuous, convex, and monotonic: if $x^h, \bar{x}^h \in X^h$ with $x^h < \bar{x}^h$, then $x^h \prec^h \bar{x}^h$.

A3. The bank is owned by the households: for $h \in H$, $\theta^h \geq 0$, and $\sum_{h \in H} \theta^h = 1$.

A4. For $h \in H$ we have:

1. The correspondence β^h is lower hemi-continuous and closed.
2. Monetary needs are bounded: There exist continuous functions $\underline{f}^h : \mathbb{R}_+^{LS} \times X^h \to -\mathbb{R}_+^S$, $\bar{f}^h : \mathbb{R}_+^{LS} \times X^h \to \mathbb{R}_+^S$ such that $b^h \in \beta^h(p, x^h)$ implies $\underline{f}^h(p, x^h) \leq b^h$ and $\min\{\bar{f}^h(p, x^h), b^h\} \in \beta^h(p, x^h)$.
3. The correspondence β^h is positive homogeneous in prices. Consider $p, \bar{p} \in \mathbb{R}_+^{LS}$ and $\lambda \geq 0$ such that $\bar{p}_{\bar{s}_t} = \lambda p_{\bar{s}_t}$ and, for $s_t \in S \backslash \{\bar{s}_t\}$, $\bar{p}_{s_t} = p_{s_t}$. Consider $x^h \in X^h$. Then, it holds that $b^h \in \beta^h(p, x^h)$ implies $\bar{b}^h \in \beta^h(\bar{p}, x^h)$, where $\bar{b}^h_{\bar{s}_t} = \lambda b^h_{\bar{s}_t}$ and, for $s_t \in S \backslash \{\bar{s}_t\}$, $\bar{b}^h_{s_t} = b^h_{s_t}$.
4. The correspondence β^h satisfies the following convexity property: For $p \in \mathbb{R}_+^{LS}$, for $x^h, \bar{x}^h \in X^h$, it holds that $b^h \in \beta^h(p, x^h)$ and $\bar{b}^h \in \beta^h(p, \bar{x}^h)$ implies $\lambda b^h + (1 - \lambda)\bar{b}^h \in \beta^h(p, \lambda x^h + (1 - \lambda)\bar{x}^h)$ for all $\lambda \in [0, 1]$.
5. For $p \in \mathbb{R}_+^{LS}$, for $x^h \in X^h$, $x^h \leq e^h$ implies $0 \in \beta^h(p, x^h)$.
6. For $p \in \mathbb{R}_+^{LS}$, for $\ell s_t \in L \times S$ with $p_{\ell s_t} = 0$, for $x^h, \bar{x}^h \in X^h$ with $\bar{x}^h = x^h + \varepsilon e_{\ell s_t}$ for some $\varepsilon > 0$, it holds that $\beta^h(p, x^h) \subset \beta^h(p, \bar{x}^h)$.

A5. Only the bank can create money: If $x \in \prod_h X^h$ satisfies $\sum_h x^h = \sum_h e^h$ and, for some $p \in \mathbb{R}_+^{LS}$, for all $h \in H$, $b^h \in \beta^h(p, x^h)$, then $\sum_{h \in H} b^h \geq 0$.

A6. $p_{s_{-1}} \gg 0$.

The Assumptions A1, A2, A3, and A6 are standard. A4.1 is a standard continuity assumption that is satisfied by cash-in-advance technologies for instance. We require β^h to be closed rather than to be upper hemi-continuous, since the latter assumption is quite strong for correspondences that are not compact-valued and would be violated by cash-in-advance technologies. Assumption A4.2 puts lower and upper bounds on monetary needs. A natural choice for the functions $\underline{f}^h, \bar{f}^h : \mathbb{R}^{LS}_+ \times X^h \to \mathbb{R}^S_+$ would be

$$\underline{f}^h_{s_t}(p, x^h) = -p_{s_t}(e^h_{s_t} - x^h_{s_t})^+, \; s_t \in S,$$
$$\bar{f}^h_{s_t}(p, x^h) = p_{s_t}(x^h_{s_t} - e^h_{s_t})^+, \quad s_t \in S.$$

Assumption A4.3 is a standard homogeneity assumption, and A4.4 a standard convexity assumption. In A4.5, we require that when a consumption bundle involves only sales, no bank loan is needed. This is natural, since making the sales should result in a bank deposit rather than a loan. Similarly, for a consumption bundle involving only purchases, non-negative bank loans are required. A4.6 states that additional consumption of a commodity with a zero price does not require additional money balances.[2] A5 requires that all attainable allocations involve non-negative aggregate monetary holdings.

6 Parametrized prices and rationing schemes

We will not only establish that sticky price equilibria exist, but also that there is an abundance of such equilibria. In fact, we will argue that the set of sticky price equilibria is $|S_T|$-dimensional. To prove this result, it is convenient to choose a suitable parametrization of price and rationing variables. For $\ell s_t \in L \times S$, we define the set of parameters $R_{\ell s_t}$ by

$$R_{\ell s_t} = \mathbb{R}_+, \qquad \text{if } s_t \in N^{\mathrm{a}}_\ell,$$
$$R_{\ell s_t} = [0, 1], \quad \text{if } s_t \in N^{\mathrm{s}}_\ell,$$

with typical element $\rho_{\ell s_t}$. Next, we define $R = \prod_{\ell s_t \in L \times S} R_{\ell s_t}$ with typical element ρ.

When $s_t \in N^{\mathrm{a}}_\ell$, supply and demand are equilibrated by the price $p_{\ell s_t} \in \mathbb{R}_+$. We parametrize this price by the variable $\rho_{\ell s_t} \in \mathbb{R}_+$ and set $p_{\ell s_t}(\rho) = \rho_{\ell s_t}$.

At s_t in N^{s}_ℓ, the market for commodity ℓ is equilibrated by one-sided rationing. We will parametrize the relevant rationing schemes by a single parameter $\rho_{\ell s_t} \in [0, 1]$. We define $\underline{z}_{\ell s_t}(\rho)$ and $\bar{z}_{\ell s_t}(\rho)$ in such a way that the following properties hold: $\underline{z}_{\ell s_t}(\rho) = 0$ if $\rho_{\ell s_t} = 0$, at equilibrium ρ may lead to supply rationing in the market for commodity ℓs_t only if $\rho_{\ell s_t} < 1/2$, but is irrelevant for supply rationing when $\rho_{\ell s_t} \geq 1/2$. Similarly, $\bar{z}_{\ell s_t}(\rho) = 0$ if $\rho_{\ell s_t} = 1$, at equilibrium $\rho_{\ell s_t}$ may lead to demand rationing in the market for commodity ℓs_t only if $\rho_{\ell s_t} > 1/2$, but is irrelevant for demand rationing when

[2] If this commodity helps in saving on transaction costs, more consumption could actually decrease the amount of money balances needed, a possibility that is allowed for in the current formulation.

$\rho_{\ell s_t} \leq 1/2$. The rationing scheme on supply induced by ρ is more negative, so less restrictive for choice, when $\rho_{\ell s_t}$ increases; the induced rationing scheme on demand is less positive, so more restrictive for choice, when $\rho_{\ell s_t}$ increases. In this way, $\rho_{\ell s_t}$ clears markets by means of rationing in a way analogously to a price variable. Low values of $\rho_{\ell s_t}$ tend to induce excess demand, and high values of $\rho_{\ell s_t}$ tend to induce excess supply, completely analogous to the response of excess demand to the price of a commodity.

We now make the parametrization explicit. Since consumption sets are bounded from below, the market clearing conditions imply that the set of attainable allocations of consumption bundles, the set of $x \in \prod_{h\in H} X^h$ such that $\sum_{h\in H} x^h = \sum_{h\in H} e^h$, is bounded. Let $\bar{c} \in \mathbb{R}^{LS}_{+}$ be such that at every attainable allocation the excess consumption $x^h_{\ell s_t} - e^h_{\ell s_t}$ of every household h is strictly less than $\bar{c}_{\ell s_t}$ and strictly more than $-\bar{c}_{\ell s_t}$. Moreover, we choose $\bar{c} \geq \sum_{h\in H}(e^h - \underline{x}^h)$, where $\underline{x}^h$ is a lower bound for X^h, which implies that consumption equal to $\bar{c}_{\ell s_t}$ by a single household is not compatible with feasibility of the allocation.

Consider some $\rho \in R$. We define, for $\ell \in L$, for $s_t \in N^{\mathrm{a}}_{\ell}$,

$$\begin{aligned} p_{\ell s_t}(\rho) &= \rho_{\ell s_t}, \\ \underline{z}_{\ell s_t}(\rho) &= -\bar{c}_{\ell s_t}, \\ \bar{z}_{\ell s_t}(\rho) &= \bar{c}_{\ell s_t}, \end{aligned}$$

for $s_t \in N^{\mathrm{s}}_{\ell}$,

$$\begin{aligned} p_{\ell s_t}(\rho) &= p_{\ell s_t^-}(\rho), \\ \underline{z}_{\ell s_t}(\rho) &= \max\{-2\bar{c}_{\ell s_t}\rho_{\ell s_t}, -\bar{c}_{\ell s_t}\}, \\ \bar{z}_{\ell s_t}(\rho) &= \min\{2\bar{c}_{\ell s_t} - 2\bar{c}_{\ell s_t}\rho_{\ell s_t}, \bar{c}_{\ell s_t}\}. \end{aligned}$$

In case $s_0 \in N^{\mathrm{s}}_{\ell}$, the above definition should be read as requiring $p_{\ell s_0^-}(\rho) = p_{\ell s_{-1}}$.

Notice that this recursive definition implies $p_{\ell s_{t'}}(\rho) = \rho_{\ell s_t(s_{t'})}$, where $\ell s_{t'}$ is a commodity with a sticky price, t is the date where this price is set, and $s_t(s_{t'})$ the prevailing date-event. In case $t = -1$, it holds that $p_{\ell s_{t'}}(\rho) = p_{\ell s_{-1}}$. An important observation is that for each commodity ℓs_t there is exactly one instrument, parametrized by $\rho_{\ell s_t}$, to clear its market, either a price, or a rationing scheme on supply, or a rationing scheme on demand.

Definition 6.1 A *parametrized sticky price equilibrium* for the economy $\mathcal{E}$ is $(\rho^*, q^*, v^*, x^*, a^*, b^*)$ in $R \times \mathbb{R}^{S^+} \times \mathbb{R}^{HS} \times \mathbb{R}^{HLS} \times \mathbb{R}^{HS^+} \times \mathbb{R}^{HS}$ such that

(a) for $h \in H$, (x^{*h}, a^{*h}, b^{*h}) is $\preceq^h$-maximal on $\gamma^h(p(\rho^*), q^*, r, \underline{z}(\rho^*), \bar{z}(\rho^*), v^{*h})$,
(b) commodity markets clear, $\sum_{h\in H} x^{*h} = \sum_{h\in H} e^h$,
(c) Arrow security markets clear, $\sum_{h\in H} a^{*h} = 0$,
(d) the no-arbitrage conditions hold, for $s_t \in S$, $\sum_{s_{t+1}\in s_t^+} q^*_{s_{t+1}} = 1/(1 + r_{s_t})$,
(e) for $h \in H$, for $s_t \in S$, dividends satisfy $v^{*h}_{s_t} = \theta^h r_{s_t} \sum_{h\in H} b^{*h}_{s_t}$.

The notion of parametrized sticky price equilibrium is more convenient than the one of sticky price equilibrium for a number of reasons. The number of free variables in a parametrized sticky price equilibrium is less than the number in a sticky price equilibrium and is equal to the number of market clearing conditions. None of the equilibrium conditions in a parametrized sticky price equilibrium involves conditionals. All the variables in a parametrized sticky price equilibrium are real numbers; extended real numbers are avoided.

The next result shows that a parametrized sticky price equilibrium induces a sticky price equilibrium in a straightforward way. In fact, all that we need to do is to replace supply rationing schemes that are equal to the lower bound $-\bar{c}_{\ell s_t}$ by $-\infty$ and demand rationing schemes equal to the upper bound $\bar{c}_{\ell s_t}$ by $+\infty$. To this end, we define $\underline{z}^{\infty}_{\ell s_t}(\rho) = \underline{z}_{\ell s_t}(\rho)$ if $\underline{z}_{\ell s_t}(\rho) > -\bar{c}_{\ell s_t}$ and $\underline{z}^{\infty}_{\ell s_t}(\rho) = -\infty$ if $\underline{z}_{\ell s_t}(\rho) = -\bar{c}_{\ell s_t}$, and $\bar{z}^{\infty}_{\ell s_t}(\rho) = \bar{z}_{\ell s_t}(\rho)$ if $\bar{z}_{\ell s_t}(\rho) < \bar{c}_{\ell s_t}$ and $\bar{z}^{\infty}_{\ell s_t}(\rho) = +\infty$ if $\bar{z}_{\ell s_t}(\rho) = \bar{c}_{\ell s_t}$.

Theorem 6.2 *If $(\rho^*, q^*, v^*, x^*, a^*, b^*)$ is a parametrized sticky price equilibrium, then $(p(\rho^*), q^*, \underline{z}^{\infty}(\rho^*), \bar{z}^{\infty}(\rho^*), v^*, x^*, a^*, b^*)$ is a sticky price equilibrium.*

Proof We verify that $(p(\rho^*), q^*, \underline{z}^{\infty}(\rho^*), \bar{z}^{\infty}(\rho^*), v^*, x^*, a^*, b^*)$ satisfies Conditions (a)–(h) of Definition 2.2.

To show (a), it is sufficient to show that replacing a rationing scheme on supply equal to $-\bar{c}_{\ell s_t}$ by $-\infty$ and a rationing scheme on demand equal to $\bar{c}_{\ell s_t}$ by $+\infty$ is not going to give opportunities to any household h to acquire $(x^h, a^h, b^h) \in \gamma^h(p(\rho^*), q^*, \underline{z}^{\infty}(\rho^*), \bar{z}^{\infty}(\rho^*), v^{*h})$ such that $x^{*h} \prec^h x^h$. Suppose, on the contrary, there is such a household h and a corresponding (x^h, a^h, b^h). Since x^* is an attainable consumption bundle, it holds that $-\bar{c} \ll x^{*h} - e^h \ll \bar{c}$. It follows that $-\bar{c} \ll \lambda(x^h - e^h) + (1 - \lambda)(x^{*h} - e^h) \ll \bar{c}$ for λ strictly positive and sufficiently close to zero, and therefore $\underline{z}(\rho^*) \leq \lambda(x^h - e^h) + (1 - \lambda)(x^{*h} - e^h) \leq \bar{z}(\rho^*)$. Since $\gamma^h(p(\rho^*), q^*, r, \underline{z}^{\infty}(\rho^*), \bar{z}^{\infty}(\rho^*), v^{*h})$ is convex, we find that $(\lambda x^h + (1 - \lambda)x^{*h}, \lambda a^h + (1 - \lambda)a^{*h}, \lambda b^h + (1 - \lambda)b^{*h}, v^{*h}) \in \gamma^h(p(\rho^*), q^*, r, \underline{z}(\rho^*), \bar{z}(\rho^*))$. By convexity of $\preceq^h$ we have that $x^{*h} \prec^h \lambda x^h + (1 - \lambda)x^{*h}$, which contradicts that (x^{*h}, a^{*h}, b^{*h}) is $\preceq^h$-maximal on $\gamma^h(p(\rho^*), q^*, r, \underline{z}(\rho^*), \bar{z}(\rho^*), v^{*h})$.

Conditions (b), (c), (d), (e), and (f) hold trivially.

For $s_t \in N^{\mathrm{a}}_{\ell}$, $\underline{z}_{\ell s_t}(\rho^*) = -\bar{c}_{\ell s_t}$ and $\bar{z}_{\ell s_t}(\rho^*) = \bar{c}_{\ell s_t}$, so $\underline{z}^{\infty}_{\ell s_t}(\rho^*) = -\infty$ and $\bar{z}^{\infty}_{\ell s_t}(\rho^*) = +\infty$, and we obtain Condition (g).

By definition, for any value of $\rho \in R$, it is not the case that simultaneously $\underline{z}_{\ell s_t}(\rho) > -\bar{c}_{\ell s_t}$ and $\bar{z}_{\ell s_t}(\rho) < \bar{c}_{\ell s_t}$, and therefore, it is not the case that simultaneously $\underline{z}^{\infty}_{\ell s_t}(\rho) > -\infty$ and $\bar{z}^{\infty}_{\ell s_t}(\rho) < +\infty$. This proves that Condition (h) is satisfied. □

By Theorem 6.2, if we show that parametrized sticky price equilibria exist, we have shown the existence of sticky price equilibria. Theorem 6.3 shows the converse of Theorem 6.2. Up to irrelevant values of non-binding rationing schemes, all sticky price equilibria are obtained when restricting attention to parametrized sticky price equilibria.

Theorem 6.3 *If $(p^*, q^*, \underline{z}^*, \bar{z}^*, v^*, x^*, a^*, b^*)$ is a sticky price equilibrium, then $(\rho^*, q^*, v^*, x^*, a^*, b^*)$ is a parametrized sticky price equilibrium, where for $s_t \in N^{\mathrm{a}}_{\ell}$,*

$$\rho^*_{\ell s_t} = p^*_{\ell s_t},$$

and for $s_t \in N^{\mathrm{s}}_\ell$,

$$\begin{array}{lll} \textit{if } \underline{z}^*_{\ell s_t} < -\bar{c}_{\ell s_t} \textit{ and } \bar{z}^*_{\ell s_t} > \bar{c}_{\ell s_t}, & \textit{then } \rho^*_{\ell s_t} = 1/2, \\ \textit{if } \underline{z}^*_{\ell s_t} \geq -\bar{c}_{\ell s_t}, & \textit{then } \rho^*_{\ell s_t} = \underline{z}^*_{\ell s_t} / -2\bar{c}_{\ell s_t}, \\ \textit{if } \bar{z}^*_{\ell s_t} \leq \bar{c}_{\ell s_t}, & \textit{then } \rho^*_{\ell s_t} = (2\bar{c}_{\ell s_t} - \bar{z}^*_{\ell s_t})/2\bar{c}_{\ell s_t}. \end{array}$$

Proof We show that $(\rho^*, q^*, v^*, x^*, a^*, b^*)$ satisfies all the conditions of a parametrized sticky price equilibrium.

Condition (a) follows since

$$(x^{*h}, a^{*h}, b^{*h}) \in \gamma^h(p(\rho^*), q^*, r, \underline{z}(\rho^*), \bar{z}(\rho^*), v^{*h}) \subset \gamma^h(p^*, q^*, r, \underline{z}^*, \bar{z}^*, v^{*h}).$$

Conditions (b), (c), (d), and (e) hold trivially. □

Theorems 6.2 and 6.3 imply that there is no loss of generality to restrict attention to parametrized sticky price equilibria.

7 Multiplicity of sticky price equilibria

We show that the degree of multiplicity of sticky price equilibria is $|S_T|$ by showing that parametrized sticky price equilibria exist that satisfy $|S_T|$ additional restrictions on top of the equilibrium conditions. To make our formulation as simple as possible, we assume that at every terminal date-event $s_T \in S_T$ there is at least one commodity with a flexible price. We select for each date-event s_T one such commodity, denoted by $\ell(s_T)s_T$, and define the set of those commodities by $L^* = \{\ell(s_T)s_T \mid s_T \in S_T\}$. The $|S_T|$ additional restrictions are formulated by choosing an arbitrary vector $\alpha \in \mathbb{R}^{S_T}_{++}$ and imposing, for $s_T \in S_T$, $\rho_{\ell(s_T)s_T} = \alpha_{s_T}$ on top of the equilibrium conditions. The additional restrictions pin down the nominal prices of $|S_T|$ commodities.

Theorem 7.1 *For all* $s_T \in S_T$, *assume* $s_T \in \cup_{\ell \in L} N^{\mathrm{a}}_\ell$. *Let* L^* *be a set with one flexible price commodity for each terminal date-event. For each choice of* $\alpha \in \mathbb{R}^{S_T}_{++}$, *there is a parametrized sticky price equilibrium* $(\rho^*, q^*, v^*, x^*, a^*, b^*)$ *satisfying, for all* $\ell s_T \in L^*$, $\rho^*_{\ell s_T} = \alpha_{s_T}$.

Proof See the "Appendix". □

By Theorem 6.2 the following corollary follows at once.

Corollary 7.2 *For all* $s_T \in S_T$, *assume* $s_T \in \cup_{\ell \in L} N^{\mathrm{a}}_\ell$. *Let* L^* *be a set with one flexible price commodity for each terminal date-event. For each choice of* $\alpha \in \mathbb{R}^{S_T}_{++}$, *there is a sticky price equilibrium* $(p^*, q^*, \underline{z}^*, \bar{z}^*, v^*, x^*, a^*, b^*)$ *satisfying, for all* $\ell s_T \in L^*$, $p^*_{\ell s_T} = \alpha_{s_T}$.

What is the intuition underlying the results of Theorem 7.1 and Corollary 7.2? Counting equations and unknowns in Definition 6.1 of a parametrized sticky price equilibrium reveals that there are $|L||S|$ commodity market clearing conditions and

$|L||S|$ variables $\rho_{\ell s_t}$. Moreover, there are $|S^+|$ asset market clearing conditions and $|S^+|$ asset prices q_{s_t}. There are $|S| + |S_T|$ budget constraints in (2.1), leading to $|S| + |S_T|$ Walras' laws, inducing $|S| + |S_T|$ degrees of freedom for equilibrium. The policy of the bank involves setting $|S|$ interest rates, leading to $|S|$ no-arbitrage conditions in Definition 6.1, reducing the degrees of freedom for equilibrium by $|S|$, resulting in $|S_T|$ degrees of indeterminacy.

One way to lift the $|S_T|$ degrees of indeterminacy has been suggested by Magill and Quinzii (2014a, b) in a model without price stickiness. Their suggestion essentially boils down to choosing $|S_T|$ additional instruments, in their case by having the bank not only controlling the short-term interest rates, but in addition the interest rates on bonds with longer maturity. Although it is not completely evident that the needed controllability requirements carry over to the sticky price framework, the general principle that more instruments are needed to lower the degree of indeterminacy holds. But since the number $|S_T|$ should be thought of as huge, the date-event tree should in principle include all date-events on which agents can condition their actions, the extent to which additional instruments help to reduce nominal and real indeterminacy remains an open issue.

8 Degrees of nominal and real indeterminacy

Corollary 7.2 shows that for each choice of $\alpha \in \mathbb{R}^{S_T}_{++}$ there is a sticky price equilibrium with, for all $\ell s_T \in L^*$, $p^*_{\ell s_T} = \alpha_{s_T}$. The equilibrium nominal price level at terminal date-events is arbitrary, irrespective of the interest rate policy by the bank.

The case where all prices are flexible has been studied in Drèze and Polemarchakis (2001). The absence of imperfections in price formation and the homogeneity assumptions imposed on the transactions technology imply that the whole analysis there can be done in terms of present-value prices. Each equilibrium in terms of present-value prices leads to an S_T-dimensional set of equilibria.

To illustrate this fact, consider, for the sake of concreteness, an economy with two periods, and let $(p^0_{s_0}, (p^0_{s_1})_{s_1 \in S_1})$ be present-value equilibrium prices. In terms of nominal prices, we have that

$$\begin{aligned} p^0_{s_0} &= p_{s_0}, \\ p^0_{s_1} &= q^0_{s_1} p_{s_1} = q_{s_1} p_{s_1}, \quad s_1 \in S_1. \end{aligned}$$

By the homogeneity assumption on the transactions technology, we find that the present-value equilibrium prices $(p^0_{s_0}, (p^0_{s_1})_{s_1 \in S_1})$ induce the $(S_1 - 1)$-dimensional set of nominal equilibrium prices $(p_{s_0}, (p_{s_1})_{s_1 \in S_1})$ given by

$$\begin{aligned} p_{s_0} &= p^0_{s_0}, \\ p_{s_1} &= \tfrac{1}{q_{s_1}} p^0_{s_1}, \quad s_1 \in S_1, \end{aligned}$$

whenever $q \in \mathbb{R}^{S_1}_{++}$ satisfies $\sum_{s_1 \in S_1} q_{s_1} = 1$, so induces $S_1 - 1$ degrees of freedom.

Whenever $(p_{s_0}, (p_{s_1})_{s_1 \in S_1})$ are nominal equilibrium prices, so is $(\lambda p_{s_0}, (\lambda p_{s_1})_{s_1 \in S_1})$ for any $\lambda > 0$, which adds one degree of freedom and shows that each equilibrium in terms of present-value prices leads to an S_1-dimensional set of equilibria. All these equilibria induce the same equilibrium allocation, so there is only nominal indeterminacy and no real indeterminacy of equilibrium in this case.

The same S_1-dimensional set of equilibria results from Corollary 7.2, though the parametrization chosen there is different. According to Corollary 7.2, one can choose a flexible price commodity $\ell(s_1)$ for each date-event $s_1 \in S_1$, a vector of price levels $\alpha \in \mathbb{R}^{S_1}_{++}$, and have an equilibrium with nominal prices given by $p_{\ell(s_1)} = \alpha_{s_1}$.

Although the bank cannot control the price level by interest rate policy, it can control expected inflation. To illustrate this, suppose the economy is stationary, meaning that present-value equilibrium prices are constant up to a discount factor δ,

$$p^0_{s_1} = \pi_{s_1} \delta p^0_{s_0}, \quad s_1 \in S_1,$$

where π_{s_1} is the probability of occurrence of state s_1. We have that the harmonic mean of period one prices is equal to

$$\begin{aligned} H(p_{S_1}) &= \frac{1}{\sum_{s_1 \in S_1} \frac{\pi_{s_1}}{p_{s_1}}} = \frac{1}{\sum_{s_1 \in S_1} \frac{\pi_{s_1} q_{s_1}}{p^0_{s_1}}} = \frac{1}{\sum_{s_1 \in S_1} \frac{q_{s_1}}{\delta p_{s_0}}} \\ &= \frac{1}{\frac{1}{\delta p_{s_0}(1+r_{s_0})}} = \delta p_{s_0}(1 + r_{s_0}). \end{aligned}$$

Expected inflation is equal to $\delta(1 + r_{s_0})$. Higher nominal interest rates lead to higher expected inflation. The variance of inflation, however, can be arbitrarily high and is not controlled by the interest rate. Whenever there are two or more date-events in period one, arbitrarily high inflation rates are caused by values of q_{s_1} arbitrarily close to zero. Since expected inflation is controlled, high inflation in some date-events has to be compensated by low inflation or even deflation in other date-events, as it holds that $\sum_{s_1 \in S} q_{s_1} = 1/(1 + r_{s_0})$.

This reasoning extends to the general model with $T + 1$ periods.

These conclusions change when the set of sticky price commodities is non-empty. By Corollary 7.2, there is still an S_T-dimensional set of equilibria. However, the route to demonstrate this result via proving the existence of an equilibrium in terms of present-value prices and next generating an S_T-dimensional set of nominal equilibrium prices by appropriate choices of q and λ is blocked. Multiplications of all prices by λ violates price stickiness if $s_0 \in \cup_{\ell \in L} N^{\mathrm{s}}_{\ell}$, whereas it can easily happen that no choice for q is consistent with price stickiness. The existence proof of Theorem 7.1 is therefore by means of correspondences that are formulated in terms of nominal prices.

To illustrate the implications of Theorem 7.1 and Corollary 7.2, consider again an economy with two periods, one commodity per date-event, but now a sticky price in period 0, and flexible prices in period 1. By Corollary 7.2 one can choose an arbitrary vector of prices $\alpha \in \mathbb{R}^{S_1}_{++}$ and have an equilibrium with nominal prices given by $p_{s_1} = \alpha_{s_1}$ for $s_1 \in S_1$. Since the period 0 price is sticky, this result implies that it is not even possible to control expected inflation by nominal interest rate policy. Expected inflation can be arbitrarily high or low, irrespective of nominal interest rates, as the vector α can be arbitrarily chosen.

How are markets equilibrated in such an economy? The equilibrium state prices q_{s_1} should satisfy $\sum_{s_1 \in S_1} q_{s_1} = 1/(1 + r_{s_0})$ and, since all prices in period 1 are flexible, should be such that at equilibrium present-value prices $p^0_{s_1} = q_{s_1} p_{s_1}$ there is zero aggregate net trade across future date-events. Relatively high values of α_{s_1} go together with low prices q_{s_1} and vice versa. When the vector α has high values across the board, with a sticky price in period 0, the only channel to generate zero intertemporal aggregate trade is demand rationing in period 0. When expected inflation is high and the price in period 0 does not adjust, consumers will face demand rationing in period 0. Vice versa, with low expected inflation, or even expected deflation, a sticky price in period 0 leads to lack of demand and supply rationing in period 0.

In the simple case with one commodity per date-event and a sticky price in period 0, we can generate $S_1 - 1$ degrees of nominal indeterminacy by an appropriate choice of the asset prices q in the following way. Let $(p^0_{s_0}, (p^0_{s_1})_{s_1 \in S_1})$ be present-value equilibrium prices. Since the price in period 0 is assumed to be sticky, it holds that $p^0_{s_0} = p_{s_0} = p_{s_{-1}}$. Any choice of $q \in \mathbb{R}^{S_1}_{++}$ satisfying $\sum_{s_1 \in S_1} q_{s_1} = 1/(1 + r_{s_0})$ leads to nominal equilibrium prices

$$p_{s_1} = \frac{1}{q_{s_1}} p^0_{s_1}, \quad s_1 \in S_1,$$

at unchanged rationing schemes and allocation of commodities, thereby generating $S_1 - 1$ degrees of nominal indeterminacy.

Consider a particular choice for asset prices, say $\bar{q} \in \mathbb{R}^{S_1}_{++}$ satisfying $\sum_{s_1 \in S_1} \bar{q}_{s_1} = 1/(1 + r_{s_0})$, and let $\bar{p}_{S_1}$ be the corresponding nominal commodity prices in period 1. For any $\lambda > 0$ it holds by Corollary 7.2 that there is a sticky price equilibrium with $p_{S_1} = \lambda \bar{p}_{S_1}$. Due to the requirement $\sum_{s_1 \in S_1} \bar{q}_{s_1} = 1/(1 + r_{s_0})$, the present-value equilibrium prices corresponding to different values of λ are all distinct. Since p_{s_0} is sticky, it also holds that price ratios $(p^0_{s_1}/p^0_{s_0})_{s_1 \in S_1}$ are distinct for distinct values of λ. There is a one-dimensional set of equilibria exhibiting real indeterminacy as an increase or a decrease of λ leads on average to overall increases or decreases in present-value prices for future commodities, thereby affecting the budget set.

The general message, however, is that in the presence of price stickiness there are S_T degrees of real indeterminacy, so all indeterminacy is real. To show such a result, we make the assumption that at every date-event there is at least one commodity with a sticky price. The next result demonstrates that sticky price equilibria corresponding to different choices for α have present-value prices which are not proportional to each other.

Theorem 8.1 *For all $s_t \in S$, assume $s_t \in \cup_{\ell \in L} N^{\mathrm{s}}_{\ell}$, and for all $s_T \in S_T$, assume $s_T \in \cup_{\ell \in L} N^{\mathrm{a}}_{\ell}$. Let L^* be a set with one flexible price commodity for each terminal date-event. Let $(p, q, \underline{z}, \bar{z}, v, x, a, b)$ and $(\bar{p}, \bar{q}, \underline{z}', \bar{z}', \bar{v}, \bar{x}, \bar{a}, \bar{b})$ be sticky price equilibria such that $p_{L^*} = \alpha$ and $\bar{p}_{L^*} = \bar{\alpha}$. If $\alpha \neq \bar{\alpha}$, then there is no $\lambda \geq 0$ such that $p^0 = \lambda \bar{p}^0$.*

Proof Assume $p^0 = \lambda \bar{p}^0$ for some $\lambda \geq 0$. In the sequel, we make repeatedly use of the facts shown in the proof of Theorem 7.1 that according to (11.9) $p, \bar{p} \gg 0$ and according to (11.11) $q, \bar{q} \gg 0$.

Let $\ell \in L$ be such that $s_0 \in N^{\mathrm{s}}_\ell$. We have that

$$p^0_{\ell s_0} = p_{\ell s_0} = p_{\ell s_{-1}} = \bar{p}_{\ell s_0} = \bar{p}^0_{\ell s_0}.$$

These inequalities show that $\lambda = 1$, so $p^0 = \bar{p}^0$, and in particular $p_{s_0} = \bar{p}_{s_0}$.

Next, we prove by induction on t that for all $s_t \in S_t$, $p_{s_t} = \bar{p}_{s_t}$ and $q^0_{s_t} = \bar{q}^0_{s_t}$.

Consider some $s_1 \in S_1$ and let $\ell \in L$ be such that $s_1 \in N^{\mathrm{s}}_\ell$. We have that

$$\begin{aligned} p^0_{\ell s_1} &= q^0_{s_1} p_{\ell s_1} = q^0_{s_1} p_{\ell s_0}, \\ \bar{p}^0_{\ell s_1} &= \bar{q}^0_{s_1} \bar{p}_{\ell s_1} = \bar{q}^0_{s_1} \bar{p}_{\ell s_0} = \bar{q}^0_{s_1} p_{\ell s_0}. \end{aligned}$$

Since $p^0_{\ell s_1} = \bar{p}^0_{\ell s_1}$, the above inequalities yield $q^0_{s_1} = \bar{q}^0_{s_1}$. We also have the equalities

$$\begin{aligned} p^0_{s_1} &= q^0_{s_1} p_{s_1}, \\ \bar{p}^0_{s_1} &= \bar{q}^0_{s_1} \bar{p}_{s_1} = q^0_{s_1} \bar{p}_{s_1}. \end{aligned}$$

Since $p^0_{s_1} = \bar{p}^0_{s_1}$, we find that $p_{s_1} = \bar{p}_{s_1}$.

Assume, for some $t \in \{1, \ldots, T-1\}$, we have shown that $p_{s_t} = \bar{p}_{s_t}$ and $q_{s_t} = \bar{q}_{s_t}$ for all $s_t \in S_t$. We complete the proof by showing that $p_{s_{t+1}} = \bar{p}_{s_{t+1}}$ and $q^0_{s_{t+1}} = \bar{q}^0_{s_{t+1}}$ for all $s_{t+1} \in S_{t+1}$.

Consider some $s_{t+1} \in S_{t+1}$ and let $\ell \in L$ be such that $s_{t+1} \in N^{\mathrm{s}}_\ell$. We have that

$$\begin{aligned} p^0_{\ell s_{t+1}} &= q^0_{s_{t+1}} p_{\ell s_{t+1}} = q^0_{s_{t+1}} p_{\ell s_t(s_{t+1})} = q_{s_{t+1}} q^0_{s_t(s_{t+1})} p_{\ell s_t(s_{t+1})}, \\ \bar{p}^0_{\ell s_{t+1}} &= \bar{q}^0_{s_{t+1}} \bar{p}_{\ell s_{t+1}} = \bar{q}^0_{s_{t+1}} \bar{p}_{\ell s_t(s_{t+1})} = \bar{q}_{s_{t+1}} \bar{q}^0_{s_t(s_{t+1})} \bar{p}_{\ell s_t(s_{t+1})} = \bar{q}_{s_{t+1}} q^0_{s_t(s_{t+1})} p_{\ell s_t(s_{t+1})}. \end{aligned}$$

Since $p^0_{\ell s_{t+1}} = \bar{p}^0_{\ell s_{t+1}}$, the above inequalities yield $q_{s_{t+1}} = \bar{q}_{s_{t+1}}$, and since $q^0_{s_t(s_{t+1})} = \bar{q}^0_{s_t(s_{t+1})}$, we find that $q^0_{s_{t+1}} = \bar{q}^0_{s_{t+1}}$. We also have the equalities

$$\begin{aligned} p^0_{s_{t+1}} &= q^0_{s_{t+1}} p_{s_{t+1}}, \\ \bar{p}^0_{s_{t+1}} &= \bar{q}^0_{s_{t+1}} \bar{p}_{s_{t+1}} = q^0_{s_{t+1}} \bar{p}_{s_{t+1}}. \end{aligned}$$

Since $p^0_{s_{t+1}} = \bar{p}^0_{s_{t+1}}$, we find that $p_{s_{t+1}} = \bar{p}_{s_{t+1}}$, which completes the induction step.

It follows that $\alpha = p_{L^*} = \bar{p}_{L^*} = \bar{\alpha}$, which completes the proof. □

9 Extensions

In a framework like ours, with multiple commodities at each date-event, there is no unique way to define price levels or inflation. We have taken the easiest definition for the price level, which defines the price level at a date-event to be equal to the price of an arbitrarily chosen commodity with a flexible price. A modest generalization would be to go from price levels to “activity” levels, which would naturally be parametrized by the variable ρ. In this case, the arbitrarily chosen commodity might be one with a sticky price, in which case the activity level would correspond to the amount of supply rationing or demand rationing. This seemingly modest extension of Theorem 7.1 and Corollary 7.2 does not hold.

Consider for the sake of simplicity an economy with two periods, zero nominal interest rates, a single date-event in period 1, and one commodity per date-event. Assume that the price in period 0 is flexible and the price in period 1 is sticky. Stickiness of the price in period 1 coupled with a zero nominal interest rate implies that the present-value price of the future commodity is equal to the price of the current commodity. To show the existence of a sticky price equilibrium with $\rho_{s_1} = 1/2$, we have to show existence of a sticky price equilibrium with no rationing in the future and, since there is a flexible price in period 0, no rationing in the present. We therefore have to show the existence of an equilibrium in the Arrow–Debreu model where the prices of the two arbitrarily chosen commodities are equal to each other. Generically, such an equilibrium does not exist, which proves that Theorem 7.1 and Corollary 7.2 cannot be extended in this way. Still, there is an $|S_1|$-dimensional set of equilibria in this example, with a fixed amount of either supply or demand rationing in the market of the commodity at date-event s_1, parametrized by the price of the commodity at date-event s_0.

Section 4 presents a concrete example, illustrating the point of the previous paragraph. In that example, given the interest rate policy, the amount of rationing is uniquely determined and all indeterminacy of equilibrium is nominal.

Rather than choosing a commodity with a flexible price for each terminal date-event, we might choose commodities with a flexible price at intermediate date-events. Such an extension can be proven by the same approach as in the proof of Theorem 7.1, with the obvious modifications.

Rather than choosing a single commodity at terminal date-events, one may define a price level at a terminal date-event, for instance by taking some weighted sum of the prices at that terminal date-event. It follows immediately from Corollary 7.2 that there are sticky price equilibria with arbitrarily high price levels at each terminal date-event.

The interest rate policy by the bank can depend on any exogenous shock as it is an arbitrary function of date-events in S. Since the date-events in S need not be restricted to payoff relevant shocks, the interest rate policy could even depend on sunspots or be random. Our current approach, however, does not allow the interest rate to depend on past endogenous variables. Many papers in the macroeconomic literature have stressed the importance of such interest rate rules, see Woodford (2003) for a detailed discussion of this literature. Since interest rate rules are in general not compatible with the assumption that interest rates are restricted to some compact set, such an extension poses challenging equilibrium existence issues as it is not straightforward how the endogenous variables of the economy should be compactified.

Although the equilibrium existence problem is challenging with interest rate rules, there is no reason to expect that the $|S_T|$-dimensional multiplicity of equilibrium will be lost as a result. The reason is that the imposition of $|S|$ interest rate rules will lead to $|S|$ no-arbitrage conditions, exactly the same as the number of conditions following from our approach with exogenous interest rate policies by the bank. In the macroeconomic literature, this is confirmed by the results in Benhabib et al. (2001), who show that indeterminacy even results under active interest rate feedback rules like for instance Taylor rules.

10 Conclusion

We have presented a general equilibrium model that has the same level of generality as the Arrow–Debreu model and that incorporates the main desiderata of the macroeconomic literature known as the new neoclassical synthesis. Agents form rational expectations on prices of commodities and assets, interest rates, supply constraints, and demand constraints, in a stochastically developing multi-period economy. Commodity prices are allowed to be sticky, implying that monetary policy has non-trivial real consequences. At date-events where a commodity price is flexible, it does not adjust mechanically to some measure of disequilibrium, but is set at a market clearing level corresponding to the forces of supply and demand. Since price stickiness involves nominal prices, the model contains a general formulation of the monetary transaction technology.

The main result of the paper is that rational expectations are compatible with an $|S_T|$-dimensional set of equilibria and that in the presence of price stickiness this indeterminacy of equilibrium is real rather than nominal. This poses serious challenges to the issue of how households coordinate their expectations on one particular equilibrium and if they succeed in coordinating their expectations, on which equilibrium that will be. Under strong stationary assumptions and with flexible prices, it might seem natural that they coordinate on an equilibrium where future inflation is deterministic. Without stationary assumptions, and in the presence of price stickiness, such equilibria do generally not exist, and the issue of equilibrium selection becomes even more prominent.

11 Appendix: Proof of Theorem 7.1

How does one prove the existence of a sticky price equilibrium as defined in Definition 2.2? The first problem to be taken care of is a continuity problem. The budget correspondence γ^h may fail to be continuous at $(p, q, r, \underline{z}, \bar{z}, v^h)$. To facilitate the study of continuity, we introduce budget correspondences $\hat{\gamma}^h$ and $\tilde{\gamma}^h$ next.

It is convenient to introduce the set $\bar{Q}$ of prices of Arrow securities that are arbitrage-free,

$$\bar{Q} = \left\{ (q, r) \in \mathbb{R}_+^{S^+} \times \mathbb{R}_+^S \mid \forall s_t \in S, \sum_{s_{t+1} \in s_t^+} q_{s_{t+1}} = \frac{1}{1 + r_{s_t}} \right\}.$$

We define the correspondence $\hat{\gamma}^h : \mathbb{R}_+^{LS} \times \bar{Q} \times -\mathbb{R}_+^{LS} \times \mathbb{R}_+^{LS} \times \mathbb{R}_+^S \to X^h \times \mathbb{R}^{S^+} \times \mathbb{R}^S$ by

$$
\begin{aligned}
\hat{\gamma}^h(p, q, r, \underline{z}, \bar{z}, v^h) = \{(x^h, a^h, b^h) \in X^h \times \mathbb{R}^{S^+} \times \mathbb{R}^S \mid \\
& b^h \in \beta^h(p, x^h), \\
& \underline{z} \leq x^h - e^h \leq \bar{z}, \\
\tilde{p}x^h + r\tilde{b}^h \leq \tilde{p}e^h + \tilde{w}^h, \\
a^h_{s_{T+1}} = p_{s^-_{T+1}}(x^h_{s^-_{T+1}} - e^h_{s^-_{T+1}}) + r_{s^-_{T+1}} b^h_{s^-_{T+1}} - v^h_{s^-_{T+1}}, \quad s_{T+1} \in S_{T+1}, \\
a^h_{s_t} = p_{s^-_t}\left(x^h_{s^-_t} - e^h_{s^-_t}\right) + r_{s^-_t} b^h_{s^-_t} - v^h_{s^-_t} + \textstyle\sum_{s_{t+1} \in s^+_t} q_{s_{t+1}} a^h_{s_{t+1}}, \quad s_t \in S \backslash \{s_0\}\},
\end{aligned}
$$

and the correspondence $\tilde{\gamma}^h : \mathbb{R}^{LS}_+ \times \mathbb{R}^S_+ \times -\mathbb{R}^{LS}_+ \times \mathbb{R}^{LS}_+ \times \mathbb{R}_+ \to X^h \times \mathbb{R}^S$ by

$$
\begin{aligned}
\tilde{\gamma}^h(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h) = \big\{(x^h, \tilde{b}^h) \in X^h \times \mathbb{R}^S \mid \tilde{b}^h \in \beta^h(\tilde{p}, x^h), \\
\underline{z} \leq x^h - e^h \leq \bar{z}, \\
\tilde{p}x^h + r\tilde{b}^h \leq \tilde{p}e^h + \tilde{w}^h\big\}.
\end{aligned}
$$

The two differences between $\gamma^h(p, q, r, \underline{z}, \bar{z}, v^h)$ and $\hat{\gamma}^h(p, q, r, \underline{z}, \bar{z}, v^h)$ are the inequality rather than the equality in the budget constraint, and the use of real numbers rather than extended real numbers for rationing schemes. The existence proofs are such that the use of extended real numbers for rationing schemes can be avoided. The inequality in the budget constraint is introduced to ensure that $\hat{\gamma}^h$ is non-empty valued. Indeed, by A4, it holds that $0 \in \beta^h(p, e^h)$. Let a^h solve the recursive system of equations (3.3) for $b^h = 0$. Then we have $(e^h, a^h, 0) \in \hat{\gamma}^h(p, q, r, \underline{z}, \bar{z}, v^h)$. The correspondence γ^h on the other hand can be empty valued. Empty values for $\gamma^h(p, q, r, \underline{z}, \bar{z}, v^h)$ could for instance occur when $\bar{z} = 0$ and v^h is strictly positive.

The correspondence $\tilde{\gamma}^h$ is a reformulation of the correspondence $\hat{\gamma}^h$ in present-value terms and omits the determination of the a^h variables. The proofs of equilibrium existence require continuity properties of both the correspondences $\hat{\gamma}^h$ and $\tilde{\gamma}^h$, which extend similar continuity properties for non-monetary economies provided in Drèze (1975) and Herings (1996a).

Lemma 11.1 *The correspondence $\tilde{\gamma}^h$ is lower hemi-continuous and closed at any $(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h) \in \mathbb{R}^{LS}_+ \times \mathbb{R}^S_+ \times -\mathbb{R}^{LS}_+ \times \mathbb{R}^{LS}_+ \times \mathbb{R}_+$ satisfying $\tilde{p}\underline{z} < 0$ or $\tilde{w}^h > 0$.*

Proof Let $(\tilde{p}_n, r_n, \underline{z}_n, \bar{z}_n, \tilde{w}^h_n)_{n \in \mathbb{N}}$ be a sequence of points in $\mathbb{R}^{LS}_+ \times \mathbb{R}^S_+ \times -\mathbb{R}^{LS}_+ \times \mathbb{R}^{LS}_+ \times \mathbb{R}_+$ converging to $(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h)$. Let $(x^h, \tilde{b}^h)$ be an element of $\tilde{\gamma}^h(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h)$. The correspondence $\tilde{\gamma}^h$ is lower hemi-continuous at $(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h)$ if there is a sequence $(x^h_n, \tilde{b}^h_n)_{n \in \mathbb{N}}$ such that $(x^h_n, \tilde{b}^h_n) \in \tilde{\gamma}^h(\tilde{p}_n, r_n, \underline{z}_n, \bar{z}_n, \tilde{w}^h_n)$ and $(x^h_n, \tilde{b}^h_n) \to (x^h, \tilde{b}^h)$.

We consider two cases, 1. $\tilde{p}x^h + r\tilde{b}^h < \tilde{p}e^h + \tilde{w}^h$ and 2. $\tilde{p}x^h + r\tilde{b}^h = \tilde{p}e^h + \tilde{w}^h$ and $[\tilde{p}\underline{z} < 0$ or $\tilde{w}^h > 0]$.

Case 1. $\tilde{p}x^h + r\tilde{b}^h < \tilde{p}e^h + \tilde{w}^h$.
We define the sets $L^+ = \{\ell s_t \in L \times S \mid x^h_{\ell s_t} > e^h_{\ell s_t}\}$, $L^0 = \{\ell s_t \in L \times S \mid x^h_{\ell s_t} = e^h_{\ell s_t}\}$, and $L^- = \{\ell s_t \in L \times S \mid x^h_{\ell s_t} < e^h_{\ell s_t}\}$. For $n \in \mathbb{N}$, for $\ell s_t \in L^-$, let

$\lambda^h_{\ell s_t,n} = \underline{z}_{\ell s_t,n}/(x^h_{\ell s_t} - e^h_{\ell s_t})$, then $\lambda^h_{\ell s_t,n} \geq 0$ since $x^h_{\ell s_t} - e^h_{\ell s_t} < 0$ and $\underline{z}_{\ell s_t,n} \leq 0$. For $n \in \mathbb{N}$, for $\ell s_t \in L^+$, let $\lambda^h_{\ell s_t,n} = \bar{z}_{\ell s_t,n}/(x^h_{\ell s_t} - e^h_{\ell s_t})$, then $\lambda^h_{\ell s_t,n} \geq 0$ since $x^h_{\ell s_t} - e^h_{\ell s_t} > 0$ and $\bar{z}_{\ell s_t,n} \geq 0$. Finally, let $\lambda^h_n = \min(\{\lambda^h_{\ell s_t,n} \mid \ell s_t \in L^- \cup L^+\} \cup \{1\})$. Clearly, $0 \leq \lambda^h_n \leq 1$. We define $x^h_n = e^h + \lambda^h_n(x^h - e^h)$. Since $x^h, e^h \in X^h$ and by the convexity of X^h it holds that $x^h_n \in X^h$. Moreover,

$$
\begin{aligned}
x^h_{\ell s_t,n} - e^h_{\ell s_t} &= \lambda^h_n\Big(x^h_{\ell s_t} - e^h_{\ell s_t}\Big) \geq \lambda^h_{\ell s_t,n}\Big(x^h_{\ell s_t} - e^h_{\ell s_t}\Big) = \underline{z}_{\ell s_t,n}, && \ell s_t \in L^-,\\
x^h_{\ell s_t,n} - e^h_{\ell s_t} &= \lambda^h_n\Big(x^h_{\ell s_t} - e^h_{\ell s_t}\Big) \leq 0 \leq \bar{z}_{\ell s_t,n}, && \ell s_t \in L^-,\\
x^h_{\ell s_t,n} - e^h_{\ell s_t} &= \lambda^h_n\Big(x^h_{\ell s_t} - e^h_{\ell s_t}\Big) = 0 \text{ and so } \underline{z}_{\ell s_t,n} \leq x^h_{\ell s_t,n} - e^h_{\ell s_t} \leq \bar{z}_{\ell s_t,n}, && \ell s_t \in L^0,\\
x^h_{\ell s_t,n} - e^h_{\ell s_t} &= \lambda^h_n\Big(x^h_{\ell s_t} - e^h_{\ell s_t}\Big) \leq \lambda^h_{\ell s_t,n}\Big(x^h_{\ell s_t} - e^h_{\ell s_t}\Big) = \bar{z}_{\ell s_t,n}, && \ell s_t \in L^+,\\
x^h_{\ell s_t,n} - e^h_{\ell s_t} &= \lambda^h_n\Big(x^h_{\ell s_t} - e^h_{\ell s_t}\Big) \geq 0 \geq \underline{z}_{\ell s_t,n}, && \ell s_t \in L^+.
\end{aligned}
$$

Further,

$$
\begin{aligned}
\lambda^h_{\ell s_t,n} &= \tfrac{\underline{z}_{\ell s_t,n}}{x^h_{\ell s_t} - e^h_{\ell s_t}} \to \tfrac{\underline{z}_{\ell s_t}}{x^h_{\ell s_t} - e^h_{\ell s_t}} \geq \tfrac{x^h_{\ell s_t} - e^h_{\ell s_t}}{x^h_{\ell s_t} - e^h_{\ell s_t}} = 1, \quad \ell s_t \in L^-,\\
\lambda^h_{\ell s_t,n} &= \tfrac{\bar{z}_{\ell s_t,n}}{x^h_{\ell s_t} - e^h_{\ell s_t}} \to \tfrac{\bar{z}_{\ell s_t}}{x^h_{\ell s_t} - e^h_{\ell s_t}} \geq \tfrac{x^h_{\ell s_t} - e^h_{\ell s_t}}{x^h_{\ell s_t} - e^h_{\ell s_t}} = 1, \quad \ell s_t \in L^+.
\end{aligned}
$$

So $\lambda^h_n \to 1$ and therefore $x^h_n = e^h + \lambda^h_n(x^h - e^h) \to e^h + x^h - e^h = x^h$.

Since β^h is lower hemi-continuous, there exists a sequence $(\tilde{b}^h_n)_{n\in\mathbb{N}}$ such that $\tilde{b}^h_n \in \beta^h(\tilde{p}_n, x^h_n)$ and $\tilde{b}^h_n \to \tilde{b}^h$. Moreover, $\tilde{p}_n x^h_n + r_n\tilde{b}^h_n - \tilde{p}_n e^h - \tilde{w}^h_n \to \tilde{p}x^h + r\tilde{b}^h - \tilde{p}e^h - \tilde{w}^h < 0$. Therefore, for n sufficiently large, $\tilde{p}_n x^h_n + r_n\tilde{b}^h_n - \tilde{p}_n e^h - \tilde{w}^h_n < 0$, so $(x^h_n, \tilde{b}^h_n) \in \tilde{\gamma}^h(\tilde{p}_n, r_n, \underline{z}_n, \bar{z}_n, \tilde{w}^h_n)$ and lower hemi-continuity of $\tilde{\gamma}^h$ follows.

Case 2. $\tilde{p}x^h + r\tilde{b}^h = \tilde{p}e^h + \tilde{w}^h$ and $[\tilde{p}\underline{z} < 0$ or $\tilde{w}^h > 0]$.
Let $\alpha \in (0, 1]$ be such that $e^h + \alpha\underline{z}_n \in X^h$ for all n sufficiently large. Such an α exists since $e^h \in \text{int}(X^h)$, so α can be chosen such that $e^h + \alpha\underline{z} \in \text{int}(X^h)$, and $\underline{z}_n \to \underline{z}$. We define $\hat{e}^h_n = e^h + \alpha\underline{z}_n$. For n sufficiently large, $\hat{e}^h_n$ has the following properties,

$$
\hat{e}^h_n \in X^h,\ \underline{z}_n \leq \alpha\underline{z}_n = \hat{e}^h_n - e^h \leq 0 \leq \bar{z}_n. \tag{11.1}
$$

Since $\hat{e}^h_n \leq e^h$ it holds by A4.5 that $0 \in \beta^h(\tilde{p}_n, \hat{e}^h_n)$. For n sufficiently large we have

$$
\tilde{p}_n\hat{e}^h_n + r_n 0 < \tilde{p}_n e^h + \tilde{w}^h_n.
$$

The strict inequality follows since, for n sufficiently large, $\tilde{p}\underline{z} < 0$ implies $\tilde{p}_n\hat{e}^h_n < \tilde{p}_n e^h$ and $r_n 0 \leq \tilde{w}^h_n$, whereas $\tilde{w}^h > 0$ implies $\tilde{p}_n\hat{e}^h_n \leq \tilde{p}_n e^h$ and $r_n 0 < \tilde{w}^h_n$. Moreover, when we define $\hat{e}^h = e^h + \alpha\underline{z}$, then we have that $\hat{e}^h_n \to \hat{e}^h$,

$$
\hat{e}^h \in X^h,\ \underline{z} \leq \alpha\underline{z} = \hat{e}^h - e^h \leq 0 \leq \bar{z},\ \text{and } \tilde{p}\hat{e}^h + r0 < \tilde{p}e^h + \tilde{w}^h.
$$

Consider the sequence $(x_n^h, \tilde{b}_n^h)_{n\in\mathbb{N}}$ as defined in Case 1. It may be assumed that the elements of this sequence satisfy

$$x_n^h \in X^h,\ \underline{z}_n \leq x_n^h - e^h \leq \bar{z}_n,\ x_n^h \to x^h, \quad \text{and } \tilde{b}_n^h \in \beta^h(\tilde{p}_n, x_n^h). \tag{11.2}$$

If $\tilde{p}_n x_n^h + r_n \tilde{b}_n^h > \tilde{p}_n e^h + \tilde{w}_n^h$, then define μ_n^h by

$$\mu_n^h = \frac{\tilde{p}_n e^h + \tilde{w}_n^h - \tilde{p}_n \hat{e}_n^h}{\tilde{p}_n x_n^h + r_n \tilde{b}_n^h - \tilde{p}_n \hat{e}_n^h} \tag{11.3}$$

and if $\tilde{p}_n x_n^h + r_n \tilde{b}_n^h \leq \tilde{p}_n e^h + \tilde{w}_n^h$, then define $\mu_n^h = 1$. It holds that $0 \leq \mu_n^h \leq 1$. We define $\hat{x}_n^h = \hat{e}_n^h + \mu_n^h(x_n^h - \hat{e}_n^h)$ and $\hat{b}_n^h = \mu_n^h \tilde{b}_n^h$. Using the convexity of X^h, it holds that $\hat{x}^h \in X^h$.

By (11.1) and (11.2),

$$\begin{aligned} \hat{x}_n^h - e^h &= \mu_n^h(x_n^h - e^h) + (1 - \mu_n^h)(\hat{e}_n^h - e^h) \geq \underline{z}_n, \\ \hat{x}_n^h - e^h &= \mu_n^h(x_n^h - e^h) + (1 - \mu_n^h)(\hat{e}_n^h - e^h) \leq \bar{z}_n. \end{aligned}$$

If $\tilde{p}_n x_n^h + r_n \tilde{b}_n^h > \tilde{p}_n e^h + \tilde{w}_n^h$, then by (11.3)

$$\begin{aligned} \tilde{p}_n \hat{x}_n^h + r_n \hat{b}_n^h &= \tfrac{\tilde{p}_n e^h + \tilde{w}_n^h - \tilde{p}_n \hat{e}_n^h}{\tilde{p}_n x_n^h + r_n \tilde{b}_n^h - \tilde{p}_n \hat{e}_n^h}(\tilde{p}_n x_n^h + r_n \tilde{b}_n^h) + \tfrac{\tilde{p}_n x_n^h + r_n \tilde{b}_n^h - \tilde{p}_n e^h - \tilde{w}_n^h}{\tilde{p}_n x_n^h + r_n \tilde{b}_n^h - \tilde{p}_n \hat{e}_n^h} \tilde{p}_n \hat{e}_n^h \\ &= \tilde{p}_n e^h + \tilde{w}_n^h, \end{aligned}$$

and if $\tilde{p}_n x_n^h + r_n \hat{b}_n^h \leq \tilde{p}_n e^h + \tilde{w}_n^h$ then because $\mu_n^h = 1$

$$\tilde{p}_n \hat{x}_n^h + r_n \hat{b}_n^h = \tilde{p}_n x_n^h + r_n \tilde{b}_n^h \leq \tilde{p}_n e^h + \tilde{w}_n^h.$$

Since $\hat{e}_n^h \leq e^h$, it holds that $0 \in \beta^h(\tilde{p}_n, x_n^h)$ by A4.5. By A4.4 it holds that

$$\hat{b}_n^h = \mu_n^h \tilde{b}_n^h + (1 - \mu_n^h)0 \in \beta^h(\tilde{p}_n, \mu_n^h x_n^h + (1 - \mu_n^h)\hat{e}_n^h) = \beta^h(\tilde{p}_n, \hat{x}_n^h).$$

It follows that $(\hat{x}_n^h, \hat{b}_n^h) \in \tilde{\gamma}^h(\tilde{p}_n, r_n, \underline{z}_n, \bar{z}_n, \tilde{w}_n^h)$. Using $(x_n^h, \tilde{b}_n^h) \to (x^h, \tilde{b}^h)$ and $\tilde{p}x^h + r\tilde{b}^h = \tilde{p}e^h + \tilde{w}^h$,

$$\frac{\tilde{p}_n e^h + \tilde{w}_n^h - \tilde{p}_n \hat{e}_n^h}{\tilde{p}_n x_n^h + r_n \tilde{b}_n^h - \tilde{p}_n \hat{e}_n^h} \to \frac{\tilde{p}e^h + \tilde{w}^h - \tilde{p}\hat{e}^h}{\tilde{p}x^h + r\tilde{b}^h - \tilde{p}\hat{e}^h} = 1$$

and so $\mu_n^h \to 1$. Consequently, $\hat{x}_n^h \to \hat{e}^h + (x^h - \hat{e}^h) = x^h$ and $\hat{b}_n^h \to \tilde{b}^h$. Lower hemi-continuity of $\tilde{\gamma}^h$ follows.

Let $(\tilde{p}_n, r_n, \underline{z}_n, \bar{z}_n, \tilde{w}_n^h)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}_+^{LS} \times \mathbb{R}_+^S \times -\mathbb{R}_+^{LS} \times \mathbb{R}_+^{LS} \times \mathbb{R}_+$ converging to $(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h)$. For $n \in \mathbb{N}$, let $(x_n^h, \tilde{b}_n^h)$ be an element of $\tilde{\gamma}^h(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h)$ converging to $(x^h, \tilde{b}^h)$. The correspondence $\tilde{\gamma}^h$ is closed at $(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h)$ if $(x^h, \tilde{b}^h) \in \tilde{\gamma}^h(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h)$. Since X^h is closed by A1, it holds that $(x^h, \tilde{b}^h) \in X^h \times \mathbb{R}^S$. Since β^h is closed by A4.1, we have $\tilde{b}^h \in \beta^h(\tilde{p}, x^h)$. The usual continuity arguments imply that $\underline{z} \leq x^h - e^h \leq \bar{z}$, and $\tilde{p}x^h + r\tilde{b}^h \leq \tilde{p}e^h + \tilde{w}^h$. It follows that $(x^h, \tilde{b}^h) \in \tilde{\gamma}^h(\tilde{p}, r, \underline{z}, \bar{z}, \tilde{w}^h)$. □

The assumptions in Lemma 11.1 are such that for every household there is a consumption bundle in the budget set which is strictly less expensive than the household's total income. This cheaper-point assumption is well known in general equilibrium theory, see Debreu (1959), and crucial to show lower hemi-continuity of the budget correspondence. A similar assumption is made in Lemma 11.2.

Lemma 11.2 *The correspondence $\hat{\gamma}^h$ is lower hemi-continuous and closed at any $(p, q, r, \underline{z}, \bar{z}, v^h) \in \mathbb{R}^{LS}_+ \times \bar{Q} \times -\mathbb{R}^{LS}_+ \times \mathbb{R}^{LS}_+ \times \mathbb{R}^S_+$ satisfying $\tilde{p}\underline{z} < 0$ or $\tilde{q}v^h > 0$.*

Proof Let $(p_n, q_n, r_n, \underline{z}_n, \bar{z}_n, v^h_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^{LS}_+ \times \bar{Q} \times -\mathbb{R}^{LS}_+ \times \mathbb{R}^{LS}_+ \times \mathbb{R}^S_+$ converging to $(p, q, r, \underline{z}, \bar{z}, v^h)$. Let (x^h, a^h, b^h) be an element of $\hat{\gamma}^h(p, q, r, \underline{z}, \bar{z}, v^h)$. The correspondence $\hat{\gamma}^h$ is lower hemi-continuous at $(p, q, r, \underline{z}, \bar{z}, v^h)$ if there is a sequence $(x^h_n, a^h_n, b^h_n)_{n\in\mathbb{N}}$ such that $(x^h_n, a^h_n, b^h_n) \in \hat{\gamma}^h(p_n, q_n, r_n, \underline{z}_n, \bar{z}_n, v^h_n)$ and $(x^h_n, a^h_n, b^h_n) \to (x^h, a^h, b^h)$.

For $s_t \in S$, we define $\tilde{q}_{s_t} = q^0_{s_t}/(1 + r_{s_t})$, $\tilde{p}_{s_t} = \tilde{q}_{s_t} p_{s_t}$, $\tilde{b}^h_{s_t} = \tilde{q}_{s_t} b^h_{s_t}$, $\tilde{v}^h_{s_t} = \tilde{q}_{s_t} v^h_{s_t}$, and $\tilde{w}^h_{s_t} = \sum_{s_t\in S} \tilde{v}^h_{s_t}$.

We consider two cases, 1. $\tilde{p}x^h + r\tilde{b}^h < \tilde{p}e^h + \tilde{w}^h$ and 2. $\tilde{p}x^h + r\tilde{b}^h = \tilde{p}e^h + \tilde{w}^h$, $[\tilde{p}\underline{z} < 0$ or $\tilde{w}^h > 0]$.

Case 1. $\tilde{p}x^h + r\tilde{b}^h < \tilde{p}e^h + \tilde{w}^h$.
We define the sequence $(x^h_n)_{n\in\mathbb{N}}$ as in Case 1 of Lemma 11.1. It holds that $x^h_n \in X^h$, $\underline{z}_n \leq x^h_n - e^h \leq \bar{z}_n$, and $x^h_n \to x^h$. Since β^h is lower hemi-continuous, there exists a sequence $(b^h_n)_{n\in\mathbb{N}}$ such that $b^h_n \in \beta^h(p_n, x^h_n)$ and $b^h_n \to b^h$. Moreover, $\tilde{p}_n x^h_n + r_n\tilde{b}^h_n - \tilde{p}_n e^h - \tilde{w}^h_n \to \tilde{p}x^h + r\tilde{b}^h - \tilde{p}e^h - \tilde{w}^h < 0$. Therefore, for n sufficiently large, $\tilde{p}_n x^h_n + r_n\tilde{b}^h_n - \tilde{p}_n e^h - \tilde{w}^h_n < 0$. We define a^h_n by means of the recursive system of equations (3.3) as determined by (p_n, q_n, r_n, v^h_n) and (x^h_n, b^h_n). It is straightforward to demonstrate that $a^h_n \to a^h$. Clearly, for n sufficiently large, $(x^h_n, a^h_n, b^h_n) \in \hat{\gamma}^h(p_n, q_n, r_n, \underline{z}_n, \bar{z}_n, v^h_n)$ and lower hemi-continuity of $\hat{\gamma}^h$ follows.

Case 2. $\tilde{p}x^h + \tilde{r}b^h = \tilde{p}e^h + \tilde{w}^h$, $[\tilde{p}\underline{z} < 0$ or $\tilde{w}^h > 0]$.
Consider the sequence $(x^h_n, a^h_n, b^h_n)_{n\in\mathbb{N}}$ as defined in Case 1 and the sequence $(\hat{e}^h_n)_{n\in\mathbb{N}}$ as defined in Case 2 of Lemma 11.1. For $n \in \mathbb{N}$, for $s_t \in S$, we define $\tilde{b}^h_{s_t,n} = \tilde{q}_{s_t,n} b^h_{s_t,n}$. If $\tilde{p}_n x^h_n + r_n\tilde{b}^h_n > \tilde{p}_n e^h + \tilde{w}^h_n$, then define μ^h_n by

$$\mu^h_n = \frac{\tilde{p}_n e^h + \tilde{w}^h_n - \tilde{p}_n\hat{e}^h_n}{\tilde{p}_n x^h_n + r_n\tilde{b}^h_n - \tilde{p}_n\hat{e}^h_n} \tag{11.4}$$

and if $\tilde{p}_n x^h_n + r_n\tilde{b}^h_n \leq \tilde{p}_n e^h + \tilde{w}^h_n$, then define $\mu^h_n = 1$. Next, we define $\hat{x}^h_n = \hat{e}^h_n + \mu^h_n(x^h_n - \hat{e}^h_n)$ and $\hat{b}^h_n = \mu^h_n b^h_n$. We define $\hat{a}^h_n$ by means of the recursive system of equations (3.3) as determined by (p_n, q_n, r_n, v^h_n) and $(\hat{x}^h_n, \hat{b}^h_n)$.

By exactly the same arguments as in the proof of Case 1 of Lemma 11.1 it follows that $(\hat{x}^h_n, \hat{a}^h_n, \hat{b}^h_n) \in \hat{\gamma}^h(p_n, q_n, r_n, \underline{z}_n, \bar{z}_n, v^h_n)$ and $(\hat{x}^h_n, \hat{a}^h_n, \hat{b}^h_n) \to (x^h, a^h, b^h)$. Lower hemi-continuity of $\hat{\gamma}^h$ follows.

Let $(p_n, q_n, r_n, \underline{z}_n, \bar{z}_n, v^h_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^{LS}_+ \times \bar{Q} \times -\mathbb{R}^{LS}_+ \times \mathbb{R}^{LS}_+ \times \mathbb{R}^S_+$ converging to $(p, q, r, \underline{z}, \bar{z}, v^h)$. For $n \in \mathbb{N}$, let (x^h_n, a^h_n, b^h_n) be an element of $\hat{\gamma}^h(p, q, r, \underline{z}, \bar{z}, v^h)$ converging to (x^h, a^h, b^h). The correspondence $\hat{\gamma}^h$ is closed at $(p, q, r, \underline{z}, \bar{z}, v^h)$ if $(x^h, a^h, b^h) \in \hat{\gamma}^h(p, q, r, \underline{z}, \bar{z}, v^h)$. Since X^h is closed by A1, it holds that $(x^h, a^h, b^h) \in X^h \times \mathbb{R}^{S^+} \times \mathbb{R}^S$. Since β^h is closed by A4.1, we have $b^h \in \beta^h(p, x)$. The usual continuity arguments imply that

$$\begin{aligned}
&\underline{z} \leq x^h - e^h \leq \bar{z},\\
&\tilde{p}x^h + \tilde{r}b^h \leq \tilde{p}e^h + \tilde{q}v^h,\\
&a^h_{s_{T+1}} = p_{s_{T+1}^-}\left(x^h_{s_{T+1}^-} - e^h_{s_{T+1}^-}\right) + r_{s_{T+1}^-} b^h_{s_{T+1}^-} - v^h_{s_{T+1}^-}, \quad s_{T+1} \in S_{T+1},\\
&a^h_{s_t} = p_{s_t^-}\left(x^h_{s_t^-} - e^h_{s_t^-}\right) + r_{s_t^-} b^h_{s_t^-} - v^h_{s_t^-} + \textstyle\sum_{s_{t+1}\in s_t^+} q_{s_{t+1}} a^h_{s_{t+1}}, \quad s_t \in S\backslash\{s_0\}.
\end{aligned}$$

It follows that $(x^h, a^h, b^h) \in \hat{\gamma}^h(p, q, r, \underline{z}, \bar{z}, v^h)$. □

Proof of Theorem 7.1 Step 1. Compactification.
Fix some $u \geq \max\{1, \max_{s_T\in S_T} \alpha_{s_T}, \max_{\ell\in L} p_{\ell s_{-1}}\}$. For $\ell s_t \in (L \times S)\backslash L^*$, we define $\bar{R}_{\ell s_t} = R_{\ell s_t} \cap [0, u]$ and $\bar{R} = \prod_{\ell s_t\in(L\times S)\backslash L^*} \bar{R}_{\ell s_t} \times \{\alpha\}$. Since $u \geq 1$, the definition of $\bar{R}$ imposes no restrictions on $\rho_{\ell s_t}$ when $s_t \in N^{\mathrm{s}}_\ell$. Since $u \geq \max_{s_T\in S_T} \alpha_{s_T}$ and $u \geq \max_{\ell\in L} p_{\ell s_{-1}}$, we have that $p(\bar{R}) \subset [0, u^S]$, where $u^S \in \mathbb{R}^S$ is the vector with all components equal to u.

For $h \in H$, we define the compact set

$$\bar{X}^h = \{x^h \in X^h \mid x^h - e^h \leq \bar{c}\}.$$

Choose $\underline{f}^h$ and $\bar{f}^h$ as in A4. Since $[0, u^S] \times \bar{X}^h$ is compact and $\underline{f}^h$ and $\bar{f}^h$ are continuous, $\cup_{h\in H}\underline{f}^h([0, u^S] \times \bar{X}^h)$ and $\cup_{h\in H}\bar{f}^h([0, u^S] \times \bar{X}^h)$ are compact. Let $b^- \in \mathbb{R}^S$ be a lower bound for the former and $b^+ \in \mathbb{R}^S$ be an upper bound for the latter set. We define

$$\begin{aligned}
B^h &= \{b^h \in \mathbb{R}^S \mid b^- \leq b^h \leq b^+\}, \quad h \in H,\\
V^{\mathrm{c}} &= \left\{v \in \mathbb{R}^S_+ \mid \forall s_t \in S,\ v_{s_t} \leq r_{s_t} \textstyle\sum_{h\in H} b^+_{s_t}\right\},\\
V^h &= V^{\mathrm{c}}, \quad h \in H.
\end{aligned}$$

For $h \in H$, for $s_t \in S$, we define $A^h_{s_t}$ recursively as a compact, convex set containing all $a^h_{s_t^+} \in \mathbb{R}^{s_t^+}$ such that (3.3) is satisfied for some $p \in [0, u^S]$, $(q, r) \in \bar{Q}$, $v^h \in V^h$, $x^h \in \bar{X}^h$, $a^h_{(s_t^+)^+} \in A^h_{s_t^+}$, and $b^h \in B^h$. Next, we define $A^h = \prod_{s_t\in S} A^h_{s_t}$.
Step 2. Formulation of the fixed point correspondence.
For $\ell s_t \in (L \times S)\backslash L^*$, we define

$$\begin{aligned}
\mu_{\ell s_t}(x) = \Bigg\{&\bar{\rho}_{\ell s_t} \in \bar{R}_{\ell s_t} \mid \bar{\rho}_{\ell s_t} \sum_{h\in H}\left(x^h_{\ell s_t} - e^h_{\ell s_t}\right)\\
&\geq \rho_{\ell s_t} \sum_{h\in H}\left(x^h_{\ell s_t} - e^h_{\ell s_t}\right) \text{ for all } \rho_{\ell s_t} \in \bar{R}_{\ell s_t}\Bigg\}.
\end{aligned}$$

We define the correspondence $\mu : \bar{X} \to \bar{R}$ by setting $\mu(x) = \prod_{\ell s_t \in (L \times S) \setminus L^*} \mu_{\ell s_t}(x) \times \{\alpha\}$.

For $s_t \in S$, we define $Q_{s_t} = \{q_{s_t^+} \in \mathbb{R}_+^{s_t^+} \mid \sum_{s_{t+1} \in s_t^+} q_{s_{t+1}} = 1/(1 + r_{s_t})\}$ and the correspondence $\nu_{s_t} : A_{s_t} \to Q_{s_t}$ by

$$\nu_{s_t}(a_{s_t^+}) = \left\{ \bar{q}_{s_t^+} \in Q_{s_t} \mid \bar{q}_{s_t^+} \sum_{h \in H} a^h_{s_t^+} \geq q_{s_t^+} \sum_{h \in H} a^h_{s_t^+} \right\}, \quad a_{s_t^+} \in A_{s_t}.$$

For $h \in H$, we define the (single-valued) correspondence $o^h : B^h \to V^h$ by

$$o^h_{s_t}(b^h) = \left\{ \theta^h r_{s_t} \sum_{h' \in H} b^{h'}_{s_t} \right\}, \quad s_t \in S.$$

We define the correspondence $\bar{\gamma}^h : \bar{R} \times Q \times V^h \to \bar{X}^h \times A^h \times B^h$ by

$$\bar{\gamma}^h(\rho, q, v^h) = \hat{\gamma}^h(p(\rho), q, r, \underline{z}(\rho), \bar{z}(\rho), v^h) \cap (X^h \times A^h \times B^h),$$
$$(\rho, q, v^h) \in \bar{R} \times Q \times V^h.$$

The construction of the functions $\underline{z}$ and $\bar{z}$ guarantees that any consumption bundle in $\hat{\gamma}^h(p(\rho), q, r, \underline{z}(\rho), \bar{z}(\rho), v^h)$ belongs to $\bar{X}^h$. The construction of B^h ensures that any $\preceq^h$-optimal consumption bundle in $\hat{\gamma}^h(p(\rho), q, r, \underline{z}(\rho), \bar{z}(\rho), v^h)$ can be purchased by means of bank loans in B^h. Finally, the choice of A^h is such that any pair consisting of a consumption bundle in $\hat{\gamma}^h(p(\rho), q, r, \underline{z}(\rho), \bar{z}(\rho), v^h)$ and a bank loan in B^h can be financed by asset market transactions in A^h.

For all $q \in Q$, there is $s_T \in S_T$ such that $q^0_{s_T} > 0$ and therefore

$$\frac{q^0_{s_T}}{1 + r_{s_T}} p_{\ell(s_T)s_T}(\rho) = \frac{q^0_{s_T}}{1 + r_{s_T}} \alpha_{s_T} > 0,$$

so

$$\sum_{s_t \in S} \frac{q^0_{s_t}}{1 + r_{s_t}} p_{s_t}(\rho) \underline{z}_{s_t}(\rho) \leq -\frac{q^0_{s_T}}{1 + r_{s_T}} \alpha_{s_T} \bar{c}_{\ell(s_T)s_T} < 0,$$

and it follows by Lemma 11.2 that $\hat{\gamma}^h$ is lower hemi-continuous at any $(p(\rho), q, r, \underline{z}(\rho), \bar{z}(\rho), v^h)$ with $(\rho, q, v^h) \in \bar{R} \times Q \times V^h$. For all $(\rho, q, v^h) \in \bar{R} \times Q \times V^h$, $\text{int}(\hat{\gamma}^h(p(\rho), q, r, \underline{z}(\rho), \bar{z}(\rho), v^h)) \cap \text{int}(X^h \times A^h \times B^h) \neq \emptyset$, so it follows from Hildenbrand (1974), Problem 6, p. 35, that $\bar{\gamma}^h$ is lower hemi-continuous. Since $\bar{X}^h \times A^h \times B^h$ is compact, and $\hat{\gamma}^h$ is closed by Lemma 11.2, we have that $\bar{\gamma}^h$ has a closed graph, so is upper hemi-continuous. We define $\delta^h(\rho, q, v^h)$ as the set of $\preceq^h$-maximal elements on $\bar{\gamma}^h(\rho, q, v^h)$. An application of the maximum theorem demonstrates that δ^h is an upper hemi-continuous correspondence.

Consider the correspondence $\varphi : R \times Q \times V \times \bar{X} \times A \times B \to R \times Q \times V \times \bar{X} \times A \times B$ defined by

$$\varphi(\rho, q, v, x, a, b) = \mu(x) \times \nu(a) \times o(b) \times \delta(\rho, q, v),$$
$$(\rho, q, v, x, a, b) \in R \times Q \times V \times \bar{X} \times A \times B,$$

where all correspondences involved in the definition of φ are defined as the obvious products.

STEP 3. Existence of a fixed point.
Since the domain of φ is a non-empty, compact, and convex set, and φ is an upper hemi-continuous convex-valued correspondence, all conditions of Kakutani's fixed point theorem are satisfied. The correspondence φ has a fixed point $(\rho^*, q^*, v^*, x^*, a^*, b^*) \in \mu(x^*)\times\nu(a^*)\times o(b^*)\times\delta(\rho^*, q^*, v^*)$. We define $p^* = p(\rho^*)$, $\underline{z}^* = \underline{z}(\rho^*)$, $\bar{z}^* = \bar{z}(\rho^*)$, and $z^* = x^* - e$. Moreover, we define present-value prices $\tilde{q}^*$, $\tilde{p}^*$, $\tilde{b}^*$, $\tilde{v}^*$, and $\tilde{w}^*$ by setting, for $s_t \in S$, $\tilde{q}^*_{s_t} = q^{0*}_{s_t}/(1 + r_{s_t})$, $\tilde{p}^*_{s_t} = \tilde{q}^*_{s_t} p^*_{s_t}$, $\tilde{b}^*_{s_t} = \tilde{q}^*_{s_t} b^*_{s_t}$, $\tilde{v}^*_{s_t} = \tilde{q}^*_{s_t} v^*_{s_t}$, and $\tilde{w}^* = \sum_{s_t \in S} \tilde{v}^*_{s_t}$.

It holds by definition of the various correspondences involved that

1. for every $h \in H$, $(x^{*h}, a^{*h}, b^{*h}) \in \delta^h(\rho^*, q^*, v^{*h})$,
2. for every $s_t \in S$, $\sum_{s_{t+1} \in s_t^+} q^*_{s_{t+1}} = 1/(1 + r_{s_t})$,
3. for every $h \in H$, $s_t \in S$, $v^{*h}_{s_t} = \theta^h r_{s_t} \sum_{h' \in H} b^{*h'}_{s_t}$,
4. for every $s_T \in S_T$, $\rho^*_{\ell(s_T)s_T} = \alpha_{s_T}$.

By summing the equalities in 3. over all households, we obtain

$$\text{for every } s_t \in S, \quad \sum_{h \in H} v^{*h}_{s_t} = r_{s_t} \sum_{h \in H} b^{*h}_{s_t}. \tag{11.5}$$

Notice that $(x^{*h}, a^{*h}, b^{*h}) \in \delta^h(\rho^*, q^*, v^{*h})$ implies (x^{*h}, a^{*h}, b^{*h}) is $\preceq^h$-maximal on $\bar{\gamma}^h(\rho^*, q^*, v^{*h}) = \hat{\gamma}^h(p^*, q^*, r, \underline{z}^*, \bar{z}^*, v^{*h}) \cap (X^h \times A^h \times B^h)$. As argued before, the construction of A^h and B^h now implies that

$$\text{for every } h \in H, (x^{*h}, a^{*h}, b^{*h}) \text{ is } \preceq^h -\text{maximal on } \hat{\gamma}^h(p^*, q^*, r, \underline{z}^*, \bar{z}^*, v^{*h}). \tag{11.6}$$

STEP 4. Properties of the fixed point: commodities not in L^*.
Consider $\ell s_t \in (L \times S)\backslash L^*$. Suppose $\sum_{h \in H} z^{*h}_{\ell s_t} < 0$. By definition of $\mu_{\ell s_t}$, we have $\rho^*_{\ell s_t} = 0$, so $p^*_{\ell s_t} = 0$ if $s_t \in N^{\mathrm{a}}_\ell$ and $\underline{z}^*_{\ell s_t} = 0$ if $s_t \in N^{\mathrm{s}}_\ell$. In the latter case, $x^{*h}_{\ell s_t} \geq e^h_{\ell s_t} + \underline{z}^*_{\ell s_t} = e^h_{\ell s_t}$ for all $h \in H$ by definition of $\bar{\gamma}^h$, so $\sum_{h \in H} z^{*h}_{\ell s_t} \geq 0$, leading to a contradiction. In the former case, it holds that $\bar{z}^*_{\ell s_t} = \bar{c}_{\ell s_t}$ since $s_t \in N^{\mathrm{a}}_\ell$. By A2 and A4.6 it holds that $x^{*h}_{\ell s_t} = e^h_{\ell s_t} + \bar{c}_{\ell s_t}$ for all $h \in H$, which implies $\sum_{h \in H} z^{*h}_{\ell s_t} > 0$, leading to a contradiction. Consequently,

$$\text{for every } \ell s_t \in (L \times S)\backslash L^*, \quad \sum_{h \in H} z^{*h}_{\ell s_t} \geq 0.$$

Consider $\ell s_t \in (L \times S)\backslash L^*$. Suppose $\sum_{h \in H} z^{*h}_{\ell s_t} > 0$. By definition of $\mu_{\ell s_t}$, we have $\rho^*_{\ell s_t} = u$ if $s_t \in N^{\mathrm{a}}_\ell$ and $\rho^*_{\ell s_t} = 1$ if $s_t \in N^{\mathrm{s}}_\ell$. In the latter case, $x^{*h}_{\ell s_t} \leq e^h_{\ell s_t} + \bar{z}^*_{\ell s_t} = e^h_{\ell s_t}$

for all $h \in H$ by definition of $\bar{\gamma}^h$, so $\sum_{h \in H} z^{*h}_{\ell s_t} \leq 0$, leading to a contradiction. In the former case, $p^*_{\ell s_t} = u$. Consequently,

$$\text{for every } \ell s_t \in L \times S,\ s_t \in N^{\mathrm{s}}_{\ell},\ \sum_{h \in H} z^{*h}_{\ell s_t} = 0, \tag{11.7}$$

$$\text{for every } \ell s_t \in (L \times S) \backslash L^* \text{ such that } s_t \in N^{\mathrm{a}}_{\ell},\ \begin{array}{r} \sum_{h \in H} z^{*h}_{\ell s_t} \geq 0, \\ \sum_{h \in H} z^{*h}_{\ell s_t} > 0 \Rightarrow p^*_{\ell s_t} = u. \end{array} \tag{11.8}$$

Step 5. Properties of the fixed point: commodity prices.
Consider $\ell s_t \in (L \times S) \backslash L^*$ such that $s_t \in N^{\mathrm{a}}_{\ell}$. Suppose $p^*_{\ell s_t} = 0$. Then we have $\sum_{h \in H} z^{*h}_{\ell s_t} > 0$ since $\bar{z}^*_{\ell s_t} = \bar{c}_{\ell s_t}$ by definition of $\bar{z}$ and $x^{*h}_{\ell s_t} = e^h_{\ell s_t} + \bar{c}_{\ell s_t}$ by A2 and A4.6. By (11.8) it holds that $p^*_{\ell s_t} = u$, leading to a contradiction. Consequently,

$$\text{for every } \ell s_t \in (L \times S) \backslash L^* \text{ such that } s_t \in N^{\mathrm{a}}_{\ell},\ p^*_{\ell s_t} > 0.$$

Now, it holds that

$$p^* \gg 0, \tag{11.9}$$

since the price of a commodity $\ell s_T \in L^*$ is equal to $\alpha_{s_T} > 0$ and the price of a commodity $\ell s_{t'}$ with a sticky price is set in previous date-event $s_t(s_{t'})$, so $p^*_{\ell s_{t'}} = p^*_{\ell s_t(s_{t'})} > 0$. This argument makes use of the fact that $p_{s_{-1}} \gg 0$ by A6 in case $t = -1$.

Step 6. Properties of the fixed point: assets.
We now prove by induction that

$$\text{for every } s_t \in S^+,\ \sum_{h \in H} a^{*h}_{s_t} \leq 0.$$

By definition of $\bar{\gamma}^h$ we have $q^*_{s_0^+} a^{*h}_{s_0^+} \leq 0$. Let $1^{s_0^+}_{s_1}$ denote the s_1-th unit vector in $\mathbb{R}^{s_0^+}$. It holds that

$$0 \geq q^*_{s_0^+} \sum_{h \in H} a^{*h}_{s_0^+} \geq \frac{1}{1 + r_{s_0}} 1^{s_0^+}_{s_1} \sum_{h \in H} a^{*h}_{s_0^+} = \frac{1}{1 + r_{s_0}} \sum_{h \in H} a^{*h}_{s_1},$$

where the second inequality follows by definition of ν_{s_0}. We have shown that for all $s_1 \in s_0^+$, $\sum_{h \in H} a^{*h}_{s_1} \leq 0$, or equivalently $\sum_{h \in H} a^{*h}_{s_0^+} \leq 0$.

We show next that if, for some $t \in \{1, \ldots, T\}$, for some $s_t \in S_t$, $\sum_{h \in H} a^{*h}_{s_t} \leq 0$, then for all $s_{t+1} \in s_t^+$, $\sum_{h \in H} a^{*h}_{s_{t+1}} \leq 0$, or equivalently $\sum_{h \in H} a^{*h}_{s_t^+} \leq 0$. By definition of $\bar{\gamma}^h$ we have

$$\sum_{h \in H} q^*_{s_t^+} a^{*h}_{s_t^+} = \sum_{h \in H} \left(a^{*h}_{s_t} - p^*_{s_t^-} z^{*h}_{s_t^-} - r_{s_t^-} b^{*h}_{s_t^-} + v^{*h}_{s_t^-} \right) \leq 0,$$

where we use the induction hypothesis, the fact that $p^*_{s_t^-} \sum_{h \in H} z^{*h}_{s_t^-} \geq 0$ by (11.7) and (11.8), and (11.5). By definition of ν_{s_t} it holds that

$$0 \geq q^*_{s_t^+} \sum_{h \in H} a^{*h}_{s_t^+} \geq \frac{1}{1 + r_{s_t}} 1^{s_t^+}_{s_{t+1}} \sum_{h \in H} a^{*h}_{s_t^+} = \frac{1}{1 + r_{s_t}} \sum_{h \in H} a^{*h}_{s_{t+1}}, \quad s_{t+1} \in s_t^+,$$

so $\sum_{h \in H} a^{*h}_{s_t^+} \leq 0$. This completes the induction step.

Step 7. Properties of the fixed point: commodities in L^*.
Consider $s_T \in S_T$. It holds that

$$p^*_{s_T} \sum_{h \in H} z^{*h}_{s_T} = \sum_{h \in H} a^{*h}_{s_T^+} - \sum_{h \in H} (r_{s_T} b^{*h}_{s_T} - v^{*h}_{s_T}) = \sum_{h \in H} a^{*h}_{s_T^+} \leq 0,$$

where the first equality follows from the definition of $\bar{\gamma}^h$ and the second equality from (11.5). By (11.7), (11.8), and (11.9) we have the following result,

$$\text{for every } s_T \in S_T, \quad \sum_{h \in H} z^{*h}_{\ell(s_T)s_T} \leq 0. \tag{11.10}$$

Step 8. Properties of the fixed point: asset prices.
For $s_{T+1} \in S_{T+1}$ we have $q^*_{s_{T+1}} = 1/(1 + r_{s_{T+1}}) > 0$. Suppose $q^*_{s_t} = 0$ for some $s_t \in S \backslash \{s_0\}$. Let s_T be a period T successor of s_t. Then $\tilde{p}^*_{s_T} = \tilde{q}^*_{s_T} p^*_{s_T} = 0$, so by the by now familiar argument, $x^{*h}_{\ell(s_T)s_T} = e^h_{\ell(s_T)s_T} + \bar{c}_{\ell(s_T)s_T}$ for all $h \in H$, leading to a contradiction to (11.10). We have shown that

$$q^* \gg 0. \tag{11.11}$$

Step 9. Properties of the fixed point: optimality of choices.
Consider $\ell \in L$ and $s_T \in N^{\mathrm{a}}_\ell$. For all $h \in H$ we have that

$$\begin{aligned}
e^h_{\ell s_T} + \bar{z}^*_{\ell s_T} &= e^h_{\ell s_T} + \bar{c}_{\ell s_T} \\
&\geq e^h_{\ell s_T} + \textstyle\sum_{h' \in H} (e^{h'}_{\ell s_T} - \underline{x}^{h'}_{\ell s_T}) \\
&\geq e^h_{\ell s_T} + \textstyle\sum_{h' \in H} (x^{*h'}_{\ell s_T} - \underline{x}^{h'}_{\ell s_T}) \\
&= x^{*h}_{\ell s_T} + \textstyle\sum_{h' \in H \backslash \{h\}} (x^{*h'}_{\ell s_T} - \underline{x}^{h'}_{\ell s_T}) + (e^h_{\ell s_T} - \underline{x}^h_{\ell s_T}) \\
&> x^{*h}_{\ell s_T},
\end{aligned} \tag{11.12}$$

where the first equality uses $s_T \in N^{\mathrm{a}}_\ell$, the first inequality uses the definition of $\bar{c}_{\ell s_T}$, the second inequality uses (11.10), and the final inequality uses A1 and the definition of $\underline{x}^h$.

Suppose, for some $h \in H$, $\tilde{p}^* x^{*h} + r^* \tilde{b}^{*h} < \tilde{p}^* e^h + \tilde{w}^{*h}$. Consider a sequence $(x^h_n)_{n \in \mathbb{N}}$ of points in $\bar{X}^h$ such that x^h_n converges to x^{*h}, $x^h_{\ell s_T, n} > x^{*h}_{\ell s_T}$ if $s_T \in N^{\mathrm{a}}_\ell$, and $x^h_{\ell s_t, n} = x^{*h}_{\ell s_t}$ otherwise. Inequality (11.12) guarantees that $x^h_n - e^h \leq \bar{z}^*$ for n

sufficiently large. Since β^h is lower hemi-continuous, there exists a sequence $(b_n^h)_{n\in\mathbb{N}}$ of points in $\beta^h(p^*, x_n^h)$ such that $b_n^h \to b^{*h}$. For n sufficiently large it holds that $\tilde{p}^* x_n^h + r^*\tilde{b}_n^h < \tilde{p}^* e^h + \tilde{w}^{*h}$, and one can therefore choose $a_n^h \in \mathbb{R}^{S^+}$ such that $(x_n^h, a_n^h, b_n^h) \in \hat{\gamma}^h(p^*, q^*, r, \underline{z}^*, \bar{z}^*, v^{*h})$. Since $\preceq^h$ is monotonic, this contradicts that (x^{*h}, a^{*h}, b^{*h}) is $\preceq^h$-maximal on $\hat{\gamma}^h(p^*, q^*, r, \underline{z}^*, \bar{z}^*, v^{*h})$. Consequently,

$$\text{for every } h \in H,\ \tilde{p}^* x^{*h} + r^*\tilde{b}^{*h} = \tilde{p}^* e^h + \tilde{w}^{*h}, \tag{11.13}$$

and using (11.6) it follows that

$$\text{for every } h \in H,\ (x^{*h}, a^{*h}, b^{*h}) \text{ is } \preceq^h \text{-maximal on } \gamma^h(p^*, q^*, r, \underline{z}^*, \bar{z}^*, v^{*h}),$$

so (a) of Definition 6.1 holds.

STEP 10. Lifting the upper bound on prices.
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of points such that $u_n \geq \max\{1, \max_{s_T\in S_T} \alpha_{s_T}, \max_{\ell\in L} p_{\ell s_{-1}}\}$ and $u_n \to \infty$. Let $\bar{R}_n$, A_n, B_n, and V_n denote the corresponding compactified sets of endogenous variables as constructed in Step 1 and let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of fixed point correspondences as constructed in Step 2. Let $(\rho_n^*, q_n^*, v_n^*, x_n^*, a_n^*, b_n^*)_{n\in\mathbb{N}}$ be a corresponding sequence of fixed points, whose existence is shown in Step 3. For $n \in \mathbb{N}$, we define $p_n^* = p(\rho_n^*)$, $\underline{z}_n^* = \underline{z}(\rho_n^*)$, $\bar{z}_n^* = \bar{z}(\rho_n^*)$, and $z_n^* = x_n^* - e$. Moreover, we define the present-value variables $\tilde{q}_n^*$, $\tilde{p}_n^*$, $\tilde{b}_n^*$, $\tilde{v}_n^*$, and $\tilde{w}_n^*$ by setting, for $s_t \in S$, $\tilde{q}_{s_t,n}^* = q_{s_t,n}^{0*}/(1+r_{s_t})$, $\tilde{p}_{s_t,n}^* = \tilde{q}_{s_t,n}^* p_{s_t,n}^*$, $\tilde{b}_{s_t,n}^* = \tilde{q}_{s_t,n}^* b_{s_t,n}^*$, $\tilde{v}_{s_t,n}^* = \tilde{q}_{s_t,n}^* v_{s_t,n}^*$, and $\tilde{w}_n^* = \sum_{s_t\in S} \tilde{v}_{s_t,n}^*$.

Suppose for every $n \in \mathbb{N}$ there is $\ell s_t \in L \times S$ such that $\sum_{h\in H} z_{\ell s_t,n}^{*h} > 0$. We will derive a contradiction in Steps 11–13.

STEP 11. Construction of a convergent subsequence.
For every $n \in \mathbb{N}$, for every $h \in H$, let $\underline{b}_n^h$ be a minimal element of $\beta^h(p_n^*, x_n^{*h})$, so there is no $b^h \in \beta^h(p_n^*, x_n^{*h})$ with $b^h < \underline{b}_n^h$. Since β^h is closed by A4.1, such a $\underline{b}_n^h$ exists. We define $\tilde{\underline{b}}_n^h$ by setting $\tilde{\underline{b}}_{s_t,n}^h = \tilde{q}_{s_t,n}^* \underline{b}_{s_t,n}^h$ for $s_t \in S$. We have

$$\text{for every } n \in \mathbb{N},\ \text{for every } h \in H,\ r\tilde{\underline{b}}_n^h = r\tilde{b}_n^{*h}, \tag{11.14}$$

since it clearly holds that $r\tilde{\underline{b}}_n^h \leq r\tilde{b}_n^{*h}$, whereas a strict inequality would lead to a contradiction as in the proof of inequality (11.11).

We divide $\tilde{p}_n^*$, $\tilde{\underline{b}}_n$, $\tilde{v}_n^*$, and $\tilde{w}_n^*$ by $\|\tilde{p}_n^*\|_\infty$ and denote the resulting variables by $\hat{p}_n$, $\hat{\underline{b}}_n$, $\hat{v}_n$, and $\hat{w}_n$. We claim that these variables are bounded. It is obvious that the sequence $(\hat{p}_n)_{n\in\mathbb{N}}$ is bounded.

We show next that the sequence $(\hat{b}_n)_{n\in\mathbb{N}}$ is bounded and start by showing that this sequence is bounded from below. We have that $\underline{b}_n^h \in \beta^h(p_n^*, x_n^{*h})$ and $\underline{f}^h(p_n^*, x_n^{*h}) \leq \underline{b}_n^h$. By the homogeneity as stated in A4.3, it holds that $\hat{\underline{b}}_n^h \in \beta^h(\hat{p}_n, x_n^{*h})$. By A4.2 it holds that $\underline{f}^h(\hat{p}_n, x_n^{*h}) \leq \hat{\underline{b}}_n^h$. We define $P = \{p \in \mathbb{R}_+^{LS} \mid \|p\|_\infty = 1\}$. Since the set $P \times \bar{X}^h$ is compact and $\underline{f}^h$ is continuous, the set $\underline{f}^h(P \times \bar{X}^h)$ is compact, so bounded from below. It follows that the sequence $(\hat{\underline{b}}_n)_{n\in\mathbb{N}}$ is bounded from below. To show that this sequence is bounded from above, we show first that $\hat{\underline{b}}_n^h \leq \bar{f}^h(\hat{p}_n, x_n^{*h})$. Suppose not, then by A4.2 we have $\hat{b}_n^h = \min\{\bar{f}^h(\hat{p}_n, x_n^{*h}), \hat{\underline{b}}_n^h\} \in \beta^h(\hat{p}_n, x_n^{*h})$ and $\hat{b}_n^h < \hat{\underline{b}}_n^h$. We define $b_n^h \in \mathbb{R}^S$ by setting

$$b_{s_t,n}^h = \frac{\|\tilde{p}_n^*\|_\infty}{\tilde{q}_{s_t,n}^*}\hat{b}_{s_t,n}^h, \quad s_t \in S.$$

Notice that $\tilde{q}_{s_t,n}^* > 0$ by (11.11), so b_n^h is well-defined. Then, it holds by A4.3 that $b_n^h \in \beta^h(p_n^*, x_n^{*h})$, whereas $b_n^h < \underline{b}_n^h$, a contradiction to the choice of $\underline{b}_n^h$. Consequently, it holds that $\hat{\underline{b}}_n^h \leq \bar{f}^h(\hat{p}_n, x_n^{*h})$. Since the set $P \times \bar{X}^h$ is compact and $\bar{f}^h$ is continuous, the set $\bar{f}^h(P \times X^h)$ is compact, so bounded from above. It follows that the sequence $(\hat{\underline{b}}_n)_{n\in\mathbb{N}}$ is bounded from above.

Since, by (11.14), for $s_t \in S$, $\hat{v}_{s_t,n}^h = \theta^h r_{s_t} \sum_{h'\in H} \hat{\underline{b}}_{s_t,n}^{h'}$, and $(\hat{b}_n)_{n\in\mathbb{N}}$ is bounded, it follows that the sequence $(\hat{v}_n)_{n\in\mathbb{N}}$ is bounded. Next, since $\hat{w}^h = \sum_{s_t\in S} \hat{v}_{s_t}^h$, it follows that the sequence $(\hat{w}_n)_{n\in\mathbb{N}}$ is bounded.

The boundedness of the various sequences of endogenous variables implies that, without loss of generality, $(\hat{p}_n, \tilde{q}_n^*, \underline{z}_n^*, \bar{z}_n^*, \hat{w}_n, x_n^*, \hat{\underline{b}}_n)$ is a convergent sequence with limit, say, $(\hat{p}, \tilde{q}^*, \underline{z}^*, \bar{z}^*, \hat{w}, x^*, \hat{\underline{b}})$. Moreover, without loss of generality, there is $\ell s_t \in L \times S$ such that for all $n \in \mathbb{N}$, $z_{\ell s_t,n}^{*h} > 0$. By (11.8) we have that $s_t \in N_\ell^{\mathrm{a}}$ and $p_{\ell s_t,n}^* = u_n$.

Step 12. Continuity of demand at the limit.
We argue next that $\hat{p}\underline{z}^* < 0$. We achieve this by showing that there is a commodity ℓs_t such that $s_t \in N_\ell^{\mathrm{a}}$ and $\hat{p}_{\ell s_t} > 0$.

Consider $\ell s_{t'} \in L \times S$ such that $\hat{p}_{\ell s_{t'}} = 1$. If $s_{t'} \in N_\ell^{\mathrm{a}}$, then we are done. If $s_{t'} \in N_\ell^{\mathrm{s}}$, then let s_t be the date-event where the price of $\ell s_{t'}$ is set. If $t \geq 0$, then $s_t \in N_\ell^{\mathrm{a}}$ and

$$\hat{p}_{\ell s_t} = \lim_{n\to\infty} \hat{p}_{\ell s_t,n} \geq \lim_{n\to\infty} \hat{p}_{\ell s_{t'},n} = \hat{p}_{\ell s_{t'}} = 1,$$

where the inequality uses the fact that $p_{\ell s_t,n}^* = p_{\ell s_{t'},n}^*$ and $\tilde{q}_{\ell s_t,n}^* \geq \tilde{q}_{\ell s_{t'},n}^*$. It follows that $\hat{p}_{\ell s_t} = 1$. If $t = -1$, then

$$\hat{p}_{\ell s_{t'}} = \lim_{n\to\infty} \frac{\tilde{p}_{\ell s_{t'},n}^*}{\|\tilde{p}_n^*\|_\infty} = \lim_{n\to\infty} \frac{\tilde{q}_{s_{t'},n}^* p_{\ell s_{t'},n}^*}{\|\tilde{p}_n^*\|_\infty} = \lim_{n\to\infty} \frac{\tilde{q}_{s_{t'},n}^* p_{\ell s_{-1}}}{\|\tilde{p}_n^*\|_\infty} = 1,$$

so $\lim_{n\to\infty} \|\tilde{p}_n^*\|_\infty = \tilde{q}_{s_{t'}}^* p_{\ell s_{-1}}$. Let s_T be a date-event such that $\tilde{q}_{s_T}^* > 0$. Then, for every $n \in \mathbb{N}$,

$$\|\tilde{p}_n^*\|_\infty \geq \tilde{q}_{s_T,n}^* p_{\ell(s_T)s_T}^* = \tilde{q}_{s_T,n}^* \alpha_{s_T},$$

so $\tilde{q}^*_{s_{t'}} p_{\ell s_{-1}} = \lim_{n\to\infty} \|\tilde{p}^*_n\|_\infty \geq \tilde{q}^*_{s_T}\alpha_{s_T} > 0$ and

$$\hat{p}_{\ell(s_T)s_T} = \frac{\tilde{q}^*_{s_T}\alpha_{s_T}}{\tilde{q}^*_{s_{t'}} p_{\ell s_{-1}}} > 0.$$

Since $s_T \in N^{\mathrm{a}}_{\ell(s_T)}$, we have shown that $\hat{p}\underline{z}^* < 0$.

Let Ξ^h be the set of all points in the sequence $(\hat{p}_n, r, \underline{z}^*_n, \bar{z}^*_n, \hat{w}^h_n)_{n\in\mathbb{N}}$ and its limit $(\hat{p}, r, \underline{z}^*, \bar{z}^*, \hat{w}^h)$. Let $\underline{B}^h$ be a compact set containing all the points in the sequence $(\hat{\underline{b}}^h_n)_{n\in\mathbb{N}}$ and its limit $(\hat{\underline{b}}^h)$ in its interior. We define $\underline{\gamma}^h : \Xi^h \to \bar{X}^h \times \underline{B}^h$ by $\underline{\gamma}^h(\xi^h) = \tilde{\gamma}^h(\xi^h) \cap (\bar{X}^h \cap \underline{B}^h)$. Since $\tilde{\gamma}^h$ is lower hemi-continuous on Ξ^h by Lemma 11.1 and for all $\xi^h \in \Xi^h$, $\mathrm{int}(\tilde{\gamma}^h(\xi^h)) \cap \mathrm{int}(\bar{X}^h \cap \underline{B}^h) \neq \emptyset$, it follows from Hildenbrand (1974), Problem 6, p. 35, that $\underline{\gamma}^h$ is lower hemi-continuous. Since $\bar{X}^h \times \underline{B}^h$ is compact and $\tilde{\gamma}^h$ is closed by Lemma 11.1, we have that $\underline{\gamma}^h$ has a closed graph, so is upper hemi-continuous. We define $\underline{\delta}^h : \Xi^h \to \bar{X}^h \times \underline{B}^h$ by defining $\underline{\delta}^h(\xi^h)$ as the set of $\preceq^h$-maximal elements on $\underline{\gamma}^h(\xi^h)$. An application of the maximum theorem demonstrates that $\underline{\delta}^h$ is an upper hemi-continuous correspondence.

Step 13. Deriving a contradiction.
Since for $n \in \mathbb{N}$, $(x^{*h}_n, \hat{b}^h_n) \in \underline{\delta}^h(\hat{p}_n, r, \underline{z}^*_n, \bar{z}^*_n, \hat{w}^h_n)_{n\in\mathbb{N}}$, it follows by upper hemi-continuity of $\underline{\delta}^h$ that $(x^{*h}, \hat{b}^h) \in \underline{\delta}^h(\hat{p}, r, \underline{z}^*, \bar{z}^*, \hat{w}^h)$.

Consider $\ell s_t \in L \times S$ such that for all $n \in \mathbb{N}$, $p^*_{\ell s_t,n} = u_n$ and let s_T be a successor of s_t in period T or equal to s_t in case $t = T$. We show that $\hat{p}_{\ell(s_T)s_T} = 0$. First, consider the case where $\tilde{q}^*_{s_t} = 0$. Then, it holds that $\tilde{q}^*_{s_T} = 0$ and $p_{\ell(s_T)s_T} = \alpha_{s_T}$. Moreover, $\|\tilde{p}^*_n\|_\infty$ is bounded away from zero, since $\|\tilde{p}^*_n\|_\infty \geq \tilde{q}^*_{\bar{s}_T,n}\alpha_{\bar{s}_T} \to \tilde{q}^*_{\bar{s}_T}\alpha_{\bar{s}_T} > 0$, where $\bar{s}_T$ is chosen such that $\tilde{q}^*_{\bar{s}_T} > 0$. We therefore have that

$$\hat{p}_{\ell(s_T)s_T} = \lim_{n\to\infty} \hat{p}_{\ell(s_T)s_T,n} = \lim_{n\to\infty} \frac{\tilde{q}^*_{s_T,n} p^*_{\ell(s_T)s_T,n}}{\|\tilde{p}^*_n\|_\infty} = 0.$$

Second, consider the case where $\tilde{q}^*_{s_t} > 0$. Since $p^*_{\ell s_t,n} = u_n$, it follows that $\tilde{p}^*_{\ell s_t,n} = \tilde{q}^*_{s_t,n} p^*_{\ell s_t,n} \to \infty$ as $n \to \infty$, so $\|\tilde{p}^*_n\|_\infty \to \infty$ as $n \to \infty$. We therefore have that

$$\hat{p}_{\ell(s_T)s_T} = \lim_{n\to\infty} \hat{p}_{\ell(s_T)s_T,n} = \lim_{n\to\infty} \frac{\tilde{q}^*_{s_T,n} p^*_{\ell(s_T)s_T,n}}{\|\tilde{p}^*_n\|_\infty} = \frac{\tilde{q}^*_{s_T}\alpha_{s_T}}{\lim_{n\to\infty}\|\tilde{p}^*_n\|_\infty} = 0.$$

Consider $(x^h, b^h) \in \underline{\delta}^h(\hat{p}, r, \underline{z}^*, \bar{z}^*, \hat{w}^h)$. Since $\hat{p}_{\ell(s_T)s_T} = 0$, we have that

$$x^h_{\ell(s_T)s_T} = e^h_{\ell(s_T)s_T} + \bar{z}^*_{\ell(s_T)s_T} = e^h_{\ell(s_T)s_T} + c_{\ell(s_T)s_T}.$$

Since $\underline{\delta}^h$ is upper hemi-continuous at $(\hat{p}, r, \underline{z}^*, \bar{z}^*, \hat{w}^h)$, we have that

$$\sum_{h\in H} z^{*h}_{\ell(s_T)s_T,n} \to \sum_{h\in H} c_{\ell(s_T)s_T} > 0,$$

a contradiction to (11.10).

STEP 14. Market clearing for commodities.
We have shown that there is $n \in \mathbb{N}$ such that for every $\ell s_t \in L \times S$, $\sum_{h\in H} z^{*h}_{\ell s_t,n} \leq 0$. We fix such an n and omit it from the notation. Suppose $\sum_{h\in H} z^{*h} < 0$. It holds that

$$\tilde{p}^* \sum_{h\in H} z^{*h} < 0,$$

where the inequality uses (11.9) and (11.11). At the same time, we have by (11.13) and (11.5) that

$$\begin{aligned}\tilde{p}^* \textstyle\sum_{h\in H} z^{*h} &= r^* \textstyle\sum_{h\in H} \tilde{b}^{*h} - \tilde{w}^{*h} \\ &= \textstyle\sum_{s_t\in S} \tilde{q}^*_{s_t} \left(r_{s_t} \sum_{h\in H} b^{*h} - \sum_{h\in H} v^{*h}\right) = 0.\end{aligned}$$

We have derived a contradiction. Consequently, it holds that $\sum_{h\in H} x^{*h} = \sum_{h\in H} e^h$ and we have shown (b) of Definition 6.1.

STEP 15. Market clearing for assets.
We use induction to show (c) of Definition 6.1. For every $s_{T+1} \in S_{T+1}$, by definition of γ^h, $h \in H$,

$$\sum_{h\in H} a^{*h}_{s_{T+1}} = p^*_{s^-_{T+1}} \sum_{h\in H} \left(x^{*h}_{s^-_{T+1}} - e^h_{s^-_{T+1}}\right) + r_{s^-_{T+1}} \sum_{h\in H} b^{*h}_{s^-_{T+1}} - \sum_{h\in H} v^{*h}_{s^-_{T+1}} = 0,$$

where the last equality uses (b) of Definition 6.1 and (11.5).

Assume, for some $t \in \{0, \ldots, T\}$, we have shown that for every $s_{t+1} \in S_{t+1}$, $\sum_{h\in H} a^{*h}_{s_{t+1}} = 0$. For every $s_t \in S_t$, by definition of γ^h, $h \in H$,

$$\begin{aligned}\sum_{h\in H} a^{*h}_{s_t} = p^*_{s^-_t} \sum_{h\in H} \left(x^{*h}_{s^-_t} - e^h_{s^-_t}\right) + r_{s^-_t} \sum_{h\in H} b^{*h}_{s^-_t} - \sum_{h\in H} v^{*h}_{s^-_t} \\ + \sum_{s_{t+1}\in s^+_t} q^*_{s_{t+1}} \sum_{h\in H} a^{*h}_{s_{t+1}} = 0,\end{aligned}$$

where the last equality uses (b) of Definition 6.1, (11.5), and the induction hypothesis. This completes the proof of Theorem 7.1. □

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