The Intertemporal Keynesian Cross

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Abstract

This paper develops a novel approach to analyzing the transmission of shocks and policies in many existing macroeconomic models with nominal rigidities. Our approach is centered around a network representation of agents' spending patterns: nodes are goods markets at different times, and flows between nodes are agents' marginal propensities to spend income earned in one node on another one. Since, in general equilibrium, one agent's spending is another agent's income, equilibrium demand in each node is described by a recursive equation with a special structure, which we call the intertemporal Keynesian cross (IKC). Each solution to the IKC corresponds to an equilibrium of the model, and the direction of indeterminacy is given by the network's eigenvector centrality measure. We use results from Markov chain potential theory to tightly characterize all solutions. In particular, we derive (a) a generalized Taylor principle to ensure bounded equilibrium determinacy; (b) how most shocks do not affect the net present value of aggregate spending in partial equilibrium and nevertheless do so in general equilibrium; (c) when heterogeneity matters for the aggregate effect of monetary and fiscal policy. We demonstrate the power of our approach in the context of a quantitative Bewley-Huggett-Aiyagari economy for fiscal and monetary policy.

1 Introduction

One of the most important questions in macroeconomics is that of how shocks are propagated and amplified in general equilibrium. Recently, shocks that have received particular attention are those that affect households in heterogeneous ways—such as tax rebates, changes in monetary policy, falls in house prices, credit crunches, or increases in economic uncertainty or income inequality. New empirical evidence has been brought to bear, and new decision-theoretic models have been developed, to quantify and understand the causal effect of these shocks on aggregate consumption.[1] Yet both cross-sectional empirical evidence and decision-theory models deliver

[1]For empirical evidence using cross-sectional variation, see Johnson, Parker and Souleles (2006) for tax rebates, on or Fagereng, Holm and Natvik (2016) on unanticipated income shocks. For mostly partial equilibrium decision-theory

partial equilibrium estimates of aggregate effects. A significant question that remains is how partial equilibrium effects translate into general equilibrium outcomes, once markets clear and the endogenous responses of monetary and fiscal policy are taken into account. This is the question we study in this paper.

It is intuitive that there should be commonalities in the partial-to-general equilibrium adjustment mechanism. In a closed economy, whether consumption falls because of a fall in house prices, a credit crunch, or a fiscal retrenchment, common mechanisms are likely to operate to restore equality between equilibrium demand and output. One important such adjustment mechanism, highlighted by Keynes (1936), is the feedback between consumption and incomes: as consumption falls, incomes across the population falls, triggering further adjustments in consumption. Modern models used for monetary and fiscal policy analysis tend to emphasize other mechanisms instead: those that arise from the endogenous reaction of policy to the fall in economic activity. These models typically feature representative agents who smooth their consumption responses to income shocks over their infinite lifetime, rendering the consumption-income feedback quantitatively irrelevant. But a newer vintage of models incorporates much larger marginal propensities to consume out of transitory income shocks, which empirical evidence has consistently shown to be an important feature of the data. In these models, consumption-income feedbacks have the potential to play a much more important role for general equilibrium adjustment.

We show that, in a broad class of models with nominal rigidities, the question of how to go from fundamental shocks to equilibrium outcomes is answered by the solution to one simple recursive equation,

$$d\mathbf{Y}=\partial\mathbf{Y}+\mathbf{M}\cdot d\mathbf{Y} \tag{1}$$

In (1), the vector $d\mathbf{Y}$ represents the general equilibrium impulse response to the shock–the change in the time path of output induced by the shock, which is the ultimate object of interest to macroeconomic analysis. $\partial\mathbf{Y}$ represents the partial equilibrium impulse response, whose definition we establish precisely below, and which in our benchmark case corresponds to the effect of the shock holding interest rates and quantities constant, as in the aforementioned literature. Finally, $\mathbf{M}$ is a matrix of general equilibrium feedbacks, encapsulating the response of private individuals to changes in incomes and interest rates, as well as endogenous policy responses, notably that of monetary policy. In a benchmark case in which real interest rates are held constant by monetary policy ('constant-r'), the entries of the $\mathbf{M}$ matrix correspond to households' average marginal propensities to consume, at each date, for income received at other dates, weighted by the incidence of income across the population. This renders the $\mathbf{M}$ matrix potentially observable directly, just as $\partial\mathbf{Y}$ can be, opening the promise of conducting general equilibrium analysis using a sufficient statistic approach under minimal model assumptions.

Keynes (1936) proposed an answer similar to our equation (1), focusing exclusively on the income-consumption feedback mechanism and the immediate response of output to shocks. In

models, see Auclert (2017) on the effects of monetary policy, Kaplan and Violante (2014) on tax rebates, and Berger, Guerrieri, Lorenzoni and Vavra (2015) on house prices.

his formulation,

$$dY = \partial Y + MPC \cdot dY \tag{2}$$

where ∂Y is the impulse to aggregate demand, dY is the equilibrium change in output, and MPC is the aggregate marginal propensity to consume out of income. Equation (2) can be solved to obtain $dY = (1 - MPC)^{-1}\partial Y$, where $(1 - MPC)^{-1} = 1 + MPC + MPC^2 + \dots$ is called the *multiplier* and reflects the accumulated consumption feedback amplifying the original impulse. Although this traditional 'Keynesian cross' embodies a useful intuition, it has a number of weaknesses. It is static, not dynamic. It does not require that budget constraints be satisfied: an impulse to demand from government spending comes out of thin air, rather than being offset by taxation or a cut in spending at some other date. It does not directly correspond to any standard, microfounded model. It does not give a role to monetary policy. It assumes that the response of the economy is determinate—an assumption modern monetary theory has shown is only guaranteed under particular types of policy rules.

We show that, nevertheless, going back to these intuitions is useful to understand modern business cycle theory, and furthermore the parallel delivers novel results that had escaped the literature until now. First, when monetary policy maintains the real interest rate constant, the $\mathbf{M}$ matrix in (1) corresponds to a weighted average of marginal propensities to consume. This captures the static consumption-income feedbacks emphasized by Keynes, as well as richer intertemporal interactions: for instance, some income earned in period 1 will be spent in period 2, from which some income will be spent again in period 1. Budget constraints also impose a number of important modifications to the static view. First, it is not generally possible to simply solve for $d\mathbf{Y}$ in (1) without further considerations. The crucial observation here is that all income is eventually consumed in net present value terms–the dynamic $\mathbf{M}$ matrix has an eigenvalue of 1, making inversion of $I - \mathbf{M}$ impossible in general. On the other hand, partial equilibrium shocks on their own do not create present value. In our dynamic case, we show that equation (1) can be solved, but that it generally has multiple solutions.[2] This indeterminacy is inherent to models with nominal rigidities, and for the first time, we are sharply able to characterize it, by showing how it relates to properties of the $\mathbf{M}$ matrix.

To understand the amplification mechanisms inherent to our model is useful to think of time periods as nodes of a network, as in figure 1. Each new unit of income generated at a given node is spent, partly on itself and partly on every other node, according to relationships given by the matrix $\mathbf{M}$. This round of spending generates additional income at each node, which is again spent according to the same pattern, and so on. The final outcome for the distribution of income across nodes is our main object of interest. It is the general equilibrium effect on consumption after the intertemporal Keynesian cross has run its course. If the matrix $\mathbf{M}$ is written in net present value units—as we generally do—then the fact that all income is spent in net present value terms means that $\mathbf{M}$ is a left-stochastic matrix, with columns summing to one. It can then be interpreted as

[2] Intutitively, in the scalar version (2), budget constraints impose that $\partial Y = 0$ and $MPC = 1$, so any value of dY is a solution.

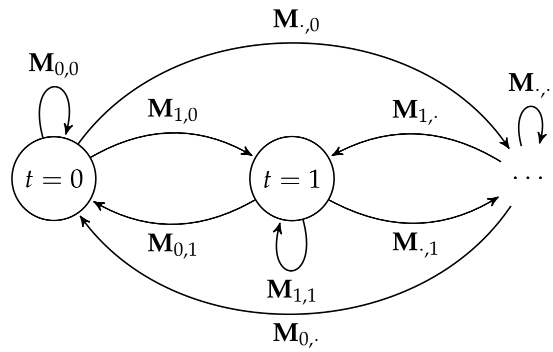

Figure 1: Interpretating the economy as a network

the transition matrix for a Markov chain, with dimension equal to the number of periods of the model. The stationary distribution of this Markov chain, if it exists, delivers the dimension of indeterminacy: if agents expect aggregate income to increase in proportion to this distribution over time, then their collective consumption responses are such that it sustain exactly this additional increase in income. Sometimes, however, when $\mathcal{T} = \infty$, the only stationary 'distributions' diverge with time—an outcome that is natural to rule out, as the literature typically does. In this case, equilibrium is uniquely pinned down.

We first show that, under constant-r monetary policy, whether or not the equilibrium is determinate depends on the economy's $\mathbf{M}$ matrix. Specifically, to the characteristic function of the limiting distribution implied by the final columns of the $\mathbf{M}$ matrix—the time path of the response of consumption to an income shock far into the future. Intuitively, if income increases at a point in time tend to be collectively spent by agents later on—say because agents tend to wait to receive income before spending it due to borrowing constraints—then the economy is naturally determinate because only explosive patterns of income would self-sustain. Relaxing the assumption of constant real interest rates, we establish a New Taylor principle for equilibrium determinacy. When monetary policy endogenously raises interest rates in response to increases in economic activity (directly or indirectly through the effect of activity on inflation), agents collectively postpone their spending decisions, placing the economy in a determinate case. The Taylor principle characterizes exactly by how much nominal interest rates need to increase. The answer depends on the natural tendency of the economy to delay consumption relative to income, and the elasticity of the timing of consumption to changes in real interest rates. To the best of our knowledge, this is the first general Taylor principle ever derived in a heterogeneous agent setting. Our result also provides a new intuition for why the Taylor principle is necessary to ensure bounded equilibrium determinacy in standard New Keynesian models.

We next show conditions under which partial equilibrium impulse responses can end up having non-zero net present value aggregate output implications, and relate it to the benefit of frontloading macroeconomic policies. The intuition there is that, when an economy is determinate, a policy that initially has a positive effect on aggregate demand but has a negative effect later on–such as a 'cash for clunkers' program of temporary subsidies to car purchases–might nevertheless be a net positive because the later bust will be tempered by the rise in income occurring in the earlier period, boosting spending later on and offsetting the negative impulse. Finally, by exhibiting two benchmarks for fiscal and monetary policy, we show that our results allow us to quantify precisely when heterogeneity matters for the aggregate effect of policy, shedding new light on an important question in the literature.

We illustrate all of our results in the context of an heterogeneous agent New Keynesian model, belonging to the new state of the art vintage of models used for policy analysis that are consistent with marginal propensities to consume observed in the data. We first show that the economy is determinate under constant real interest rates, provided income risk is not too countercyclical. If income risk becomes very countercyclical, however, agents expecting future falls in income start reducing their consumption today, an outcome that can become self-fulfilling. This relates our finding to those of Ravn and Sterk (2017), among others, who have shown the possibility of self-fulfilling increases in unemployment in a model with explicit search and matching. We next show that the Taylor principle is weakened in our baseline environment, relative to the representative agent case, and relate this to the amount of liquidity issued by the government. In the limit case of perfect liquidity, we recover the representative agent benchmark of a responsiveness of 1, while the economy can be determinate even under constant nominal interest rates in the limit as liquidity vanishes. We then study the magnitude of government spending multipliers, a topic that has received tremendous attention in the literature. We first establish a benchmark result for a balanced budget multiplier of 1 under the constant-r case, under a specific but natural assumption about tax adjustment. This generalizes Woodford (2011)'s landmark result to the case with heterogeneous agents, providing a useful benchmark for the quantitative literature going forward (see for example Hagedorn, Manovskii and Mitman (2017)). We show that, relative to this benchmark, multipliers are increased when the government delays levying the taxes. We relate this to our frontloading result: delaying taxes implies an additional fiscal stimulus that translates into a positive equilibrium output effect, and show that this effect can be quantitatively large–a stark departure from ricardian equivalence. Finally, we study monetary policy, comparing our results to the benchmark 'as if' case of Werning (2015). Contemporaneous monetary policy shocks are amplified relative to the representative agent case. This is almost entirely due to the income effect on consumption–those who gains from current fall in interest rates have higher marginal propensities to consume than those who lose—validating the result in Auclert (2017). Finally, we study forward guidance, and show that the effects are attenuated. This validates the result in McKay, Nakamura and Steinsson (2016), but for a different reason: those who gains from future falls in interest rates tend to be constrained and cannot increase their consumption today in anticipation of lower future payments, while those who lose are less constrained and therefore more responsive.

Related literature. Our paper relates to several strands of the literature. First, it is related to a recent literature on Heterogeneous Agent New Keynesian Models, which has studied, inter alia, monetary policy and forward guidance (McKay and Reis (2016), Werning (2015), Kaplan, Moll and Violante (2016)), fiscal multipliers (Hagedorn et al. (2017)), or the role of precautionary savings in amplifying fluctuations (Ravn and Sterk (2013), den Hann, Rendahl and Riegler (2015), Bayer, Lütticke, Pham-Dao and von Tjaden (2015), Challe, Matheron, Ragot and Rubio-Ramirez (2014), Heathcote and Perri (2016)). We show that the that the $\mathbf{M}$ matrix encapsulates all that is needed to compare across these models. Second, our determinacy results relate to a long literature that has studied determinacy in New Keynesian models. In standard representative agent results (Woodford (2003), Bullard and Mitra (2002)), the Taylor principle is independent of features of the economic environment, except the slope of the Phillips curve and the discount factor of households. Recently, limited forms of Taylor principles have been obtained for economies with richer heterogeneity, eg in Galí, López-Salido and Vallés (2007), Bilbiie (2008), Coibion and Gorodnichenko (2011), or Ravn and Sterk (2017), the results depend on deeper features of the economy environment.Third, we related to a literature on sufficient statistics for heterogeneous agent models (Auclert (2017), Berger et al. (2015), Auclert and Rognlie (2016)). These papers have obtained sufficient statistics for partial equilibrium effects ($\partial \mathbf{Y}$). Our key innovation here is to pave the way for taking this methodology to the general equilibrium, by obtaining an $\mathbf{M}$ that is theoretically observable under the constant-real rate policy. Finally, our network related results relate to a literature on network theory and business cycles, in the tradition of Long and Plosser (1983), or more recently Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi (2012). An important distinction between this literature, which models production networks with intermediate inputs, and our results is that they work with some fixed factor of production rendering the Leontieff matrix $I - \mathbf{M}$ invertible, whereas in our case it is generally not invertible. Instead, we make heavy use of the theory of infinite dimensional Markov chains, as covered in Kemeny, Snell and Knapp (1976), to obtain solutions to our leading equation.

2 Baseline model

In this section, we introduce a benchmark Heterogeneous Agent New Keynesian model that we will use for our numerical illustrations throughout the paper. This is one of many models that can be characterized by an Intertemporal Keynesian Cross, as we discuss further in the next section.

Model setup Time is discrete and runs from $t = 0$ to $\mathcal{T}$, with the infinite horizon case $\mathcal{T} = \infty$ having special interest. The economy is populated by a continuum measure 1 of ex-ante identical agents who face no aggregate uncertainty, but may face idiosyncratic uncertainty. Individuals can be in various idiosyncratic ability states e_i, and transition between those states according to a Markov process with fixed transition matrix Π. We assume that the mass of worker type i in idiosycratic state e_i is always equal to $\pi(e_i)$, the probability of e_i in the stationary distribution of Π. Ability levels are normalized to be one on average: $\mathbb{E}_e[e] = \sum_{e_i} \pi(e_i) e_i = 1$.

Agents. Agents have time-0 utility over consumption and labor supply plans given by separable preferences

$$\mathbb{E}\left[\sum_{t\geq 0}\beta^t\left\{u_t\left(c_{it}\right)-v\left(n_{it}\right)\right\}\right] \tag{3}$$

Each period t, an agent with incoming real wealth a_{it-1} and realized earnings ability e_{it} enjoys the consumption of a generic consumption good c_{it} and gets disutility from working n_{it} hours. Consumption goods have price p_t and the nominal wage per unit of ability is w_t. The agent receives a lump sum transfer from the government t_t, pays a marginal tax rate τ_t on labor income, can trade in real bonds that deliver a net real interest rate r_t, and faces a borrowing constraint. Specifically, his budget constraint in period t in units of date-t consumption goods is

$$c_{it}+a_{it} = \left(1+r_t\right)a_{it-1}+t_t+\left(1-\tau_t\right)\frac{w_t}{p_t}e_{it}n_{it} \quad \forall t,i \tag{4}$$

$$a_{it} \geq \underline{a_t} \tag{5}$$

The agent maximizes (3) by choice of c_{it} and a_{it}, subject to (4), the borrowing constraint (5), and, when $\mathcal{T}<\infty$, a terminal condition $a_{i\mathcal{T}}=0$. By contrast, due to frictions in the labor market, the agent is restricted to supply n_{it} hours. Hence the agent takes total net of tax income $z_{it} = t_t+\left(1-\tau_t\right)\frac{w_t}{p_t}e_{it}n_{it}$ as given. From the household's perspective, therefore, equation (4) simplifies to

$$c_{it}+a_{it}=\left(1+r_t\right)a_{it-1}+z_{it} \quad \forall t,i \tag{4'}$$

Maximization of (3) subject to (4') and (5) constitutes the core of the standard incomplete market model of consumption and savings.

Labor market. Following standard practice in the New Keynesian sticky-wage literature, labor hours n_{it} are determined by aggregate union labor demand: when employment is l_t, households must supply

$$n_{it}=\Gamma\left(e_{it},l_t\right)$$

As in Werning (2015) and Auclert and Rognlie (2016), the Γ function—which we refer to as the gross incidence function—allocates hours among heterogeneous households as a function of aggregate employment. That function respects aggregation, in that skill-weighted hours worked are always equal to aggregate labor demand

$$\mathbb{E}_e\left[e\Gamma\left(e,l\right)\right]=l \quad \forall l$$

Households are off their labor supply curves because the nominal wage w_t is partially sticky. In appendix we formalize this wage stickiness by assuming that that there is a continuum of unions $j\in[0,1]$, each providing a union-specific task l_{jt} and employing each of the workers, and each only occasionally resetting their wage in a Calvo fashion. In a steady-state with constant employment

l, this formulation implies that the real wage $\frac{w^*}{p^*}$ is at a markup over some average across all households of the marginal rate of substitution between consumption and hours.

Outside of steady-state, we make the assumption that wage resets take place under the assumption that the consumption distribution remains at its steady-state level. We then show that this problem leads to a simple New Keynesian Phillips curve for aggregate wage growth, $\pi_t^w \equiv \log\left(\frac{W_t}{W_{t-1}}\right)$, given to first order by

$$\pi_t^w = \lambda_t \left(\frac{1}{\psi}\widehat{l_t} - (w_t - p_t)\right) + \beta_t \pi_{t+1}^w \tag{6}$$

where $l_t \equiv \log(l_t/l)$, ψ is a constant, and κ_t, β_t are deterministic functions of time, with $\beta_t = \beta$ when $\mathcal{T} = \infty$.

Our assumption that the consumption distribution remains at its steady-state level when wages are reset implies that, outside of steady state, wealth effects on labor supply are absent. This is important to obtain some of our results mapping partial to general equilibrium effects—a point we return to further in section 3.6.

Production of final goods. Firms operate a simple, time-invariant technology

$$y_t = l_t \tag{7}$$

They are perfectly competitive and set prices flexibly, so that the final goods price is given by

$$p_t = w_t \tag{8}$$

and profits are 0, justifying why no dividends enter households' budget constraints in (4).

Hence, goods price inflation $\pi_t \equiv \log\left(\frac{P_t}{P_{t-1}}\right)$ and aggregate wage inflation are identical at all times, $\pi_t = \pi_t^w$, and the Philips curve (6) can also be written as

$$\pi_t = \kappa_t \widehat{y_t} + \beta_t \pi_{t+1} \tag{9}$$

where $\kappa_t = \frac{\lambda_t}{\psi}$ and $\widehat{y_t} \equiv \log(y_t/y)$, where y in steady state output.

Government. In each period, the government has rule for its outstanding amount of real debt $b_t = \overline{b_t}$, and spending spending $g_t = \overline{g_t}$. It then adjusts a combination of the lump-sum t_t and the marginal tax rate τ_t so as to satisfy its budget constraint

$$\tau_t \frac{w_t}{p_t} l_t - t_t = (1 + r_t)\, b_{t-1} + g_t - b_t \equiv rev_t \tag{10}$$

where rev_t represents net government revenue. Specifically, we assume that there exists a sequence τ_t^r and a constant φ such that tax revenue and the lump-sum rebate are, respectively,

$$\tau_t \frac{w_t}{p_t} l_t = \tau_t^r y_t + (1-\varphi)\, rev_t \tag{11}$$

$$t_t = \tau_t^r y_t - \varphi \cdot rev_t \tag{12}$$

At time t, $\tau_t^r \in [0,1]$ corresponds to the share of GDP that the government would use for transfers to households if it did not require revenue to pay for spending or interest on the debt. In addition, $\varphi \in [0,1]$ specifies how the government adjusts taxes vs transfers at the margin when it requires an extra unit of revenue. $\varphi = 0$ correponds to the case where it only raises taxes, $\varphi = 1$ where it only lowers transfers. When $\tau_t^r = \varphi$, all changes to rev_t affect households' *net-of-tax incomes* $t_t + (1-\tau_t) \frac{w_t}{p_t} e_{it} n_{it}$ in proportion, irrespective of their skill level e_{it}.

Monetary policy sets the nominal interest rate i_t by following one of two rules described below. Given the path for good prices p_t, the real interest rate at t (the price of date-$t+1$ goods in units of date-t goods) is then equal to

$$1 + r_t \equiv (1 + i_t) \frac{p_t}{p_{t+1}} \tag{13}$$

The first rule we consider is a constant rule for the real interest rate

$$r_t = \bar{r}_t \tag{Constant-r}$$

Such a rule has proved useful in the equilibrium analysis of both representative- and heterogeneous-agent New Keynesian models,[3] and we will show that it facilitates simple results in our context as well. We will also consider the consequences of assuming that monetary policy follows a conventional Taylor rule

$$i_t = \bar{i}_t + \phi_\pi \pi_t + \phi_y \widehat{y}_t \tag{Taylor rule}$$

where, at this stage, ϕ_π and ϕ_y are unrestricted, and we provide conditions for equilibrium determinacy in section 5.2.

Definition 1. Given exogenous sequences for preferences $\{u_t\}$, borrowing constraints $\{\underline{a_t}\}$, fiscal policy $\left\{\overline{b_t}, \overline{g_t}, \tau_t^r\right\}$, and monetary policy $\{\overline{r_t}, \overline{i_t}\}$, a *general equilibrium* in this economy is a path for prices $\{p_t, w_t, r_t, i_t\}$, aggregates $\{y_t, l_t, c_t, b_t, g_t, t_t\}$, individual allocations rules $\{c_t(a,e), n_{it}\}$, and joint distributions over assets and productivity levels $\{\Psi_t(a,e)\}$, such that households optimize, unions optimize, firms optimize, monetary and fiscal policy follow their rules, and the goods and bond markets clear:

$$g_t + \int c_t(a,e)\, d\Psi_t(a,e) = y_t \tag{14}$$

$$\int a\, d\Psi_t(a,e) = b_t \tag{15}$$

[3] See for example Woodford (2011) or McKay et al. (2016),

Parameters	Description	Value
ν	Elasticity of intertemporal substitution	0.5
β	Discount factor	0.962
r	Real interest rate	4%
π	Inflation rate	0%
b/y	Government debt to GDP	140%
$\underline{a}$	Borrowing constraint	0
g/y	Government spending to GDP	20%
τ^r	Tax rate for lump-sum	17.5%
τ	Marginal tax rate	38%
κ	Slope of the Phillips Curve	0.1

Table 1: Calibrated parameters

This completes the description of our environment. The model cannot be solved analytically and one has to resort to numerical simulations for solutions. However, in the next section, we show that we can make conceptual progress in characterizing impulse responses from steady state.

Calibration. For our numerical simulations, we stay as close as possible to a standard calibration for a Huggett model. We consider the model in infinite horizon ($\mathcal{T} = \infty$), and assume that households have constant CES utility over consumption $u(c) = \frac{c^{1-\nu^{-1}}}{1-\nu^{-1}}$ with a standard value of $\nu = \frac{1}{2}$. We set a steady-state value for output of $y = 1$, a steady-state inflation rate of $\pi = 0$, and assume a slope for the Phillips curve of $\kappa = 0.1$. Conditional on these choices, the specific form of labor disutility $v(n)$ and the price rigidity parameter θ are unimportant.

We follow the recent incomplet markets literature in calibrating the gross income process to features of earnings changes from W2 data, and the tax function to features of the US tax-and-transfer system. In this spirit, our gross income process is the same as in Kaplan et al. (2016), while we follow Auclert and Rognlie (2016) and assume that the lump-sum transfer component of taxation represents $\tau^r = 17.5\%$ of GDP.

We assume that all individual hours change proportion to aggregate employment, $\Gamma(e,l) = l$, and set the marginal tax adjustment parameter to $\varphi = \tau^r$. Together, these two assumptions imply that changes in aggregate income l_t affect the net-of-tax incomes of all households in proportion. We refer to this as the *constant incidence* case. Werning (2015) established this case as a benchmark for the analysis of monetary policy, and we will show that it is a natural benchmark for the analysis of fiscal policy as well.

We follow McKay et al. (2016) and assume that households cannot borrow ($\underline{a} = 0$) and that government debt is $\frac{b}{y}$ =140% of output—an assumption meant to capture the amount of liquid assets in the US economy more broadly. We target an equilibrium real interest rate of $r = 4\%$, and assume that government spending is $\frac{g}{y} = 20\%$ of output. Together, these assumptions imply an overall marginal tax rate of $\tau = \tau^r + (1-\tau^r)\left(r\frac{b}{y} + \frac{g}{y}\right) = 38\%$ and a lump sum of $\frac{t}{y} = \tau^r\left(1 - \left(r\frac{b}{y} + \frac{g}{y}\right)\right) = 13\%$. Finally, we choose the discount factor β so as to achieve our target real interest rate in the economy at steady state. Table 1 summarizes our calibration.

3 Deriving the Intertemporal Keynesian Cross

We assume that the economy starts in a given steady state with $r > 0$. We index all sequences that are exogenous in our definition of general equilibrium by a single parameter ϵ. Our objective is to understand the equilibrium response of the economy, starting from steady state, to a small change in this ϵ. This encompasses a broad class of shocks that these models enable us to study: shocks to preferences (u_t), borrowing constraints ($\underline{a_t}$), fiscal policy ($\overline{b_t}, \overline{g_t}, \tau_t^r$) and monetary policy ($\overline{r_t}, \overline{i_t}$). Note further that this formalism captures both unexpected shocks at date 0 (for example $\overline{r_0} > 0$, a contemporaneous monetary policy shock) as well as expected shocks that agents anticipate ahead of time (for example $\overline{r_T} > 0$ for $T > 0$, a forward guidance shock).

In a setup such as ours, the shocks we consider all have complex general equilibrium effects. This complexity mainly stems from intertemporal feedbacks: a shock to income in the future reverberates, via agents' consumption smoothing decisions, to the entire path of consumption. Changes in real interest rates—as induced by endogenous inflation or responses of monetary policy—have a similar feature: they elicit complex patterns of intertemporal substitution responses, as well as income effects that are both difficult to characterize theoretically and hard to map to data. When these feedback mechanisms are strong enough, the model may even feature multiple equilibria.

We break through this complexity by defining a simpler notion of equilibrium which we call the *exogenous y equilibrium* (or *EYE*) that shuts down these effects in response to the shocks we consider. We show that this definition nests a simple notion of *partial equilibrium* (PE) that the literature has considered. This PE effect is conceptually a lot simpler, and is an object that can conceivably be measured in data following the lead of a recent 'sufficient statistic' literature.

Next, we consider how the general equilibrium (GE) effect relates to the PE effect. We show that, to first order, the differences between PE and GE are captured by a simple linear mapping summarizing all the intertemporal feedbacks described above. We characterize how the **M** matrix varies under different policy rules, and show that they relate to households' marginal propensities to consume out of income and interest rates. This opens up the possibility of mapping the **M** matrix to observables as a key element of model validation.

We finally briefly illustrate the usefulness of this approach for model comparison by showing how alternative models can all be summarized through the **M** matrix, before turning to the solution of the model in the next section.

3.1 The exogenous y equilibrium (EYE)

We start by defining our notion of an exogenous y equilibrium.

Definition 2. Given a value of ϵ, and a path for output supply $\{y_t\}$, an ***exogenous y equilibrium* (EYE)** is a path for prices $\{p_t, w_t, r_t, i_t\}$, aggregates $\{y_t, l_t, c_t, b_t, g_t, t_t\}$, individual allocations

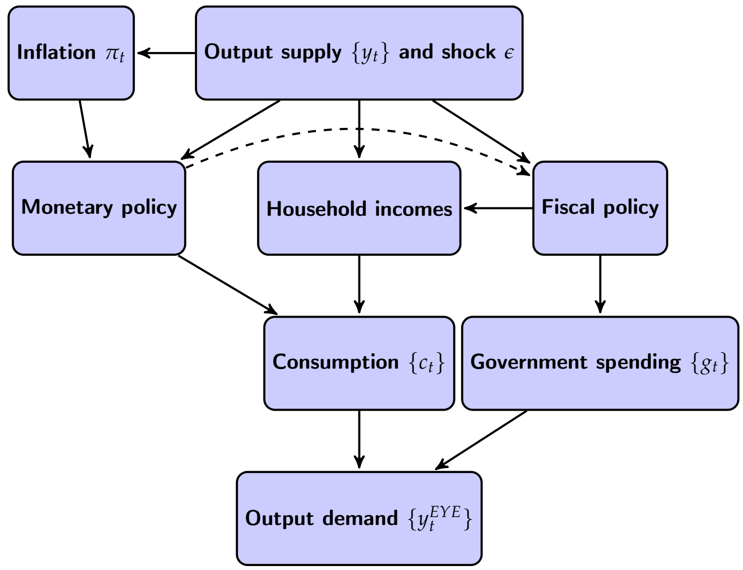

Figure 2: Uniqueness of EYE mapping $y^{EYE}(\{y_t\})$

rules $\{c_t(a,e), n_{it}\}$, and joint distributions over assets and productivity levels $\{\Psi_t(a,e)\}$, such that households optimize, unions optimize, firms optimize, and monetary and fiscal policy follow their rules. We define $y_t^{EYE} \equiv c_t + g_t$ as the resulting path of output demand.

Definition 2 drops the goods market clearing condition $y_t = c_t + g_t$ and instead assumes that the path of y_t is given. This shuts off most of the complex intertemporal feedbacks in the model, since there is now a simple directional flow from a path of output supply $\{y_t\}$ to a path of aggregate output demand $\{y_t^{EYE}\}$. The following lemma shows formally that there exists such a mapping.

Lemma 1. *There is a unique path for output demand $\{y_t^{EYE}\}$ corresponding to a given path for output supply $\{y_t\}$ and shock ϵ. This defines a mapping $y_t^{EYE}(\{y_t\};\epsilon)$.*

The proof of lemma 1 is illustrated in figure 2. Given ϵ and a path for output supply, firm optimization implies that employment is $l_t = y_t$, by (7). Given this, union optimization implies an updated path for aggregate nominal wages $\{w_t\}$ and individual employment $\{n_{it}\}$. In turn, final good firm optimization determines the path for prices $p_t = w_t$, and therfore inflation $\{\pi_t\}$. Next, given $\{y_t, \pi_t\}$ and the shock sequence, we obtain the paths for real interest rates $\{r_t\}$ and the taxes $\{t_t, \tau_t\}$ that households face. Together, these deliver all of the elements that enter as an input into households' decision problem: a state-by-state path for aftertax incomes and a path for real interest rates. Households optimally choose their consumption plans given these paths, and aggregating those decisions using the wealth distribution at every point we obtain the partial equilibrium path $\{c_t\}$, while using the fiscal rule we obtain the path for government spending $\{g_t\}$. Summing the two, we obtain output demand y_t^{EYE}.

The only difference between definitions 1 and 2 is the requirement of goods market clearing. We therefore obtain the following lemma.

Lemma 2. *A general equilibrium is an exogenous y equilibrium that satisfies the additional requirement that goods market clear at every point in time, that is, a path* $\{y_t\}$ *that solves the system of equations*

$$y_s = y_s^{EYE}\left(\{y_t\};\epsilon\right) \quad \forall s \tag{16}$$

When (16) holds for all t, it follows from Walras's law that (15) holds for all t as well.

A key object for our analysis will be that of an EYE that has constant output equal to the steady state value $\{y_t\} = \{y\}$. We define the sequence $\{\partial y_t\}$ as

$$\partial y_t \equiv \frac{\partial y_t^{EYE}\left(\{y\},\epsilon\right)}{\partial \epsilon} d\epsilon \tag{17}$$

In the next section, we relate this to the partial equilibrium household problem. In the following subsection we then show the sense in which ∂y_t is sufficient for the general equilibrium effect.

3.2 Partial equilibrium interpretation

A large literature has studied partial equilibrium household problems. We adapt this standard definition to our context below.

Definition 3. Given exogenous sequences for preferences $\{u_t\}$, borrowing constraints $\{\underline{a_t}\}$, real interest rates $\{r_t\}$ and a stochastic process for net incomes $\{z_t(e)\}$, a *partial equilibrium for households* is a set of individual allocations $\{c_t(a,e)\}$ and joint distributions over assets and productivity levels $\{\Psi_t(a,e)\}$, such that households optimize, and the evolution of Ψ_t is consistent with household policies. We write aggregate consumption as $c_t^{PE} \equiv \int c_t(a,e)\, d\Psi_t(a,e)$.

For each of our shocks, we can now describe ∂y_t in terms of a particular c_t^{PE} response to a change in inputs.

Proposition 1. *For each of the following shocks, starting from steady state, we have*

$$\partial y_t = dc_t^{PE} + d\overline{g_t}$$

where $d\overline{g_t}$ *is the perturbation to government spending and* dc_t^{PE} *is the first order response of partial equilibrium consumption to the following changes:*

a) For preferences $\{u_t\}$ *or borrowing constraints* $\{\underline{a_t}\}$*, to the shocks themselves*

b) For fiscal policy $\left\{\tau_t^r, \overline{g_t}, \overline{b_t}\right\}$*, to a change in net labor income*

$$dz_{it} = y\left(1-\omega_{it}\right) d\tau_t^r - \left(\left(1-\varphi\right)\omega_{it} + \varphi\right) drev_t \tag{18}$$

where $\omega_{it} \equiv \frac{e_{it} n_{it}}{y} = \frac{e_{it}\Gamma(e_{it},y)}{y}$ *is i's relative gross income (with* $\mathbb{E}_I[\omega_{it}] = 1$*), and* $drev_t = (1+r_t)\,d\overline{b_{t-1}} + d\overline{g_t} - d\overline{b_t}$ *is the change in required government revenue induced by the new fiscal path*

c) *For monetary policy* $\{\overline{i_t}, \overline{r_t}\}$*, to a change in real interest rates* $dr_t = d\overline{r_t}$ *or* $dr_t = d\overline{i_t}$*, and a change in net labor income (18), where* $drev_t = \overline{b_{t-1}}dr_t$ *is the change in required government revenue induced by the new path for interest rates*

Proposition 1 shows that, for shocks that affect the household problem directly, the EYE response ∂y_t corresponds exactly to the traditional partial equilibrium consumption demand response dc_t^{PE}. For fiscal shocks, EYE includes two components: the direct change in government spending $\overline{dg_t}$, if any, and the partial equilibrium consumption response to the change in taxes. For shocks to the monetary rule, EYE combines two effects on household partial equilibrium consumption: the effect of changing r_t, and the effect of changing taxes due to the government's changing interest obligations.

This reveals a practical advantage of EYE: calculating ∂y_t requires only the ability to calculate partial equilibrium consumption responses, sidestepping the complexity of general equilibrium. Since only such computations are involved, we choose to refer to ∂y_t as the **partial equilibrium effect** of a shock. For a fiscal or monetary shock, this concepts includes the consumption impact of the tax changes required to balance the government budget.

The major advantage of adopting this definition of a partial equilibrium effect is that $\{\partial y_t\}$ will turn out to be sufficient to determine the general equilibrium output effect of any shock—something that alternative definitions could not deliver. Under our definition, all partial equilibrium effects also share the following unifying property:

Lemma 3. *All partial equilibrium effects* ∂y_t *have zero present value, that is,*

$$\sum_{t=0}^{\mathcal{T}} \frac{\partial y_t}{(1+r)^t} = 0 \tag{19}$$

The proof of lemma 3, in appendix C.1.2, follows from the fact, in our definition of EYE, all agents respect their budget constraints. By our assumption of constant y, the shocks we consider do not on their own generate any additional income in the aggregate, even though usually are redistributing between agents. Therefore, they cannot lead to a change in the present value of aggregate spending, even though they can affect its path over time.

Lemma 19 relates the popular idea that fiscal stimulus programs such as cash for clunkers "steal demand from the future". With our notion of EYE this is true of *all* shocks, including redistributive shocks, government spending shocks and monetary policy shocks.

Note finally that ∂y_t has a simple interpretation as the aggregate goods demand impulse to shock in a small open economy (interpreting changes in monetary policy as changes in the world interest rate), provided the economy starts out in a steady state with a zero net international asset position.

Empirical moments for partial equilibrium effects. An important benefit of the partial equilibrium concept is that, since all intertemporal effects have been shut down, it is either measurable directly (as in the case of a self-financing government spending change), or corresponds to simple data moments. For example, Auclert and Rognlie (2017a) show that, for a redistributive tax rate shock $d\tau_0^r$ taking place at date 0,

$$\partial y_t = \text{Cov}_I\left(mpc_{i0t}, z_{i0}\right)\frac{d\tau_r^0}{1-\tau_r^0} \tag{20}$$

where mpc_{i0t} is agent i's average marginal propensity to consume at date t for shocks received at date 0, and z_{i0} is agent i's net income at date 0. Conveptually (20) is measurable in data containing information on marginal propensities to consume and incomes. Relatedly, Auclert (2017) shows sufficient statistics for the effect of a time-0 monetary policy shock, Guerrieri and Lorenzoni (2017) have one for the effect of a deleveraging shock, and Berger et al. (2015) for the effect of a house price shock.[4] These empirical moments promise discipline for general equilibrium models, but share the property that they all correspond to partial equilibrium effects. We next show the sense in which partial equilibrium effects are sufficient for the general equilibrium effect.

3.3 General equilibrium and the Intertemporal Keynesian Cross

Define $y_t(\epsilon)$ to be a function mapping each shock ϵ to a general equilibrium. (If there are multiple equilibria for a given ϵ, this can be any differentiable selection). Write $dy_t \equiv \frac{dy_t}{d\epsilon}d\epsilon$. We can now relate $\{dy_t\}$ to the partial equilibrium $\{\partial y_t\}$ from the previous section.

Totally differentiating (16), we obtain

$$dy_s = \partial y_s + \sum_{t=0}^{\mathcal{T}} m_{s,t}dy_t \quad \forall s \tag{21}$$

where

$$m_{s,t} \equiv \frac{\partial y_s^{EYE}\left(\{y\},\epsilon\right)}{\partial y_t}$$

For any given $\{\partial y_s\}$, a general equilibrium $\{dy_t\}$ is a solution to the $\mathcal{T}\times\mathcal{T}$ linear system in (21).

We can prove the following analogue of lemma 3:

Lemma 4. *For any t, the present value of $m_{s,t}$ discounted to date t equals one:*

$$\sum_{s=0}^{\mathcal{T}} \frac{m_{s,t}}{(1+r)^{s-t}} = 1 \tag{22}$$

[4]House prices shocks are outside of the scope of our model, but many papers study their effect on the aggregate demand for consumption without modeling the general equilibrium feedbacks between income and consumption.

As with lemma 21, this comes from the observation that all income earned in the economy is spent at some point in time. Incrementing y_t by one therefore also increases the date-t present value of goods demand across all periods by one.

The Intertemporal Keynesian Cross. Our main result, the Intertemporal Keynesian Cross, is an expression of equation (21) in vector form. Since lemmas 3 and 4 will prove to be important, it is convenient to rescale quantities so that they have especially simple vector interpretations. We therefore write $dY_t \equiv \frac{1}{(1+r)^t} dy_t$, $\partial Y_t \equiv \frac{1}{(1+r)^t} \partial y_t$, and $M_{s,t} \equiv \frac{1}{(1+r)^{t-s}} m_{s,t}$. All other capital letters will refer to present value concepts as well.

Define the $\mathcal{T}$-dimensional vector $\partial \mathbf{Y} \equiv (\partial Y_0, \partial Y_1, \cdots)$ and the $\mathcal{T} \times \mathcal{T}$ matrix $\mathbf{M} \equiv (M_{s,t})$, recalling that $\mathcal{T}$ can be infinite. The system of equations (21) delivers our main proposition.

Proposition 2 (The Intertemporal Keynesian Cross.)**.** *To first order, for any shock, impulses to partial equilibrium output $\partial \mathbf{Y}$ and general equilibrium output $d\mathbf{Y}$ satisfy*

$$d\mathbf{Y} = \partial \mathbf{Y} + \mathbf{M} d\mathbf{Y} \tag{23}$$

The matrix $\mathbf{M}$ is column stochastic (it has a left eigenvector with eigenvalue 1, $\mathbf{1}'\mathbf{M} = \mathbf{1}'$), and the vector $\partial \mathbf{Y}$ has mean zero, $\mathbf{1}'\partial \mathbf{Y} = 0$.

We are interested in characterizing the solution(s) to equation (23). Lemma 1 implies that for any shock, $\partial \mathbf{Y}$ is unique. Further, it implies that if $d\mathbf{Y}$ is unique, all equilibrium objects locally also are. The question of local equilibrium muliplicity is therefore reduced to the question of whether, for a given $\partial \mathbf{Y}$, there is a unique solution $d\mathbf{Y}$ to equation (23). Characterizing when this is true is a major topic of sections 4 and 5 of this paper. For now, we observe:

Corollary 1. *When equilibrium is locally unique, $\partial \mathbf{Y}$ is **sufficient** for $d\mathbf{Y}$: shocks that have the same partial equilibrium effect also have the same general equilibrium effect. Moreover, there exists a unique matrix $\mathcal{G}$, invariant across shocks, such that*

$$d\mathbf{Y} = \mathcal{G} \partial \mathbf{Y}$$

We call (23) the Intertemporal Keynesian Cross because it relates any impulse to aggregate demand $\partial \mathbf{Y}$ to its general equilibrium effect $d\mathbf{Y}$. The fact that there exists a simple mapping is very important: it implies that all shocks affect aggregate output *through* their partial equilibrium effect, irrespective of the source of that shock. Clearly, the matrix $\mathcal{G}$ relates to $\mathbf{M}$, even though inverting $I - \mathbf{M}$ is generally infeasible.

What is the benefit of this approach? It allows us to split conceptually the analysis of general equilibrium into two parts. First, how does the policy affect the economy in partial equilibrium? ($\partial \mathbf{Y}$) Second, how do partial equilibrium effects translate into general equilibrium effects? ($\mathcal{G}$). It shows clearly why the underlying mechanisms of adjustment from partial to general equilibrium are common, once partial equilibrium is defined in the way that we do. This provides useful discipline, both for empirical work and for theoretical research going forward.

Network interpretation. Since **M** is a Markov chain, we can interpret equation (23) as an equation for flows on a network

$$dY_s = \partial Y_s + \sum_{t=0}^{\mathcal{T}} M_{s,t} dY_t \quad \forall s \tag{24}$$

In this network, nodes are time periods, dY_t are values of demand at each node, and $M_{s,t} \equiv \frac{\partial y_s^{EYE}(\{y\},\epsilon)}{(1+r)^{s-t}\partial y_t}$ is the economy's aggregate marginal propensity to spend out at date s out of aggregate income at date t, which represents the strenght of flows across the network, with lemma 22 showing that all flows are conserved, ie, $\sum_{s=0}^{\mathcal{T}} M_{s,t} = 1$. Equation (24) then says that the value of demand at node s must be equal to the exogenous value ∂Y_s plus the sum of entering flows from other periods. We will come back to this useful analogy when discussing the solution in section 4.

3.4 Structure of the M matrix and relation to empirical MPCs

We now turn to the relationship between the **M** matrix and determinants of household behavior and policy.

The M matrix under a constant-r rule. We first characterize the **M** matrix under the assumption that the monetary policy rule enforces an exogenous path for the real interest rate.

Lemma 5. *Assume that monetary policy follows the rule (Constant-r). Then $\mathbf{M} = \mathbf{M}^Y$, where the $\mathbf{M}^Y$ matrix is defined as*

$$M_{s,t}^Y \equiv \frac{\mathbb{E}_I\left[mpc_{ist}\right]}{(1+r)^{s-t}} \tag{25}$$

where $mpc_{ist} \equiv \frac{\mathbb{E}_0[\partial c_{is}]}{\partial y_t}$ is i's average marginal propensity to consume income received at date s.

The reason why the **M** matrix is so simple under the assumption of constant real interest rates is that the only way in which an aggregate income change affects aggregate spending is through the spending patterns of agents who receive that extra income. By contrast, if income changes generate endogenous changes in real interest rates, they will elicit aggregate income and substitution effects. We turn to this more complex case next.

For our benchmark economy, we compute the $\mathbf{M}^Y$ matrix numerically, and plot it in figure 3. Consider first the term $M_{0,0}^Y$ is the upper left corner. According to (25), this term is

$$M_{0,0}^Y = \mathbb{E}_I\left[\frac{\partial c_{i0}}{\partial y_0}\right] = \mathbb{E}_I\left[\frac{\partial c_{i0}}{\partial z_{i0}}\frac{\partial z_{i0}}{\partial y_0}\right] \tag{26}$$

where z_{i0} is the net income of agent i at date 0. Equation (26) shows that $M_{0,0}^Y$ is the average marginal propensity to consume out of individuals out of net income, $\frac{\partial c_{i0}}{\partial z_{i0}}$, weighted by the incidence of aggregate income for individual net income, $\gamma_{i0} \equiv \frac{\partial z_{i0}}{\partial y_0}$. In our benchmark case of equal incidence of net incomes $n_{i0} = l_0 = y_0$, it is easy to show that $\gamma_{i0} = \frac{z_{i0}}{\mathbb{E}_I[z_{i0}]}$. The intuition is that every additional dollar of GDP goes to the household sector: the government cuts the tax rate and

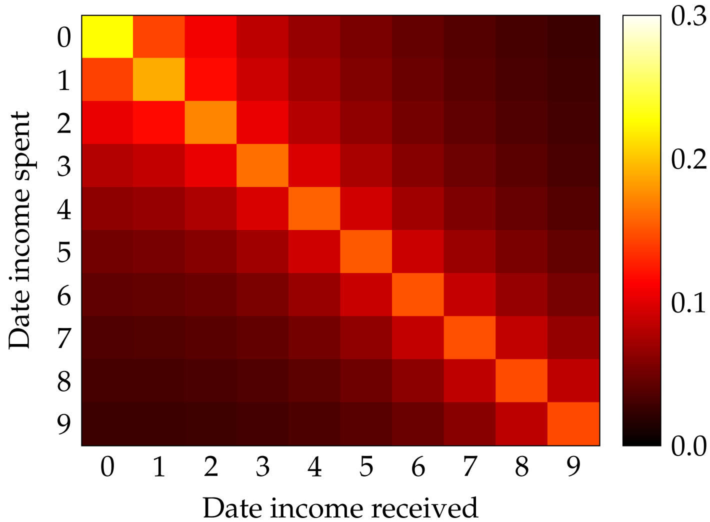

Figure 3: Numerical **M** matrix for our HANK economy

raises the lump-sum in such a way that all indidviduals are affected in proportion to their current income z_{i0}. Hence, in that case, $M^{Y}_{0,0}$ is just the net-income-weighted average of MPCs. In our calibration that number is 0.25, consistent with the empirical evidence of how households spend tax rebates.

The **M** matrix contains other terms, however. First, it tells us how a contemporaneous increase in GDP ends up being spent over time (the left column). Second, it tells us how future increases in GDP are spent. In this case, what matters is both future MPCs and expected incidence of a future increase in aggregate income.

A salient feature of this matrix is that it is close to diagonal and that, especially after the first few periods, columns tend to have the same structure shifted down along the diagonal. We say that the **M** matrix is asymptotically self-similar and will exploit this feature in section 5.1.

The M **matrix under a Taylor rule.** Under more standard rules for monetary policy such as (Taylor rule), the **M** matrix also reflects the way in which spending is shifted towards other dates to the endogenous reaction of real interest rates to aggregate income changes at a given point in time. Define the matrix Φ with element (s,t) given by $\frac{\partial r_s}{\partial \widehat{y}_t}$, capturing the effect that an increase in current income $\widehat{y}_t$ has on the real interest rate in period s. Assume further that $\mathcal{T}=\infty$, so that the Phillips curve (6) is simply

$$\pi_t = \kappa \widehat{y}_t + \beta \pi_{t+1} \tag{27}$$

Combining (Taylor rule), (27), and the approximation $i_t = r_t + \pi_{t+1}$ to the Fisher equation (13), we see that the matrix Φ has elements

$$\Phi_{s,t} \equiv \frac{\partial r_s}{\partial \widehat{y}_t} = \begin{cases} 0 & s > t \\ \left(\phi_y + \kappa\phi_\pi\right) & s = t \\ \left(\phi_\pi - \beta^{-1}\right)\beta^{t-s}\kappa & s < t \end{cases}$$

Output increases raise real interest rates contemporaneously by both the direct effect ϕ_y and the indirect effect due to inflation $\kappa\phi_\pi$. Future output increases affect both inflation both today and tomorrow. The former effect leads to an increase in nominal interest rates due to the Taylor rule, while the latter leads to a decline in real interest rates via the Fisher equation. We can then use the chain rule to obtain:

Lemma 6. *Assuming that monetary policy follows (Taylor rule), then*

$$\mathbf{M} = \mathbf{M}^Y + \mathbf{M}^R\Phi$$

where

$$M^R_{s,t} \equiv \frac{1}{(1+r)^{s-t}}\mathbb{E}_I\left[mpcr_{ist}\right]$$

and $mpcr_{ist} \equiv \frac{\mathbb{E}_0[\partial c_{is}]/y}{\partial r_t}$ *is i's average consumption response at date s for small changes in real interest rates at date t.*

The $\mathbf{M}$ matrix now captures both the direct household spending reaction, as well as the indirect spending reaction induced by a change in real interest rates via the endogenous monetary policy rule. Since the household sector intermediates the effect of changes in real interest rates on aggregate demand, its matrix of reactions to real interest rates $\mathbf{M}^R$ now matters. Note that changes in real interest rates are just another type of partial equilibrium shock, and so all columns of $\mathbf{M}^R$ sum to 0. Therefore, conservation of flows is still satisfied.[5]

Using the M matrix for model validation. Lemmas 6 and 5 show the centrality of the matrix of marginal propensities to consume (contained in $\mathbf{M}^Y$) in determining the general equilibrium $\mathbf{M}$ matrix. Together with $\partial\mathbf{Y}$, for which sufficient statistics already exist, the $\mathbf{M}^Y$ matrix is therefore a key determinant of equilibrium. While the literature already understands that it is important for aggregate models of fiscal and monetary policy to match average marginal propensities to consume as estimated in data (i.e., $M^Y_{0,0}$), these lemmas show that it is also important, in principle, to match data on how steeply these MPCs decay over time after agents receive transfers, as well as their anticipated spending ahead of transfers.

To date, limited empirical evidence exists on such dynamic MPCs (see Broda and Parker (2014) for an example estimation following the 2008 tax rebates). Our theory shows why it is important

[5] Specifically, $\mathbf{1}'\mathbf{M} = 1'\mathbf{M}^Y + 1'\mathbf{M}^R\Phi = \mathbf{1}' + 0$

to gather more information on these, as they promise to discipline general equilibrium models going forward.[6]

3.5 The M matrix as a model comparison tool

Even though we have derived Proposition 2 in the context of a specific model, it should be clear that a similar equation holds in any model in which a function y^{EYE} can be defined. In particular, we can easily relax assumptions about preferences as well as population structure—for example, introducing life-cycle considerations, overlapping generations, many different permanent types of agents, alternative asset market structures (agents that can insure their idiosyncratic shocks or hand to mouth agents that have no access to any asset).The main important ingredient is budet constraints, which holds in all models. This allows us to use the **M** matrix as a tool to compare across models.

Representative-agent economy. An infinitely-lived representative agent economy in which $\beta(1+r) = 1$ has $\mathbf{M}^Y$ matrix

$$\mathbf{M}^Y = \begin{bmatrix} m & m & m & \cdots \\ m\beta & m\beta & m\beta & \cdots \\ m\beta^2 & m\beta^2 & m\beta^2 & \cdots \\ m\beta^3 & m\beta^3 & m\beta^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{28}$$

with $m = 1 - \beta$ is the marginal propensity to consume. This reflects the fact that the consumption of each agent obeys the permanent income hypothesis, so that its marginal propensity to consume is the same in every period, no matter when the income is received.

OLG economy. In appendix B.1 we describe an OLG economy, inhabited by young and old agents at a given point in time with discount factor β. We show that, under constant-R monetary policy, the M^Y matrix for that economy is

$$\mathbf{M}^Y = \begin{bmatrix} m & 0 & & & \\ 1-m & m & 0 & & \\ 0 & 1-m & \ddots & 0 & \\ \vdots & 0 & & m & 0 \\ 0 & \vdots & & 1-m & \ddots \end{bmatrix} \tag{29}$$

where $m \equiv \frac{1}{1+\beta}$ is the marginal propensity to consume of a young generation.

[6] Our theory also suggests the importance of gathering information on the time path of household responses to changes in interest rates, which is important for the transmission mechanism through endogenous monetary policy.

Hand to mouth agents. An important part of the New Keynesian literature has argued that the presence of hand-to-mouth agents, who just consume their income in every period, significantly alters the transmission mechanism of monetary and fiscal policy (see for example Galí et al. (2007) or Bilbiie (2008)). Consider an economy whose $\mathbf{M}$ matrix would be $\mathbf{M}^{Yn}$ without hand to mouth agents, and assume that there are a fraction μ of such agents, each with a unit elasticity to aggregate income $\Gamma(e, l) = l$. Then, the economy's $\mathbf{M}^{Y}$ matrix is

$$\mathbf{M}^{Y} = (1 - \mu)\,\mathbf{M}^{Yn} + \mu\mathbf{I} \tag{30}$$

For example, if the economy is a Two-Agent New Keynesian model (TANK) with permanent income and HtM agents, as in the literature cited above, then as the fraction of hand to mouth agents increase, the $\mathbf{M}^{Y}$ matrix moves away from the 'flat' pattern in (28) toward a diagonal pattern. Intuitively, this increases the amount of current-income amplification. Note, however, that the $\mathbf{M}^{Y}$ matrix depicted in figure 3 also contains positive elements close to the diagonal. There is a significant debate in the literature as to whether heterogeneous agent models such as the one we write down can be correctly approximated by two-agent models of this kind. This result suggests exactly what the TANK approximation misses: feedbacks from periods around the shocks, that agents without access to financial markets are insensitive to, while agents with some ability to smoothe across periods are able to respond to.

3.6 Discussion

As we have stressed in this section, our approach has a number of distinct advantages over a standard computational approach, which would be to simply solve for general equilibrium directly. It gives us insights into the equilibrium response, illustrating the common adjustment mechanisms across shocks. The $\mathbf{M}$ matrix can be used for model validation as well as for model comparison. In the next section we will also be able to show that it helps characterize when the model has a unique equilibrium.

On the other hand, this approach also comes at the cost of a number of assumptions. Some of these assumptions are common to general equilibrium models: agents satisfy budget constraints and interact via markets in which they fact the same prices. The most important limiting assumption we have made is the absence of wealth effects on labor supply: agents desired labor supply does not change as their income changes. This assumption had two parts: first, wage rigidities impose that agents do not choose how much they work in the short run, and second, unions do not take the consumption distribution into account when resetting wages, eliminating medium-run wealth effects on labor supply as well. By contrast, our assumption of wage rigidities can be easily relaxed, provided we maintain absence of wealth effects on labor supply.[7]

[7] GHH preferences and flexible prices would be an alternative, but this setup has problematic implications for aggregate multipliers (see Auclert and Rognlie (2017b)). We therefore prefer wage rigidites in this context.

4 Solving the IKC

Having derived the IKC in Proposition 2 in the context of our HANK economy and having discussed its applicability to wide range of alternative models in Section 3.5, we can now turn to its solution. Section 4.1 provides an existence result and Section 4.2 establishes conditions that ensure (local) determinacy of the solution.

To be widely applicable, this section only relies on the IKC (23) and not on any other aspects of the model. It is therefore essential to state all assumptions on the objects entering the IKC. We assume that $\mathbf{M}$ satisfies the assumptions of our column-stochastic matrix in $\mathbb{R}^{(\mathcal{T}+1)\times(\mathcal{T}+1)}$, $\sum_{t=0}^{\mathcal{T}} M_{t,s} = 1$, and that $\partial\mathbf{Y} \in \mathbb{R}^{T+1}$ has zero net present value (NPV), that is, $\sum_{t=0}^{\mathcal{T}} \partial Y_t = 0$. In order to be able to use tools from Markov chain theory, in this section, we further assume that $\mathbf{M}$ is non-negative $M_{t,s} \geq 0$.[8] We allow $\mathcal{T}$ to be either finite or infinite, in which case any sum is well-defined whenever each the associated sequence of partial sums converges. Similarly, the product of two infinite-dimensional matrices is well-defined whenever each element, itself an infinite sum, is well-defined. All proofs of results in this section are contained in Appendix C.

4.1 General solution of the IKC

Two main complications arise when trying to solve the IKC (23) for $d\mathbf{Y}$. First, the IKC cannot be solved by simply inverting $(1-\mathbf{M})$ to solve for $d\mathbf{Y}$ since $1-\mathbf{M}$ is generally not invertible: in many cases it has a right-eigenvalue of 1. For instance, when $\mathcal{T} < \infty$, this is an immediate consequence of the stochasticity of $\mathbf{M}$ (Lemma 4). This property of the IKC causes indeterminacy, which we will study in great detail below.

Second, an equation of the form of the IKC (23) does not necessarily admit any solution, even with the assumptions in place so far. For example, suppose the MPC matrix $\mathbf{M}$ is the identity matrix. In that case, no solution $d\mathbf{Y}$ exists whenever $\partial\mathbf{Y} \neq 0$. Clearly, such an MPC matrix would not be a very convincing description of an aggregate economy for it would imply that any additional income dY_t earned in some period t is entirely spent in that period. In other words, there is no money being spent *across* periods. We now introduce a restriction on the MPC matrix $\mathbf{M}$ that rules out such strict "within period" spending.

Assumption 1. $\mathbf{M}$ *is irreducible and noncyclic: (i) for each* $s,t \in \{0,\ldots,T\}$ *there exists an* $m \in \mathbb{N}$ *such that* $(\mathbf{M}^m)_{t,s} > 0$*; (ii) for each* $s \in \{0,\ldots,T\}$, $M_{ss} > 0$.

Assumption 1 has two parts. According to part (i), $\mathbf{M}$ needs to be such that, for any two periods s and t, an increase in aggregate income in period s will be partially spent in period t after m iterations. To give an example of $m > 1$, take a world in which additional aggregate income in period $t+1$ is spent in period t but not period $t-1$. Yet, since spending in period t is equal to income in period t, it is true—after two iterations—that some additional income in period $t+1$

[8] While this assumption is always verified under a constant-r monetary policy rule, and we have verified it numerically in other cases, we currently do not have a proof that it is always verified.

will be spent in period $t-1$ despite the lack of a direct link. According to part (ii), additional aggregate income earned in a given period will at least partially be spent in that same period.

In describing these properties, we use terminology from the theory of Markov chains. In fact, when T is finite, the two conditions in Assumption 1 establish the existence of a unique stationary distribution $\mathbf{v}$ of $\mathbf{M}$, satisfying $\mathbf{M}\mathbf{v} = \mathbf{v}$, and the convergence of $\mathbf{M}^k\mathbf{w}$ to the stationary distribution for an arbitrary intitial distribution $\mathbf{w}$, as $k \to \infty$.

Assumption 1 is verified in our benchmark model of section 2, as well as, for example, in the two-agent New Keynesian model described by (30), except in the limit case where the fraction of hand to mouth agents becomes $\mu = 1$.

As we mentioned, an equation like the IKC (23) has a natural degree of indeterminacy, deeply rooted in the stochasticity of $\mathbf{M}$: If $d\mathbf{Y}$ is a solution, then so is $d\mathbf{Y} + \mathbf{v}$ whenever $\mathbf{v}$ is a vector that satisfies $\mathbf{M}\mathbf{v} = \mathbf{v}$. In cases with a finite horizon, $T < \infty$, such a vector $\mathbf{v}$ can only be equal to the stationary distribution of $\mathbf{M}$ (up to scale). When the horizon is infinite, however, this is no longer true as there can be cases in which $\mathbf{v}$ can no longer normalized to sum to 1. We therefore define the more general concept of a "regular" vector.

Definition 4. A vector $\mathbf{v} \in \mathbb{R}^{T+1}$ is *regular* if and only if $\mathbf{M}\mathbf{v} = \mathbf{v}$.

Intuitively, $\mathbf{v}$ can be thought of as "self-sustaining demand": When the economy expects a path for incomes $\{v_t\}$, then agents find it optimal to demand $\{v_t\}$. At this point, it is worth reminding the reader that such self-sustaining demand $\mathbf{v}$ is inherent in many model with nominal rigidities and by no means special to our setup.[9] In fact, as we shall see in the next sections, dealing with this indeterminacy is at the heart of many New Keynesian models.

We are now in a position to precisely characterize the solution to the IKC (23). In particular, the following result characterizes all possible solutions $d\mathbf{Y}$ that have finite NPV $\sum_{t=0}^{\infty} d\mathbf{Y}_t$.[10]

Theorem 1 (Solving the Intertemporal Keynesian Cross)**.** *Let* $\mathbf{M}$ *satisfy Assumption 1. Any finite-NPV solution* $d\mathbf{Y} \in \mathbb{R}^{T+1}$ *to the IKC* (23) *is given by*

$$d\mathbf{Y} = \sum_{k=0}^{\infty} \mathbf{M}^k \partial\mathbf{Y} + \mathbf{v}, \tag{31}$$

where the infinite sum $\sum_{k=0}^{\infty} \mathbf{M}^k \partial\mathbf{Y}$ *is finite-valued and* $\mathbf{v} \in \mathbb{R}^{T+1}$ *is a regular vector. Moreover, any* $d\mathbf{Y} \in \mathbb{R}^{T+1}$ *of the form in* (31) *is a solution to the IKC.*

Theorem 1 justifies the "Keynesian Cross" part in the IKC. Whereas the solution to the Old-Keynesian cross, dY, is a *scalar-valued* infinite sum,

$$dY = \partial Y + MPC \cdot \partial Y + MPC^2 \cdot \partial Y + \ldots,$$

[9] In fact, even if the economy has flexible, but nominal prices, such a degree of indeterminacy still exists as all prices can be uniformly scaled up or down.

[10] There exist contrived examples when some solutions $d\mathbf{Y}$ can have infinite NPV, yet, as we discuss in the next section, those are not of interest to us.

the solution to the intertemporal Keynesian cross is a *vector-valued* sum

$$d\mathbf{Y} = \partial\mathbf{Y} + \mathbf{M}\partial\mathbf{Y} + \mathbf{M}^2\partial\mathbf{Y} + \ldots + \mathbf{v}$$

with an additional vector $\mathbf{v}$ that could potentially be nonzero.

We already discussed how $\mathbf{v}$ captures indeterminacy due to self-sustaining demand, but what does the first term $\sum_k \mathbf{M}^k\partial\mathbf{Y}$ capture intuitively in the IKC? Consider a given PE shock $\partial\mathbf{Y}$ such as the government promising to purchase more in some periods and less in others. For simplicity, assume that $\mathbf{M}$ is transient (see the proof of Theorem 1). To evaluate this term, consider its t-th element, which we can write as

$$\sum_{s=0}^{T} \underbrace{\sum_{k=0}^{\infty}(\mathbf{M}^k)_{t,s}}_{\text{total time } t \text{ demand caused by time } s \text{ income}} \times \underbrace{\partial Y_s}_{\text{PE income shock at time } s}.$$

This term is summing over all possible periods s. For each period s, it adds up the PE shock ∂Y_s and the total spending effect it has on period t. Notice that this total effect can be very large (e.g. much larger than 1): Since one household's spending is another household's income, any PE shock ∂Y_s may travel around the "MPC network" in various ways until it is spent in period t.

4.2 Determinacy

Solutions to the IKC are generally indeterminate, and so are linearized solutions to the general model of Section 2. We now explain two ways in which there can be determinacy: First, in infinite horizon economies, it can be that there is only a single solution among all possible solutions that does not diverge in current values. Building on Woodford (2003) we call this solution *locally determinate*. Second, one can impose an ad-hoc rule that output $d\mathbf{Y}_t$ needs to obey, limiting the degree to which there can be multiplicity. As we show in Appendix D, these emerge naturally when the government's budget constraint (10) is allowed to be violated off-equilibrium, so that there is fiscal dominance. For this section, we assume that the economy is at a steady state with a constant gross interest rate $R > 1$, and define $r = \log R > 0$. In addition, we assume that the MPC matrix $\mathbf{M}$ does not asymptote to an identity matrix.

Assumption 2. *If* $T = \infty$, $\mathbf{M}$ *is such that* $\limsup_{t\to\infty} M_{t,t} < 1$.

Local determinacy in infinite horizon economies. In an infinite horizon economy, there can be situations in which the vector of self-sustaining demand $\mathbf{v}$ is unbounded in current value terms, that is, $e^{rt}v_t$ is unbounded. Intuitively, any multiplicity would require agents to expect an ever increasing amount of income to generate enough demand for that income.

Definition 5. A vector $\mathbf{w} \in \mathbb{R}^\infty$ is *bounded in current values* if $e^{rt}w_t$ is bounded in t. Otherwise, $\mathbf{w}$ is *unbounded in current values.*

These definitions are useful to define the notion of local determinacy.

Definition 6. A solution to the IKC (31) is *locally determinate (or locally unique)* if it is the only solution $d\mathbf{Y}$ that is bounded in current values. Otherwise, it is *locally indeterminate*.

Our definition of local determinacy follows the one in Woodford (2003), who introduced a similar notion of uniqueness in the context of the standard New-Keynesian model. There, it is shown that the introduction of a sufficiently responsive Taylor rule, that is, one that satisfies the Taylor principle, can generate local determinacy.

Why would a Taylor rule help with determinacy, through the lens of our approach? According to our definition in (21), the Taylor rule is a part of the MPC matrix $\mathbf{M}$: For example, when a sufficiently responsive Taylor rule is in place, any additional rise in income dY_t earned in a period t is not merely spent according to the individual agents MPC's across periods; it also raises real interest rates in period, incentivizing individuals to shift their spending into the future. Thus, a Taylor rule can direct the spending of any extra dollars earned in a given period into the future. Any such "spending in the future", however, is precisely what requires self-sustaining demand to increase over time, possibly to the extent that $e^{rt}v_t$ is unbounded.

To characterize how much spending in the future there is after an increase in aggregate income, we define the following *demand shift index* λ_t^*.

Definition 7. Let $\mathbf{M}$ be an MPC matrix and define for each $t \geq 0$ the *characteristic function* p_t : $\mathbb{R} \to (0, \infty]$,

$$p_t(\lambda) \equiv \sum_s e^{-\lambda(s-t)} M_{s,t}. \tag{32}$$

The *demand shift index* λ_t^* *of* $\mathbf{M}$ *at time* t is defined as the unique non-trivial solution to $p_t(\lambda) = 1$, whenever such a solution exists, else we set $\lambda_t^* = 0$.

To see that λ_t^* is well-defined, observe first that $p_t(\lambda)$ is really the expectation of $e^{-\lambda(s-t)}$ under the probability distribution given by the t-th column of $\mathbf{M}$. As $e^{-\lambda(s-t)}$ is convex in s for any t, $p_t(\lambda)$ is necessarily a convex function. Moreover, $p_t(0) = 1$. Since $\mathbf{M}$ is irreducible, it must be the case that $M_{s,t}$ puts positive weight on time periods other than $s = t$. Thus, the characteristic function $p_t(\lambda)$ is strictly convex (whenever finite) and therefore $p_t(\lambda) = 1$ can only ever have a single solution other than $\lambda = 0$—thus, λ_t^* is well defined.

Intuitively, λ_t^* is a measure of how much of any additional income earned in period t would be spend in the future: When the t-th column $(M_{s,t})_s$ puts a lot of weight on future time periods, this means coefficients for $e^{-\lambda(s-t)}$ with $s > t$ are large, pushing down $p_t(\lambda)$ for larger λ's, which in turn raises λ_t^*. This is illustrated in Figure 4: More weight on the future "tilts" $p_t(\lambda)$ to the right and therefore increases λ_t^*.

As it turns out, the demand shift index λ_t^* is a powerful tool that allows us to summarize how much demand is pushed into the future of any additional aggregate income earned in a given period. We now relate λ_t^* for far out time periods t to whether or not the equilibrium in our economy (and therefore the solution of the IKC) is locally determinate.

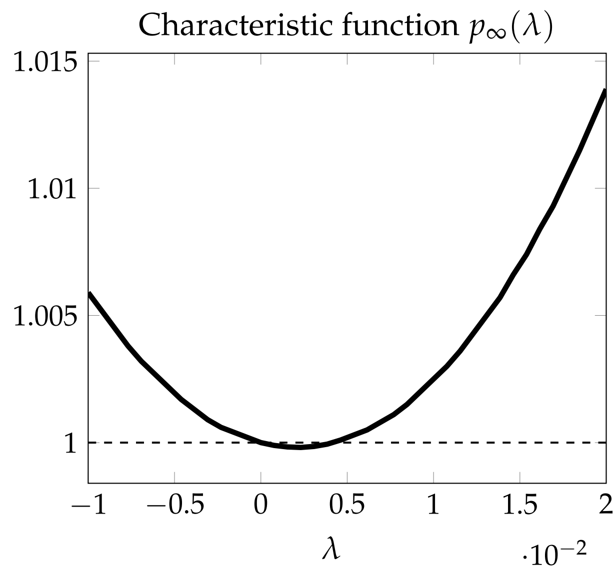

Figure 4: The characteristic function $p_\infty(\lambda)$ in our benchmark calibration with (Constant-r).

Theorem 2 (Local determinacy)**.** *Let* $\mathbf{M}$ *satisfy Assumptions 1 and 2. Suppose* $d\mathbf{Y}$ *is a solution to the IKC that is bounded in current values.*

(a) $d\mathbf{Y}$ *is locally determinate if* $\liminf_{t\to\infty} \lambda_t^* > -r$.

(b) $d\mathbf{Y}$ *is locally indeterminate if* $\limsup_{t\to\infty} \lambda_t^* < -r$.

A simple sufficient condition for (a) is $\liminf_{t\to\infty} p_t(-\tilde{r}) > 1$ *for some* $\tilde{r} < r$*; a simple sufficient condition for (b) is* $\limsup_{t\to\infty} p_t(-r) < 1$.

Theorem 2 sharply characterizes when a given bounded solution $d\mathbf{Y}$ is locally unique: Precisely when $\mathbf{M}$ tends to shift demand into the future to a sufficiently large degree. How large? Precisely enough so that the sum of the *current* value spending pattern, $p_t(-r) = \sum_s e^{r(s-t)} M_{s,t}$ is strictly above 1 for all sufficiently large t. As we explain in detail in Section 7, local determinacy in our general framework can also be induced by policies such as interest rate rules, if the economy is not already locally determinate. Foreshadowing these results, note that a more responsive Taylor rule for instance *changes* the $\mathbf{M}$ matrix, pushing the demand shift index λ_t^* out to the right and shifting demand more into the future. According to Theorem 2, larger demand shift indices always bring the economy closer to determinacy.

To prove this remarkable result, we once more build on various results in Markov chain potential theory and on Foster-Lyapunov conditions to establish whether a nonzero self-sustatining demand vector $\mathbf{v}$ is bounded in current values or not. Intuitively, the proof works since the decay rate of any such $\mathbf{v}$ for large time periods is bounded below by $\liminf_{t\to\infty} \lambda_t^*$ and above by $\limsup_{t\to\infty} \lambda_t^*$. So when for instance $\liminf_{t\to\infty} \lambda_t^* > -r$, then any such $\mathbf{v}$ must necessarily be explosive in current values—and thus $d\mathbf{Y}$ is locally determinate.

While this is a powerful local uniqueness result, one may wonder whether one can characterize the solution some more if it is indeed locally unique. As it turns out, for this it is necessary to distinguish MPC matrices $\mathbf{M}$ according to "how strongly" they shift demand into the future.

Proposition 3 (Characterizing the locally unique solution)**.** *Let* $\mathbf{M}$ *satisfy Assumptions 1 and 2.*

(a) If $\liminf_{t\to\infty} \lambda_t^* > -r$ *and* $\limsup_{t\to\infty} \lambda_t^* < 0$*, the unique locally determinate solution* $d\mathbf{Y}$ *(if it exists) is given by*

$$d\mathbf{Y} = \left(1 + \mathbf{v}\mathbf{a}'\right) \sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y} \tag{33}$$

where $\mathbf{v}$ *is the unique (up to scale) stationary measure of* $\mathbf{M}$ *with eigenvalue 1, and is positive and bounded; and* $\mathbf{a} \in \mathbb{R}^{T+1}$ *a unique vector (up to an additive constant).*

(b) If $\liminf_{t\to\infty} \lambda_t^* > 0$*, the unique locally determinate solution* $d\mathbf{Y}$ *(if it exists) is given by*

$$d\mathbf{Y} = \sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}.$$

In this case, the infinite MPC sum $\sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}$ *can have nonzero net present value.*

Proposition 3 establishes that, in the intertemporal Keynesian cross, the infinite MPC sum is generally not enough to characterize the locally unique output response $d\mathbf{Y}$. Indeed, the solution is a combination of the infinite sum and a self-sustaining demand term. There is, however an exception: If $\mathbf{M}$ pushes out demand in present value terms (case (b)), the locally determinate solution $d\mathbf{Y}$ is just the simple infinite sum.

Interestingly, the split in (a) and (b) according to how strongly $\mathbf{M}$ pushes demand out into the future is precisely reflected in whether the infinite sum can generate positive NPV responses by itself, or not. In appendix 29 we present a simple analytical example in which the economy falls into the "strong" case (b) so that the infinite sum is indeed sufficient to characterize the unique locally determinate solution.

Determinacy through ad-hoc rules. We have established local determinacy results for the uniqueness of solutions that are bounded in current values in infinite horizon economies. However, in some cases, such as finite horizon economies or otherwise indeterminate economies, one may wonder whether other approaches to achieving determinacy exist that can be studied using our approach. We now explain one such approach, namely an ad-hoc approach, amounting to the imposition of an addition condition on $d\mathbf{Y}$. In Appendix D, we illustrate how such an ad-hoc constraint can be derived from the assumption of fiscal dominance in the fiscal theory of the price level. For simplicity, we explain the ad-hoc approach only for finite horizon economies. The analysis is similar in infinite horizon models.

Suppose the additional restriction on $d\mathbf{Y}$ is given by $\mathbf{a}' d\mathbf{Y} = 0$, where α is a T-dimensional vector, and T is the finite horizon of the economy. Then, the unique solution is described as follows.

Proposition 4 (Uniqueness with ad-hoc rule)**.** *The unique solution in a finite horizon economy with selection rule $\alpha' d\mathbf{Y} = 0$ is given by*

$$d\mathbf{Y} = \left(1 - \frac{\mathbf{v}\mathbf{a}'}{\mathbf{a}'\mathbf{v}}\right) \sum_{k=0}^{T} \mathbf{M}^k \partial \mathbf{Y}.$$

The easiest and arguably most intuitive selection rule is to set final period output to zero, $d\mathbf{Y}_T = 0$. This mirrors local determinacy in infinite horizon economies, where at a locally determinate solution, $d\mathbf{Y}_t \to 0$ (since such a $d\mathbf{Y}$ is bounded in current values). But alternatives exist. For instance, in some models of the fiscal theory of the price level, simple averages appear as well, $\sum_t dY_t = 0$.[11]

5 Determinacy and the new Taylor principle

5.1 The asymptotically self-similar IKC

As we have just seen, local determinacy of a GE response depends crucially on the far-out shape of the MPC matrix $\mathbf{M}$. In this section, we consider an important subset of MPC matrices $\mathbf{M}$, whose columns are asymptotically identical, just shifted down by one row. We call such $\mathbf{M}$ matrices self-similar, formally defined as follows.

Definition 8. Let $\mathbf{M}$ be an infinite-dimensional matrix $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ with well-defined characteristic functions $\{p_t(\lambda)\}_t$ (see (32)). $\mathbf{M}$ is *(asymptotically) self-similar* if there exists a continuous function $p(\lambda)$, the *asymptotic characteristic polynomial*, such that $p_t(\lambda) \to p(\lambda)$ as $t \to \infty$, pointwise for all $\lambda \in \mathbb{R}_-$.

Observe that we only require convergence for $\lambda < 0$ since more is not needed for the following result. In practice, however, most models generate MPC matrices where $p_t(\lambda)$ converges for all $\lambda \in \mathbb{R}$. We can state the indeterminacy result for self-similar matrices. Define the scalar

$$\mu \equiv p\left(-r\right)$$

which corresponds to the area under the curve of the matrix of a far-out column of the M matrix, once expressed in current values relative to the time of the shock.

Corollary 2. *Let* $\mathbf{M}$ *be a self-similar matrix satisfying Assumptions 1 and 2. In that case, the sequence of demand shift indices λ_t^* converges to a limit $\lambda^* \in \mathbb{R}$. Moreover, let $d\mathbf{Y}$ be a solution to the IKC that is bounded in current values. Then:*

(a) $d\mathbf{Y}$ is locally determinate if $\lambda^* > -r$ or $\mu > 1$.

(b) $d\mathbf{Y}$ is locally indeterminate if $\lambda^* < -r$ or $\mu < 1$.

[11] For papers using this approach to analyzing and resolving indeterminacy, see Leeper (1991), Woodford (1995), Cochrane (2011), or Caramp and Silva (2017).

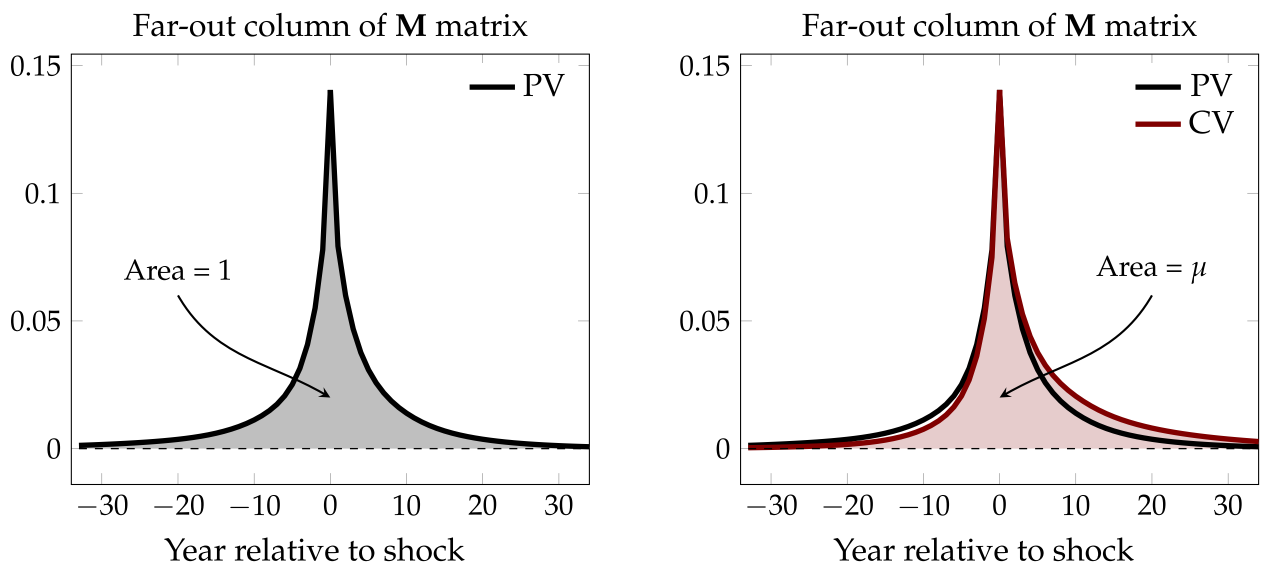

Figure 5: Simple criterion for multiplicity: μ vs 1

Self-similar MPC matrices play a prominent role because models with idiosyncratic risk and incomplete markets, such as our main model of section 2, naturally generate such MPC matrices.

Figure 5 illustrates this criterion in the context of our benchmark model under a (Constant-r) rule. The left panel shows a column of $\mathbf{M} = \mathbf{M}^Y$ for a shock at some date $T \gg 0$, centered around date T. By construction, since $\sum M_{s,T} = 1$, we know that the area under this curve is 1. The right panel plots these in current values relative to the time of the shock The area $\mu = \sum M_{s,T} (1+r)^{s-T}$ may in principle be greater or lower than 1 depending on the balance of weights to either side of T. When $\mu > 1$, weights tend to fall towards the future ("demand is pushed to the future"), generating determinacy. The criterion is very intuitive and broadly applicable to models with incomplete markets as a simple test of determinacy. It also yields intuitive comparative statics, as described below.

5.2 A New Taylor Principle

So far, we have provided generic conditions for determinacy that do not exploit specific features of the $\mathbf{M}$ matrix. We apply these below to understand determinacy in an economy under (Constant-r), for which lemma 5 showed that $\mathbf{M} = \mathbf{M}^Y$. We now consider the consequences of the structure imposed by monetary policy that follows (6), as described in 6.

Let $p^R(\lambda) \equiv \lim_{t\to\infty} \sum_s e^{-\lambda(s-t)} M^R_{s,t}$ bet the asymptotic characteristic polynomial of the (real) interest rate response matrix $\mathbf{M}^R$. By construction, $p^R(0) = 0$. Moreover, typically, higher interest rates push demand into the future, so that $p^R(-r) > 0$.

We obtain the following generalized Taylor principle. Define

$$\begin{aligned} \mu^Y &= p^Y(-r) \\ \mu^R &= p^R(-r) \end{aligned}$$

Theorem 3 (Generalized Taylor Principle.)**.** *Let* $\mathbf{M}$ *be a self-similar matrix satisfying Assumptions 1 and 2. Let* $\mathbf{M}^R$ *be a self-similar matrix whose columns sum to zero. Assume* $\mathbf{M}^{joint}$ *is similar to a (non-negative) column-stochastic matrix. Then,*

(a) if ϕ_y, ϕ_π *are such that* $\phi_\pi > 1 - \frac{1-\beta}{\kappa}\left(\phi_y + \frac{1-\mu^Y}{\mu^R}\right)$ *any equilibrium is locally determinate;*

(b) if ϕ_y, ϕ_π *are such that* $\phi_\pi < 1 - \frac{1-\beta}{\kappa}\left(\phi_y + \frac{1-\mu^Y}{\mu^R}\right)$ *any equilibrium is locally indeterminate.*

In particular, when the Taylor rule is only based on inflation, $\phi_y = 0$*, the threshold responsiveness* ϕ_π *for determinacy is equal to* $1 - \frac{1-\beta}{\kappa}\left(\phi_y + \frac{1-\mu^Y}{\mu^R}\right)$.

Our result in Theorem 3 succinctly summarizes when how strong a Taylor rule is required to be in order to induce determinacy in any macroeconomic model that can be described by a self-similar MPC matrix $\mathbf{M}$ and a self-similar interest rate response matrix $\mathbf{M}^R$. It stresses the role of the values μ^Y and μ^R. We explain the intuition behind both for the simple case where $\phi_y = 0$.

The position of μ^Y relative to 1 is a measure of how much demand is shifted into the future. In the last subsection, we saw that under a constant real interest rate, 1 is precisely the threshold μ^Y has to exceed for determinacy. Theorem 3 adds to this result that where μ^Y lies relative to 1 precisely determines whether the Taylor rule has to respond to inflation ϕ_π more or less than one-for-one. If μ^Y is large compared to 1 and prices are not too flexible, that is, κ is small, it may even be that the economy is determinate with a Taylor rule that does not respond to inflation at all, $\phi_\pi = 0$.

The magnitude of μ^R roughly captures how responsive the economy is to real interest rates. To illustrate why this is important, imagine an economy with $\mu^Y < 1$. How responsive does the Taylor rule have to be to shift enough demand out into the future and generate determinacy? This is exactly where μ^R matters: if the economy's responsiveness to real interest rates μ^R were large (say infinite), even a small increase in ϕ_π relative to 1 will be enough to ensure determinacy. Thus, the threshold for ϕ_π declines in μ^R when $\mu^Y < 1$, and vice versa if $\mu^Y > 1$.

5.3 Determinacy with constant-r in our model

We start by considering the case of our model with monetary policy described by (Constant-r). In this case, the position of μ^Y relative to 1 matters. Figure 6 shows the value of μ^Y in our model under various assumptions about two key parameters: the degree of "cyclicality of income risk" $\gamma = \frac{\partial^2 \log \Gamma}{\partial \log l \partial \log e}$ and the amount of aggregate liquidity $\frac{b}{y}$.

The dot on each curve represents our benchmark calibration detailed in table 1, with $\Gamma(e, l) = l$ so that $\gamma = 0$ and $\frac{b}{y} = 140\%$. In this benchmark, $\mu^Y = 1.08$, so that the economy with constant-r is determinate. This therefore contrasts with the case of a representative agent, for which the

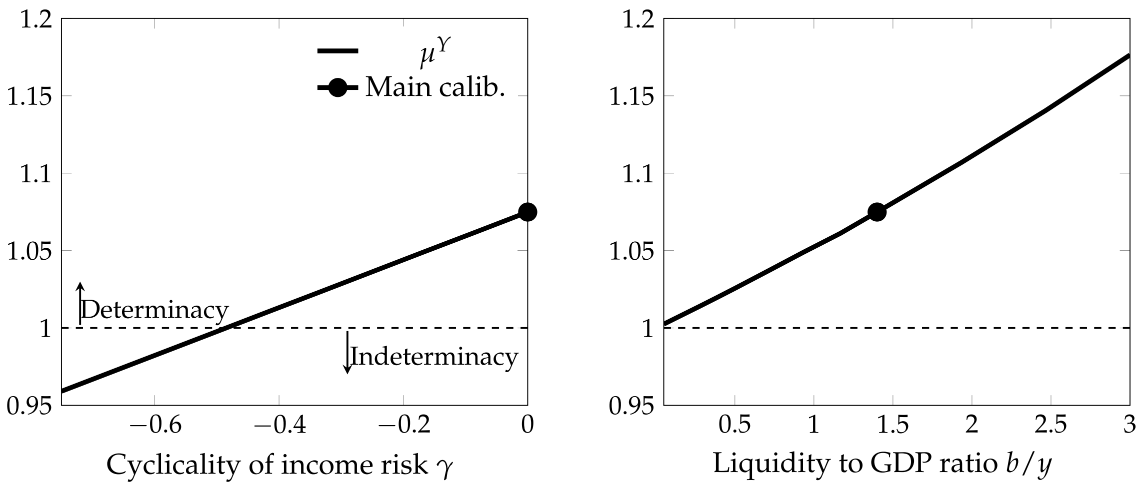

Figure 6: Determinants of determinacy: Cyclicality of income risk and liquidity.

economy is just indeterminate with constant real interest rates. The intuition is that, relative to that case, agents in our benchmark incomplete markets model tend to spend incomes after they receive it, in large part because of borrowing constraints.

The intuition that more stringent borrowing constraints always push towards determinacy is incorrect, however. Consider the left panel, which illustrates what happens as we make income risk more countercyclical, taking γ to more negative values. In this case, agents with low skills e are disproportionately hurt when employment l falls. Suppose that agents expect employment to fall in future periods. Then they get collectively worried about the future and end up cutting consumption today as well. Symetrically, if they expect the economy to improve, they cut precautionary savings and increase their spending immediately. The outcome is an economy that tends to spend in anticipation of income—far-out **M** columns are tilted towards the past, and this tends to generate indeterminacy with (Constant-r). Figure 7 further illustrates this in two extreme cases: one indeterminate case with $\gamma = -0.75$ and one determinate case with $\gamma = 0$. Observe that the impulse responses in the former case are more tilted to the past than those in the latter case. The resulting multiplicity vector $\mathbf{v}$ (plotted in current value terms of the right panel of figure 7), is explosive under $\gamma = 0$ but converges to zero under $\gamma = -0.75$. What self-sustains this vector in that case is the following feedback mechanism: an expectation of improvement of the economy tomorrow generates even more spending today, so the pattern of demand over time self sustains very precisely. This feedback loop between precautionary savings and the level of economic activity under nominal rigidities has been highlighted by several authors before (see Ravn and Sterk 2013, den Hann et al. 2015, Bayer et al. 2015, Challe et al. 2014, or Heathcote and Perri 2016), but had never been related to a matrix of MPCs until now.

The right panel of figure 6 plots another interesting comparative statics: that of the level of liquidity $\frac{b}{y}$. For each value of $\frac{b}{y}$ on the left axis, we recalibrate the value of β to hit the constant interest rate $r = 4\%$, and then plot the resulting μ^Y. Here, it is always the case that $\mu^Y > 1$ so that

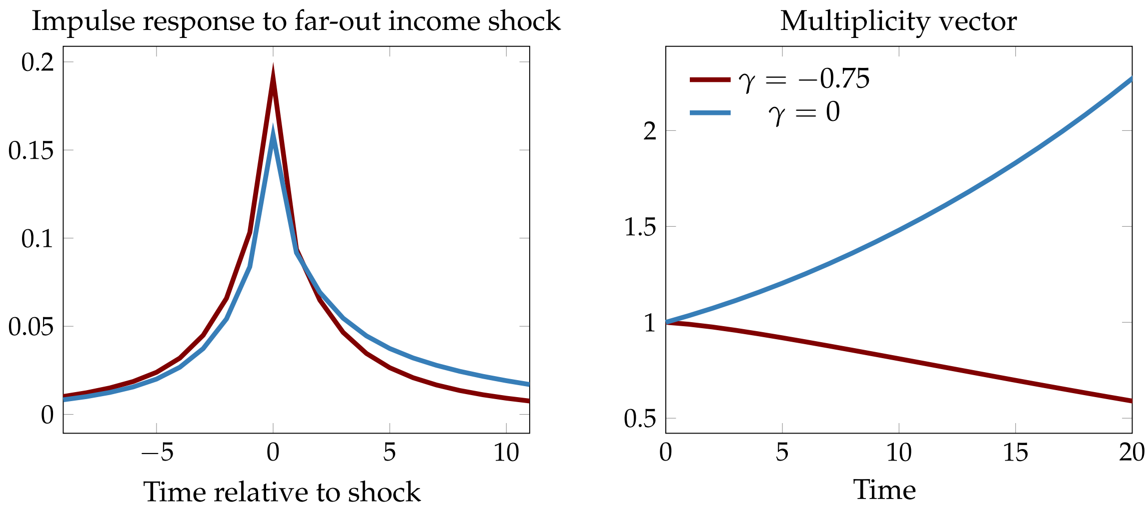

Figure 7: Understanding multiplicity.

the economy is determinate. Further, μ^Y smoothly converges to 1 as liquidity disappears and all agents become nearly completely constrained, with MPCs equal to 1 for each of them.

5.4 Applying the New Taylor Principle

We next turn to the case when monetary policy in our model is described by (Taylor rule). We consider a case where $\phi_y = 0$ and plot

$$\phi_\pi^* = 1 - \frac{1-\beta}{\kappa}\left(\phi_y + \frac{1-\mu^Y}{\mu^R}\right)$$

for the two comparative statics of the model described in the previous section, maintaining the calibration of table 1 except as explicitly mentioned.

The left panel of figure 6 illustrates the same force at play: across all these experiments μ^R is constant, and therefore, the more countercylical income risk is (the more negative γ), the higher ϕ_y needs to be to ensure determinacy. When $\gamma = -0.75$ (the extreme case to the left of the graph), the Taylor rule coefficient needs to be almost 1.1 to ensure determinacy. This illustrates the fact that monetary policy has to fight off the natural tendency of the economy to indeterminacy, by responding more strongly in terms of interest rates.

By constrast, the right panel of figure 8 illustrates a new force: as the economy has less and less liquidity, agents become almost completely unresponsive to changes in real interet rates, and further μ^R limits to zero faster than μ^Y limits to 1. Hence the economy behaves in a determinate way, as if r was constant, even if monetary policy varies the nominal interest rate substantially in response to developments in inflation.

A conclusion from this analysis is that important factors are likely to affect the degree to which

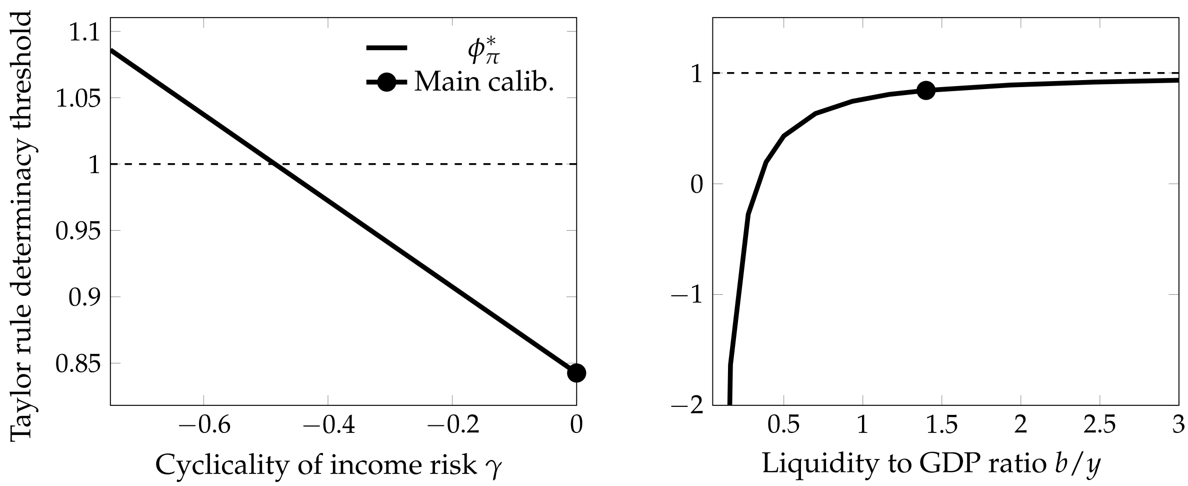

Figure 8: Determinants of the Taylor rule thresholds.

monetary policy needs to lean against the wind to ensure economic stability. For example it has been widely argued (Clarida, Galí and Gertler (2000)) that monetary policy has small responses to inflation in the pre-1980 period. However, to the extent that the amount of liquidity at the time was also limited, figure 8 suggests that this may not have been a cause of instability. A further exploration of this topic is an interesting question for future research.

6 When are shocks a net stimulus?

Consider the response to a government spending shock in our benchmark model. The government spends an extra unit today, and simultaneously increases debt, withdrawing it at a rate of 20% per period.[12] Figure 9 shows the outcome of this experiment. In partial equilibrium, households cut consumption, with some anticipation of the taxes and a large effect at the time the taxes are levied–an effect that fades out over time. As per lemma 3, the net present value of the impulse reponse is clearly 0. However, in general equilibrium, the impact effect is greater that 1, and further persists for a number of periods. It never goes below 0.

This experiment lays bare one of the most important questions one may ask about aggregate demand: Where does it come from? How can it be, that a *zero net present value* partial equilibrium shock $\partial\mathbf{Y}$ causes a *nonzero* net present value general equilibrium response $d\mathbf{Y}$? What features of the MPC network $\mathbf{M}$ make this possible? And what features of $\partial\mathbf{Y}$ decide whether $d\mathbf{Y}$ has positive net present value vs. negative net present value?

[12]Specifically, $d\overline{g_0} = 1$, $d\overline{b_0} = 1$, $d\overline{b_{t+1}} = \rho^t d\overline{b_t}$ for $\rho = 0.2$ and $t > 0$.

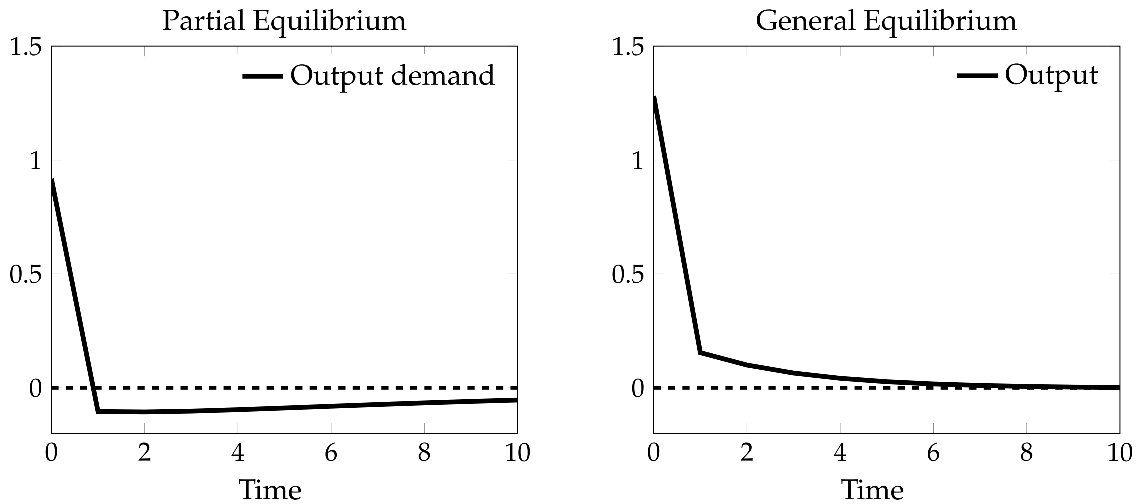

Figure 9: Fiscal policy: the effect of government spending with delayed taxes

6.1 Theory

The crucial "symmetry breaking" ingredient to answering these questions is going to be time: We show that if the MPC matrix is *ordered*, in that later increases in aggregate incomes are spent later that earlier ones, then a partial equilibrium impulse $\partial \mathbf{Y}$ causes a positive NPV GE response $d\mathbf{Y}$ precisely if it is *front-loaded*; and vice-versa a negative NPV response $d\mathbf{Y}$ if it is *back-loaded*. We define these concepts formally as follows.

Definition 9. Let $\mathbf{M} \in \mathbb{R}^{(T+1)\times(T+1)}$ be a column-stochastic matrix, and let $\mathbf{v}', \mathbf{v} \in \mathbb{R}^{T+1}$ be two vectors.

a) $\mathbf{M}$ is *ordered*, if for any $t' > t$, $(M_{s,t'})_s$ first-order stochastically dominates $(M_{s,t})_s$.

b) $\mathbf{v}'$ is *more* front-loaded than $\mathbf{v}$, *if for any* t, $\sum_{\tau\leq t} \mathbf{v}'_\tau \geq \sum_{\tau \leq t} \mathbf{v}_\tau$. *In this case,* $\mathbf{v}$ *is* more back-loaded than $\mathbf{v}'$.

c) $\mathbf{v}'$ is front-loaded *if it is more front-loaded than a vector of zeros.* $\mathbf{v}$ *is* back-loaded *if it is more back-loaded than a vector of zeros.*

This allows us to state the following main result on the origins of aggregate demand.

Proposition 5 (Ordered matrices and font-loading)**.** *Let* $\mathbf{M}$ *be an ordered matrix satisfying Assumptions 1 and 2. Let* $d\mathbf{Y}$ *and* $d\mathbf{Y}'$ *be locally unique GE responses to PE impulses* $\partial\mathbf{Y}$ *and* $\partial\mathbf{Y}'$. *If* $\partial\mathbf{Y}'$ *is more front-loaded than* $\partial\mathbf{Y}$, *then*

$$d\mathbf{Y}' \text{ is more front-loaded than } d\mathbf{Y}.$$

In particular, if $\partial\mathbf{Y}'$ *is front-loaded,* $d\mathbf{Y}$ *has non-negative initial impact* $dY_0 \geq 0$ *and non-negative net present value* $\sum_t dY_t \geq 0$. *The opposite holds if* $\partial\mathbf{Y}'$ *is back-loaded:* $dY_0 \leq 0$ *and* $\sum_t dY_t \leq 0$.

The proposition makes clear what the key feature is that generates nonzero net present value aggregate demand: how front-loaded the partial equilibrium shock is $\partial \mathbf{Y}$. In particular, the *only* difference between a contractionary and an expansionary PE shock is how front-loaded they are, not their net present value (which is zero for both).

Of course, Proposition 5 requires the MPC matrix $\mathbf{M}$ to be ordered. While we have not been able to obtain an analytical proof for the general model of section 2, it is a condition that we have easily checked to be true in all our analytical and numerical applications, and that we conjecture to be true more generally.

6.2 IKC iterations and tatonnement to equilibrium

Figure 10 illustrates a 'tatonnement' process that happens to deliver general equilibrium as an outcome in the fiscal multiplier scenario of figure 9: it plots the partial sums $\sum_{k=0}^{K} \mathbf{M}^k \partial \mathbf{Y}$ for various K's. In this case, this process converges to the general equilibrium outcome $d\mathbf{Y}$. This corresponds to case b) in theorem 3, for which the Markov chain represented by $\mathbf{M}$ is transient and iterations converge to the general equilibrium outcome. In this process, we see that the negative demand values initially generated by the partial equilibrium effect (owing to the taxes being levied) ultimately never materialize. The intuition is as follows: as agents receive more income in period 0 due to higher government spending, they spend this additional income in period 0 and 1. This mitigates the fall in demand due to the lower taxes in period 1, ultimately raising income in period 1, which is then spent in period 2, and so on. Once the intertemporal keynesian cross has run its course, the initial negative demand has been pushed out to infinity and is not featured at all in the general equilibrium impulse response.[13]

7 When does heterogeneity matter for fiscal and monetary policy?

In this section, we conclude by using our new tools to answer an important question in the literature: when does heterogeneity matter for the aggregate effect of monetary and fiscal policy? We do so by exhibiting two benchmarks for which heterogeneity does not matter, in the sense that a fiscal shock or a monetary shock delivers the same answer as under the representative agent case. By looking how, in partial equilibrium, a given economy departs from a benchmark, we are able to characterize the precise conditions under which heterogeneity matters.

7.1 Fiscal policy

We start by defining a balanced budget fiscal policy.

[13] Note however that, in case a) of theorem 3, which features a recurrent Markov chain—so that the economy's natural tendency to push demand into the future is not as strong—this specific process will not converge to the unique equilibrium.

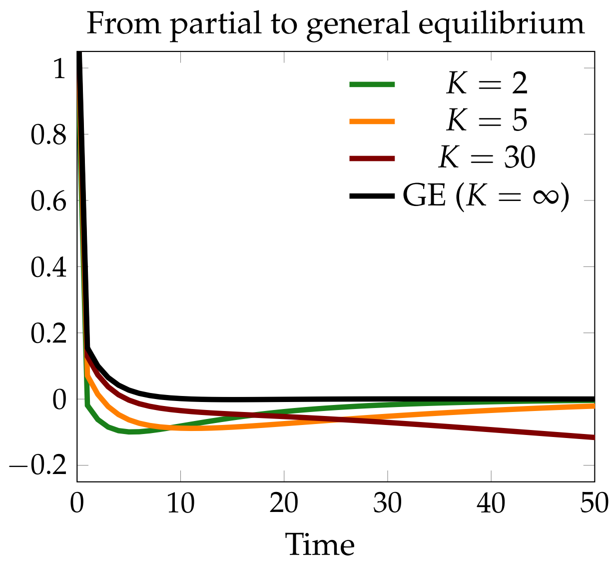

Figure 10: From PE to GE: Demonstrating the iteration.

Definition 10. A balanced budget government spending shock is a shock to government spending $d\overline{g_t}$ that does not lead to a change in debt at any point in time: $d\overline{b_t} = 0$ for all t. In equilibrium, the government adjusts taxes and transfer to raise revenue in the period of the shock, $drev_t = d\overline{g_t}$.

Proposition 6 (Benchmark multiplier of 1)**.** *Assume a) equal incidence of both gross $\Gamma(e, l) = l$, b) equal incidence of net incomes $\varphi = \tau^r$, and c) monetary policy follows (Constant-r). Then the general equilibrium effect of a balanced-budget increase in government spending is equal to the path of spending itself. This is also the representative agent response.*

$$d\mathbf{Y} = \partial\mathbf{G}$$

Proof. Let $\overline{g_t}$ be a new path for spending. Conjecture an equilibrium in which the entire stochastic process for agent post tax incomes $z_t(e_{it})$ remains unchanged at their initial level, for every t. Since monetary policy follows (Constant-r), the real interest rate is also constant for all agents. Preferences and borrowing constraints have not changed, and therefore each agent has an identical consumption path, so aggregate consumption path follows the same path as before. The government's additional revenue requirement is $drev_t = d\overline{g_t}$ at every t. Under our conjecture, GDP also changes by $dy_t = d\overline{g_t}$ at every t. Hence, the fiscal rule (12) with $\varphi = \tau^r$ implies that the lump sum t_t is unchanged at every t. Moreover, with $\varphi = \tau^r$ the fiscal rule (11) also implies that tax revenue $d(\tau_t y_t) = dy_t = d\overline{g_t}$. Total tax revenue goes up exactly by enough to pay for the extra government spending. Putting this together and using equal incidence $\Gamma(e, l) = l$, agent incomes all change by $dz_{it} = 0 + e_{it} d((1 - \tau_t) y_t) = e_{it}(dy_t - d(\tau_t y_t)) = 0$, confirming the initial equilibrium. □

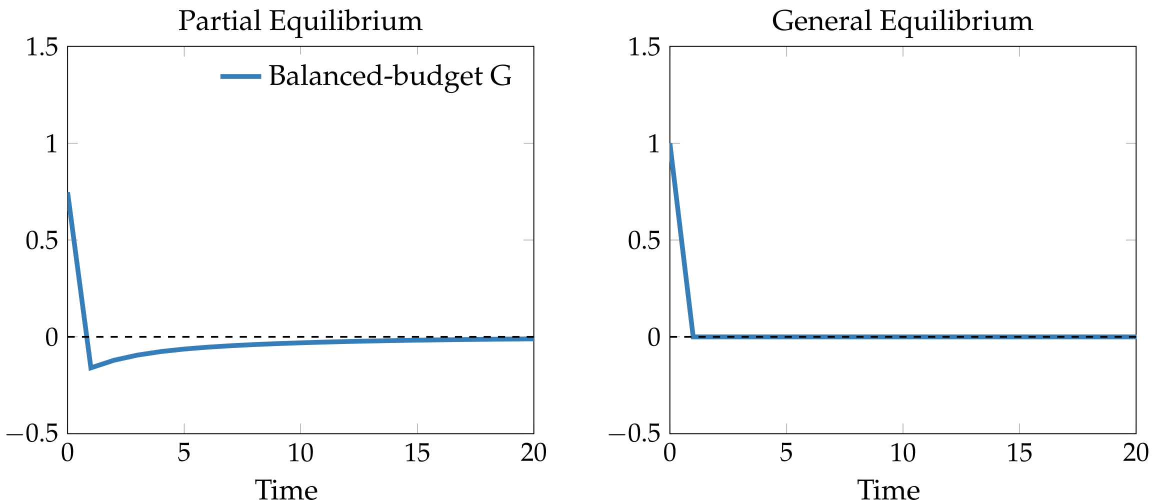

Figure 11: Fiscal policy: balanced-budget multiplier

Proposition 6 delivers a benchmark of one for the fiscal multiplier—a surprising result, given all of the heterogeneity. This is true no matter what the path for government spending is. As is well known, this benchmark of 1 is the same as the one that obtains in the standard New Keynesian model under a constant-r monetary policy (see, for example, Woodford (2011)), and would of course obtain under the representative agent version of our model. Hence, in this case, heterogeneity is irrelevant for understanding the general equilibrium effect of fiscal policy.

A recent literature has computed fiscal multipliers in heterogeneous agent models (see for example Hagedorn et al. (2017)), finding multipliers sometimes smaller and sometimes larger than 1. Proposition 6 establishes exactly when a multiplier of 1 obtains. The next section then quantities a force that can lead to larger multipliers.

Balanced budget vs. delayed taxes. Consider a government spending shock financed by delayed taxes, such as that of figure 11. Using the linearity of our $\mathcal{G}$ operator, it is natural to decompose this shock as the combined effect of a balanced budget fiscal shock (which has multiplier of 1) $\partial \mathbf{G}^{bb}$ and an effect from a tax rebate today, with higher taxes in the future $\partial \mathbf{C}^{taxrebate}$. Applying linearity and proposition 1, we know that $d\mathbf{Y} = \partial \mathbf{G} + \mathcal{G} \partial \mathbf{C}^{taxrebate}$. Applying proposition 5, we further know that—since a tax rebate shock has frontloaded spending—the effect of $\mathcal{G} \partial \mathbf{C}^{taxrebate}$ is positive. The more the taxes are delayed, the greater the fiscal multiplier, as illustrated in figure 12. Quantitatively, this shows that departures from Ricardian equivalence can be largely amplified in general equilibrium.

7.2 A benchmark result for monetary policy

Next, we define a new benchmark for monetary policy shocks. This benchmark corresponds to what we label the modified substitution effect. [...to be defined. Proposition: the modified sub-

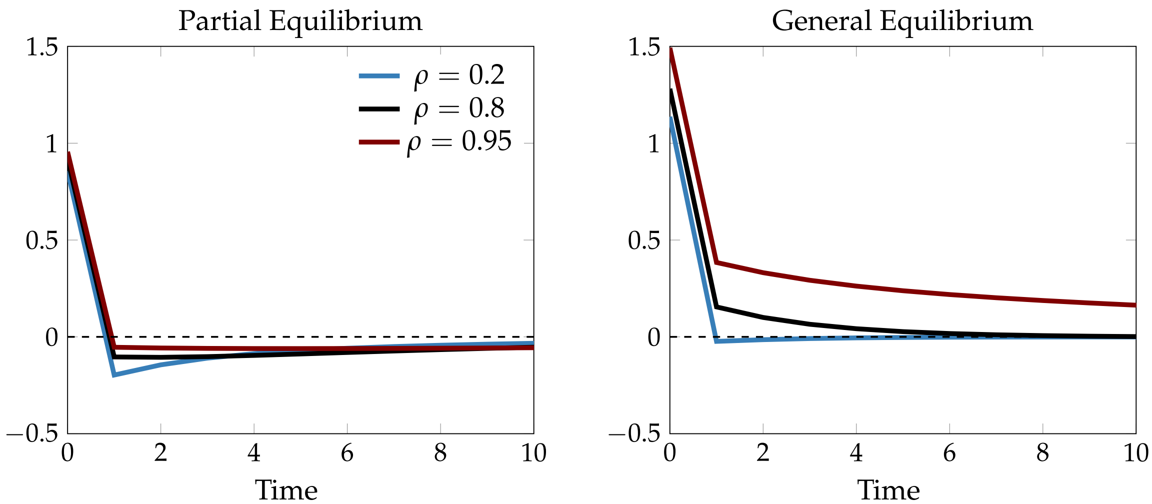

Figure 12: General equilibrium effect of increasingly delaying taxes

stitution effect is the representative agent effect. This relates to Werning (2015), but is different because it does not require his case of $\sigma = 1$ or zero liquidity, under which the modified income effects cancel.].

We can also see how the modified substitution effect corresponds to the more standard substitution effect of interest rate changes. In partial equilibrium, using Slutsky's equations, we define the subtitution effect as

$$\partial \mathbf{Y} = \partial \mathbf{C}^{inc} + \partial \mathbf{C}^{sub}$$

we then consider the general equilibrium effect, in turn for contemporaneous monetary policy shocks and for forward guidance shocks.

Standard monetary policy shocks. Standard monetary policy shocks operate, in our model, both due to income and to substitution effects. In general equilibrium, the substitution effects are almost exactly the representative agent response, and the income effects contribute to amplify the response by around 50%. This confirms the results in Auclert (2017), who argued that income effects were likely to operate in the same direction as subtitution effects and to amplify monetary policy under heterogeneous agents.

Forward guidance shocks. Figure 14 repeats the exercise for a forward guidance shock: a change in interest rates at date $t = 10$. Here, we find again that the income effect is what makes the difference to the representative agent case, but this time in the other direction, generating dampening of forward guidance in general equilibrium. The reason here is as follows: consider a contractionary monetary policy shock that will raise interest rates in the future. Agents who will benefit from this increase in interest rates tend to be relatively unconstrained savers, who can immediately increase their consumption in response. By contrast, agents who will lose from this

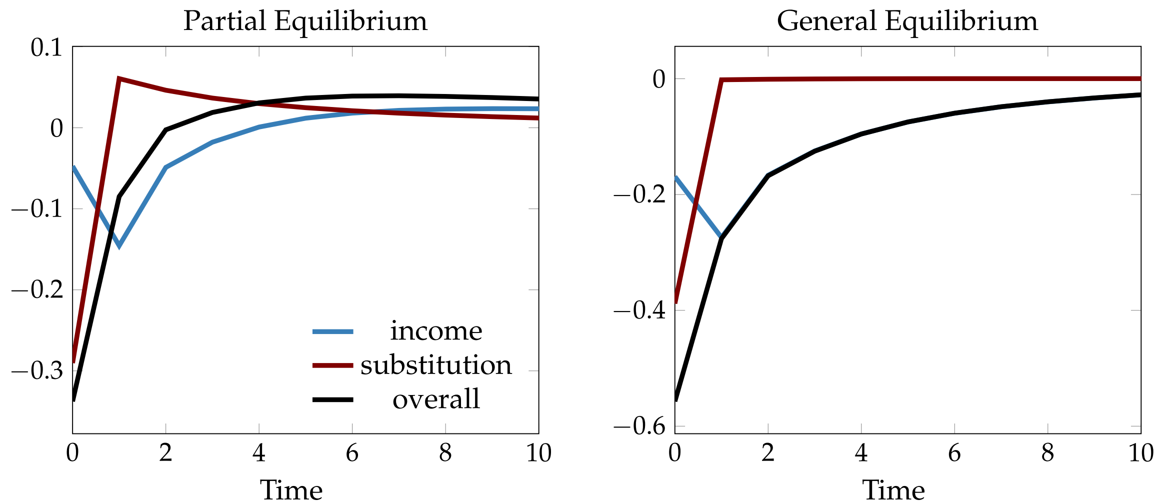

Figure 13: Decomposing standard monetary policy shocks

increase in interest rates tend to be more constrained, and do not lower their consumption further. This generates an asymmetry for the income effect that results in an overall dampening of monetary policy.

8 Conclusion

[To be added.]

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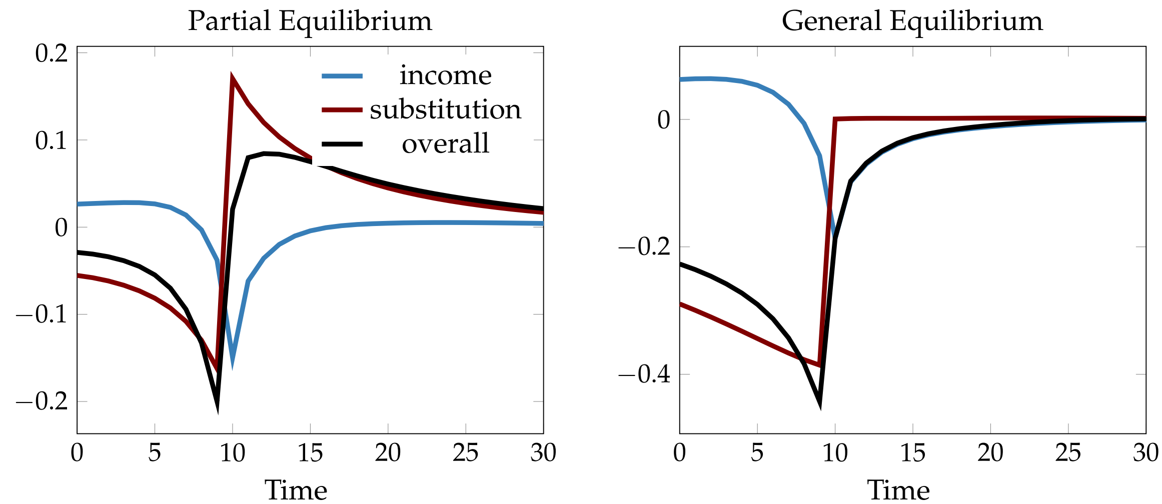

Figure 14: Decomposing forward guidance shocks

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A Union wage setting

A.1 Details of the labor market

We assume that the worker provides n_{ijt} hours of work to each of a continuum of unions indexed by $j \in [0,1]$, so that his total labor effort is

$$n_{it} \equiv \int_0^1 n_{jt}^i dj$$

The agent gets paid a nominal market wage of w_{jt} per efficient unit of work in union j. Hence his total nominal gross earnings are

$$w_t e_{it} n_{it} \equiv \int_0^1 w_{jt} e_{it} n_{ijt} dj$$

Each union j aggregates efficient units of work into a union-specific task l_{jt}, such that

$$l_{jt} = \sum_{e_i} \pi\left(e_{it}\right) e_{it} n_{ijt} \tag{34}$$

Aggregate employment l_t, in turn, is an aggregate of union-specific tasks that are imperfect substitutes for one another,

$$l_t \equiv \left(\int_0^1 l_{jt}^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}}$$

Each union sets its own wage w_{jt} per unit of task provided. We assume that nominal wages w_{jt} are partially rigid, and only reset occasionally in a Calvo fashion: each period, any given union j keeps its wage constant with probability θ. At every point in time, the wage index

$$w_t = \left(\int_0^1 w_{jt}^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}$$

serves as a reference for final goods firms's demand for tasks. Conditional on employment l_t, this demand is

$$l_{jt} = \left(\frac{w_{jt}}{w_t} \right)^{-\epsilon} l_t$$

In turn, union j, when producing l_{jt} tasks, pays all of its workers the same wage w_{jt} and demands that those with skill e_{it} work n_{ijt} hours according to a particular rule Γ, so that

$$n_{ijt} = \Gamma\left(e_{it}, l_{jt}\right) \tag{35}$$

The Γ function must be consistent with aggregation (34) for any l_{jt}. A simple example is $\Gamma\left(e,l\right) = l$. In this case, each union member works $n_{ijt} = l_{jt}$ hours irrespective of their skill or their consumption level.

A wage-resetting union maximizes an average of the utilities (3) of each of its workers, with each worker receiving weight λ^i. In appendix A, we write down the wage setting problem of the union conditional on aggregate employment l_t and a real wage $\frac{w}{p_t}$. We show that in a steady state in which employment is constant at l^*, prices are constant at p^* and all resetting unions set wages at w^*, the consumption distribution $\Psi\left(c\right)$ statisfies

$$\frac{w^*}{p^*} = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_{e,\Psi(c)}\left[\lambda\left(e,c\right) \Gamma_l\left(e,l^*\right) l^* \frac{\frac{dv^i}{dn}\left(\Gamma\left(e,l^*\right)\right)}{e} \right]}{\mathbb{E}\left[\int \lambda\left(e,c\right) \gamma_l\left(e,l^*\right) l^* \frac{\epsilon - \frac{\Gamma(e,l^*)}{\Gamma_l(e,l^*) l^*}}{\epsilon - 1} \frac{du}{dc}\left(c\right) \right]} \tag{36}$$

Intuitively, union wage-setting ensures that the real wage $\frac{w^*}{p^*}$ is at a markup over some average across union members of the marginal rate of susbtitution between consumption and hours.

The wage setting rule (36) requires knowledge of the consumption distribution in the population c^{*i}. Away from steady-state, we assume that unions reset wages under a behavioral approximation that this distribution remains constant. The underlying assumption is that it is too complicated for the union to keep track of each member's asset position over time when that distribution is changing. Practically, this implies that the union ignores income effects on labor supply when resetting wages away from steady-state. Below we show that this problem leads to a simple New Keynesian Phillips curve for aggregate wage growth, $\pi_t^w \equiv \log\left(\frac{W_t}{W_{t-1}}\right)$, given to first order by

$$\pi_t^w = \lambda_t \left(\frac{1}{\psi} \widehat{l}_t - \left(w_t - p_t\right) \right) + \beta_t \pi_{t+1}^w \tag{37}$$

where $l_t \equiv \log(l_t/l^*)$, ψ is a constant, and κ_t, β_t are deterministic functions of time, with $\beta_t = \beta$ when $\mathcal{T} = \infty$.

A.2 Derivation of the general case

Consider the problem of a union j. The union supplies l_{jt} units of efficient work by aggregating labor from each of its members according to (34), by allocating work across its living members acording to the rule in (35), that is $e_t^i n_t^i = \gamma\left(l_{jt}, e_t^i\right)$.

Consider first the general case. Given a path for agregates $\{l_t, P_t, W_t\}$, wages at every union $\{W_{jt}\}$, and an assumed distribution for the consumption of every agent $\{c_t^i\}$, the problem that the union k solves when it gets a change to reset its wage W_{kt} is

$$\max_{W_{kt}} \sum_{\tau\geq 0} (\beta\theta)^\tau \int \lambda^i \left(u^i\left(c_{t+\tau}^i;\theta\right) - v^i\left(\int n_{jt+\tau}^i dj;\theta\right)\right)$$

where λ^i is the weight assigned to member i in social utility.

The first-order condition for this problem is

$$\sum_{\tau\geq 0} (\beta\theta)^\tau \int \lambda^i \frac{\frac{du^i}{dc}\left(c_{t+\tau}^i\right)}{P_{t+\tau}} \left(\gamma\left(l_{kt+\tau}, e_{t+\tau}^i\right) - \epsilon\gamma_l\left(l_{kt+\tau}, e_{t+\tau}^i\right) l_{kt+\tau}\right) di$$
$$+ \sum_{\tau\geq 0} (\beta\theta)^\tau \int \lambda^i \epsilon \frac{\frac{dv^i}{dn}\left(\int n_{jt+\tau}^i dj\right)}{e_{t+\tau}^i W_{kt}} \gamma_l\left(l_{kt+\tau}, e_{t+\tau}^i\right) l_{kt+\tau} di = 0$$

which we can rewrite as

$$\sum_{\tau\geq 0} (\beta\theta)^\tau \int \lambda^i \frac{du^i}{dc}\left(c_{t+\tau}^i\right) \gamma_l\left(l_{kt+\tau}, e_{t+\tau}^i\right) l_{kt+\tau}$$
$$\left[\left(\epsilon - \frac{\gamma\left(l_{kt+\tau}, e_{t+\tau}^i\right)}{\gamma_l\left(l_{kt+\tau}, e_{t+\tau}^i\right) l_{kt+\tau}}\right) \frac{W_{kt}}{P_{t+\tau}} - \epsilon \frac{\frac{dv^i}{dn}\left(\int n_{jt+\tau}^i dj\right)}{\frac{du^i}{dc}\left(c_{t+\tau}^i\right) e_{t+\tau}^i}\right] di \quad = \quad 0$$

Hence all resetting unions choose the same wage $W_{kt} = \overline{W_t}$, satisfying the implicit equation

$$\overline{W_t} = \frac{\epsilon}{\epsilon - 1} \frac{\sum_{\tau\geq 0} (\beta\theta)^\tau \int \lambda^i \gamma_l\left(l_{kt+\tau}, e_{t+\tau}^i\right) l_{kt+\tau} \frac{\frac{dv^i}{dn}\left(\int n_{jt+\tau}^i dj\right)}{e_{t+\tau}^i} di}{\sum_{\tau\geq 0} (\beta\theta)^\tau \int \lambda^i \gamma_l\left(l_{kt+\tau}, e_{t+\tau}^i\right) l_{kt+\tau} \left(\frac{\epsilon - \frac{\gamma\left(l_{kt+\tau}, e_{t+\tau}^i\right)}{\gamma_l\left(l_{kt+\tau}, e_{t+\tau}^i\right) l_{kt+\tau}}}{\epsilon - 1}\right) \frac{\frac{du^i}{dc}\left(c_{t+\tau}^i\right)}{P_{t+k}} di} \tag{38}$$

In the special case where γ is linear in l, so that $\gamma_l l = \gamma$, (38) simplifies to

$$\overline{W_t} = \frac{\epsilon}{\epsilon - 1} \frac{\sum_{\tau \geq 0} (\beta\theta)^\tau \int \lambda^i \gamma \left(l_{kt+\tau}, e^i_{t+\tau}\right) \frac{\frac{dv^i}{dn}\left(\int n^i_{jt+\tau} dj\right)}{e^i_{t+\tau}} di}{\sum_{\tau \geq 0} (\beta\theta)^\tau \int \lambda^i \gamma \left(l_{kt+\tau}, e^i_{t+\tau}\right) \frac{\frac{du^i}{dc}\left(c^i_{t+\tau}\right)}{P_{t+k}} di}$$

In a steady-state in which the price level is P^*, the nominal reset wage is equal to the constant $\overline{W} = W^*$, labor is the constant l^*, and the consumption distribution is c^{*i}, (38) yields

$$\epsilon \int \lambda^i \gamma_l \left(l^*, e^i\right) l^* \frac{\frac{dv^i}{dn}\left(\frac{1}{e^i}\gamma\left(l^*, e^i\right)\right)}{e^i} di = \frac{W^*}{P^*} \int \lambda^i \left(\epsilon \gamma_l \left(l^*, e^i\right) l^* - \gamma\left(l^*, e^i\right)\right) \frac{du^i}{dc}\left(c^{*i}\right) di \tag{39}$$

which is equation (36). In general equilibrium, equation (39) implicitly defines the level of steady-state employment l^*, where the consumption distribution is consistent with a real earnings process given by $\frac{W^*}{P^*}\gamma\left(l^*, e^i\right)$. We linearize around such a steady state.

From (38), we can write the reset wage relative to the current wage W_t as

$$\frac{\overline{W_t}}{W_t} = \frac{\epsilon \sum_{\tau \geq 0} (\beta\theta)^\tau \int \lambda^i \gamma_l \left(l_{kt+\tau}, e^i_{t+\tau}\right) l_{kt+\tau} \frac{\frac{dv^i}{dn}\left(\int n^i_{jt+\tau} dj\right)}{e^i_{t+\tau} W_t} di}{\sum_{\tau \geq 0} (\beta\theta)^\tau \int \lambda^i \left(\epsilon \gamma_l \left(l_{kt+\tau}, e^i_{t+\tau}\right) l_{kt+\tau} - \gamma\left(l_{kt+\tau}, e^i_{t+\tau}\right)\right) \frac{\frac{du^i}{dc}\left(c^i_{t+\tau}\right)}{P_{t+k}} di} = \frac{G_t}{H_t} \tag{40}$$

where G_t, H_t satisfy the recursions

$$G_t = \epsilon \int \lambda^i \gamma_l \left(l_{kt}, e^i_t\right) l_{kt} \frac{\frac{dv^i}{dn}\left(\int n^i_{jt} dj\right)}{e^i_t W_t} di + \beta\theta \frac{W_{t+1}}{W_t} G_{t+1} \tag{41}$$

$$H_t = \int \lambda^i \left(\epsilon \gamma_l \left(l_{kt}, e^i_t\right) l_{kt} - \gamma\left(l_{kt}, e^i_t\right)\right) \frac{\frac{du^i}{dc}\left(c^i_t\right)}{P_t} di + \beta\theta H_{t+1} \tag{42}$$

The wage index W_t at time t satisfies the recursion

$$W_t^{1-\epsilon} = \theta W_{t-1}^{1-\epsilon} + (1-\theta)\left(W_t^*\right)^{1-\epsilon} \tag{43}$$

Take logs on both sides of (43) and (40) and rearrange to get

$$\pi^w_t \equiv w_t - w_{t-1} = \frac{1-\theta}{\theta}\left(w^*_t - w_t\right) = \frac{1-\theta}{\theta}\left(g_t - h_t\right) \tag{44}$$

Consider the steady state where $l_t = l^*$, $P_t = P^*$, $W_t = W_{jt} = W^*$ for all j and the distribution of consumption is at its steady state c^{*i}. From (35), all workers with productivity e^i_t work equally at each firm,

$$n^i_{jt} = \frac{\gamma\left(l^*, e^i_t\right)}{e^i_t} \quad \forall j$$

Then (41)–(42) deliver

$$G_t^* = \Phi_t \epsilon \int \lambda^i \gamma_l \left(l^*, e^i\right) l^* \frac{\frac{dv^i}{dn}\left(\frac{\gamma\left(l^*, e^i\right)}{e^i}\right)}{e_t^i W^*}$$

$$H_t^* = \Phi_t \int \lambda^i \left(\epsilon \gamma_l \left(l^*, e^i\right) l^* - \gamma\left(l^*, e^i\right)\right) \frac{\frac{du^i}{dc}\left(c^{*i}\right)}{P^*} di$$

where $\Phi_t = \sum_{\tau=0}^{T-t} (\beta\theta)^\tau$. Using (39), we obtain

$$G_t^* = C\Phi_t$$

$$H_t^* = C\Phi_t$$

with $C = \epsilon \int \lambda^i \gamma_l \left(l^*, e^i\right) l^* \frac{\frac{dv^i}{dn}\left(\frac{1}{e^i}\gamma\left(l^*, e^i\right)\right)}{e^i W^*} di = \int \lambda^i \left(\epsilon \gamma_l \left(l^*, e^i\right) l^* - \gamma\left(l^*, e^i\right)\right) \frac{\frac{du^i}{dc}\left(c^{*i}\right)}{P^*} di.$

Approximating (41)–(42) around G_t^*, H_t^*, and defining $\widehat{g}_t \equiv \log\left(\frac{G_t}{G_t^*}\right)$ and $\widehat{g}_t \equiv \log\left(\frac{H_t}{H_t^*}\right)$, we obtain

$$\widehat{g}_t = \frac{1}{\Phi_t} a_t + \left(1 - \frac{1}{\Phi_t}\right)\left(w_{t+1} - w_t + \widehat{g_{t+1}}\right) \tag{45}$$

$$\widehat{h}_t = \frac{1}{\Phi_t} b_t + \left(1 - \frac{1}{\Phi_t}\right)\widehat{h_{t+1}} \tag{46}$$

Where we have defined

$$a_t \equiv \int \omega^{Gi} \left\{ \left(\frac{\gamma_{ll}\left(l^*, e^i\right) l^*}{\gamma_l\left(l^*, e^i\right)} + 1\right)\left(-\epsilon\left(w_{kt} - w_t\right) + \widehat{l}_t\right) + \frac{1}{\psi^i\left(e^i\right)} \frac{\gamma_l\left(l^*, e^i\right) l^*}{\gamma\left(l^*, e^i\right)} \widehat{l}_t - w_t \right\} di$$

In this expression, ψ^i is the Frisch elasticity of labor supply of individual i in state e^i, and individual weights are the following transformation of social weights:

$$\omega^{Gi} = \frac{\lambda^i \gamma_l \left(l^*, e^i\right) l^* \frac{\frac{dv^i}{dn}\left(\frac{\gamma\left(l^*, e^i\right)}{e^i}\right)}{e^i}}{\int \lambda^i \gamma_l \left(l^*, e^i\right) l^* \frac{\frac{dv^i}{dn}\left(\frac{1}{e^i}\gamma\left(l^*, e^i\right)\right)}{e^i} di}$$

Similarly,

$$b_t = \int \omega^{Hi} \left(\left\{ \frac{\epsilon}{\epsilon - 1} \left(\frac{\gamma_{ll}\left(l^*, e^i\right) l^*}{\gamma_l\left(l^*, e^i\right)} + 1\right) - \frac{1}{\epsilon - 1}\left(\frac{\gamma_l\left(l^*, e^i\right) l^*}{\gamma\left(l^*, e^i\right)}\right) \right\} \left(-\epsilon\left(w_{kt} - w_t\right) + \widehat{l}_t\right) - \frac{1}{\sigma\left(e^i\right)} \widehat{c_t^i} - p_t \right) di$$

with

$$\omega^{Hi} = \frac{\lambda^i \left(\epsilon \gamma_l \left(l^*, e^i\right) l^* - \gamma\left(l^*, e^i\right)\right) \frac{du^i}{dc}\left(c^{*i}\right)}{\int \lambda^i \left(\epsilon \gamma_l \left(l^*, e^i\right) l^* - \gamma\left(l^*, e^i\right)\right) \frac{du^i}{dc}\left(c^{*i}\right) di}$$

Our behavioral assumption for the union consists in assuming that $\int \omega^{Hi} \frac{1}{\sigma^i(e^i)} \widehat{c_t^i} di = 0$, that is, the consumption distribution remains at its steady-state. Using this assumption to simplify b_t, we can solve for the difference

$$\begin{aligned} a_t - b_t &= \left(\int \omega^{Gi} \frac{1}{\psi^i\left(e^i\right)} \frac{\gamma_l\left(l^*, e^i\right) l^*}{\gamma\left(l^*, e^i\right)} di \right) \widehat{l_t} - \left(w_t - p_t\right) \\ &\quad + \left(\int \left\{ \omega^{Gi} \frac{1}{\epsilon - 1} \left(\frac{-\gamma_{ll}\left(l^*, e^i\right) l^*}{\gamma_l\left(l^*, e^i\right)} \right) \right\} di \right) \left(\widehat{l_t} - \epsilon\left(\overline{w_t} - w_t\right) \right) \\ a_t - b_t &= \frac{1}{\psi} l_t - \left(w_t - p_t\right) - \epsilon\nu\left(\overline{w_t} - w_t\right) \end{aligned} \tag{47}$$

where

$$\begin{aligned} \frac{1}{\psi} &\equiv \int \omega^{Gi} \frac{1}{\psi^i\left(e^i\right)} \frac{\gamma_l\left(l^*, e^i\right) l^*}{\gamma\left(l^*, e^i\right)} di + \nu \\ \nu &\equiv \int \omega^{Gi} \frac{1}{\epsilon - 1} \left(\frac{-\gamma_{ll}\left(l^*, e^i\right) l^*}{\gamma_l\left(l^*, e^i\right)} \right) di \end{aligned}$$

We finally obtain the wage Phillips Curve as follows. Start with equation (44)

$$\pi_t^w = w_t - w_{t-1} = \frac{1-\theta}{\theta}\left(\overline{w_t} - w_t\right) = \frac{1-\theta}{\theta}\left(g_t - h_t\right) = \frac{1-\theta}{\theta}\left(\widehat{g_t} - \widehat{h_t}\right) \tag{48}$$

where the last line follows $G_t^* = H_t^*$. Next, combine (48), (45), (46) and (47) into

$$\begin{aligned} \overline{w_t} - w_t &= \widehat{g_t} - \widehat{h_t} \\ &= \frac{1}{\Phi_t}\left(a_t - b_t\right) + \left(1 - \frac{1}{\Phi_t}\right)\left(w_{t+1} - w_t + \widehat{g_{t+1}} - \widehat{h_{t+1}}\right) \\ &= \frac{1}{\Phi_t}\left(\frac{1}{\psi} l_t - \left(w_t - p_t\right) - \epsilon\nu\left(\overline{w_t} - w_t\right)\right) + \left(1 - \frac{1}{\Phi_t}\right)\left(w_{t+1} - w_t + \widehat{g_{t+1}} - \widehat{h_{t+1}}\right) \end{aligned}$$

solve out for $\overline{w_t} - w_t$

$$\overline{w_t} - w_t = \left(1 + \frac{\epsilon\nu}{\Phi_t}\right)^{-1} \left\{ \frac{1}{\Phi_t}\left(\frac{1}{\psi} l_t - \left(w_t - p_t\right)\right) + \left(1 - \frac{1}{\Phi_t}\right)\left(w_{t+1} - w_t + \widehat{g_{t+1}} - \widehat{h_{t+1}}\right) \right\}$$

Wage inflation π_t^w is then

$$\begin{aligned} w_t - w_{t-1} &= \frac{1-\theta}{\theta}\left(1 + \frac{\epsilon\nu}{\Phi_t}\right)^{-1} \left\{ \frac{1}{\Phi_t}\left(\frac{1}{\psi} l_t - \left(w_t - p_t\right)\right) + \left(1 - \frac{1}{\Phi_t}\right)\left(\left(\frac{1}{\theta} - 1\right)\left(w_{t+1} - w_t\right) + \frac{1-\theta}{\theta}\left(\widehat{g_{t+1}} - \right.\right.\right. \\ &= \left(1 + \frac{\epsilon\nu}{\Phi_t}\right)^{-1} \left\{ \frac{1}{\Phi_t} \frac{1-\theta}{\theta}\left(\frac{1}{\psi} l_t - \left(w_t - p_t\right)\right) \right\} + \left(1 - \frac{1}{\Phi_t}\right)\frac{1}{\theta}\left(w_{t+1} - w_t\right) \end{aligned}$$

This gives us our final expression for the wage Phillips curve

$$\pi_t^w = \lambda_t \left(\frac{1}{\psi} \widehat{l}_t - (w_t - p_t) \right) + \beta_t \pi_{t+1}^w$$

where

$$\begin{aligned}
\lambda_t &\equiv \left(1 + \frac{\epsilon \nu}{\Phi_t}\right)^{-1} \left(\frac{1}{\Phi_t} \frac{1-\theta}{\theta} \right) \\
\beta_t &\equiv \left(1 - \frac{1}{\Phi_t}\right) \frac{1}{\theta} \\
\Phi_t &\equiv \sum_{\tau=0}^{T-t} (\beta\theta)^\tau
\end{aligned}$$

which is equation (9). Note in particular that $\Phi_t \to \frac{1}{1-\beta\theta}$ as $T \to \infty$, and therefore $\beta_t \to \beta$.

B Alternative economies

B.1 OLG economy

Consider an infinite-horizon model with overlapping generations of agents living for two periods each. At every time t live two generations of equal size: an old generation born at time $t-1$, and a young generation born at time t. The old consume their savings s_t. The young work n_t hours at a nominal market wage W_t, pay real taxes t_t, and can buy consumption goods with price P_t or save in nominal bonds with price $\frac{1}{1+i_t}$. Each young person born at time t solves the maximization problem

$$\begin{aligned}
\max \quad & u(c_{t,t}) + \beta u(c_{t,t+1}) \\
& P_t c_{t,t} + \frac{s_{t+1}}{1+i_t} = W_t n_t - P_t t_t \\
& P_{t+1} c_{t,t+1} = s_{t+1}
\end{aligned}$$

where u has constant elasticity of substitution σ, $u(c) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$. Only the current young supply labor, and they work as many hours n_t as firms demand given the real wage $\frac{W_t}{P_t}$. For the purpose of this illustrative example, we assume that wages are perfectly rigid at a constant level ($\theta = 1$), and normalize this level to 1 for convenience:

$$W_t = 1$$

Hence firm price setting (8) implies that the nominal price level is constant at $P_t = 1$. This equation also implies that the nominal and the real interest rate are equal at all times, $R_t = 1 + i_t$. We assume that monetary policy sets a constant path for the nominal interest rate i_t equal to $\beta^{-1} - 1$. This is equivalent to our (Constant-r) assumption.

In each period, the government only levies taxes t_t on the current young . Its budget constraint is therefore

$$\frac{b_{t+1}}{R_t} + P_t t_t = b_t + P_t g_t \tag{49}$$

At $t = 0$ one generation is born old, and the government endows these agents with b_0 bonds.

Given a path for government policy $\{b_t, g_t, t_t, i_t\}$ satisfying (49) at each point in time and the terminal condition $\left(\prod_{s=0}^{\infty} \frac{1}{R_s}\right) b_\infty = 0$, as well as a fixed level for the nominal wage $W_t = 1$, an equilibrium in this economy is a path for prices $\{P_t, R_t\}$, and quantities $c_{-1,0}$ and $\{y_t, n, c_{t,t}, c_{t,t+1}\}$, such that households optimize, firms optimize, and the goods and bond markets clear:

$$\begin{aligned} g_t + c_{t,t} + c_{t-1,t} &= y_t \\ s_t &= b_t \end{aligned}$$

Solving for equilibrium. Given a feasible path for policy $\{b_t, g_t, t_t, i_t\}$, equilibrium is determined as follows. Household's optimal intertemporal choice given the real wage $\frac{W_t}{P_t} = 1$ and the real interest rate $R_t = \beta^{-1}$ determines the path of aggregate savings s_{t+1},

$$\frac{s_{t+1}}{P_{t+1}} = \frac{1}{1+\beta}\{n_t - t_t\} \tag{50}$$

Bond market clearing imposes that $\frac{s_{t+1}}{P_{t+1}} = \frac{b_{t+1}}{P_{t+1}}$, while the government budget constraint requires $\frac{b_{t+1}}{P_{t+1}} = \frac{1}{\beta}\left(\frac{b_t}{P_t} + g_t - t_t\right)$. Solving out these equations for y_t, we obtain the equilibrium output and employment level as a function of current government policy and the outstanding debt stock,

$$y_t = n_t = \frac{1}{\beta}\left((1+\beta)\left(\frac{b_t}{P_t} + g_t\right) - t_t\right) \qquad \forall t \tag{51}$$

Equation (51) shows neatly how output is determined by government policy. Examining equation (51) leads us to a number of important remarks.

Equilibrium determinacy. Notice that equilibrium is uniquely pinned down in our economy, despite the fact that the path of nominal interest rates is constant. This negates a classic result in monetary economics, dating back to Wicksell (1898) and more recently Sargent and Wallace (1975), according to which equilibrium is indeterminante under a pegged interest rate. Section 4.2 will shed light on this rather mysterious result.

Government spending multipliers: benchmark is 1, higher if frontloaded. Equation (51) shows that an increase in current spending g_t paid for by current taxes t_t increases output one-for-one: the tax-financed spending multiplier is one.

$$\left.\frac{dy_t}{dg_t}\right|_{dg_t = dt_t} = 1 \tag{52}$$

Moreover, an increase in current spending paid for by *future* taxes has more than one-for-one effect on current output. It also has an effect on future output, since the current government deficit increases future debt. Indeed, along a path with $dt_t = 0$, it is immediate to show that

$$\left.\frac{dy_{t+k}}{dg_t}\right|_{dt_t=\ldots=dt_{t+k}=0} = (1+\beta)\left(\frac{1}{\beta}\right)^{k+1} > 1 \tag{53}$$

Of course, this process cannot go on indefinitely since the government must eventually raise taxes to satisfy its intertemporal budget constraint. However, this observation illustrates a key property of equilibrium multipliers, which is that they are raised by frontloading spending relative to taxes. In section 6.1, we provide a substantial generalization of both of these results.

The intertemporal Keynesian Cross. It is simple to show that the present value impulse responses $dY_t = \beta^t dy_t$ must satisfy the following equilibrium conditions

$$\begin{aligned} dY_0 &= \frac{1}{1+\beta}\{dY_0 - dT_0\} + dG_0 \\ dY_t &= \frac{\beta}{1+\beta}\{dY_{t-1} - dT_{t-1}\} + \frac{1}{1+\beta}\{dY_t - dT_t\} + dG_t \end{aligned}$$

which is our key Intertemporal Keynesian Cross equation (23), where in this case

$$\mathbf{M} = \begin{bmatrix} m & 0 & & & \\ 1-m & m & 0 & & \\ 0 & 1-m & \ddots & 0 & \\ \vdots & 0 & & m & 0 \\ 0 & \vdots & & 1-m & \ddots \end{bmatrix} \qquad \partial\mathbf{Y} = \begin{bmatrix} dG_0 - m dT_0 \\ dG_1 - (1-m)\,dT_0 - m dT_1 \\ \vdots \\ dG_t - (1-m)\,dT_{t-1} - m dT_t \\ \vdots \end{bmatrix} \tag{54}$$

and $m \equiv \frac{1}{1+\beta}$ is the static MPC. These equations come together with a restriction from assumption **??** that the government budget constraint is always satisfied, that is,

$$\sum_{t\geq 0} dG_t = \sum_{t\geq 0} dT_t$$

This infinite-dimensional linear system can therefore be written more compactly as (23). Lemmas 4 and **??** can then easily be verified. Any increase in government spending must result in an increase in current or future taxes. These taxes lower incomes, to which households respond by cutting consumption, to the point that the present-value effect must be nil. For example, consider two alternative schemes for dG_0: one in which current taxes are raised, and one in which taxes are only raised after t periods. In the first case, if the government spends $dG_0 = 1$ and raises taxes on the current generation, this in turn lowers spending by m today and by $\beta m = 1 - m$ tomorrow, hence

$$\partial \mathbf{Y}^{G,tax} = \begin{bmatrix} 1-m & -(1-m) & 0 & \cdots & \cdots \end{bmatrix}' dG_0 \tag{55}$$

In the second case, we have

$$\partial \mathbf{Y}^{G,tax} = \begin{bmatrix} 1 & 0 & \cdots & -m & -(1-m) & 0 & \cdots \end{bmatrix}' dG_0 \tag{56}$$

The government spends 1 at time 0, but the partial-equilibrium offset from private spending only occurs over the two periods during which the generation affected by the tax increase actually lives.

C Main proofs

C.1 Section 3 proofs

C.1.1 Proof of proposition 1

In an EYE given constant $y_t = y$, inflation is constant at $\pi_t = 0$ from (9). This, combined with the observation that $y_t = y$, implies from both (Constant-r) or (Taylor rule), that the real interest rate only changes by $dr_t = d\overline{r_t}$ (respectively $di_t = d\overline{r_t}$). Given flexible prices and no productivity shocks, real wages are constant at $\frac{w_t}{p_t} = 1$. Given $l = y$, household hours are constant at their steady state level $n_{it} = \Gamma(e_{it}, y)$.

The change in fiscal or monetary policy induces a change in government revenue of $drev_t = (1+r_t)\, d\overline{b_{t-1}} + \overline{b_{t-1}} dr_t + d\overline{g_t} - d\overline{b_t}$ per period. We now see how this affects household net incomes $z_{it} = t_t + (1-\tau_t)\, e_{it} n_{it}$. From (11) and (12) we have

$$y d\tau_t = y d\tau_t^r + (1-\varphi)\, drev_t$$

and

$$dt_t = y d\tau_t^r - \varphi \cdot drev_t$$

Hence net incomes are affected by

$$\begin{aligned} dz_{it} &= dt_t - e_{it}\Gamma(e_{it}, l)\, d\tau_t \\ &= y d\tau_t^r - \varphi \cdot drev_t - \frac{e_{it}\Gamma(e_{it}, y)}{y}\left(y d\tau_t^r + (1-\varphi)\, drev_t\right) \\ &= y d\tau_t^r (1-\omega_i) - (\varphi + (1-\varphi)\,\omega_i)\, drev_t \end{aligned} \tag{57}$$

where $\omega_i = \frac{e_{it}\Gamma(e_{it}, y)}{y}$ is household gross income relative to the mean. This delivers the proposition. Equation (57) shows that shocks to the tax rate $d\tau_t^r$ are just redistributive, with $d\tau_t^r > 0$ increasing the incomes of agents below the mean and reducing those of agents above the mean. Meanwhile, any dollar change in $\mathbb{E}_I [dz_{it}] = -drev_t$, so that any unit change in government revenue in a quarter ends up reducing household incomes by a unit on average, with φ determining whether this is paid mostly via a lump-sum or via a proportional reduction in incomes.

C.1.2 Proof of lemma 3

According to our partial equibrium definition 3, each household takes z_{it} and maximizes (3) subject to

$$c_{it} + a_{it} = (1 + r_t)\, a_{it-1} + z_{it} \quad \forall t, i \tag{4'}$$

Let $q_t \equiv \prod_{s=0}^{t} \frac{1}{1+r_s}$. Along any realized path, households satisfy

$$(1 + r_0)\, a_{i,-1} + \left(\sum_{t=0}^{\mathcal{T}} q_t z_{it} \right) = \left(\sum_{t=0}^{\mathcal{T}} q_t c_{it} \right) + q_{\mathcal{T}} a_{i\mathcal{T}}$$

Taking expectations at date 0 for a given household, we have,

$$(1 + r_0)\, a_{i,-1} + \left(\sum_{t=0}^{\mathcal{T}} q_t \mathbb{E}_0[z_{it}] \right) = \left(\sum_{t=0}^{\mathcal{T}} \mathbb{E}_0[c_{it}] \right) + q_{\mathcal{T}} \mathbb{E}_0[a_{i\mathcal{T}}]$$

Next, taking the population mean $\mathbb{E}_I$ of both sides gives, using iterated expectations $\mathbb{E}_I[\mathbb{E}_0[\cdot]] = \mathbb{E}_I[\cdot]$, and noting that asset market clearing implies $\mathbb{E}_I\left[a_{i,-1}\right] = b_{-1}$ as well as $\mathbb{E}\left[a_{i,T}\right] = b_{\mathcal{T}}$ (the level of government debt outstanding), we have

$$(1 + r_0)\, b_{-1} + \left(\sum_{t=0}^{\mathcal{T}} q_t \mathbb{E}_I[z_{it}] \right) = \left(\sum_{t=0}^{\mathcal{T}} q_t \mathbb{E}_I[c_{it}] \right) + q_{\mathcal{T}} b_{\mathcal{T}} \tag{58}$$

But, from the definition of labor incomes and constant-y,

$$\begin{aligned} \mathbb{E}_I[z_{it}] &= t_t + (1 - \tau_t)\, \mathbb{E}_I\left[e_{it} n_{it}\right] \\ &= t_t + (1 - \tau_t)\, y \\ &= y - rev_t \\ &= y - g_t + b_t - (1 + r_t)\, b_{t-1} \end{aligned}$$

Now, telescoping the sum

$$\sum_{t=0}^{\mathcal{T}} q_t \left(b_t - (1 + r_t)\, b_{t-1} \right) = -\,(1 + r_0)\, b_{-1} + q_{\mathcal{T}} b_{\mathcal{T}}$$

and therefore (58) is simply

$$\sum_{t=0}^{\mathcal{T}} q_t \left(c_t + g_t - y \right) = 0$$

Hence, in our definition of an EYE $\partial y_t = dc_t^{PE} + \partial g_t$, any shock must satisfy

$$\sum_{t=0}^{\mathcal{T}} q_t d\left(c_t + g_t \right) + \sum_{t=0}^{\mathcal{T}} dq_t \times \left(c_t + g_t - y \right) \quad = \quad 0$$

Hence

$$\sum_{t=0}^{\mathcal{T}} q_t \partial y_t + \sum_{t=0}^{\mathcal{T}} dq_t \times 0 = 0$$

where the first term follows from our definition of ∂y_t and the second term from goods market clearing at every date before the shock.

C.1.3 Proof of lemma 4

[Analogous to lemma 3, to be added.]

C.2 Section 4 proofs

C.2.1 Proof of Theorem 1

The proof of Theorem 1 heavily relies on results and terminology from Markov chain potential theory, in particular the book by Kemeny et al. (1976), henceforth KSK. To use these results, we regard $\mathbf{M}$ as the transition matrix of a Markov chain and identify $\mathbf{M}$ with the associated Markov chain. Notice that by Assumption 1, $\mathbf{M}$ is either recurrent or transient.

First, suppose $\mathbf{M}$ is recurrent. Consider the direction where we are given a bounded regular vector $\mathbf{v}$ and that $\mathbf{v}_0 \equiv \sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}$ is finite-valued. Since $\partial \mathbf{Y}$ has zero NPV, this lets us apply Theorem 9-15 in KSK to show that $\mathbf{v}_0$ is a particular solution to the IKC. Since $\mathbf{v}$ is regular, $d\mathbf{Y} = \mathbf{v}_0 + \mathbf{v}$ solves the IKC, too. Conversely, suppose $d\mathbf{Y}$ is a finite-NPV solution to the IKC. Let $\lambda = \sum_{t=0}^{\infty} dY_t \in \mathbb{R}$ be $d\mathbf{Y}$'s NPV. Then, if $\mathbf{M}$ is ergodic, this means that $d\mathbf{Y} = \sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y} + \mathbf{v}$ by Theorem 9-53, where $\mathbf{v}$ is unique regular vector with $\mathbf{1}'\mathbf{v} = \lambda$. If $\mathbf{M}$ is a null chain, this means that $d\mathbf{Y} = \sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}$ by Corollary 9-17.

Second, suppose $\mathbf{M}$ is transient. In that case, $\partial \mathbf{Y}$ is always a charge and $\sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}$ is always a finite-valued solution to the IKC, see Theorem 8-3. This also means that any vector of the form $\sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y} + \mathbf{v}$ where $\mathbf{v}$ is a regular vector solves the IKC. Conversely, if $d\mathbf{Y}$ is a finite-valued solution, then $(I - \mathbf{M})(d\mathbf{Y} - \sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}) = 0$ so $d\mathbf{Y}$ and $\sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}$ differ precisely by a regular vector, implying (31).

C.2.2 Three helpful lemmas for determinacy

The following three lemmas turn out to be helpful for the proofs of our main results.

Lemma 7. *Under Assumptions 1 and 2, the following holds:*

a) if $\liminf_{t\to\infty} \lambda_t^* > 0$*, there exists a* $\lambda > 0$ *such that* $\limsup p_t(\lambda) < 1$.

b) if $\limsup_{t\to\infty} \lambda_t^* < 0$*, there exists a* $\lambda < 0$ *such that* $\limsup p_t(\lambda) < 1$.

Proof. We prove part 1 only. Part 2 is analogous. Define $\lambda_\chi = \chi \liminf_{t\to\infty} \lambda_t^*$ for all $\chi \in [0, 1]$. By concavity, it follows that for all sufficiently large t, say $t \geq \underline{t}$, $p_t(\lambda_{0.5}) < 1$. Suppose it were the case that $p_t(\lambda_{0.5}) \to 1$ (or along some subsequence; the argument is analogous). Then, by concavity, $p_t(\lambda_\chi) \to 1$ for all χ. In fact, convergence (for large t) of $\chi \mapsto p_t(\lambda_\chi)$ is uniform, since each $p_t(\lambda_\chi)$ is bounded below by either by a line connecting point $(0.5, p_t(\lambda_{0.5}))$ with $(0,1)$ or with $(1, p_t(\lambda_1))$. Moreover, since $p_t(\lambda)$ interpreted as a polynomial in e^λ is analytic, the limit is analytic too. But this means it must be that, point wise, $M_{s,t} \to 0$ unless $s = t$. A contradiction. □

Lemma 8. *Let* $\mathbf{M}$ *be the (column-stochastic) transition matrix of a Markov chain with regular measure* $\mathbf{v} = (v_t)$, *that is,* $\mathbf{M}\mathbf{v} = \mathbf{v}$.

a) If $f : \mathbb{N}_0 \to \mathbb{R}_+$ *is a strictly increasing function such that* $\sum_s f(s) M_{s,t} > f(t)$ *for all* t *outside a finite set of states, then* $\sum_{t=0}^\infty f(t) v_t = \infty$.

b) If $f : \mathbb{N}_0 \to \mathbb{R}_+$ *is a strictly decreasing function such that* $\sum_s f(s) M_{s,t} < f(t)$ *for all* t *outside a finite set of states, then* $\sum_{t=0}^\infty f(t) v_t = \infty$.

Proof. We prove part 1. Part 2 is analogous. We are given that there exists a $\underline{t} \geq 0$ such that $\sum_s f(s) M_{s,t} > f(t)$ for all $t \geq \underline{t}$. Assume by contradiction that $\sum_{t=0}^\infty f(t) v_t < \infty$. In that case, we can split up the sum into two finite pieces,

$$\sum_{t=0}^{\infty} f(t) v_t = \sum_{t=0}^{\underline{t}} f(t) v_t + \sum_{t=\underline{t}+1}^{\infty} f(t) v_t, \tag{59}$$

which can both be bounded below.

For the first sum, observe that $v_t = \sum_s M_{t,s} v_s$, thus

$$\sum_{t=0}^{\underline{t}} f(t) v_t = \sum_{s=0}^{\underline{t}} \sum_{t=0}^{\underline{t}} f(t) M_{t,s} v_s + \sum_{s=\underline{t}+1}^{\infty} \sum_{t=0}^{\underline{t}} f(t) M_{t,s} v_s.$$

Here, note that the second term is bounded above,

$$\sum_{s=\underline{t}+1}^{\infty} \sum_{t=0}^{\underline{t}} f(t) M_{t,s} v_s \quad < \quad f(\underline{t}) \sum_{s=\underline{t}+1}^{\infty} \sum_{t=0}^{\underline{t}} M_{t,s} v_s, \tag{60}$$

where we used the monotonicity of f. Further, by the regularity of $\mathbf{v}$ and the column-stochasticity of $\mathbf{M}$ we obtain,

$$\begin{aligned} \sum_{s=\underline{t}+1}^{\infty} \sum_{t=0}^{\underline{t}} M_{t,s} v_s &= \sum_{t=0}^{\underline{t}} \left(v_t - \sum_{s=0}^{\underline{t}} M_{t,s} v_s \right) = \sum_{t=0}^{\underline{t}} v_t - \sum_{s=0}^{\underline{t}} \sum_{t=0}^{\underline{t}} M_{t,s} v_s \\ &= \sum_{t=0}^{\underline{t}} v_t - \sum_{s=0}^{\underline{t}} \left(1 - \sum_{t=\underline{t}+1}^{\infty} M_{t,s} \right) v_s = \sum_{s=0}^{\underline{t}} \sum_{t=\underline{t}+1}^{\infty} M_{t,s} v_s, \end{aligned} \tag{61}$$

so that, when substituting back into (60), we arrive at

$$\sum_{s=\underline{t}+1}^{\infty} \sum_{t=0}^{\underline{t}} f(t) M_{t,s} v_s < f(\underline{t}) \sum_{s=0}^{\underline{t}} \sum_{t=\underline{t}+1}^{\infty} M_{t,s} v_s < \sum_{s=0}^{\underline{t}} \sum_{t=\underline{t}+1}^{\infty} f(t) M_{t,s} v_s. \tag{62}$$

Thus, using (62), the first sum of (59) is bounded above by

$$\sum_{t=0}^{\underline{t}} f(t) v_t < \sum_{s=0}^{\underline{t}} \sum_{t=0}^{\underline{t}} f(t) M_{t,s} v_s + \sum_{s=0}^{\underline{t}} \sum_{t=\underline{t}+1}^{\infty} f(t) M_{t,s} v_s = \sum_{s=0}^{\underline{t}} \sum_{t=0}^{\infty} f(t) M_{t,s} v_s.$$

The second sum of (59) is also bounded above,

$$\sum_{t=\underline{t}+1}^{\infty} f(t) v_t < \sum_{t=\underline{t}+1}^{\infty} \sum_{s=0}^{\infty} f(s) M_{s,t} v_t$$

where we used the fact that for large t, $\sum_s f(s) M_{s,t} > f(t)$. After relabeling of indices s and t, we then find that

$$\sum_{t=0}^{\infty} f(t) v_t < \sum_{s=0}^{\underline{t}} \sum_{t=0}^{\infty} f(t) M_{t,s} v_s + \sum_{t=\underline{t}+1}^{\infty} \sum_{s=0}^{\infty} f(s) M_{s,t} v_t = \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} f(s) M_{s,t} v_t = \sum_{t=0}^{\infty} f(t) v_t,$$

which is a contradiction. □

Lemma 9. *Let* $\mathbf{M}$ *be the (column-stochastic) transition matrix of a recurrent Markov chain with unique (up to scale) stationary measure* $\mathbf{v} = (v_t)$*. If* $f : \mathbb{N}_0 \to \mathbb{R}_+$ *is a strictly increasing function such that there exists* $\epsilon > 0$ *with* $\sum_s f(s) M_{s,t} < f(t)(1-\epsilon)$ *for all* t *outside a finite set of states, then* $\sum_{t=0}^{\infty} f(t) v_t < \infty$*.*

Proof. This proof is analogous to the proof of Proposition 1.4 in Hairer (2016). Let $\underline{t}$ be such that $\sum_s f(s) M_{s,t} < f(t)(1-\epsilon)$ for all $t > \underline{t}$. Define $f_N = f \wedge N$ for $N > 0$ and G_N by $G_N(t) = \sum_t f_N(s) M_{s,t} - f_N(t)$. G_N is negative outside a finite set and satisfies $G_N(t) \to \sum_t f(s) M_{s,t} - f(t) \equiv G(t)$ for all t (pointwise). By definition $\sum_t G_N(t) v_t = 0$ and by Fatou's lemma we therefore have

$$0 = \liminf_{N\to\infty} \sum_t G_N(t) v_t \le \sum_t G(t) v_t = \sum_{t \le \underline{t}} G(t) v_t + \sum_{t > \underline{t}} G(t) v_t < \sum_{t \le \underline{t}} G(t) v_t - \epsilon \sum_{t > \underline{t}} f(t) v_t.$$

The sum $\sum_{t=0}^{\infty} f(t) v_t$ is then bounded above by

$$\sum_{t=0}^{\infty} f(t) v_t < \epsilon^{-1} \sum_{t \le \underline{t}} G(t) v_t + \sum_{t \le \underline{t}} f(t) v_t < \infty.$$

□

C.2.3 Proof of Theorem 2

Again we use the terminology in Kemeny et al. (1976). For part (a), distinguish two cases: **M** transient and **M** recurrent. If **M** is transient, Theorem XXX in Kemeny et al. (1976) proves that for any vector **v** with finite present value, $|1'\mathbf{v}| < \infty$, $\sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{v}$ is finite-valued. This immediately implies that any right-eigenvector with eigenvalue 1 of **M** must have infinite present value $|1'\mathbf{v}|$, or the sum $1'\mathbf{v}$ must be otherwise ill-defined. In particular, this rules out a case where **v** is bounded in current values since that would immediately imply summability of the elements of **v**. Thus, in that case, $d\mathbf{Y}$ is locally determinate. If **M** is recurrent, let **v** be the unique (up to scale) stationary measure of **M**. Choose $\epsilon > 0$ so that $-r + \epsilon \in (-r, \liminf \lambda_t^*)$ and define $f(t) = e^{(r-\epsilon)t}$. By the strict convexity of $p_t(\lambda)$, it must then be that for sufficiently far out states $t \geq \underline{t}$ for some $\underline{t}$, $\sum_s f(s) M_{s,t} > f(t)$. Applying Lemma 8 below, we obtain that $\sum_{t=0}^{\infty} f(t) v_t = \infty$, so that $e^{rt} v_t = e^{\epsilon t} f(t) v_t$ is necessarily unbounded (or else $f(t) v_t$ were summable). Therefore, $d\mathbf{Y}$ is locally determinate.

Consider part (b). First notice that according to Lemma 7, there exists a $\lambda < 0$ with $\limsup_{t\to\infty} p_t(\lambda) < 1$. Second, **M** is necessarily recurrent, since $1 - \delta \equiv \limsup_{t\to\infty} p_t(\lambda) < 1$ implies that $\sum_s e^{-\lambda s} M_{s,t} - e^{-\lambda t} < -\delta e^{-\lambda t} < -\delta$ for sufficiently large t (and by negativity of λ). A straightforward application of Proposition 1.3 in Hairer (2016) establishes that **M** is recurrent.

Now, by convexity of $p_t(\lambda)$, the fact that $\limsup_{t\to\infty} \lambda_t^* < -r$ and $\limsup_{t\to\infty} p_t(\lambda) < 1$, it follows that for any $\epsilon \in (0, r)$, $\limsup_{t\to\infty} p_t(-r) = 1 - \delta < 1$ for some other $\delta > 0$. Defining $f(t) = e^{rt}$, this can be written as $\sum f(s) M_{s,t} - f(t) < -\delta f(t)$ for all sufficiently large t. Applying Lemma 9 below, we then obtain that $\sum_{t=0}^{\infty} f(t) v_t < \infty$ in this case, or, $f(t) v_t \to 0$. Therefore, v_t is bounded in current values and $d\mathbf{Y}$ is locally indeterminate.

The proof of the simple sufficient conditions is immediate.

C.2.4 Proof of Proposition 3

By Lemma 7, there exists a $\lambda < 0$ with $\limsup_{t\to\infty} p_t(\lambda) \equiv 1 - \delta < 1$.

Part (a): Define $f : \{0, 1, 2, \ldots\} \to \mathbb{R}_+$, $f(t) \equiv e^{-\lambda t}$. Then, $\limsup_{t\to\infty} p_t(\lambda) < 1$ implies that for all t sufficiently large,

$$\sum_{s=0}^{\infty} f(s) M_{s,t} < f(t) - \delta f(t).$$

By Proposition 1.3 in Hairer (2016), **M** is recurrent. In that case, the stationary measure of **M** is unique up to scale. The fact that any solution takes the form (33) is a direct consequence of the general IKC solution in Theorem 1. Finally, an application of Theorem XXX in Kemeny et al. (1976) proves that $\sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}$ has zero NPV.

Part (b): First, $\liminf_{t\to\infty} \lambda_t^* > 0$, together with $\limsup_{t\to\infty} p_t(\lambda) < 1$ and concavity of all p_t, implies that $\lambda \in (0, \liminf_{t\to\infty} \lambda_t^*)$. Thus, for sufficiently large t, say $t \geq \underline{t}$, $p_t(\lambda) < 1 - \epsilon$, where

$\epsilon \in (0, 1 - \limsup_{t\to\infty} p_t(\lambda))$, meaning

$$\sum_{s=0}^{\infty} e^{-\lambda(s-t)} M_{s,t} < 1 - \epsilon, \quad \forall t \geq \underline{t}. \tag{63}$$

Using Proposition 1.3 from Hairer (2016), this shows that $\mathbf{M}$ is transient. For that case, Kemeny et al. (1976) prove that $\mathbf{N} \equiv \sum_{k=0}^{\infty} \mathbf{M}^k$ is finite-valued (Theorem XXX), that $N_{s,t} \leq N_{s,s}$ for any $s, t \geq 0$ (Theorem XXX), and that $N_{s,s} \leq \left(1 - Pr_s\left(\tau_{\{s\}} < \infty\right)\right)^{-1}$ (Theorem XXX). Here, $Pr_s(A)$ denotes the probability that the Markov chain associated with $\mathbf{M}$ moves from initial state s to set $A \subset \{0, 1, 2, \ldots\}$; and τ_A denotes the hitting time of set A once one period has passed, that is, if (X_n) denotes the Markov chain, $\tau_A = \min\{n \geq 1 \,|\, X_n \in A\}$. We now prove that, under our assumptions, $N_{t,t}$ is uniformly bounded (above) in $t \geq 0$. Once that result is established, it follows immediately that $d\mathbf{Y}$ is bounded, entry by entry. Uniqueness follows from Theorem 2.

To prove that $N_{t,t}$ is uniformly bounded above, we again define the function $f : \{0, 1, 2, \ldots\} \to \mathbb{R}_+$, $f(t) \equiv e^{-\lambda t}$. We establish the bound in three steps. First, pick two times t_0, t_1 with $t_1 > t_0 \geq \underline{t}$ and define $A_0 \equiv \{0, 1, \ldots, t_0\}$. Then,[14]

$$\begin{aligned} f(t_0)^{-1} f(t_1) &> f(t_0)^{-1} \epsilon f(t_1) + f(t_0)^{-1} \sum_{s=0}^{\infty} e^{-\lambda s} M_{s,t} \\ &\geq f(t_0)^{-1} \sum_{s=0}^{t_0} e^{-\lambda s} M_{s,t} + f(t_0)^{-1} \sum_{s=t_0+1}^{\infty} e^{-\lambda s} M_{s,t} \\ &\geq Pr_{t_1}(A_0) + \sum_{s=t_0+1}^{\infty} f(t_0)^{-1} f(s) M_{s,t}, \end{aligned}$$

which by iteration, proves that $f(t_0)^{-1} f(t_1) \geq Pr_{t_1}(\tau_{A_0} < \infty)$. Since $f(t_0)^{-1} f(t_1) \leq e^{-\lambda} \equiv 1 - \delta$, this proves that

$$Pr_{t_1}(\tau_{A_0} < \infty) < 1 - \delta \quad \forall t_0, t_1 : \, t_1 > t_0 \geq \underline{t}. \tag{64}$$

Second, observe that (63) implies

$$\sum_{s=t+1}^{\infty} \left(e^{-\lambda(s-t)} - 1\right) M_{s,t} < \sum_{s=0}^{\infty} \left(e^{-\lambda(s-t)} - 1\right) M_{s,t} < -\epsilon$$

such that

$$Pr_t\left(\{t+1, t+2, \ldots\}\right) = \sum_{s=t+1}^{\infty} M_{s,t} \geq \sum_{s=t+1}^{\infty} \left(1 - e^{-\lambda(s-t)}\right) M_{s,t} > \epsilon \quad \forall t \geq \underline{t}. \tag{65}$$

Finally, we bring conditions (64) and (65) together to establish uniform boundedness of $N_{t,t}$.

[14] This argument is analogous to one made in the proof of Proposition 1.3 in Hairer (2016).

Pick any $t_0 \geq \underline{t}$. Then,

$$\begin{aligned} 1 - N_{t_0,t_0}^{-1} \leq Pr_{t_0}\left(\tau_{\{t_0\}} < \infty\right) &= Pr_{t_0}(\{t_0\}) + \sum_{t_1<t_0} Pr_{t_0}(\{t_1\}) Pr_{t_1}(\tau_{\{t_0\}} < \infty) + \sum_{t_1>t_0} Pr_{t_0}(\{t_1\}) \underbrace{Pr_{t_1}(\tau_{\{t_0\}} < \infty)}_{\leq Pr_{t_1}(\tau_{A_0}<\infty)} \\ &< 1 - Pr_{t_0}(\{t_0+1, t_0+2, \ldots\}) + Pr_{t_0}(\{t_0+1, t_0+2, \ldots\}) \cdot (1-\delta) \\ &= 1 - \delta \cdot Pr_{t_0}(\{t_0+1, t_0+2, \ldots\}) \\ &< 1 - \delta\epsilon. \end{aligned}$$

Thus, $N_{t_0,t_0} < (\delta\epsilon)^{-1}$ for $t_0 \geq \underline{t}$, where ϵ and δ are independent of t_0. This establishes that $\mathbf{N}$ is not only finite-valued but also bounded, element-by-element.

C.2.5 Proof of Proposition 5

We show this result in 2 steps: First, we show that any ordered matrix $\mathbf{M}$ preserves the "front-loaded" ordering under matrix multiplication. Second, we prove the main result.

Lemma 10. *Let $\mathbf{M} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ be an ordered, column-stochastic matrix. Let $\mathbf{v}' \in \mathbb{R}^{\mathbb{N}}$ be more front-loaded than $\mathbf{v} \in \mathbb{R}^{\mathbb{N}}$. Then, $\mathbf{M}\mathbf{v}'$ is more front-loaded than $\mathbf{M}\mathbf{v}$.*

Proof. □

Now we prove the main result. Notice that from Lemma 10 it immediately follows that $\sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}'$ is more front-loaded than $\sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}$. This establishes the result when $\mathbf{M}$ is transient (case (b) in Proposition 3). When $\mathbf{M}$ is recurrent, $\mathbf{w} \equiv \sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}$ and $\mathbf{w}' \equiv \sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y}'$ have zero NPV. In particular, this implies that eventually, for some $t \geq T$ for some T, $w_t' \geq w_t$. Let $\mathbf{v} \gg 0$ be the unique eigenvector of $\mathbf{M}$ with eigenvalue 1 , so we can write

$$d\mathbf{Y} = \sum_{k=0}^{\infty} \mathbf{M}^k \partial \mathbf{Y} + \lambda \mathbf{v},$$

and similarly for $d\mathbf{Y}'$. We have $\lambda \geq \lambda'$, and therefore $d\mathbf{Y}'$ is more front-loaded than $d\mathbf{Y}$.

C.2.6 Proof of Theorem 3

The idea behind the proof of Theorem 3 is an application of Corollary 2. In particular, we first compute the asymptotic characteristic polynomial $p^{joint}(\lambda)$ of the joint MPC matrix $\mathbf{M}^{joint}$ and then check under what conditions $p^{joint}(-r) \gtrless 1$.

First, the joint MPC matrix is given by

$$M_{s,t}^{joint} = M_{s,t} + M_{s,t}^R \left(\phi_y + \kappa\phi_\pi\right) + (\phi_\pi - \beta^{-1})\kappa \sum_{t'=0}^{t-1} (\beta R)^{t-t'} M_{s,t'}^R,$$

with characteristic polynomial

$$p_t^{joint}(\lambda) = p_t(\lambda) + p_t^R(\lambda)\left(\phi_y + \kappa\phi_\pi\right) + (\phi_\pi - \beta^{-1})\kappa \sum_{s=0}^{\infty}\sum_{t'=0}^{t-1} e^{-\lambda(s-t)} e^{-(\rho-r)(t-t')} M^R_{s,t'},$$

where we use the notation $\rho \equiv -\log\beta$. By exchanging the order of summation this simplifies to

$$p_t^{joint}(\lambda) = p_t(\lambda) + p_t^R(\lambda)\left(\phi_y + \kappa\phi_\pi\right) + (\phi_\pi - \beta^{-1})\kappa \sum_{t'=0}^{t-1} e^{-(\rho-r-\lambda)(t-t')} p^R_{t'}(\lambda).$$

Since $\mathbf{M}$ and $\mathbf{M}^R$ were assumed to be self-similar, for large t and $\lambda < \rho - r$ this converges to

$$p^{joint}(\lambda) = p(\lambda) + p^R(\lambda)\left(\phi_y + \kappa\phi_\pi\right) + (\phi_\pi - \beta^{-1})\kappa \frac{p^R(\lambda)}{(\beta R)^{-1}e^{-\lambda} - 1}.$$

Rearranging, $p^{joint}(-r) \gtrless 1$ if and only if

$$\frac{1-\beta}{\kappa}\phi_y + \phi_\pi \gtrless 1 + \frac{1-\beta}{\kappa}\frac{1-p(-r)}{p^R(-r)}.$$

The determinacy result then follows by applying Corollary 2.

D Fiscal theory of the price level

To be added.