SUSTAINABLE UNSECURED DEBT

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Abstract

We show that, in the absence of collateral or legal sanctions available to creditors, debt is sustainable at a competitive equilibrium by the debtor's reputation for repayment. Under incomplete markets, when the interest rate is recurrently negative (net of growth), self-insurance is more costly than borrowing and repayments on loans are enforced by the implicit threat of loosing the risk-sharing advantages of debt contracts. Private debt credibly circulates as a form of inside money and, in general, is not valued as a speculative bubble. We establish existence of a competitive equilibrium with self-enforcing debt under a suitable hypothesis of gains from trade.

1. Introduction

Conventional wisdom asserts that debt is unsustainable when not secured by collateral or by sanctions that creditors can exercise against debtors upon default. We instead argue that creditors can entirely rely on the debtors' mere reputation for repayment. We identify the implicit enforcement mechanism and show the existence of a competitive equilibrium with unsecured debt. In general, debt is not valued as a speculative bubble but fairly reflects the value of future repayments.

When creditors have no legal rights whatsoever, debtors are able to borrow only if they can maintain a reputation for repaying their loans (Eaton and Gersovitz [16]). In a seminal paper, Bulow and Rogoff [13] present their celebrated critique of the reputational theory of unsecured debt. The loss of reputation cannot prevent debtors from continuing to save in financial markets after default. When they maintain access to cash-in-advance insurance contracts, then reneging on their debt obligations becomes profitable eventually. Indeed, debtors that have reached the upper bound on liabilities may prefer to declare bankruptcy and to divert saved repayments to acquire cash-in-advance contracts. This is feasible and, as shown by Bulow and Rogoff [13], generates higher utility as long as the upper bound on debt is strictly positive. Creditors anticipate debtors' incentives to default and provide no loans at all.

In contrast with this conventional view, Hellwig and Lorenzoni [19] show that debt is sustainable in economies with a complete asset market when interest rates are determined endogenously to clear markets. Their insight relies on an equivalence between speculative bubbles and self-enforcing debt (a variation of Kocherlakota

[23]'s Bubble Equivalence Theorem). At a competitive equilibrium with a speculative bubble, an asset with no intrinsic value is used as store of value and provides liquidity services to individuals for intertemporal consumption smoothing, exactly as money does in Bewley [9, 10]'s monetary economy. At a competitive equilibrium with self-enforcing debt, instead, each individual issues private debt and this is valued in the market as a speculative bubble. In other terms, the privileges of issuing the speculative bubble are attributed to individuals as opposed to being initially entailed in an asset in positive net supply. In fact, incentives to default disappear because individuals are allowed to exactly roll-over outstanding debt period by period and, so, no effective repayment is enforced.[1]

We revisit the classic issue of whether reputation for repayment alone can sustain debt in competitive markets when saving provides self-insurance upon default. We treat unsecured debt as being uncontingent and so we focus on economies with incomplete markets. Our main objective is to show that, in the presence of uninsurable risks, reputation provides an effective mechanism for debt enforcement. Differently from a situation with complete markets, debt cannot be rolled over and, so, does not circulate as a speculative bubble. This reveals a substantive failure of Bulow and Rogoff [13]'s claim.[2]

How can debt be self-enforcing? Consider a situation in which the value of a claim into the debtor's entire future income is finite, as assumed by Bulow and Rogoff [13]. This provides an upper bound on the debtor's repayment capacity (the natural debt limit) and rules out the debt roll-over regime occurring in Hellwig and Lorenzoni [19]. Upon default no further debt can be issued and the debtor will have to rely on self-insurance. When markets are complete, saved repayments can be invested in specific portfolios of securities replicating equivalent risk-sharing advantages of debt contracts. As an interest rate also accrues on these investments, saved repayments more than compensate for the cost of self-insurance. When markets are incomplete, however, contingent claims for an equivalent insurance might not be tradable and, without issuing debt, risk-sharing contacts can only be acquired at a higher cost. Therefore, the loss due to the increased cost for self-insurance might well exceed the gain from default and this provides an implicit incentive to debt repayment.

When a risk-free bond is the only available security, for instance, consumption smoothing can only be obtained via uncontingent claims. When interest rate remains recurrently negative for a long phase with some probability (though somehow positive on average), the cost of self-insurance grows prohibitively high. This is so because saving depreciates at a negative interest rate. When debt can be issued,

[1] In the working paper version, Bulow and Rogoff [13] explicitly rule out these *Ponzi-type reputational contracts* by assuming a finite market value of the debtor's entire future income.

[2] In Bloise et al. [12] we identify sufficient conditions for the extension of Bulow and Rogoff [13]'s claim to incomplete markets. We also provide examples where their claim fails. Here we further refine the debt enforcement mechanism and we develop the analysis in a competitive equilibrium framework. Conditions for sustainable debt emerge endogenously.

instead, insurance obtains at a sensible lower cost, as outstanding debt can be refinanced at a negative interest rate. Hence, debt is a superior instrument and repayments are implicitly enforced by the threat of loosing borrowing privileges.

We show that, under incomplete markets, debt roll-over in general fails at a competitive equilibrium with self-enforcing debt. To this purpose, we develop a novel method based on the dominant root (Perron-Frobenius) approach.[3] Long-term interest rate is only ambiguously identified under incomplete markets and dominant roots provide bounds on its estimates. When long-term interest rate is unambiguously positive, debt is exploding in a roll-over regime. On the opposite, it is disappearing over time, along with trade, when long-term interest rate is unambiguously negative. Neither case is consistent with a competitive equilibrium in which trade persists. When long-term interest rate is never positive and occasionally negative, debt is sometimes refinanced at a discount and, in a roll-over regime, additional debt is in fact sustainable at equilibrium, a contradiction. As a result, a necessary condition for persistent debt roll-over is that long-term interest rate be unambiguously zero. This imposes excessively severe limitations, as interest rate will need to adjust upwards and downwards to clear markets.

We also establish the existence of a competitive equilibrium with self-enforcing debt. Our approach is innovative and exploits the dominant root to show that trade occurs at equilibrium. We perturb the economy by introducing a legal sanction: upon default a small fraction of the endowment is confiscated. This is sufficient to enforce repayment of any debt not exceeding the present value of confiscated resources. As a result, borrowing and lending occur in the perturbed economy and, at a competitive equilibrium, a claim into each debtor's entire future income is finite. We then progressively remove the auxiliary sanction and consider the limit with no confiscation. This is a competitive equilibrium of the original economy and trade occurs under a suitable *gains from trade* hypothesis: the implicit value of a claim into each debtor's entire income is infinite at autarky. Indeed, as this claim has a finite value in the perturbed economy, autarky cannot be the limit as the perturbation is removed. The dominant root is used to convert this intuition into an accurate argument.

The absence of debt roll-over provides a relevant insight on the relation between inside money and outside money. When markets are complete, Hellwig and Lorenzoni [19] establish an equivalence between an equilibrium with self-enforcing debt (inside money) and an equilibrium with unbacked public debt (outside money). We

[3]For complete markets, a similar approach is presented by Alvarez and Jermann [4] and Hansen and Scheinkman [17]. Their purpose is to derive a lower bound for the volatility of the permanent component of asset pricing kernels. Our purpose instead is to provide necessary conditions at a competitive equilibrium with self-enforcing debt for individuals to exactly roll over existing debt period by period. The analysis of the dominant root was developed in Bloise et al. [12] for Markov pricing kernels. An extension is necessary because the Markov property generally fails at a competitive equilibrium with self-enforcing debt under incomplete markets.

show that this coincidence in general fails under incomplete markets. With uninsurable risks, an equilibrium with self-enforcing debt is distinct from Bewley [9, 10]'s monetary equilibrium. Self-enforcing debt is backed by the credible promise of future repayments, as the implicit mechanism guarantees that outstanding claims are honored. In particular, the value of a claim into the debtor's future income (the natural debt limit) is finite, as in any competitive equilibrium with full commitment. Money, instead, only circulates as a speculative bubble and requires an infinite present value of future income. A relevant implication is that self-enforcing debt is consistent with empirical tests ruling out speculative bubbles and ascertaining dynamic efficiency of the economy.

We complement our analysis with an exploration of incentives to default for exogenously given Markov pricing kernels. This is the privileged framework of the sovereign debt literature, where the pricing kernel is interpreted as the valuation of foreign investors who provide credit under full commitment.[4] Under incomplete markets, when no future borrowing is the punishment for default, we show that debt might be sustainable when foreign investors are risk-averse and risk-premia vary along the business cycle.[5] When the pricing kernel is sufficiently volatile, the foreign investors' value of a claim into the borrower's entire future income is finite (high implied interest rates), but the value of the claim becomes infinite when evaluated using other legitimate state prices (low implied interest rates). On one hand, high implied interest rates preclude debt roll-over or, according to Bulow and Rogoff [13], Ponzi-type reputational contracts. On the other hand, low implied interest rates render borrowing more appealing, and saving after default more costly. When these effects trade off, default is not profitable and debt is sustained by the mere reputation for repayment.

We briefly relate our contribution to previous studies in two separate branches of literature: sovereign debt and money. Bulow and Rogoff [13]'s objections to reputation for repayment posed a powerful challenge to the notion that the threat of exclusion from credit markets, by itself, supports sovereign borrowing. The literature evolved in three distinct directions. In a first line of research, debt repayment is sustained by direct punishments, interpreted as the outcome of interferences with the debtor's transactions upon default (for instance, Bulow and Rogoff [14]). A second line of research develops the idea that governments repay because they are worried about the repercussions of default in the credit market (for instance, Kletzer and Wright [22]). In a third line of research, finally, incentives to repay sovereign debt are created because defaults inflict broader adverse effects on a borrower's

[4]This framework is inspired by Eaton and Gersovitz [16] and extensively studied, among others, by Aguiar and Amador [1], Auclert and Rognlie [7] and Wright [27].

[5]Recent quantitative work on sovereign default risk moves away from the traditional risk-neutral pricing to provide a better understanding of risk premia, the term-structure of interest rate and movements along the business cycle (see Arellano [5], Arellano and Ramanarayanan [6] and Hatchondo et al. [18]).

reputation (for instance, Cole and Kehoe [15]). All these previous studies move from Bulow and Rogoff [13]'s critique and explore alternative, and more effective, mechanisms for debt enforcement. We instead rely on the debtor's mere reputation for repayment and show a limited validity of Bulow and Rogoff [13]'s claim with residual uninsurable risks.

In Bewley [9, 10]'s monetary economy, the (unrepresented) absence of enforcement mechanisms prevents credit markets from operating and an unbacked currency plays a relevant social role in helping individuals smooth their consumption over time. We show that, in analogous conditions, unsecured private debt also provides an alternative viable store of value for risk-sharing. In particular, when only borrowing is prevented after default, the same economy admits a monetary equilibrium and an equilibrium with self-enforcing debt. Differently, no monetary equilibrium (with persistent trade) can be supported when debt contracts are enforced by the threat of permanent exclusion from credit markets upon default, as in Kehoe and Levine [20] and Alvarez and Jermann [3]. Finally, by identifying suitable conditions in terms of gains from trade, our insights clarify the extent of mutually beneficial trades in a *default-free* safe bond under limited commitment.

The paper is organized as follows. We begin with the presentation of simple example in §2. In §3 we describe the economy and provide the concept of competitive equilibrium with self-enforcing debt. As debt sustainability depends on long-term interest rate, in §4 we present our dominant root approach. In §5 the dominant root method is used to identify a necessary condition for debt roll-over, which can only be satisfied in singular situations when markets are incomplete. In §6 we establish the existence of an equilibrium. In §7 we further explore the incentives to debt repayment for a given Markov pricing kernel. Finally, we present some concluding remarks. All proofs are collected in Appendix A.

2. Motivating Example

How can debt be sustainable when it is not secured by collateral or legal sanctions? We here present a simple example in order to provide the basic intuition and to disentangle the enforcing mechanism. When interest rate is recurrently negative, self-insurance might be too costly and debt might provide insurance services more efficiently than other instruments. Thus, debt might be implicitly secured by the threat of diminished insurance opportunities upon default, contrary to Bulow and Rogoff [13]'s claim. In the example, these conditions are determined *ad hoc* for heuristic purposes. As our general analysis illustrates, however, they will naturally emerge at a competitive equilibrium under incomplete markets.

In each period there are two states of nature, $S = \{l, h\}$, occurring with equal probability. A risk-free (discount) bond is the only security and its price is either $q_h > 1$ or $q_l < 1$. An individual can trade the risk-free bond over time, issuing debt when needed, but she cannot commit to repayment. As in Bulow and Rogoff [13], the denial of future credit is the only punishment for default. Preferences on consumption streams are given by a conventional discounted expected utility. To simplify computations, the borrower is risk-neutral and the endowment is constantly $e > 0$. We also assume that the discount factor δ lies in $(0, q_l)$. This ensures that the individual is sufficiently impatient and will never save after default. Autarky is, so, the reservation utility.

We consider a simple consumption plan in which outstanding debt remains constant over time. This is not a roll-over regime because repayments occur and, though interest rate is recurrently positive along some path, debt is not exploding. The outstanding stock of debt is $d > 0$, while consumption is, depending on the current price of the bond,

$$c_s = e + (q_s - 1)\, d > 0.$$

The feature of this plan is that, when interest is positive ($q_l < 1$), some resources are devoted to debt service; when interest rate is negative ($q_h > 1$), debt can be refinanced at no cost and excess resources are diverted to consumption. The stock of debt remains unaltered over time.[6] Is default profitable under these conditions?

Because of our simplifying assumptions, the accounting is straightforward. The expected discounted utility conditional on no default is

$$U_s(d) = e + (q_s - 1)\, d + \left(\frac{\delta}{1-\delta}\right)\left(e + \left(\frac{q_h - 1}{2}\right) d + \left(\frac{q_l - 1}{2}\right) d\right).$$

As saving is never optimal, autarky is the expected discounted utility upon default, that is,

$$U_s(0) = e + \left(\frac{\delta}{1-\delta}\right) e.$$

As a matter of fact, default is unprofitable if and only if

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{q_h - 1}{2}\right) \geq (1 - q_l) + \left(\frac{\delta}{1-\delta}\right)\left(\frac{1 - q_l}{2}\right).$$

The right hand-side is the value of saved repayments, whereas the left hand-side is the value of excess consumption afforded by refinancing debt at a negative interest rate. The condition is certainly satisfied for fixed $q_h > 1$ provided that $q_l < 1$ is sufficiently close to unity. Hence, debt is sustainable.

The example clarifies why Bulow and Rogoff [13]'s arbitrage argument fails under incomplete markets. When debt is not rolled over, repayments are necessarily enforced beginning from some contingency. By defaulting the borrower saves on these repayments at the cost of no further debt in the future. When markets are complete, saved repayments can be used to pay upfront for the same insurance as

[6] Though this is not essential to our argument, it is worth noticing that this plan is also optimal if $d > 0$ is exactly the borrowing limit.

before default and, as a result, denial of future credit bears no effective cost. This arbitrage is precluded in the example because markets are incomplete. Indeed, interest rate might remain negative for a long phase. Before default the borrower benefits from refinancing outstanding debt. After default the upfront value of a positive net consumption can be arbitrarily large, because a negative interest rate accrues on savings. Thus, default entails a large cost, whereas the gain from saved repayments might be relatively small (and in fact vanishes when $q_l = 1$).

Debt cannot be rolled over in this example. In fact, it is bounded by the least present value of future endowment (the natural debt limit), that is,

$$d \leq \sum_{t=0}^{\infty} q_l^t e = \left(\frac{1}{1 - q_l} \right) e.$$

This rules out Ponzi games and enforces recurrent repayments. In Hellwig and Lorenzoni [19], instead, borrowers simply roll-over their debt obligations and no debt repayment is enforced. This situation occurs only under specific conditions at a competitive equilibrium when markets are incomplete (see §5).

In the sovereign debt literature, beginning with Eaton and Gersovitz [16], the pricing kernel is determined by the valuation of foreign investors whose commitment is enforced by the legal system. When markets are complete, this induces a finite value of the sovereign's future income and debt is unsustainable, as established by Bulow and Rogoff [13]. Under incomplete markets, instead, foreign investors' valuation might be finite and, yet, debt might be sustainable. To accommodate this in our example, we can assume that the typical foreign lender is risk-averse and risk-premium is time-varying because of the international business cycle. The price of the bond is determined by marginal rates of substitution, that is,

$$q_h = \delta^* \left(\mu_{hh} \frac{u'(c_h^*)}{u'(c_h^*)} + \mu_{hl} \frac{u'(c_l^*)}{u'(c_h^*)} \right), \tag{2.1}$$

$$q_l = \delta^* \left(\mu_{lh} \frac{u'(c_h^*)}{u'(c_l^*)} + \mu_{ll} \frac{u'(c_l^*)}{u'(c_l^*)} \right), \tag{2.2}$$

where (c_l^*, c_h^*) are the foreign investors' consumption levels, δ^* in $(0, 1)$ is their discount factor and μ denotes the transition probabilities. By discounting, the present value of sovereign's endowment is certainly finite when computed at state prices corresponding to discounted marginal utilities. Hence, when markets are incomplete, the valuation of foreign investors might be finite even though the borrower has no incentive to default.[7]

[7] A more speculative way of explaining this property is that, under incomplete markets, the hypothesis of high implied interest rates (that is, a finite present value of the endowment) might hold true at some state prices and fail at some other state prices. Foreign investors' marginal valuation corresponds to state prices for which implied interest rates are high. This does not exclude that the value is infinite for some other state prices.

Along the lines of this simple example, in Appendix B we consider a competitive economy with two risk-neutral individuals in which gains to trade arise because of disparities in marginal utilities.[8] Depending on idiosyncratic shocks, each individual will be issuing debt, retrieving it and providing credit to the other individual. As in the previous example, debt cannot be rolled over because the natural debt limit is finite. If debt were to exceed the natural limit, its value would be growing unboundedly over time. As the debt of an individual is balanced by the credit of the other individual, this situation would necessarily imply an over-accumulation of assets for some investor and thus a violation of the transversality condition.

3. The economy

3.1. **Time and uncertainty.** The economy extends over an infinite horizon, $\mathbb{T} = \{0, 1, 2, 3, \ldots\}$. Uncertainty is represented by a probability space, $(\Omega, \mathcal{F}, \mu)$ and a filtration $(\mathcal{F}_t)_{t \in \mathbb{T}}$ of σ-algebras. To simplify, and to avoid issues of integrability, it is assumed that, for every t in $\mathbb{T}$, $\mathcal{F}_t$ is a σ-algebra generated by a *finite* partition of Ω. Given a state of nature ω in Ω, at every period t in $\mathbb{T}$, $\mu(\mathcal{F}_t(\omega)) > 0$, where $\mathcal{F}_t(\omega) = \cap \{E_t \in \mathcal{F}_t : \omega \in E_t\}$ represents the (publicly) available information. Throughout the analysis, we shall refer to any of such primitive events as a *contingency*. In the equivalent event-tree representation of uncertainty, a contingency corresponds to a date-event, or a node.

3.2. **Adapted processes.** Let L be the linear space of all adapted processes with values in $\mathbb{R}$, *i.e.*, of all maps $f : \mathbb{T} \times \Omega \to \mathbb{R}$ such that, for every t in $\mathbb{T}$, $f_t : \Omega \to \mathbb{R}$ is $\mathcal{F}_t$-measurable, and let L_t be the space of such $\mathcal{F}_t$-measurable maps. An adapted process f in L is positive (respectively, strictly positive) whenever, at every t in $\mathbb{T}$, $f_t(\omega) \geq 0$ (respectively, $f_t(\omega) > 0$) for all ω in Ω. As usual, L^+ denotes the positive cone of L.

3.3. **Preferences and endowments.** There is a finite set I of individuals. For every individual i in I, the *consumption space* C^i is L^+, the positive cone of L, and the *endowment* is e^i in C^i.

Assumption 3.1 (Endowment)**.** The endowment e^i in C^i is uniformly positive and uniformly bounded with respect to the aggregate, that is, for some sufficiently large $\epsilon_u > 0$ and some sufficiently small $\epsilon_l > 0$, at every t in $\mathbb{T}$,

$$\epsilon_l e_t \leq e_t^i \leq \epsilon_u e_t,$$

where the strictly positive process e in L^+ is the aggregate endowment.

[8] The example is rather convoluted. In general, it is hard to construct simple competitive equilibria under incomplete markets. When risk is uninsured, shocks alter the distribution of wealth and prices need to vary in order to guarantee market clearing (see, for instance, Kehoe and Levine [21, Proposition 7]). Furthermore, when debt is self-enforcing, the endogenous value of default has to be determined explicitly, adding an independent complication.

To simplify, we impose restrictive assumptions on preferences, though this is unnecessary for most of our analysis. Every individual is characterized by a canonical expected discounted utility. Preferences over the consumption space C^i are induced, at every period t in $\mathbb{T}$, by

$$U_t^i\left(c^i\right) = \mathbb{E}_t \sum_{s \in \mathbb{T}} \delta^s u^i\left(c_{t+s}^i\right),$$

where δ in $(0,1)$ is the common discount factor. A sort of Strong Inada Condition helps the arguments by ensuring strictly positive consumption.

Assumption 3.2 (Utility)**.** Per-period utility $u^i : \mathbb{R}_{++} \to \mathbb{R}$ is smooth, smoothly strictly increasing and smoothly strictly concave. Furthermore, it is bounded from above and satisfies the Strong Inada Condition, namely,

$$\lim_{c^i \to 0} u^i\left(c^i\right) = -\infty.$$

A consumption plan c^i in C^i is *individually rational* if, at every t in $\mathbb{T}$,

$$U_t^i\left(c^i\right) \geq U_t^i\left(e^i\right).$$

Notice that individual rationality is imposed at all contingencies and not only *ex ante*. An allocation c in C specifies a consumption plan c^i in C^i for every individual i in I. It is *feasible* if each consumption plan c^i in C^i is individually rational and, at every t in $\mathbb{T}$,

$$\sum_{i \in I} c_t^i \leq \sum_{i \in I} e_t^i.$$

The space of feasible allocations is denoted by $C(e)$. A simple lemma clarifies that individually rational consumption will be uniformly positive due to the Strong Inada Condition.

Lemma 3.1 (Lower bound on consumption)**.** *When the aggregate endowment e in L^+ is uniformly positive, every individually rational consumption plan c^i in C^i is also uniformly positive.*

3.4. **Competitive markets.** Short-term securities are sequentially traded in competitive markets. Available assets might not allow for all contingent transfers of resources, that is, markets are sequentially incomplete. As specific features play no direct role in our analysis, we opt for a parsimonious primitive description of the set of securities via pricing and payoff functionals. Our framework encompasses sequentially complete markets as a particular case. In some of our analysis, we restrict attention to economies in which only the risk-free bond is available.

A finite set J of securities is traded over time in the markets. At every period t in $\mathbb{T}$, security j in J is described by a market price q_t^j in L_t and a possibly contingent payoff $R_{t,t+1}^j$ in L_{t+1} at the following period. A trading plan z in Z is an adapted process in L^J, where z_t in Z_t is the portfolio of securities held in period t in $\mathbb{T}$.

Portfolios are priced by the linear functional $q_t : Z_t \to L_t$ and yield a contingent payoff according to the linear functional $R_{t,t+1} : Z_t \to L_{t+1}$. Notice that, when the risk-free bond is the only asset, the portfolio z_t is a simple random variable in L_t and the payoff functional takes the form $R_{t,t+1}(z_t) = z_t$.

At every t in $\mathbb{T}$, each individual i in I is subject to a budget constraint,

$$q_t\left(z_t^i\right) + c_t^i \leq e_t^i + v_t^i, \tag{3.1}$$

where wealth v^i in V^i (the space of adapted processes in L) evolves according to

$$v_{t+1}^i = R_{t,t+1}\left(z_t^i\right).$$

A solvency constraint imposes

$$-g_{t+1}^i \leq R_{t,t+1}\left(z_t^i\right), \tag{3.2}$$

where g^i in G^i is the adapted process of debt limits restricting trade in securities. Mandatory saving is ruled out, so we assume that debt limits are positive (that is, they belong to L^+).

A solvency constraint of the form (3.2) was initially introduced by Zhang [28] and lately adopted by Alvarez and Jermann [3]. It is an indirect portfolio restriction: an individual might hold a portfolio implying a promise to delivery at some future contingency (*i.e.*, $R_{t,t+1}\left(z_t^i\right) < 0$) only if this liability does not exceed the threshold given by debt limits. Notice that, because any portfolio with positive payoffs is still allowed (g^i lies in L^+), solvency constraints do not interfere with the traditional no arbitrage theorem and, hence, redundant securities are priced at parity. Finally, when the risk-free bond is the only security, solvency constraint (3.2) takes the simpler form

$$-g_{t+1}^i \leq z_t^i.$$

This basically restricts debt up to the minimum limit at future contingencies.

Beginning from any contingency in period t in $\mathbb{T}$, an individual maximizes expected discounted utility subject to budget and solvency constraints. Conditional on no default, the indirect utility is denoted by $J_t^i\left(v_t^i, g^i\right)$. It depends on the available initial wealth v_t^i in V_t^i, inherited from the past, and on the entire future adapted process for debt limits g^i in G^i, as well as on the process of security prices q in Q. We need to ensure that, as commitment is limited, default is never profitable.

3.5. **Not-too-tight debt limits.** In line with Bulow and Rogoff [13] and Hellwig and Lorenzoni [19], default entails the seizure of all assets and the loss of access to future borrowing opportunities. When only a risk-free bond is available, this implies that the bond cannot be sold short anymore. When multiple securities are available, a defaulter is allowed to form portfolios as long as they do not involve any future liabilities, that is, obligations to deliver at some future contingencies. Thus, all insurance contracts remain available after default provided they imply positive contingent payoffs.[9] Debt limits are set so that no debtor has an incentive to default and no lender can profit from extending credit beyond a borrower's debt limit.

Formally, as in Alvarez and Jermann [3] and Hellwig and Lorenzoni [19], debt limits that are *not too tight* allow for the maximum amount of credit that is compatible with repayment at all contingencies. This requires that, for every individual i in I, at every t in $\mathbb{T}$,

$$J_t^i\left(-g_t^i, g^i\right) = J_t^i\left(0,0\right). \tag{3.3}$$

The left hand-side is the value of market participation, beginning with the maximum sustainable debt, whereas the right hand-side is the value of default. Indeed, upon default, all assets are cleared ($\hat{v}_t^i = 0$) and no borrowing is permitted in the future ($\hat{g}^i = 0$). Debt limits are not-too-tight if the individual is indifferent between repaying and defaulting.

3.6. **Competitive equilibrium.** Given the initial distribution of wealth v_0 in V_0, a *competitive equilibrium with self-enforcing debt* consists of an allocation c in C, a price q in Q, trading plans z in Z and debt limits g in G such that the following conditions are satisfied.

(a) For every individual i in I, given initial claims v_0 in V_0, the plan $\left(c^i, z^i\right)$ in $C^i \times Z^i$ is optimal subject to budget constraints (3.1) and solvency constraints (3.2) at debt limits g^i in G^i.

(b) Commodity and financial markets clear, that is, at every t in $\mathbb{T}$,

$$\sum_{i\in I} c_t^i = \sum_{i\in I} e_t^i \text{ and } \sum_{i\in I} z_t^i = 0.$$

(c) For every individual i in I, debt limits g^i in G^i satisfy the not-too-tight condition (3.3).

This concept of equilibrium follows exactly Alvarez and Jermann [3], except that the default punishment is the denial of future credit, instead of complete autarky. When markets are complete, it coincides with the equilibrium with self-enforcing debt studied by Hellwig and Lorenzoni [19].

4. Dominant root

Whether debt is sustainable or not depends on long-term interest rate (net of growth). As proved by Bulow and Rogoff [13], when the interest rate is positive, default will necessarily occur at some contingency, unless debt is rolled over. Debt

[9]When some securities cannot be traded after default, or portfolios are further restricted by no short sale constraints, debt is implicitly secured by the cost of restricted access to some financial instruments (see Pesendorfer [25]). Admitting any trade involving no future liabilities is more in the spirit of Bulow and Rogoff [13]'s idea of *cash-in-advance* contracts after default, that is, upfront payments for future contingent deliveries.

would be exploding under a roll-over regime, a situation which is inconsistent with equilibrium, because some other individual would be over-accumulating assets and violating the necessary transversality condition. When interest rate is negative, on the other side, debt would be imploding over time, so as to disappear in the long-run. Debt is sustainable only if interest rate is neither persistently negative nor persistently positive in the long-run. This clear intuition is obscured by the fact that interest rate will be in general varying over time and across states. Thus, to identify exact conditions, we need to develop a simple theory of long-term interest rate.

We introduce an elementary dominant root approach in order to estimate the long-term interest rate. Because markets are incomplete, these estimates identify an upper bound and a lower bound only. Indeed, long-term bonds are not traded and their payoffs might not be replicable by available securities. As the pricing kernel is in general not Markovian at a competitive equilibrium, no restriction is imposed apart from the fact that securities are priced under no arbitrage. To avoid uninteresting situations, we also suppose that, at every t in $\mathbb{T}$, there is a portfolio z_t^* in Z_t such that $R_{t,t+1}(z_t^*) > 0$ at all contingencies. This is certainly satisfied when the risk-free bond is available.

As in the traditional no arbitrage theory (*e.g.*, LeRoy and Werner [24]), at every t in $\mathbb{T}$, we define the *valuation functional* $\Pi_t : L_{t+1} \to L_t$ as

$$\Pi_t(b_{t+1}) = \inf_{z_t \in Z_t} q_t(z_t)$$

subject to

$$b_{t+1} \leq R_{t,t+1}(z_t).$$

This gives the minimum expenditure to meet future obligations, conditional on available securities. Formally, this valuation defines a monotone sublinear functional.[10]

Ideally, long-term interest rate would be estimated by the dominant eigenvalue of the valuation operator, as in the Perron-Frobenius Theorem, and this approach is developed in Appendix D for Markov pricing kernels. A general theory is unfortunately unavailable under our weak assumptions and we need to provide an alternative suitable method. To ensure existence, we define dominant roots as accumulation points and this will suffice to our purposes.

As dominant roots will capture long-term interest rate net of the rate of growth of the economy, we first need to consider a suitable space. Let $L(e)$ stand for all

[10]Notice that, under no arbitrage, the cost-minimizing portfolio exists, that is, there is a portfolio z_t in Z_t such that

$$\Pi_t(b_{t+1}) = q_t(z_t)$$

and

$$b_{t+1} \leq R_{t,t+1}(z_t).$$

Indeed, remember that all σ-algebras are generated by finite partitions and, hence, the valuation functional only involves cost-minimization programs in finitely dimensional spaces.

adapted processes that are bounded by some expansion of e in L^{+}, that is,

$$L(e) = \{x \in L : |x| \leq \lambda e \text{ for some } \lambda > 0\}.$$

This space contains all streams of contingent payoffs that do not grow unboundedly relative to the aggregate endowment. The *upper dominant root* $\rho(q)$ in $\mathbb{R}_{+}$ is the greatest ρ in $\mathbb{R}_{+}$ such that, for some non-zero b in $L^{+}(e)$, at every t in $\mathbb{T}$,

$$\rho b_t \leq \Pi_t(b_{t+1}).$$

Similarly, the *lower dominant root* $\gamma(q)$ in $\mathbb{R}_{+}$ is the greatest γ in $\mathbb{R}_{+}$ such that, for some non-zero b in $L^{+}(e)$, at every t in $\mathbb{T}$,

$$\gamma b_t \leq -\Pi_t(-b_{t+1}).$$

Notice that, as the valuation functional is monotone sublinear,

$$\gamma(q) \leq \rho(q).$$

Upper and lower dominant roots are well-defined (admitting positive infinity as a value), though we cannot in general establish the existence of their associated eigen-processes. Neither we can provide an operational criterion for their computation under general pricing kernels. Heuristically, the upper dominant root estimates the minimum growth rate of savings, whereas the lower dominant root corresponds to the maximum growth rate of debt, both relative to the aggregate endowment. We intuitively discuss some properties of dominant roots and relegate a formal analysis of simple irreducible Markov pricing kernels to Appendix D.

When only a risk-free bond is traded, dominant roots admit an elementary characterization for simple Markov pricing kernels: they correspond, respectively, to the greatest and to the least price of the bond or, equivalently, to the least and to the greatest interest rate. When multiple securities are traded, this link becomes less transparent: for instance, the upper dominant root might be less than unity ($\rho < 1$) even though the interest rate is recurrently negative over time and thus the greatest price of the risk-free bond is greater than unity. When markets are incomplete, in general, the upper and the lower dominant root will be distinct.

Dominant roots act as sort of discount factors for future contingent claims. When markets are incomplete, the present value of future claims is ambiguous, because they cannot be replicated using available securities. The greatest valuation is finite when $\rho < 1$, whereas a finite least valuation occurs provided that $\gamma < 1$. This bears a relevant implication for debt sustainability. Indeed, the analysis in Bloise et al. [12] assumes an exogenously given pricing kernel satisfying $\rho < 1$ and shows that debt is unsustainable. Differently, in this paper debt is sustainable because $\rho \geq 1$ at a competitive equilibrium. When $\gamma < 1$, the least present value of the endowment is finite and this restricts the debt capacity of an individual: any debt exceeding this natural debt limit cannot be honored. When $\gamma \geq 1$, instead, any arbitrary amount of debt can be repaid in finite time. Clearly, under limited commitment, default might occur even when debt repayment is feasible.

5. Debt roll-over

The dominant root approach permits us to provide a better understanding of conditions under which debt is sustainable at a competitive equilibrium. Under complete markets, Hellwig and Lorenzoni [19] prove that self-enforcing debt limits necessarily allow borrowers to exactly roll over existing debt, that is, to exactly refinance outstanding obligations by issuing new claims. In fact, equilibrium allocations with self-enforcing private debt are equivalent to allocations that are sustained by unbacked public debt subject to no borrowing. Repayments are not enforced and private debt circulates as a speculative bubble. We show that, under incomplete markets, this roll-over property fails, in general, when interest rate is time-varying. Debt repayments are enforced by a proper reputational mechanism, violating Bulow and Rogoff [13].

We consider a competitive equilibrium with non-vanishing sustainable debt.[11] We say that debt limits allow for *persistent debt roll-over* whenever, for some individual i in I, there is an adapted process $b^i \leq g^i$ in the interior of $L^+(e)$ such that $b_0^i = g_0^i$ and, at every t in $\mathbb{T}$,

$$\Pi_t\left(-b_{t+1}^i\right) = -b_t^i.$$

This condition guarantees that, beginning from the initial period, any debt level not exceeding g_0^i in L_0^+ can be perpetually refinanced by issuing further debt subject to solvency constraints.[12] Over time the individual can repay an amount b_t^i in L_t^+ of outstanding debt by issuing additional debt up to levels b_{t+1}^i in L_{t+1}^+ (see footnote 10). Furthermore, debt roll-over is persistent because the adapted process b^i belongs to the interior of $L^+(e)$ and, hence, the amount of debt that can be refinanced does not vanish along any path relative to the aggregate endowment. An example clarifies our definition and the role of the component b^i in $L^+(e)$ distinguished from debt limits g^i in G^i.

Example 5.1 (Debt roll-over)**.** Suppose that uncertainty is governed by a Markov process on the state space $S = \{l, h\}$ and that the risk-free bond is the only asset. Also assume that the price of the risk-free bond is constantly $q = 1$, irrespectively from the Markov state s in S. Finally, the aggregate endowment is constant. Debt limits are $g_h > 0$ and $g_l > 0$, with $g_h > g_l$. In such a situation, debt roll-over occurs

[11] When debt is unsecured, expectations of future deterioration of solvency conditions might be self-fulfilling and trade might vanish in the long-run. Debt is sustainable but vanishes over time, inducing no trade in the limit. We neglect competitive equilibria of this nature and focus on those in which trade, and hence debt, occurs persistently.

[12] The initial period is used only for narrative convenience. When debt roll-over occurs from some other period, all our arguments apply to the equilibrium beginning from a future contingency.

for $(b_l, b_h) = (g_l, g_l)$. Notice that, however, the amount $g_h > 0$ of debt in state h in S cannot be rolled over, because solvency constraint would be violated in future state l in S.

Our purpose is to verify under which conditions persistent debt roll-over occurs at a competitive equilibrium. In a stationary economy, the intuition is provided by a simple situation in which the risk-free bond is the only security and its price persistently fluctuates between an upper bound $\rho > 0$ and a lower bound $\gamma > 0$. Along a path in which the price of the bond is constantly $\gamma > 0$, debt evolves according to

$$\gamma b_{t+1} = b_t.$$

The path would be exploding when $\gamma < 1$, and imploding when $\gamma > 1$. Neither case is consistent with persistent debt roll-over. Hence, $\gamma = 1$. In such a condition, any plan under no borrowing could be replicated along with the limited Ponzi game permitted by debt limits, because debt can always be refinanced at a non-positive interest rate. Supposing that $\rho > 1$, the cost of refinancing the debt would be occasionally lower, so yielding additional consumption with respect to no borrowing, a contradiction. Hence, $\rho = 1$. In other terms, persistent roll-over occurs only if interest rate is constantly zero.

Proposition 5.1 (Roll-over property)**.** *Persistent debt roll-over occurs at a competitive equilibrium only if*

$$\rho(q) = \gamma(q) = 1. \tag{5.1}$$

Necessary condition (5.1) reveals that persistent debt roll-over is a fragile property. Indeed, under incomplete markets, upper and lower dominant root will in general be distinct when the pricing kernel involves some volatility and, so, condition (5.1) will fail. We discuss this fragility in a *bounded* economy subject to *aggregate uncertainty* in which *only* a risk-free bond is traded.

Notice that, when persistent roll-over occurs at equilibrium, there is an adapted process b in the interior of $L^+(e)$ such that, at every t in $\mathbb{T}$,

$$b_t = q_t b_{t+1},$$

where q in L^+ is the price of the risk-free bond.[13] For a bounded economy, this implies that the long-term interest rate is zero along any path, that is,

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=0}^{n-1} q_{t+k}} = 1. \tag{5.2}$$

Thus, debt roll-over imposes severe restrictions at a competitive equilibrium with aggregate uncertainty: interest rate will require downward or upward adjustments

[13]This property is the established condition (A.3) in the proof of Proposition 5.1.

during phases of prosperity or recession and this flexibility is precluded by necessary condition (5.2). We first explain this heuristically and then exhibit some conditions on primitives. These conditions are sufficient but far from necessary.

During phases of prosperity, individuals will have a tendency to accumulate assets for precautionary motives, because recessions are expected in the future. Markets will clear only if these savings are balanced by a corresponding supply of bonds. To provide incentives to borrowing, interest rate will need to go through downward adjustments and, under some conditions, will be recurrently negative. More formally, notice that first-order conditions require, at every t in $\mathbb{T}$, that

$$q_t \geq \max_{i \in I} \delta \mathbb{E}_t \frac{\nabla u\left(c_{t+1}^i\right)}{\nabla u\left(c_t^i\right)}$$

Thus, under prudence (*i.e.*, marginal utility is weakly convex),

$$q_t \geq \max_{i \in I} \delta \frac{\nabla u\left(\mathbb{E}_t c_{t+1}^i\right)}{\nabla u\left(c_t^i\right)}.$$

When output declines with positive probability, expected consumption will necessarily decrease for some individual and, when individuals are sufficiently patient,

$$q_t \geq \max_{i \in I} \delta \frac{\nabla u\left(\mathbb{E}_t c_{t+1}^i\right)}{\nabla u\left(c_t^i\right)} > 1.$$

Along a path of persistent prosperity, interest rate will be recurrently negative, thus contradicting condition (5.2).

Example 5.2 (Aggregate uncertainty)**.** To identify simple assumptions on fundamentals, consider an economy in which shocks to the aggregate endowment are identically and independently distributed. Also, assume that all individuals have constant relative risk-aversion $\sigma > 0$. Let $\left(c_s^i\right)_{i \in I}$ in $\mathbb{R}_+^I$ and $\left(c_{\hat{s}}^i\right)_{i \in I}$ in $\mathbb{R}_+^I$ be, respectively, the distribution of consumptions in current state s in S and the distribution of consumptions in future state $\hat{s}$ in S. Restriction (5.2) is necessarily violated (see Claim A.1) when

$$\sum_{\hat{s} \in S} \mu_{\hat{s}} e_{\hat{s}} < \delta^{\frac{1}{\sigma}} e_s, \tag{5.3}$$

where e in $\mathbb{R}_+^S$ denotes the aggregate endowment. This condition is certainly satisfied when individuals are sufficiently patient for an appropriate choice of the current state. On the other side, in order to have trade at equilibrium with self-enforcing debt (see Proposition 6.1), the sufficient condition is

$$\min_{s \in S} \max_{i \in I} \delta \left(\frac{e_s^i}{\sum_{\hat{s} \in S} \mu_{\hat{s}} e_{\hat{s}}^i} \right)^{\sigma} > 1. \tag{5.4}$$

Apparently, both restrictions (5.3) and (5.4) can be robustly fulfilled: debt is valued at equilibrium and cannot be rolled-over.

If roll-over is unfeasible, how can unsecured debt be sustained by a truly reputational mechanism? Debt is used to reduce the volatility of consumption when averse shocks occur. However, as repayment is enforced eventually, the benefit for consumption smoothing will be exhausted at some contingency and default will become profitable. Which mechanism prevents default? As initially suggested in Bloise et al. [12], default bears the implicit cost of reduced insurance opportunities. Upon default no further debt can be issued and future insurance will require up-front payments or, using the terminology of Bulow and Rogoff [13], cash-in-advance contracts. When markets are complete, repayments saved upon default can be invested to provide resources to cover future up-front insurance costs. This in general requires access to portfolios of securities with suitable contingent payoffs. When markets are incomplete, such portfolios are only fortuitously available and the cost of providing insurance raises, overcoming the gain accruing from saved debt repayment. We shall later on provide additional intuition in a partial equilibrium framework (§7)

6. Existence

We show that, under a suitable gains to trade hypothesis, a competitive equilibrium with self-enforcing debt exists. Private debt is issued as an insurance device against income fluctuations and it trustworthy circulates as the only store of value in the economy. In general, as argued in our previous discussion (§5), debt is sustained by a proper reputational mechanism: default is unprofitable, because self-insurance is too costly, and outstanding claims are honored. This situation is distinct from a competitive equilibrium in which outside money is valued as a mere speculative bubble (Bewley [9, 10]) and it is more properly associated with a form of inside money.

We provide a proof of existence when *only a risk-free bond is traded* in an economy where intrinsic uncertainty is governed by a Markov process on the finite space S with irreducible transition $\Pi : S \to \Delta(S)$. Individual endowments oscillate according to this Markov process and, hence, all fundamentals are measurable with respect to the finite Markov state space S. This, in particular, implies that the economy cannot grow or decline over time, that is, the aggregate endowment e in L^+ is bounded. In general, at a competitive equilibrium, prices would be affected by the distribution of wealth, and possibly by future expectations, and would not be measurable with respect to the Markov state space S. Consequently, we cannot impose any Markov restriction on the pricing kernel.

The major difficulty in establishing existence relies on the fact that no trade is always a competitive equilibrium. This resembles the essential property of fiat money: when money is the only store of value and it is not worth on the market, it will not be demanded, because it bears no intrinsic value, and no intertemporal trade will occur; similarly, when all lenders expect that debtors will default, they are not willing to provide credit and no intertemporal trade will occur. We overcome this obstacle by introducing an innovative approach. Namely, we construct a perturbed economy in which debt is secured by a share of the private endowment. Trade occurs in this perturbed economy and, as the pledgeable share of the endowment vanishes, debt becomes purely self-enforcing. The dominant root plays an essential role in ensuring that trade persists in the limit.

We introduce an auxiliary economy in which, upon default, a fraction ϵ in $(0, 1)$ of the endowment is confiscated and no further borrowing is allowed. This is the economy $\mathcal{E}^\epsilon$, whereas the original economy is denoted by $\mathcal{E}^0$. A competitive equilibrium exists in the perturbed economy $\mathcal{E}^\epsilon$.[14] Debt is still unsecured, because confiscated resources are not used to satisfy creditors. However, confiscation makes default unprofitable at any level of debt that can be repaid using a fraction ϵ in $(0, 1)$ of the endowment, that is, not exceeding the least present value of confiscable resources. Indeed, why should a debtor default, and loose a fraction of the endowment, when the debt can be repaid using this fraction? The relevant implication of the perturbation is that, at any equilibrium of the perturbed economy, the least present value of the endowment is finite, irrespectively of the share of confiscable resources.

Lemma 6.1 (Finitely valued endowment)**.** *In any competitive equilibrium of the economy $\mathcal{E}^\epsilon$ for ϵ in $(0, 1)$, there is an adapted process f^ϵ in $L^+(e)$ such that, at every t in $\mathbb{T}$,*

$$f_t^\epsilon \geq e_t - \Pi_t^\epsilon\left(-f_{t+1}^\epsilon\right),$$

where e in L^+ is the aggregate endowment.

As confiscable resources vanish, equilibrium allocation cannot converge uniformly to autarky when gains to trade are available. To identify these situations precisely, we construct an implicit pricing at autarky by setting, at every t in $\mathbb{T}$,

$$q_t^0 = \max_{i \in I} \delta \mathbb{E}_t \frac{\nabla u^i\left(e_{t+1}^i\right)}{\nabla u^i\left(e_t^i\right)}.$$

The *gains to trade hypothesis* is that $\gamma\left(q^0\right) > 1$. At a competitive equilibrium of the perturbed economy, instead, $\gamma\left(q^\epsilon\right) < 1$, because the endowment would not be finitely valued otherwise, and a reversal cannot occur under uniform convergence. Hence, autarky cannot be an accumulation point.

In overlapping generations economies, a similar argument is used to establish the existence of a monetary equilibrium (see Aiyagari and Peled [2] and, recently, Barbie and Hillebrand [8]). The main difference is that monetary equilibria are Pareto

[14] Even for the perturbed economy, we cannot rely on any established theorem in the literature, because of the self-enforcing condition. We present our analysis in Appendix C. To establish existence, we truncate the economy by arbitrarily imposing default in the future and progressively remove this auxiliary restriction going to the limit.

efficient in those economies, whereas they in general fail efficiency in our economy, as well as in Bewley [9, 10]'s monetary economy. The role of the dominant root is to identify directions of efficiency that are achieved at a perturbed equilibrium and preserved in the limit. In an economy with outside assets, this issue is discussed by Bloise and Citanna [11]. We here identify conditions when only inside assets are traded and use them to ensure the existence of an equilibrium with trade.

For simple Markov processes with strictly positive transitions, the gains to trade hypothesis requires that $q_t^0 > 1$ uniformly. In such a situation, an hypothetical planner can improve upon autarky by means of a simple scheme of transfers: in every period t in $\mathbb{T}$, a small amount $\eta > 0$ is taken from any individual i in I with marginal rate of substitution equal to q_t^0 and distributed to some other individual; in the following period, the donor is compensated with an uncontingent transfer $\eta > 0$; expected utility increases because the compensation is valued more at the margin, that is, $q_t^0 > 1$; this chain of transfers can be continued indefinitely. In this interpretation, the gains to trade hypothesis guarantees a sort of time irreducibility of the economy: the transfer scheme will never be interrupted, because a potential donor will always be available. Private debt is valued at equilibrium because it allows individuals to exploit these welfare gains. On the contrary, in general, it will not be valued when similar welfare gains are not available.

Lemma 6.2 (Trade in the limit)**.** *Under the gains to trade hypothesis, as ϵ in $(0,1)$ vanishes, no sequence of equilibrium allocations in the perturbed economy $\mathcal{E}^\epsilon$ can converge to autarky uniformly.*

Unfortunately, the established lemma is not powerful enough to deliver by itself the existence of an equilibrium with trade in the limit. Indeed, it requires uniform convergence and, in general, sequences of perturbed equilibria might not converge uniformly to a limit equilibrium. However, in a Markov economy, the wealth distribution at equilibrium remains on sets that depend only on the Markov state. We exploit the lack of uniform convergence to autarky to put bounds on such sets and to extract a sequence of perturbed equilibria converging to a limit equilibrium with trade.

Let $V_s^\epsilon \subset \mathbb{R}^I$ be the space of initial distributions of wealth of the perturbed economy $\mathcal{E}^\epsilon$ for which a competitive equilibrium exits, when the economy begins from state s in S, and let $V^\epsilon \subset \mathbb{R}^{I\times S}$ be the space of initial distributions of wealth across states. Though equilibrium might not be Markov, wealth distribution over time remains in the space V^ϵ. We characterize the size of this set, defined as

$$\eta_\epsilon = \sup_{s\in S} \sup_{v_s^\epsilon \in V_s^\epsilon} \|v_s^\epsilon\|_\infty .$$

We also bound self-enforcing debt limits, using

$$\chi_\epsilon = \sup_{s\in S} \sup_{g_s^\epsilon \in G_s^\epsilon} \|g_s^\epsilon\|_\infty ,$$

where $G_s \subset \mathbb{R}^I_+$ is the space of initial debt limits at equilibrium of the perturbed economy $\mathcal{E}^\epsilon$, when it begins from state s in S.

Lemma 6.3 (Bounds)**.** *Under the gains to trade hypothesis,*

$$0 < \liminf_{\epsilon \to 0} \eta_\epsilon \leq \limsup_{\epsilon \to 0} \eta_\epsilon < \infty \tag{6.1}$$

and

$$\limsup_{\epsilon \to 0} \chi_\epsilon < \infty. \tag{6.2}$$

The most delicate implication of the established lemma is that the equilibrium set does not collapse in the limit, that is, the left hand-side of condition (6.1). Assuming not, indeed, the wealth of individuals would be becoming negligible as ϵ in $(0, 1)$ vanishes. By market clearing, this would imply a progressive contraction of equilibrium debts and credits and, thus, a uniform contraction of trades, contradicting our previous Lemma 6.2. To establish that debt limits do not explode (that is, condition (6.2)), we observe that an individual would otherwise be able to afford arbitrarily large consumption for long time, by issuing large amounts of debt, and then secure a reservation utility after default, a situation which is inconsistent with the fact that resources are limited in the economy.

We now argue that debt is sustainable at equilibrium. In particular, we show that an equilibrium with trade in the original economy can be approached as the limit of a sequence of equilibria in the perturbed economies. Because wealth equilibrium sets do not collapse, the sequence can be extracted in such a way that debts and credits do not vanish with ϵ in $(0, 1)$ and, thus, persist in the limit. A complication arises because of the not-too-tight condition for debt limits, which is instead absent in Bewley [9, 10]'s monetary economy.[15]

Proposition 6.1 (Existence)**.** *Under the gains to trade hypothesis, a non-autarkic equilibrium with self-enforcing debt exists.*

The competitive equilibrium with self-enforcing debt will in general be distinct from a Bewley [9, 10]'s monetary equilibrium of the same economy. Furthermore, when $\gamma(q) < 1$, as it happens in our examples, no speculative bubble occurs at equilibrium with self-enforcing debt and debt is valued because of implied future repayments. Inside and outside money cannot co-exist, because an outside money equilibrium necessarily requires a sort of condition (5.1). We add a few comments on monetary policy implications and welfare comparison.

[15] Under complete markets it is unnecessary to take care explicitly of the not-too-tight condition because of the equivalence established by Hellwig and Lorenzoni [19, Proposition 1]. In sequential economies with permanent exclusion from markets upon default (*e.g.*, Alvarez and Jermann [3]), the existence of competitive equilibrium is proved via Welfare Theorems and the method is not available in our economy. Neither we can use the proof in Kehoe and Levine [20, Proposition 6], because default is there precluded by a direct restriction of consumption plans.

Loosely interpreted, our contribution shows that the market is able to provide endogenous sources of liquidity. Because they do not require any legal enforcement mechanism, regulation might not be effective and self-enforcing debt might interfere with the objectives of monetary policy. For instance, when a counter-cyclical policy calls for a restrictive intervention, the inside money channel might instead provide an expansion of liquidity. Furthermore, the inside supply of liquidity might be more vulnerable to autonomous revisions of expectations and elude the stabilizing action of monetary policy. All this is devoted to future research.

An adaptation of our approach is suitable as an alternative method to establish existence of a monetary equilibrium in Bewley [9, 10]'s economy. To this purpose, the gains to trade hypothesis can be more permissive, that is, $\rho\left(q^0\right) > 1$ as opposed to $\gamma\left(q^0\right) > 1$. One way of interpreting this difference is that a risk-free bond implements only uncontingent transfers, whereas the transfer of resources might be contingent at a monetary equilibrium, because the real value of money might vary across states of nature. This seems to suggest that money is a socially superior contrivance to execute intertemporal trade. However, a welfare comparison is ambiguous due to conflicting effects: money can only circulate as a bubble and this enforces zero interest rate; private debt, on the other side, is compatible with recurrently positive interest rate; because of impatience, a positive interest rate might be less distortionary and might permit more efficient intertemporal trades, though uncontingent.

7. Markov pricing

We complement our analysis with the examination of incentives to default in a *partial* equilibrium framework. The pricing kernel is fixed exogenously and fullfils a simple Markov process.[16] We provide conditions under which default is unprofitable and, at the same time, the natural debt limit is finite. In other terms, self-enforcing debt limits exist and do not allow for roll-over. This reveals a failure of Bulow and Rogoff [13]'s theorem when some risks are uninsurable. In addition, the simplified framework allows for a better understanding of the implicit enforcement mechanism.

Our analysis might be of independent interest for the sovereign debt literature, originated by Eaton and Gersovitz [16]. In the traditional framework, the pricing kernel is given by the valuation of risk-neutral creditors. It has however been noticed that risk-neutral pricing entails risk premia which are inconsistent with empirical observations (see Arellano [5, Section D]). Our approach encompasses risk-averse creditors. This induces time variation of interest rate and risk premium through the sensitivity of the lender's stochastic discount factor (the lender's marginal rate of substitution) along the business cycle or with respect to uninsured idiosyncratic risks.

[16] We remark again that this will not in general happen at a competitive equilibrium.

Recently, Auclert and Rognlie [7] and Bloise et al. [12] showed that Bulow and Rogoff [13]'s theorem extends to incomplete markets. In Auclert and Rognlie [7], a risk-free bond is the only asset and the pricing kernel is given by a risk-neutral lender with constant discount factor. Under these conditions, because the price of the risk-free bond is constant, the dominant roots coincide and are less than unity, that is,

$$\gamma(q) = \rho(q) < 1.$$

Bloise et al. [12] study a more general framework with several securities and a time-varying pricing kernel. Under these conditions, the dominant roots are in general distinct. They show that, when $\rho(q) < 1$, debt is unsustainable, as the sovereign would profit from defaulting at some contingency, saving on repayments and paying upfront for the same insurance contract. This situation occurs when

$$\gamma(q) < \rho(q) < 1.$$

We push the analysis on default incentives further by examing the case where

$$\gamma(q) < 1 < \rho(q).$$

As $\rho(q) > 1$, interest rate can be negative for long phases, self-insurance becomes too costly and debt repayment is more convenient. As $\gamma(q) < 1$, the natural debt limit is finite and debt is not rolled-over.[17]

We assume that uncertainty is described by a Markov process on the finite state space S, with $\mu_{s,\hat{s}} > 0$ being the probability of moving from state s in S into state $\hat{s}$ in S. A finite set of securities J is traded at price q_s in $\mathbb{R}^J$ in state s in S, each delivering a payoff $R^j_{s,\hat{s}}$ in $\mathbb{R}$ in state $\hat{s}$ in S in the following period. Debt limits are given as g in $\mathbb{R}^S_+$. They are self-enforcing whenever, in every state s in S,

$$J_s(-g_s, g) = J_s(0, 0),$$

where $J_s(v_s, g)$ is the indirect utility in state s in S beginning with initial wealth v_s in $\mathbb{R}$ and subject to debt limits g in $\mathbb{R}^S_+$.

Why is debt sustainable? A comparison with complete markets helps the understanding. As argued by Hellwig and Lorenzoni [19, Theorem 1], debt is sustainable only if it can be rolled-over exactly and, as we show in Proposition 5.1, this occurs only if the dominant root is equal to unity, that is, long-term interest rate is zero. The corresponding condition when markets are incomplete would be $\gamma(q) = 1$ and, in such a situation, debt can be rolled-over without exploding over time, because long-term interest rate is not positive. However, exact roll over would in general require contingent transfers and these are not feasible due to market incompleteness.

[17] In addition, $\gamma(q) < 1$ is a necessary condition when the pricing kernel is derived from the lender's expected utility with discounting, because the valuation of infinite income streams would not be finite otherwise.

Debt roll-over is necessarily *inexact* and permits additional consumption occasionally. Hence, this situation is more favourable than no borrowing and debt-limits are too loose. For a slight contraction of $\gamma(q) < 1$, those debt limits remain loose, whereas natural debt limits are finite and are too tight. As a result, not-too-tight debt limits will exist in between.

Proposition 7.1 (Sustainable debt)**.** *Non-trivial self-enforcing debt limits exist if*

$$\gamma(q) < 1 < \rho(q),$$

provided that prices are in a sufficiently small neighbourhood around q^* in Q such that $\gamma(q^*) = 1$ and $\rho(q^*) > 1$.

We add a short comment on admitting default as in Arellano [5]. To our purposes, default enhances debt sustainability, because it increases the value of market participation for the borrower. Thus, under conditions in which debt is sustainable when default is not allowed, so will be when default can occur and the price of the bond reflects the risk of default. In the latter case, obviously, debt is sustainable in the sense that lenders are willing to supply credit, though anticipating default in some future contingencies.

Taking the approach in Aguiar and Amador [1] and Auclert and Rognlie [7, Proposition 1], debt limits can be reinterpreted as default thresholds. The pricing kernel reflects the risk of default, that is, given default thresholds g in $\mathbb{R}_+^S$,

$$\hat{q}_s(z_s) = \sum_{\hat{s} \in \hat{S}_s(z_s, g)} \pi_{s,\hat{s}} R_{s,\hat{s}}(z_s),$$

where $\hat{S}_s(z_s, g) = \{\hat{s} \in S : R_{s,\hat{s}}(z_s) \geq -g_{\hat{s}}\}$ is the set of states in which default is not expected and π in $\mathbb{R}_+^S \times \mathbb{R}_+^S$ is the matrix of state prices. We argue that, when debt is sustainable without allowing default, it is *a fortiori* sustainable when default is admitted.

Consider self-enforcing debt limits g in $\mathbb{R}_+^S$ in Proposition 7.1. As default enlarges the space of opportunities in every state s in S,

$$\hat{J}_s(g_s, g) \geq J_s(g_s, g) = J_s(0, 0),$$

where $\hat{J}_s(-g_s, g)$ and $J_s(-g_s, g)$ are, respectively, the indirect utility when default possibly occurs and when default is not allowed. It follows that debt limits under no default are too loose. On the other side, when natural debt limits $\bar{g}$ in $\mathbb{R}_+^S$ are finite, at every state s in S,

$$\hat{J}_s(\bar{g}_s, \bar{g}) < J_s(0, 0).$$

Hence, natural debt limits are too tight. By continuity, this virtually shows that some non-trivial debt limits exist under risk of default.

8. Conclusion

We have shown that, under incomplete markets, private debt is sustainable by the mere reputation for repayment. The implicit enforcement mechanism relies on a high cost of self-insurance compared with the privilege of issuing debt when interest rate is low. Private debt reflects the value of expected future repayments and, differently from Hellwig and Lorenzoni [19], does not circulate a speculative bubble. We interpret this as a genuine failure of Bulow and Rogoff [13]'s claim that lending must be supported by direct sanctions available to creditors.

We also establish the existence of a competitive equilibrium when default carries only a ban from ever borrowing in financial markets. Private debt is issued and traded as the only store of value so as to support risk-sharing. In general, debt cannot be rolled over and is allocationally distinct from outside money. Beyond ensuring existence, our strategy of proof reveals conditions for mutually beneficial trades in a default-free bond.

A distinctive feature of self-enforcing debt under complete markets is that debt can be exactly rolled over (Hellwig and Lorenzoni [19]). In other terms, debt is issued by private individuals and is valued as a speculative bubble (or, according to Bulow and Rogoff [13], as Ponzi-type reputational contracts). From an individual perspective, even if debt is rolled over, single creditors receive repayments, as when investors sell a speculative bubble and collect the expected return. Socially, however, private debt is not backed by any future repayments, whereas it in general reflects its fundamental value under incomplete markets. This bears crucial repercussions on the determinacy of equilibrium.

Under complete markets, at a competitive equilibrium, the amount of debt that can be rolled over by each single individual is totally unrelated to fundamentals. Indeed, any process of debt limits which allow for exact roll over is self-enforcing (Hellwig and Lorenzoni [19, Proposition 1]). As a result, debt privileges only derive from an unspecified process of expectations formation and, hence, competitive equilibrium is indeterminate: depending on expectations, an individual is allowed to borrow more or less and this affects the effective distribution of initial wealth. When markets are incomplete, instead, self-enforcing debt truly reflects incentives to repayment and, so, it is tied to market conditions. Though we do not establish any local determinacy, the obvious source of indeterminacy disappears when debt cannot be rolled over.

References

[1] Mark Aguiar and Manuel Amador. Chapter 11 - sovereign debt. In Elhanan Helpman Gita Gopinath and Kenneth Rogoff, editors, *Handbook of International Economics*, volume 4 of *Handbook of International Economics*, pages 647–687. Elsevier, 2014.

[2] S. Rao Aiyagari and Dan Peled. Dominant root characterization of Pareto optimality and the existence of optimal monetary equilibria in stochastic OLG models. *Journal of Economic Theory*, 54:69–83, 1991.

[3] Fernando Alvarez and Urban J. Jermann. Efficiency, equilibrium, and asset pricing with risk of default. *Econometrica*, 68(4):775–797, 2000.

[4] Fernando Alvarez and Urban J. Jermann. Using asset prices to measure the persistence of the marginal utility of wealth. *Econometrica*, 73(6):1977–2016, 2005.

[5] Cristina Arellano. Default risk and income fluctuations in emerging economies. *American Economic Review*, 98(3):690–712, 2008.

[6] Cristina Arellano and Ananth Ramanarayanan. Default and the maturity structure in sovereign bonds. *Journal of Political Economy*, 120(2):187–232, 2012.

[7] Adrien Auclert and Matthew Rognlie. Unique equilibrium in the Eaton-Gersovitz model of sovereign debt. *Journal of Monetary Economics*, 84:134–146, 2016.

[8] Martin Barbie and Marten Hillebrand. Bubbly markov equilibria. March 2017.

[9] Truman F. Bewley. The optimum quantity of money. In John Kareken and Wallace Neil, editors, *Models of Monetary Economics*. Federal Reserve Bank of Minneapolis, 1983.

[10] Truman F. Bewley. A difficulty with the optimum quantity of money. *Econometrica*, 51:1485–1504, 1983.

[11] Gaetano Bloise and Alessandro Citanna. Liquidity shortages under limited commitment. 2016.

[12] Gaetano Bloise, Herakles Polemarchakis, and Yiannis Vailakis. Sovereign debt and default incentives with uninsurable risks. *Theoretical Economics*, 12(3): 1121–1154, 2017.

[13] Jeremy Bulow and Kenneth Rogoff. Sovereign debt: Is to forgive to forget? *American Economic Review*, 79(1):43–50, 1989.

[14] Jeremy Bulow and Kenneth Rogoff. A constant recontracting model of sovereign debt. *Journal of Political Economy*, 97(1):155–178, 1989.

[15] H. Cole and P. Kehoe. Models of sovereign debt: Partial versus general reputations. *International Economic Review*, 39(1):55–70, 1998.

[16] Jonathan Eaton and Mark Gersovitz. Debt with potential repudiation: Theoretical and empirical analysis. *Review of Economic Studies*, 48(2):289–309, 1981.

[17] Lars Peter Hansen and José A. Scheinkman. Long-term risk: An operator approach. *Econometrica*, 77(1):177–234, 2009.

[18] Juan Carlos Hatchondo, Leonardo Martinez, and Cesar Sosa-Padilla. Debt dilution and sovereign default risk. *Journal of Political Economy*, 124(5):1383–1422, 2016.

[19] Christian Hellwig and Guido Lorenzoni. Bubbles and self-enforcing debt. *Econometrica*, 77(4):1137–1164, 2009.

[20] T. J. Kehoe and D. K. Levine. Debt-constrained asset markets. *Review of Economic Studies*, 60(4):865–888, 1993.

[21] Timothy J. Kehoe and David K. Levine. Liquidity constrained markets versus debt constrained markets. *Econometrica*, 69(3):575–598, 2001.

[22] Kenneth M. Kletzer and Brian D. Wright. Sovereign debt as intertemporal barter. *American Economic Review*, 90(3):621–639, 2000.

[23] N.R. Kocherlakota. Injecting rational bubbles. *Journal of Economic Theory*, 142(1):218–232, 2008.

[24] Stephen F. LeRoy and Jan Werner. *Principles of Financial Economics*. Cambridge University Press, 1st edition, 2001.

[25] Wolfgang Pesendorfer. Sovereign debt: Forgiving and forgetting reconsidered. Manuscript, Northwestern University, 1992.

[26] Manuel S. Santos and Michael Woodford. Rational asset pricing bubbles. *Econometrica*, 65(1):19–57, 1997.

[27] Mark L. J. Wright. The theory of sovereign debt and default. In Gerard Jr. Caprio, Philippe Bacchetta, James R. Barth, Takeo Hoshi, Philip R. Lane, David G. Mayes, Atif R. Mian, and Michael Taylor, editors, *Handbook of Safeguarding Global Financial Stability*, pages 187–193. Academic Press, San Diego, 2013.

[28] Harold H. Zhang. Endogenous borrowing constraints with incomplete markets. *Journal of Finance*, 52(5):2187–2209, 1997.

Appendix A. Proofs

Proof of Lemma 3.1. By individual rationality, and using Assumptions 3.1-3.2, at every t in $\mathbb{T}$,

$$\begin{aligned} u^i\left(c_t^i\right) + \left(\frac{\delta}{1-\delta}\right) \sup_{\hat{c}^i \in \mathbb{R}_+} u^i\left(\hat{c}^i\right) &\geq U_t^i\left(c^i\right) \\ &\geq U_t^i\left(e^i\right) \\ &\geq U_t^i\left(\epsilon_l e\right) \\ &\geq \left(\frac{1}{1-\delta}\right) u^i\left(\epsilon_l \eta\right), \end{aligned}$$

where $\eta > 0$ is the uniform lower bound for the aggregate endowment, that is, $e_t \geq \eta$ at every t in $\mathbb{T}$. By the Strong Inada Condition (Assumption 3.2), this suffices to prove the claim. □

Proof of Proposition 5.1. Consider any individual i in I with persistent debt roll-over and drop the index i in I in order to simplify notation. We first show that $\gamma(q) \leq 1$. Suppose that there is $\gamma > 1$ such that, for some non-zero process b in $L^+(e)$, at every t in $\mathbb{T}$,

$$\gamma b_t \leq -\Pi_t(-b_{t+1}).$$

This means that there is portfolio process Δz in Z such that, at every t in $\mathbb{T}$,

$$Q_t(\Delta z_t) \leq -\gamma b_t \leq -b_t \tag{A.1}$$

and

$$-b_{t+1} \leq R_{t,t+1}(\Delta z_t). \tag{A.2}$$

We now show that this portfolio process allows for super-replicating the optimal plan under no borrowing, thus delivering a contradiction.

Define λ in $\mathbb{R}_+$ as the greatest value satisfying $g \geq \lambda b$. Because debt limits are in the interior of $L^+(e)$, $\lambda > 0$ and, at no loss of generality, $\lambda = 1$. Thus, $g \geq b$ and, at some contingency, $g_t < \gamma b_t$, since otherwise $g \geq \gamma b$, a contradiction as $\gamma > 1$. At no loss of generality, to simplify notation, assume that $g_0 < \gamma b_0$ and, so, $b_0 > 0$. We argue that

$$V_0(-g_0, g) > V_0(-\gamma b_0, b) \geq V_0(0,0),$$

a contradiction. The first strict inequality is obvious, because $g \geq b$ and $-g_0 > -\gamma b_0$. For the other inequality, take the plan which is optimal at $(0,0)$ and replicate it at $(-\gamma b_0, b)$ by translation, that is,

$$z_t \mapsto z_t + \Delta z_t.$$

By conditions (A.1)-(A.2), this is feasible, revealing a contradiction.

The roll-over property implies $\gamma(q) \geq 1$ and, hence, we conclude that $\gamma(q) = 1$. Clearly, $\rho(q) \geq 1$. It only remains to verify that $\rho(q) \leq 1$. The roll-over component b in the interior of $L^+(e)$ satisfies conditions (A.1)-(A.2) with $\gamma = 1$. If an inequality is strict at some contingency, then the previous replication argument would imply

$$V_0(-g_0, g) > V_0(0,0),$$

a contradiction. This means that, at every t in $\mathbb{T}$,

$$b_t = \Pi_t(b_{t+1}) = -\Pi_t(-b_{t+1}). \tag{A.3}$$

Suppose that, given ρ in $\mathbb{R}_+$, there is a process d in $L^+(e)$ such that

$$\rho d_t \leq \Pi_t(d_{t+1}).$$

Let λ in $\mathbb{R}_+$ be the maximum value such that $\lambda d \leq b$ and, at no loss of generality, assume that $\lambda = 1$. Monotonicity yields, at every t in $\mathbb{T}$,

$$\rho d_t \leq \Pi_t(d_{t+1}) \leq \Pi_t(b_{t+1}) \leq b_t,$$

so that $\rho \leq 1$. This proves our claim. $\square$

Claim A.1. Under condition (5.3),

$$q_s \geq \max_{i \in I} \delta \left(\frac{c_s^i}{\sum_{\hat{s} \in S} \mu_{\hat{s}} c_{\hat{s}}^i} \right)^{\sigma} > 1.$$

Proof of Claim A.1. To obtain a contradiction, assume that, for every individual i in I,

$$\delta \left(\frac{c_s^i}{\sum_{\hat{s} \in S} \mu_{\hat{s}} c_{\hat{s}}^i} \right)^{\sigma} \leq 1.$$

This implies

$$\delta^{\frac{1}{\sigma}} \sum_{i \in I} c_s^i \leq \sum_{i \in I} \sum_{\hat{s} \in S} \mu_{\hat{s}} c_{\hat{s}}^i \leq \sum_{\hat{s} \in S} \mu_{\hat{s}} \sum_{i \in I} c_{\hat{s}}^i,$$

thus violating condition (5.3). $\square$

Proof of Lemma 6.1. Let $f^{i,\epsilon}$ in L^+ be the maximum debt that can be re-payed, beginning from each contingency, out of pledgeable resources. This is well-defined and satisfies, at every t in $\mathbb{T}$,

$$0 \leq f_t^{i,\epsilon} \leq \epsilon e_t^i - \Pi_t^\epsilon \left(-g_{t+1}^{i,\epsilon} \right).$$

At every t in $\mathbb{T}$, an individual can always repay back a debt not exceeding $f_t^{i,\epsilon}$ in L_t^+ out of share ϵ in $(0,1)$ of the endowment and, at the same time, implement the optimal plan under no borrowing with no initial wealth, so that

$$J_t^{i,\epsilon} \left(-f_t^{i,\epsilon}, g^{i,\epsilon} \right) \geq J_t^{i,\epsilon} (0, 0).$$

We claim that $g_t^{i,\epsilon} \geq f_t^{i,\epsilon}$. Indeed, supposing not, at some contingency,

$$J_t^{i,\epsilon} \left(-g_t^{i,\epsilon}, g^{i,\epsilon} \right) > J_t^{i,\epsilon} \left(-f_t^{i,\epsilon}, g^{i,\epsilon} \right) \geq J_t^{i,\epsilon} (0, 0),$$

a contradiction. Hence, adapted process $f^{i,\epsilon}$ in L^+ satisfies, at every t in $\mathbb{T}$, the recursive condition

$$f_t^{i,\epsilon} = \epsilon e_t^i - \Pi_t^\epsilon \left(-f_{t+1}^{i,\epsilon} \right).$$

This suffices to prove the claim, as e^i lies in the interior of $L^+(e)$. $\square$

Proof of Lemma 6.2. We assume uniform convergence to autarky and argue by contradiction. At no loss of generality, the price of the bond satisfies, at every t in $\mathbb{T}$,

$$q_t^\epsilon = \max_{i \in I} \delta \mathbb{E}_t \frac{\nabla u^i \left(c_{t+1}^{i,\epsilon} \right)}{\nabla u^i \left(c_t^{i,\epsilon} \right)}.$$

As convergence is uniform, for every sufficiently small ϵ in $(0,1)$,

$$b_t^0 \leq -\frac{1}{\gamma} \Pi_t^0 \left(-b_{t+1}^0 \right) \leq -\Pi_t^\epsilon \left(-b_{t+1}^0 \right),$$

where b^0 in $L^+(e)$ is the eigen-process in the limit (which exists when the pricing is Markov). Let $\lambda > 0$ be the greatest value such that $\lambda b^0 \leq f^\epsilon$ and, at no loss of generality, assume that $\lambda = 1$. Monotonicity implies

$$b_t^0 \ll e_t - \Pi_t^\epsilon\left(-b_{t+1}^0\right) \leq e_t - \Pi_t^\epsilon\left(-f_{t+1}^\epsilon\right) \leq f_t^\epsilon,$$

a contradiction. □

Proof of Lemma 6.3. Arguing by contradiction, suppose that $\liminf_{\epsilon\to 0} \eta_\epsilon = 0$. For every individual i in I, the budget constraint imposes, at every t in $\mathbb{T}$,

$$v_t^i = \left(c_t^i - e_t^i\right) + q_t v_{t+1}^i.$$

Furthermore, by first-order conditions,

$$\delta \mathbb{E}_t \frac{\nabla u^i\left(c_{t+1}^i\right)}{\nabla u^i\left(c_t^i\right)} \leq q_t,$$

with the equality when the individual is saving. Thus,

$$v_t^i \leq \left(c_t^i - e_t^i\right) + \delta \mathbb{E}_t \frac{\nabla u^i\left(c_{t+1}^i\right)}{\nabla u^i\left(c_t^i\right)} v_{t+1}^i.$$

Evaluating at a competitive equilibrium of the perturbed economy $\mathcal{E}^\epsilon$, and using the bound on wealth,

$$-\eta_\epsilon \leq \left(c_t^{i,\epsilon} - e_t^i\right) + \delta \mathbb{E}_t \frac{\nabla u^i\left(c_{t+1}^{i,\epsilon}\right)}{\nabla u^i\left(c_t^{i,\epsilon}\right)} \eta_\epsilon.$$

As the economy is bounded, by Lemma 3.1, marginal rates of substitution are uniformly bounded, so that, for some sufficiently large $\kappa > 0$,

$$\left(c_t^{i,\epsilon} - e_t^i\right)^- \leq \eta_\epsilon + \kappa \eta_\epsilon.$$

By feasibility, possibly extracting a subsequence, this implies uniform convergence to autarky, which is ruled out by Lemma 6.2. This shows that the left hand-side of condition (6.1) holds true.

We now prove that equilibrium wealth is uniformly bounded in the perturbed economy $\mathcal{E}^\epsilon$, so that the right hand-side of condition (6.1) holds true. As marginal rates of substitution are uniformly bounded (because consumption is uniformly bounded from below and from above), there exist adapted processes $\underline{q}$ and $\bar{q}$ in the interior of $L^+(e)$ such that $\underline{q} \leq q^\epsilon \leq \bar{q}$. This implies that, out of a large enough financial wealth, individual i in I can afford a consumption plan $e^i + \bar{e}^t$, where $\bar{e}^t$ in L^+ is the aggregate endowment truncated at period t in $\mathbb{T}$. By impatience, for every individual i in I,

$$\lim_{t\to\infty} U_0^i\left(e^i + \bar{e}^t\right) > U_0^i\left(\bar{e}\right) \geq U_0^i\left(c^{i,\epsilon}\right).$$

Thus, if equilibrium wealth is unbounded, some individual would be able to afford an unfeasibly large value in utility, a contradiction.

We finally show that condition (6.2) holds true. Suppose that, by an appropriate choice of the initial state s in S, there is a sequence of equilibria in the perturbed economy $\mathcal{E}^\epsilon$ such that, for some individual i in I, $\lim_{\epsilon\to 0} g_0^{i,\epsilon} = \infty$. Notice that debt limits satisfy, at every t in $\mathbb{T}$,

$$g_t^{i,\epsilon} \leq e_t^i - \Pi_t^\epsilon\left(-g_{t+1}^{i,\epsilon}\right) \leq e_t^i - \bar{\Pi}_t\left(-g_{t+1}^{i,\epsilon}\right),$$

where the pricing functional $\bar{\Pi}_t : L_{t+1} \to L_t$ is constructed using the greatest possible price process for the risk-free bond. If follows that debt limits diverge at every t in $\mathbb{T}$. Possibly extracting a subsequence, it can be assumed that the sequence of consumption plans $\left(c^{i,\epsilon}\right)_{\epsilon>0}$ in C^i converges to a consumption plan c^i in C^i. Let $\bar{c}^i$ in C^i be $c_t^i + e_t$ up to period $\hat{t}$ in $\mathbb{T}$ and $(1-\bar{\epsilon})\, e_t^i$ at any other following period t in $\mathbb{T}$, where $\bar{\epsilon}$ lies is $(0,1)$. By impatience, period $\hat{t}$ in $\mathbb{T}$ can be chosen sufficiently large so that $U_0^i\left(\bar{c}^i\right) > U_0^i\left(c^{i,\epsilon}\right)$ for every sufficiently small ϵ in $(0,\bar{\epsilon})$. Let $\bar{v}^{i,\epsilon}$ in L^+ be a financial plan supporting $\bar{c}^i$ in C^i up to period $\hat{t}$ in $\mathbb{T}$. When the individual defaults at period $\hat{t}+1$ in $\mathbb{T}$, she can secure a level of utility at least equal to $U_{\hat{t}+1}^i\left((1-\bar{\epsilon})\, e^i\right)$. Thus, we only need to verify that $\bar{v}_t^{i,\epsilon} \geq -g_t^{i,\epsilon}$ up to period $\hat{t}+1$ in $\mathbb{T}$. This is certainly satisfied as debt limits diverge and the financial plan remains bounded, thus yielding a contradiction. □

Proof of Proposition 6.1. Given a perturbation ϵ in $(0,1)$, we denote $(c^\epsilon, v^\epsilon, g^\epsilon)$ in $C \times V \times G$ a competitive equilibrium of the perturbed economy $\mathcal{E}^\epsilon$. At no loss of generality, the price of the bond is determined, at every t in $\mathbb{T}$, by

$$q_t^\epsilon = \max_{i\in I} \delta \mathbb{E}_t \frac{\nabla u^i\left(c_{t+1}^{i,\epsilon}\right)}{\nabla u^i\left(c_t^{i,\epsilon}\right)}.$$

By choosing the initial state s in S appropriately, we can extract a sequence of equilibrium plans $(c^\epsilon, v^\epsilon, g^\epsilon)_{\epsilon>0}$ in $C\times V\times G$ converging to a plan (c, v, g) in $C\times V\times G$ such that $\|v_0\|_\infty > 0$. We need to verify that plans are optimal and debt limits are self-enforcing. This would necessarily imply trade, because at least one individual will afford more than autarkic utility.

We first show that, at every t in $\mathbb{T}$, $\lim_{\epsilon\to 0} J_t^{i,\epsilon}(0,0) = J_t^i(0,0)$ and, just to simplify notation, we assume that $t = 0$. To this purpose, consider the program truncated at $\hat{t}$ in $\mathbb{T}$ and notice that, by canonical arguments,

$$\lim_{\epsilon\to 0}\left|J_0^{i,\hat{t}}(0,0) - J_0^{i,\epsilon,\hat{t}}(0,0)\right| = 0.$$

Also, continuation utility is bounded by some large $\Delta > 0$ because utility is bounded from above and utility value $U_{\hat{t}+1}\left((1-\bar{\epsilon})\, e^i\right)$ can be secured for any ϵ in $(0,\bar{\epsilon})$.

Hence, for every ϵ in $(0, \bar{\epsilon})$,

$$\left| J_0^{i,\epsilon}(0,0) - J_0^{i,\epsilon,\hat{t}}(0,0) \right| \leq \delta^{\hat{t}+1} \Delta$$

and

$$\left| J_0^{i}(0,0) - J_0^{i,\hat{t}}(0,0) \right| \leq \delta^{\hat{t}+1} \Delta.$$

This suffices to prove our claim. Similarly, we establish that, at every t in $\mathbb{T}$, $\lim_{\epsilon \to 0} J_t^{i,\epsilon}\left(-g_t^{i,\epsilon}, g^{i,\epsilon}\right) = J_t^{i,\epsilon}\left(-g_t^{i}, g^{i}\right)$.

Clearly, the plan in the limit satisfies budget and solvency constraints. Supposing that it is not optimal, for all sufficiently small ϵ in $(0,1)$, $J_0^i\left(v_0^i, g^i\right) > J_0^{i,\epsilon}\left(v_0^{i,\epsilon}, g^{i,\epsilon}\right) + \Delta$ for some $\Delta > 0$. For a sufficiently large $\hat{t}$ in $\mathbb{T}$, this implies that $J_0^{i,\hat{t}}\left(v_0^i, g^i\right) > J_0^{i,\epsilon,\hat{t}}\left(v_0^{i,\epsilon}, g^{i,\epsilon}\right) + \Delta$ for every sufficiently small ϵ in $(0,1)$, thus delivering a contradiction because the value of the truncated program varies continuously. □

Proof of Proposition 7.1. We preliminarily consider the limit case when $\gamma(q^*) = 1$ and $\rho(q^*) > 1$. To this purpose, let $\underline{g}$ in $\mathbb{R}_+^S$ be the lower dominant eigenvector (see Claim D.1). For every s in S, there exists a portfolio Δz_s in $\mathbb{R}^J$ such that

$$q_s^*(-\Delta z_s) \leq -\underline{g}_s$$

and, at every $\hat{s}$ in S,

$$-\underline{g}_{\hat{s}} \leq R_{s,\hat{s}}(-\Delta z_s).$$

Moreover, since the eigenvector is not in the market span (because $\gamma(q^*) < \rho(q^*)$), the last inequality is strict in at least one state $\hat{s}$ in S. We shall show that, for every state s in S,

$$J_s^*\left(-\underline{g}_s, \underline{g}\right) > J_s^*(0,0). \tag{A.4}$$

Let $\mathcal{S}_s$ be the space of all partial histories beginning from state s in S, and let $s(\sigma)$ be current state in S occurring at partial history σ in $\mathcal{S}_s$. The optimal plan under no borrowing and no initial wealth satisfies, at every σ in $\mathcal{S}_s$,

$$q_{s(\sigma)}^*(z_\sigma) + \left(c_\sigma - e_{s(\sigma)}\right) \leq w_\sigma$$

and, at every continuation history $\hat{\sigma} = (\hat{s}, \sigma)$ in $\mathcal{S}_s$,

$$0 \leq w_{\hat{\sigma}} = R_{s(\sigma), s(\hat{\sigma})}(z_\sigma).$$

Adding the above identified portfolio,

$$q_{s(\sigma)}^*\left(z_\sigma - \Delta z_{s(\sigma)}\right) + \left(c_\sigma - e_{s(\sigma)}\right) \leq w_\sigma - \underline{g}_{s(\sigma)}$$

and, at every continuation history $\hat{\sigma}$ in $\mathcal{S}_s$,

$$-\underline{g}_{s(\hat{\sigma})} \leq w_{\hat{\sigma}} - \underline{g}_{s(\hat{\sigma})} \leq R_{s(\sigma), s(\hat{\sigma})}\left(z_\sigma - \Delta z_{s(\sigma)}\right).$$

The last inequality is strict for at least some continuation history $\hat{\sigma}$ in $\mathcal{S}_s$. By strict monotonicity of preferences, this proves the claim.

We now show that condition (A.4) continues to hold true after a perturbation of prices such that $\gamma(q) < 1 < \rho(q)$. Provided this perturbation is sufficiently small, there exists a minimum consumption $\underline{c}$ in $\mathbb{R}^S_{++}$ such that, at every s in S,

$$\underline{g}_s \leq (e_s - \underline{c}_s) - \Pi_s(-\underline{g}).$$

The existence of a budget-feasible minimum consumption avoids complications related to unbounded utility. Consider the Bellman operator defined, at every s in S, by

$$(TJ)_s(v_s, q) = \sup u(c_s) + \delta \sum_{\hat{s} \in S} \mu_{s,\hat{s}} J_{\hat{s}}(R_{s,\hat{s}}(z_s), q)$$

subject to

$$q_s(z_s) + (c_s - e_s) \leq v_s$$

and, at every $\hat{s}$ in S,

$$-\underline{g}_s \leq R_{s,\hat{s}}(z_s).$$

This operator $T : \mathcal{J} \to \mathcal{J}$ acts on the space of all bounded maps $J : D \to \mathbb{R}$, where D contains all $v \geq -\underline{g}$ in $\mathbb{R}^S_+$ and all prices in a open neighborhood of q^0. The operator is a contraction (by Blackwell discounting) and, hence, admits a unique fixed point. Consider the feasible correspondence $F : D \twoheadrightarrow \mathbb{R}_+ \times \mathbb{R}^J$. This correspondence is continuous with non-empty compact values. By Berge's Maximum Theorem, when J in $\mathcal{J}$ is continuous, so it (TJ) in $\mathcal{J}$. Hence, the unique fixed point is continuous, which proves the claim.

As long as $\gamma(q) < 1$, natural debt limits $\bar{g}$ in $\mathbb{R}^S_{++}$ are finite and, at no loss of generality, $\underline{g} \leq \bar{g}$, because they grow unboundedly as $\gamma(q)$ approaches $\gamma(q^0) = 1$. Define the mapping $f : [\underline{g}, \bar{g}] \to [\underline{g}, \bar{g}]$ by the formula

$$J_s(-f_s(g), g) = J_s(0, 0).$$

The unique solution exists because, by Inada condition,

$$J_s(-\bar{g}_s, g) < J_s(0, 0)$$

and, by the previous characterization,

$$J_s(-\underline{g}_s, g) \geq J_s(-\underline{g}_s, \underline{g}) > J_s(0, 0).$$

Notice that mapping $f : [\underline{g}, \bar{g}] \to [\underline{g}, \bar{g}]$ is monotone. We claim that self-enforcing debt limits g in $[\underline{g}, \bar{g}]$ are determined, at every s in S, by

$$g_s = \lim_{n \to \infty} f^n_s(\underline{g}).$$

It is clear that, at every s in S,

$$J_s(-g_s, g) \geq J_s(0, 0).$$

Fix any t in $\mathbb{T}$ and let J^t be the value function corresponding to the truncated program at t in $\mathbb{T}$. By continuity,

$$J_s^t(-g_s, g) = \lim_{n\to\infty} J_s^t\left(-f_s^{n+1}(\underline{g}), f^n(\underline{g})\right).$$

Notice that, after the truncation, the no borrowing value can be secured, because, at every s in S,

$$J_s\left(-f_s^n(\underline{g}), f^n(\underline{g})\right) \geq J_s\left(-f_s^{n+1}(\underline{g}), f^n(\underline{g})\right) = J_s(0,0).$$

Therefore, for some bound $\Delta > 0$,

$$\left|J_s^t\left(-f_s^{n+1}(\underline{g}), f^n(\underline{g})\right) - J_s\left(-f_s^{n+1}(\underline{g}), f^n(\underline{g})\right)\right| \leq \delta^{t+1}\Delta$$

and

$$\left|J_s^t(-g_s, g) - J_s(-g_s, g)\right| \leq \delta^{t+1}\Delta.$$

This suffices to prove that, at every s in S,

$$J_s(-g_s, g) = \lim_{n\to\infty} J_s\left(-f_s^{n+1}(\underline{g}), f^n(\underline{g})\right) = J_s(0,0),$$

thus establishing the claim. □

Appendix B. Example

We provide a simple example of competitive equilibrium with no debt roll-over. In particular, to preserve stationarity, we assume constant marginal utilities and set prices so that one individual is indifferent in each period. The example is non-robust because of this feature. The equilibrium would in general not be stationary for a perturbation of relevant parameters.

The economy consists of two individuals with cyclic endowment. When the endowment of an individual is low, $\underline{e}$, the endowment of the other individual is high, $\bar{e}$, with $\bar{e} > \underline{e} > 0$.[18] In addition, an independent and identically distributed shock s in $S = \{u, d\}$ affects marginal utility. In particular, when income is low, marginal utility is unitary; when income is high, it is either ψ_u or ψ_d with equal probability, where

$$1 \geq \psi_u > \psi_d > 0. \tag{B.1}$$

The only asset is a discount risk-free bond. In order to ensure trade, we set prices so that the high-endowment individual is indifferent between saving and dissaving. In particular, letting δ in $(0,1)$ be the common discount factor, the price of the bond, depending on state s in S, is

$$q_s = \frac{\delta}{\psi_s}. \tag{B.2}$$

[18]Because marginal utilities are constant, high and low income play no role and are used only for narrative convenience. We assume that income is sufficiently large so as to ensure positive consumption in all states.

To guarantee that the low-endowment individual is willing to borrow, we assume that

$$\min_{s\in S} q_s > \frac{\delta}{2}\sum_{s\in S}\psi_s. \tag{B.3}$$

Under the stated conditions, we characterize the value function explicitly.

Borrowing is allowed up to debt limits $g = \left(\bar{g}, \underline{g}\right)$ in $\mathbb{R}^S_+ \times \mathbb{R}^S_+$, where $\bar{g}_s$ and $\underline{g}_s$ are the maximum sustainable debt, depending on income, in state s in S. Such debt limits are consistent, that is,

$$\bar{g}_s \leq \bar{e} + q_s \min_{s\in S} \underline{g}_s$$

and

$$\underline{g}_s \leq \underline{e} + q_s \min_{s\in S} \bar{g}_s.$$

Indeed, if not, the maximum debt would not be sustainable beginning from one of the states in S. Each individual is subject to a budget constraint

$$q_s v' + c \leq e + v,$$

whereas the holding of the risk-free bond is restricted by the solvency constraint

$$-\min_{s\in S} g'_s \leq v',$$

where g' in $\mathbb{R}^S_+$ is the maximum sustainable debt in the following period. By pricing restrictions (B.2)-(B.3), the individual will be borrowing only when income is high in the next period. As a consequence, the only relevant limit is

$$d = \min_{s\in S} \bar{g}_s.$$

Claim B.1 (Value function under borrowing)**.** The value function is, when income is low,

$$\underline{J}_s(v) = \underline{e} + v + q_s d + \frac{1}{2}\frac{\delta}{1-\delta^2}\sum_{s\in S}\psi_s\left(\bar{e} - d\right) + \frac{1}{2}\frac{\delta^2}{1-\delta^2}\sum_{s\in S}\left(\underline{e} + q_s d\right)$$

and, when income is high,

$$\bar{J}_s(v) = \psi_s\left(\bar{e} + v\right) + \frac{1}{2}\frac{\delta}{1-\delta^2}\sum_{s\in S}\left(\underline{e} + q_s d\right) + \frac{1}{2}\frac{\delta^2}{1-\delta^2}\sum_{s\in S}\psi_s\left(\bar{e} - d\right).$$

Proof. When income is low, the maximization program is

$$\max \underline{e} + v - q_s v' + \frac{\delta}{2}\sum_{s\in S}\bar{J}_s\left(v'\right)$$

subject to

$$\frac{\underline{e} + v}{q_s} \geq v' \geq -\min_{s\in S}\bar{g}_s.$$

This is equivalent to

$$\max\left(-q_s+\frac{\delta}{2}\sum_{s\in S}\psi_s\right)v'.$$

Because of condition (B.2), the only solution is $v'=-\min_{s\in S}\bar{g}_s$. Hence,

$$\begin{aligned}\underline{J}_s(v) &= \underline{e}+v+q_sd+\frac{\delta}{2}\sum_{s\in S}\bar{J}_s(-d)\\ &= \underline{e}+v+q_sd+\frac{1}{2}\frac{\delta}{1-\delta^2}\sum_{s\in S}\psi_s(\bar{e}-d)+\frac{1}{2}\frac{\delta^2}{1-\delta^2}\sum_{s\in S}(\underline{e}+q_sd),\end{aligned}$$

which is exactly the solution in the claim.

To verify the equation for high income, consider the maximization program

$$\max\psi_s(\bar{e}+v-q_sv')+\frac{\delta}{2}\sum_{s\in S}\underline{J}_s(v')$$

subject to

$$\frac{\bar{e}+v}{q_s}\geq v'\geq-\min_{s\in S}\underline{g}_s.$$

This is equivalent to

$$\max(-\psi_sq_s+\delta)v',$$

which is consistent with $v'=(\bar{e}+v)/q_s$ because of the pricing rule (B.2). Hence,

$$\begin{aligned}\bar{J}_s(v) &= \frac{\delta}{2}\sum_{s\in S}\underline{J}_s\left(\frac{\bar{e}+v}{q_s}\right)\\ &= \delta\left(\frac{\bar{e}+v}{q_s}\right)+\frac{1}{2}\frac{\delta}{1-\delta^2}\sum_{s\in S}(\underline{e}+q_sd)+\frac{1}{2}\frac{\delta^2}{1-\delta^2}\sum_{s\in S}\psi_s(\bar{e}-d)\\ &= \psi_s(\bar{e}+v)+\frac{1}{2}\frac{\delta}{1-\delta^2}\sum_{s\in S}(\underline{e}+q_sd)+\frac{1}{2}\frac{\delta^2}{1-\delta^2}\sum_{s\in S}\psi_s(\bar{e}-d),\end{aligned}$$

so proving the claim. □

To determine debt limits, we now impose the not-too-tight condition, that is,

$$J_s(-g_s)=J_s^0(0),$$

where J^0 is the value function under no borrowing (that is, when $g=0$). Notice that

$$\bar{J}_d(-d)-\bar{J}_u(-d)=(\psi_u-\psi_d)d.$$

Therefore, because of (B.1), the debt limit will be binding in state u in S, that is, $d=\bar{g}_u<\bar{g}_d$. The not-too-tight condition is thus

$$\bar{J}_u(-d)=\bar{J}_u^0(0). \tag{B.4}$$

We now show that this condition admits a non-trivial solution for some specification of parameters.

Claim B.2 (Debt limit)**.** When $\psi_u = 1$, there exists ψ_d in $(0, \delta)$ such that the not-too-tight condition (B.4) is solved by any sufficiently small $d > 0$.

Proof. By identification, condition (B.4) is satisfied if and only if

$$q_u d = \delta \left(\frac{1}{2} \frac{1}{1-\delta^2} \sum_{s \in S} q_s - \frac{1}{2} \frac{\delta}{1-\delta^2} \sum_{s \in S} \psi_s \right) d.$$

Assuming that $\psi_u = 1$ (and so that $q_u = \delta$ by condition (B.1)), the above equation becomes

$$f(\psi_d) = \left(\frac{1}{1-\delta^2} \frac{q_d + \delta}{2} - \frac{\delta}{1-\delta^2} \frac{1+\psi_d}{2} \right) = 1.$$

When $\psi_d = \delta$, then $q_d = 1$ and

$$f(\delta) = \left(\frac{1}{1-\delta^2} \frac{1+\delta}{2} - \frac{\delta}{1-\delta^2} \frac{1+\delta}{2} \right) = \frac{1-\delta}{1-\delta^2} \frac{1+\delta}{2} = \frac{1}{2}.$$

When $\psi_d \to 0$, then $q_d \to \infty$ and

$$\lim_{\psi_d \to 0} f(\psi_d) = \infty.$$

Hence, by the Intermediate Value Theorem, a solution exists in $(0, \delta)$. □

In the competitive equilibrium, the individual borrows up to $d > 0$ when income is low and saves up to $d > 0$ when income is high. This plan is optimal because the high income individual is exactly indifferent, whereas the low-income individual is constrained. Furthermore, it satisfies market clearing. A distinguished feature of this competitive equilibirium is that debt is sustainable. However, differently from Hellwig and Lorenzoni [19], debt cannot be rolled over. Indeed, notice that interest rate is strictly positive in state u in S (*i.e.*, $q_u = \delta$). Hence, when debt is refinanced along a sequence of persistent shocks u in S, its value grows unboundedly and default is eventually profitable. To enforce repayment, debt limits have to preclude roll over.

Appendix C. Perturbed equilibrium

C.1. **Preliminaries.** We here prove the existence of an equilibrium in the perturbed economy $\mathcal{E}^\epsilon$ for a given ϵ in $(0, 1)$. As some parts of the proof are rather involved, we only sketch conventional steps and expand those that require more innovative arguments. In order to simplify notation, we drop any explicit reference to the given ϵ in $(0, 1)$. To prove existence, we artificially force default at some point in time and progressively relax this additional constraint by taking the limit.

Using Lemma 3.1, by feasibility and individually rationality, consumption plans are bounded from above by $\bar{\epsilon} > 0$ and from below by $\underline{\epsilon} > 0$. We fix a lower bound $\underline{q}$ and a upper bound $\bar{q}$ in L^+ on prices such that

$$\underline{q}_t < \min_{i \in I} \delta \frac{\nabla u^i(\bar{\epsilon})}{\nabla u^i(\underline{\epsilon})} \leq \max_{i \in I} \delta \frac{\nabla u^i(\underline{\epsilon})}{\nabla u^i(\bar{\epsilon})} < \bar{q}_t. \tag{C.1}$$

The auctioneer will vary prices in the truncated interval $\bar{Q} = [0, \bar{q}] \subset L^+$.

We also fix a large upper bound $\bar{\zeta} > 0$ on the holding of the bond. This will be judiciously chosen so as not to bind and will be removed in the next step of the proof. Finally, we truncate the economy at some s in $\mathbb{T}$ and assume that a fraction ϵ in $(0, 1)$ is confiscated and that no borrowing is allowed after this period. We shall take the limit over truncations at the end of the proof.

C.2. **Optimal plans.** We assume that, for every individual i in I, $v_0^i = 0$. Given a price q in $\bar{Q}$, for each individual i in I, at every t in $\mathbb{T}$, we compute the indirect utility $\bar{J}_t^i(q)$ subject to no borrowing, and no initial wealth, when a fraction ϵ in $(0, 1)$ of the endowment is expropriated. In this program, the holding of the bond is restricted by

$$0 \leq z_t^i \leq \bar{\zeta}.$$

This indirect utility varies continuously with respect to prices.

For fixed s in $\mathbb{T}$, we also consider a truncated program where borrowing is allowed, subject to participation, up to period s in $\mathbb{T}$ and precluded thereafter. In this truncated program, the endowment $e^{i,s}$ in L^+ coincides with e^i in L^+ up to period s in $\mathbb{T}$ and with $(1 - \epsilon) e^i$ in L^+ after period s in $\mathbb{T}$. Individual i in I is subject to budget constraint and participation constraint,

$$U_t^i\left(c^i\right) \geq \bar{J}_t^i(q).$$

The holding of the bond is restricted, for every t in $\mathbb{T}_s = \{0, \ldots, s\}$, by

$$z_t^i \leq \bar{\zeta}$$

and, for every t in $(\mathbb{T}/\mathbb{T}_s)$, by

$$0 \leq z_t^i \leq \bar{\zeta}.$$

Because of the truncations, the optimal plan varies continuously with prices. Notice that continuity can be established by traditional arguments because the participation constraint is effective only over the finite horizon $\mathbb{T}_s = \{0, \ldots, s\}$. It is clear that, as no borrowing is permitted in the prosecution, no trade will occur out of this finite horizon at equilibrium.

C.3. **Adjustment process.** We construct a correspondence $F : \bar{Q} \twoheadrightarrow \bar{Q}$ by means of the rule

$$F_t(q) = \operatorname{argmax}_{\tilde{q} \in \bar{Q}} \tilde{q}_t \sum_{i \in I} z_t^i(q).$$

This correspondence is upper hemicontinuous with convex values on a compact domain and, thus, it admits a fixed point. We then argue by induction and prove that, if $\sum_{i \in I} v_t^i = 0$, then $\sum_{i \in I} z_t^i = 0$, thus implying that $\sum_{i \in I} c_t^i = \sum_{i \in I} e_t^{i,s}$ for every t in $\mathbb{T}$.

At some contingency in period t in $\mathbb{T}$, suppose that $\sum_{i \in I} z_t^i < 0$. This implies $q_t = 0$ and, by optimality, $z_t^i = \bar{\zeta} > 0$ for each individual i in I, a contradiction.

Suppose instead that $\sum_{i \in I} z_t^i > 0$. This implies $q_t = \bar{q}_t$. For some individual i in I such that $z_t^i > 0$, first-order conditions imply

$$\bar{q}_t \leq \delta \mathbb{E}_t \frac{\nabla u^i \left(c_{t+1}^i\right)}{\nabla u^i \left(c_t^i\right)}.$$

Because $c_t^i < \bar{\epsilon}$ by feasibility (indeed, $\sum_{i \in I} v_t^i = 0$ and $\sum_{i \in I} z_t^i > 0$) and $c_{t+1}^i > \underline{\epsilon}$, this violates condition (C.1). Hence, $\sum_{i \in I} z_t^i = 0$ and, thus, $\sum_{i \in I} v_{t+1}^i = 0$. Furthermore, as $z_t^i < \bar{\zeta}$ for some individual i in I, first-order conditions imply

$$\delta \mathbb{E}_t \frac{\nabla u^i \left(c_{t+1}^i\right)}{\nabla u^i \left(c_t^i\right)} \leq q_t.$$

This shows that prices remain within the bounds given by (C.1).

C.4. **Removing the upper bound.** We now argue that the upper bound $\bar{\zeta} > 0$ can be chosen so as not to bind uniformly. Exploiting convexity, and the fact that no borrowing is imposed out of a finite horizon, plans are optimal when the upper bound is removed. We now explain as the upper bound is determined.

For the program under no borrowing, we use uniform impatience. Indeed, there exists a sufficiently large β in $(0,1)$ such that, for every feasible allocation c in $C(e)$, for every individual i in I, at every t in $\mathbb{T}$,

$$u^i \left(c_t^i + e_t\right) + \delta \mathbb{E}_t U_{t+1}^i \left(\beta c^i + (1-\beta) e^{i,s}\right) > U_t^i \left(c^i\right), \tag{C.2}$$

where e in L^+ is the aggregate endowment. By an arbitrage argument (see Santos and Woodford [26, Equation (6.17)]), this implies that

$$(1-\beta) \underline{q}_t z_t^i \leq (1-\beta) q_t z_t^i \leq e_t.$$

Indeed, if not, by condition (C.2), the individual would be better off permanently contracting portfolios by factor β in $(0,1)$, increasing current consumption by e_t and modifying future consumption as $\beta c_{t+1+j}^i + (1-\beta) e_{t+1+j}^{i,s}$ so as to balance budget. As $\underline{q}$ in L^+ can be taken as constant, this proves that, for some sufficiently large $\tilde{\zeta} > 0$, $z_t^i \leq \tilde{\zeta}$ for every optimal plan under no borrowing.

We now turn on the upper bound for the principal program. By discounting, there is a sufficiently large n in $\mathbb{N}$ such that, for every individual i in I, at every t in $\mathbb{T}$,

$$U_t^i \left(\langle \bar{e}, e^i \rangle_{t+n}\right) > U_t^i (e), \tag{C.3}$$

where $\bar{e}$ in L^+ is an adapted process strictly greater than the aggregate endowment e in L^+ and $\langle x, y \rangle_t$ in L^+ denotes the adapted process coinciding with x in L^+ up to period t in $\mathbb{T}$ and with y in L^+ after period t in $\mathbb{T}$. As prices q belong to the interval $\bar{Q}$ and the upper bound $\bar{q}$ in L^+ can be taken to be bounded, we can find a sufficiently large $\hat{\zeta} > 0$ such that, beginning from any t in $\mathbb{T}$, an amount $\hat{\zeta} > 0$ of wealth is sufficient to pay for consumption $\bar{e}$ in L^+ for the following n in $\mathbb{N}$ periods.

Hence, individual i in I can implement the optimal plan under no borrowing, and no initial wealth, along with this additional large consumption. This alternative plan is budget feasible, satisfies participation constraint and, by condition (C.3), contradicts optimality of the feasible plan c^i in L^+, a contradiction. Hence, $\hat{\zeta} > 0$ provides an upper bound. Any $\zeta > \hat{\zeta} + \tilde{\zeta}$ can be used for the existence proof above.

C.5. **Relaxing truncation.** We now take the limit by relaxing the truncation s in $\mathbb{T}$. Previous steps show the existence of a truncated equilibrium prices q^s in $\bar{Q}$, with an associated optimal consumption plan $c^{i,s}$ in L^+ for every individual i in I. For fixed s in $\mathbb{T}$, given any contingency in period t in $\mathbb{T}$, we compute the indirect utility $J_t^{i,s}\left(v_t^i\right)$ subject to budget constraints, participation constraints and no borrowing after period s in $\mathbb{T}$ when initial wealth is v_t^i in L_t (as in section C.2, without imposing the upper bound $\bar{\zeta} > 0$). By convention, value is negative infinity when constraints cannot be satisfied. For every t in $\mathbb{T}$, we determine $g_t^{i,s}$ in L_t^+ as

$$J_t^{i,s}\left(-g_t^{i,s}\right) = \bar{J}_t^i, \tag{C.4}$$

where the right hand-side is the indirect utility subject to no borrowing, and no initial wealth, when the fraction ϵ in $(0,1)$ of the endowment is confiscated. A solution exists by continuity, as the participation constraint cannot be satisfied when the initial debt is too large and no borrowing is permitted eventually. Also notice that $g_t^{i,s} = 0$ for every t in $(\mathbb{T}/\mathbb{T}_s)$. The plan remains optimal when participation constraints are substituted by solvency constraints of the form

$$v_t^i \geq -g_t^{i,s}.$$

Thus, for the last steps, we only maintain not-too-tight solvency constraints (*i.e.*, satisfying condition (C.4)) and consider the limit with respect to s in $\mathbb{T}$.

Debt limits remain bounded. If not, the individual can borrow an arbitrarily large amount against the endowment, violating optimality of plans along the sequence (see the last part of the proof of Lemma 6.3 for a similar argument). Hence, possible extracting a subsequence, consumption plans, financial plans and debt limits converge. Limi

C.6. **Establishing the contradiction.** As budget feasibility is satisfied in the limit, we argue by contradiction to show that the limit plan c^i in L^+ is optimal subject to budget and solvency constraints. Supposing not, there exists an alternative budget feasible plan $\bar{c}^i$ in L^+, with an associated trading plan $\bar{z}^i$ in Z^i, yielding higher utility. (Remember that, with a single safe bond, $\bar{v}_{t+1}^i = \bar{z}_t^i$ at every t in $\mathbb{T}$.) By slightly contracting initial consumption and spreading this value over time, we can assume that budget and solvency constraints are never binding. By discounting, for some sufficiently large $\bar{t}$ in $\mathbb{T}$, we have

$$U_0^i\left(\bar{c}^i\right) + \delta^{\bar{t}+1}\mathbb{E}_0\left(U_{\bar{t}+1}^i\left((1-\epsilon)\, e^i\right) - U_{\bar{t}+1}^i\left(\bar{c}^i\right)\right) > U_0^i\left(c^i\right),$$

where c^i in L^+ is the dominated plan in the limit. For any sufficiently large s in $\mathbb{T}$, the consumption plan $\bar{c}^i$ in L^+ and the financial plan $\bar{v}^i$ in L satisfy budget and solvency constraint at every t in $\mathbb{T}_{\bar{t}} = \{0, \ldots, \bar{t}\}$. Furthermore, $\bar{v}^i_{\bar{t}+1}$ in $L_{\bar{t}+1}$ satisfies $\bar{v}^i_{\bar{t}+1} \geq -g^{i,s}_{\bar{t}+1}$. Hence, individual i in I can implement this given plan on $\mathbb{T}_{\bar{t}}$ and the optimal plan starting from wealth $\bar{v}^i_{\bar{t}+1}$ in $L_{\bar{t}+1}$ on $(\mathbb{T}/\mathbb{T}_{\bar{t}})$, so as to secure the utility value given by

$$\begin{aligned} U^i_0\left(\bar{c}^i\right) + \delta^{\bar{t}+1}\mathbb{E}_0\left(J^{i,s}_{\bar{t}+1}\left(\bar{v}^i_{\bar{t}+1}\right) - U^i_{\bar{t}+1}\left(\bar{c}^i\right)\right) &\geq \\ U^i_0\left(\bar{c}^i\right) + \delta^{\bar{t}+1}\mathbb{E}_0\left(J^{i,s}_{\bar{t}+1}\left(-g^{i,s}_{\bar{t}+1}\right) - U^i_{\bar{t}+1}\left(\bar{c}^i\right)\right) &\geq \\ U^i_0\left(\bar{c}^i\right) + \delta^{\bar{t}+1}\mathbb{E}_0\left(U^i_{\bar{t}+1}\left(\left(1-\epsilon\right)e^i\right) - U^i_{\bar{t}+1}\left(\bar{c}^i\right)\right) &> U^i_0\left(c^i\right), \end{aligned}$$

where we use the fact that the non-confiscated part of the endowment can be consumed. This shows that, for all sufficiently large s in $\mathbb{T}$, a utility greater than $U^{i,s}_0\left(c^{i,s}\right)$ is budget-affordable, a contradiction.

Appendix D. Dominant root

We provide a self-contained presentation of the dominant root method for simple Markov pricing kernels under incomplete markets. We begin with the study of an abstract operator and relate our findings to the asset pricing kernel. Our analysis integrates and expands Bloise et al. [12, Appendix C].

We consider a continuous operator $\Pi : V \to V$ on some Euclidean linear space V, endowed with its canonical norm and its canonical ordering. The operator is *strongly monotone*, that is, $v' > v''$ implies $\Pi(v') \gg \Pi(v'')$. It is also *sublinear*, that is, $\Pi(\lambda v) = \lambda \Pi(v)$, for every λ in $\mathbb{R}_+$, and $\Pi(v' + v'') \leq \Pi(v') + \Pi(v'')$. As usual, V_+ is the positive cone of the linear space V. Monotone sublinearity is the property inherited by the pricing kernel, under no arbitrage, when markets are incomplete. Strong monotonicity obtains under strictly positive Markov transitions.

Dominant roots are defined as in our analysis in §4. The *upper dominant root* $\rho(\Pi)$ is given by the greatest ρ in $\mathbb{R}_+$ such that, for some non-zero b in V_+,

$$\rho b \leq \Pi(b).$$

Analogously, the *lower dominant root* $\gamma(\Pi)$ is given by the greatest γ in $\mathbb{R}_+$ such that, for some non-zero b in V_+,

$$\gamma b \leq -\Pi(-b).$$

The upper and the lower dominant roots capture the maximum expansion rate of the operator on the positive and on the negative cone, respectively. A simple argument establishes existence of dominant roots and the associated upper eigenvector.

Claim D.1 (Dominant roots)**.** Both $\rho(\Pi)$ and $\gamma(\Pi)$ exist and satisfy

$$\gamma(\Pi) \leq \rho(\Pi).$$

Furthermore, there exists b in the interior of V_+ such that

$$\rho(\Pi)\, b = \Pi(b)\,. \tag{D.1}$$

Proof. Let Δ be the unitary simplex in V_+ and consider the map $F : \Delta \twoheadrightarrow \mathbb{R}_+$ defined by

$$F(d) = \{\rho \in \mathbb{R}_+ : \rho d \leq \Pi(d)\}\,.$$

This is upper hemicontinuous with compact values. By the Maximum Theorem, the value function $f(d) = \max_{\rho \in F(d)} \rho$ is upper semicontinuous. Its maximum $\rho(\Pi) = \max_{d \in \Delta} f(d)$ is the upper dominant root. A similar argument establishes the existence of the lower dominant root. By sublinearity,

$$-\Pi(-b) \leq \Pi(b)\,,$$

which shows that $\gamma(\Pi) \leq \rho(\Pi)$. To prove existence of an eigenvector, consider any b in Δ such that

$$\rho(\Pi)\, b \leq \Pi(b)\,.$$

When condition (D.1) fails, strong monotonicity yields

$$\rho(\Pi)\, \Pi(b) = \Pi(\rho(\Pi)\, b) \ll \Pi(\Pi(b))\,.$$

As $\Pi(b)$ lies in V_+, this contradicts the fact that $\rho(\Pi)$ is the upper dominant root. □

We also show that dominant roots are uniquely identified when eigenvectors exist. In particular, we establish that dominant roots coincide if and only if the associated eigenvector lies in the linear kernel of the sublinear operator.

Claim D.2 (Identification)**.** If there is b in the interior of V_+ such that, for some ρ in $\mathbb{R}_+$,

$$\rho b = \Pi(b)\,,$$

then $\rho(\Pi) = \rho$. Analogously, if there is b in the interior of V_+ such that, for some γ in $\mathbb{R}_+$,

$$\gamma b = -\Pi(-b)\,,$$

then $\gamma(\Pi) = \gamma$. In particular, dominant roots coincide if and only if there is b in the interior of V_+ such that, for some λ in $\mathbb{R}_+$,

$$\lambda b = \Pi(b) = -\Pi(-b)\,.$$

Proof. As the other proof is specular, to verify the second statement, consider any non-zero b^* in V_+ such that

$$\gamma(\Pi)\, b^* \leq -\Pi(-b^*)\,.$$

Let λ in $\mathbb{R}_+$ be the maximum value such that $\lambda b^* \leq b$ and, at no loss of generality, assume that $\lambda = 1$. Monotonicity yields

$$\gamma(\Pi) b^* \leq -\Pi(-b^*) \leq -\Pi(-b) \leq \gamma b,$$

which implies $\gamma(\Pi) \leq \gamma$. As $\gamma \leq \gamma(\Pi)$ by the definition of lower dominant root, the claim is proved. For the coincidence, sufficiency follows from the previous part. To establish necessity, consider any non-zero b in V_+ such that

$$\gamma(\Pi) b \leq -\Pi(-b).$$

Sublinearity implies

$$\gamma(\Pi) b \leq \Pi(b).$$

If the inequality is strict, strong monotonicity yields

$$\gamma(\Pi) \Pi(b) \ll \Pi(\Pi(b)),$$

which shows that $\gamma(\Pi) < \rho(\Pi)$ as $\Pi(b)$ lies in V_+. This is a contradiction. Hence,

$$\gamma(\Pi) b = -\Pi(-b) = \Pi(b),$$

and b is in the interior of V_+ by strong monotonicity, so proving the claim. □

We relate dominant roots to the existence of well-defined present values. Fixing a claim e in V_+, the upper present value is the solution to the recursive equation

$$f = e + \Pi(f). \tag{D.2}$$

Analogously, the lower present value is the solution to recursive equation

$$f = e - \Pi(-f). \tag{D.3}$$

We show that present values are finite if and only if dominant roots are less than unity.

Claim D.3 (Present values)**.** Given a claim e in the interior of V_+, the upper (lower) present value is finite if and only if $\rho(\Pi) < 1$ ($\gamma(\Pi) < 1$).

Proof. We show the claim for the lower present value, as the argument is analogous in the other case. Suppose that $\gamma(\Pi) \geq 1$ and that f in the interior of V_+ solves equation (D.3). Let λ be the greatest value in $\mathbb{R}_+$ such that $\lambda b \leq f$, where $\gamma(\Pi) b \leq \Pi(b)$ and b is a non-zero element of V_+. Monotone sublinearity implies

$$\lambda b \ll e - \lambda\Pi(-b) \leq e - \Pi(-\lambda b) \leq e - \Pi(-f) \leq f,$$

a contradiction. Now assume that $\gamma(\Pi) < 1$ and define, beginning with $f^0 = 0$, for every n in $\mathbb{Z}_+$,

$$f^{n+1} = e - \Pi(-f^n).$$

Clearly, $f^{n+1} \geq f^n$. If this sequence converges, we obtain the lower present value by continuity. Otherwise, it diverges and, by linear homogeneity,

$$\frac{f^n}{\|f^n\|} \leq \frac{f^{n+1}}{\|f^n\|} = \frac{e}{\|f^n\|} - \Pi\left(-\frac{f^n}{\|f^n\|}\right).$$

Possibly extracting a converging subsequence, in the limit, for some non-zero b in V_+,

$$b \leq -\Pi(-b),$$

which implies $\gamma(\Pi) \geq 1$, a contradiction. □

We are now in the condition of proving existence of a lower eigenvector. Notice that we do not show that it lies in the interior of V_+, as instead required for the identification.

Claim D.4 (Lower eigenvector)**.** There exists b in V_+ such that

$$\gamma(\Pi)\, b = -\Pi(-b). \tag{D.4}$$

Proof. Given any ϵ in $(0, 1)$, consider the perturbed operator

$$\Pi^\epsilon = \left(\frac{1-\epsilon}{\gamma(\Pi)}\right)\Pi.$$

Notice that, by linear homogeneity, $\gamma(\Pi^\epsilon) = 1 - \epsilon$. Fix a claim e in the interior of V_+ and observe that the lower present value f^ϵ in the interior of V_+ exists for the perturbed operator (Claim D.3). Therefore, there is b^ϵ in V_+, with $\|b^\epsilon\| = 1$, such that

$$b^\epsilon = \frac{e}{\|f^\epsilon\|} - \left(\frac{1-\epsilon}{\gamma(\Pi)}\right)\Pi(-b^\epsilon).$$

Going to the limit as ϵ in $(0, 1)$ vanishes, possibly extracting a subsequence, we obtain the claim because the lower present value grows unboundedly and, hence,

$$\gamma(\Pi)\, b = -\Pi(-b),$$

thus concluding the proof. □

We apply our general analysis to a Markov pricing kernel under incomplete markets. To this purpose, we assume that uncertainty is generated by a Markov process on the finite state space S, with $\mu_{s,\hat{s}} > 0$ being the probability of moving from state s in S into state $\hat{s}$ in S. A finite set of securities J is traded at price q_s in $\mathbb{R}^J$ in state s in S, each delivering a payoff $R^j_{s,\hat{s}}$ in $\mathbb{R}$ in state $\hat{s}$ in S in the following period. In state s in S, a portfolio z_s in $\mathbb{R}^J$ can be acquired at market price

$$q_s(z_s) = \sum_{j \in J} q^j_s z^j_s,$$

yielding a contingent payoff in the following period according to

$$R_s\left(z_s\right)=\left(\sum_{j\in J}R^j_{s,\hat{s}}z^j_s\right)_{\hat{s}\in S}\in\mathbb{R}^S.$$

We assume the absence of arbitrage opportunities, that is, $R_s\left(z_s\right)>0$ only if $q_s\left(z_s\right)>0$. Furthermore, we suppose that securities allows for a strictly positive transfer, that is, $R_s\left(z^f_s\right)\gg 0$ for some portfolio z^f_s in $\mathbb{R}^J$. When this fails, the current state is disconnected from some future state.

We consider the conventional valuation opertator generated by the minimum expenditure program, that is,

$$\Pi_s\left(v\right)=\min_{z_s\in\mathbb{R}^J}q_s\left(z_s\right) \tag{D.5}$$

subject to

$$v\leq R_s\left(z_s\right).$$

The specular operation is given by

$$-\Pi_s\left(-v\right)=\max_{z_s\in\mathbb{R}^J}q_s\left(z_s\right) \tag{D.6}$$

subject to

$$R_s\left(z_s\right)\leq v.$$

Under the stated assumptions, operator $\Pi:\mathbb{R}^S\to\mathbb{R}^S$ is continuous, strongly monotone and sublinear (see LeRoy and Werner [24, Chapter 5]). In particular, the cost-minimizing portfolio exists under no arbitrage. We remark that strong monotonicity obtains because all Markov transitions are strictly positive. Given an arbitrage price q in $\mathbb{R}^{J\times S}$, we denote $\rho\left(q\right)$ and $\gamma\left(q\right)$ the dominant roots of the pricing operator $\Pi:\mathbb{R}^S\to\mathbb{R}^S$.

We compute dominant roots in the relevant case when only the risk-free (discount) bond is traded: the upper dominant root is the greatest price of the bond, whereas the lower dominant root is the least price of the bond. Hence, in such a situation, the upper (lower) dominant root is less than unity if and only if interest rate is always (sometimes) strictly positive.

Claim D.5 (Safe bond only)**.** When the risk-free bond is the only asset,

$$\rho\left(q\right)=\max_{s\in S}q_s \text{ and } \gamma\left(q\right)=\min_{s\in S}q_s.$$

Proof. By no arbitrage, q lies in $\mathbb{R}^S_{++}$. By direct inspection, let b in $\mathbb{R}^S_{++}$ be given by $b_s=q_s$ at every s in S. To satisfy the constraint in (D.5), it is necessary to hold at least a quantity $\max_{\hat{s}\in S}q_{\hat{s}}$ of the risk-free bond (with unitary payoff). Thus,

$$\Pi_s\left(b\right)=\left(\max_{\hat{s}\in S}q_{\hat{s}}\right)q_s=\left(\max_{\hat{s}\in S}q_{\hat{s}}\right)b_s=\rho\left(q\right)b_s.$$

Similarly, to satisfy the reverse constraint in (D.6), it is necessary to hold no more than quantity $\min_{\hat{s}\in S} q_{\hat{s}}$ of the risk-free bond. Thus,

$$-\Pi_s(-b) = \left(\min_{\hat{s}\in S} q_{\hat{s}}\right) q_s = \left(\min_{\hat{s}\in S} q_{\hat{s}}\right) b_s = \gamma(q)\, b_s.$$

It might well be true that the upper dominant root is larger than unity, $\rho(q) > 1$, because interest rate is negative, $q_s > 1$, in some state s in S. □

We remark that conditions for coincidence of dominant roots are rather demanding when markets are incomplete. In particular, using Claim D.2, $\rho(q) = \gamma(q) = 1$ only if, in each state s in S, there exists a portfolio of securities z_s in $\mathbb{R}^J$ such that $q_s(z_s) = b_s$ and $R_s(z_s) = b$. That is, a portfolio in each state with the same payoff and with price equal to the payoff in that state. To conclude, we show continuity of valuation as asset prices vary.

Claim D.6 (Continuity)**.** For given b in $\mathbb{R}^S$, at every s in S, the minimum-cost $\Pi_s(b,q)$ is continuous in (arbitrage-free) security prices q in $\mathbb{R}^{J\times S}$.

Proof. Pick a sequence of prices $(q^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{J\times S}$ converging to q in $\mathbb{R}^{J\times S}$. Letting z_s in $\mathbb{R}^J$ be a minimum-cost portfolio in the limit, we have

$$\Pi_s^n(b) - \Pi_s(b) \le q_s^n(z_s) - q_s(z_s).$$

It follows that

$$\limsup_{n\to\infty} \Pi_s^n(b) \le \Pi_s(b).$$

In order to obtain a contradiction, assume that

$$\liminf_{n\to\infty} \Pi_s^n(b) < \Pi_s(b).$$

As the sequence $(\Pi_s^n(b))_{n\in\mathbb{N}}$ in $\mathbb{R}$ is bounded, we can assume that it converges at no loss of generality. Letting z_s^n in $\mathbb{R}^J$ be a minimum-cost portfolio at n in $\mathbb{N}$, we have

$$q_s^n(z_s^n) - q_s(z_s^n) \le \Pi_s^n(b) - \Pi_s(b).$$

Notice that, as the pricing kernel is linear, we can suppose that the there are no redundant securities, or equivalently that portfolios are taken in the quotient space. If the sequence $(z_s^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^J$ remains bounded, we can assume that it converges to z_s in $\mathbb{R}^J$ at no loss of generality. This yields

$$0 \le \lim_{n\to\infty} q_s^n(z_s^n) - q_s(z_s^n) \le \lim_{n\to\infty} \Pi_s^n(b) - \Pi_s(b) < 0,$$

a contradiction. Otherwise, the sequence $(\tilde{z}_s^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^J$ is bounded, where

$$\tilde{z}_s^n = \frac{1}{\|z_s^n\|} z_s^n.$$

Assuming convergence to $\tilde{z}_s$ in $\mathbb{R}^J$ at no loss of generality, we obtain

$$0 \leq q_s\left(\tilde{z}_s\right) = \lim_{n \to \infty} q_s^n\left(\tilde{z}_s^n\right) \leq \lim_{n \to \infty} \frac{1}{\|z_s^n\|} \Pi_s^n\left(b\right) = 0$$

and, at every $\hat{s}$ in S,

$$0 = \lim_{n \to \infty} \frac{1}{\|z_s^n\|} d_{\hat{s}} \leq \lim_{n \to \infty} R_{s,\hat{s}}\left(\tilde{z}_s^n\right) = R_{s,\hat{s}}\left(\tilde{z}_s\right).$$

No arbitrage pricing so implies that $R_{s,\hat{s}}\left(\tilde{z}_s\right) = 0$ for every $\hat{s}$ in S. By no redundancies, $\tilde{z}_s$ is the zero portfolio in $\mathbb{R}^J$, which contradicts the fact that $\|\tilde{z}_s\| = 1$. □

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