Pareto Efficiency in a General Stochastic Life-Cycle Model with Production

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First Draft: March, 2007. This version: November 13, 2008

Abstract

We study the properties of a stochastic overlapping generations model with many-period-lived agents with stochastic lifetimes, neoclassical production and capital accumulation. Our main result is a complete characterization of interim Pareto efficiency. The condition coincides with the one that holds in a two-period stochastic OLG economy. Therefore, existing Cass-type tests carry over to stochastic life-cycle models with arbitrary period length. Our results allow to assess Pareto efficiency empirically by making use of return data with observed maturities.

JEL classification: D61, D91, E13

Keywords: Stochastic OLG Model, Pareto Efficiency, Multi-Generation Model, Life-Cycle Model, Many-Period-Lived Agents

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We would like to thank Piero Gottardi and Felix Kuebler for helpful discussions.

1 Introduction

The overlapping generations (OLG) or life-cycle model has been widely used to study a variety of questions in macroeconomic theory and policy. Some of the most interesting properties of the OLG model are associated with its ability to produce inefficient equilibria in which agents hold unbacked government liabilities or fiat money.

Until recently, most theoretical research involving OLG models was conducted using highly stylized specifications, e.g. Samuelson-Diamond-like models with a lifetime of two or at most three periods. In recent years, however, macroeconomists have become increasingly interested in general equilibrium models that are realistic in the sense that the decision problems of agents in the model economy are similar in timing to the problems of agents in actual economies. Pioneers in the effort to develop such models include Auerbach and Kotlikoff (1987), who extend the Diamond (1965) model of production and capital by introducing households who live for many periods.

This paper bridges the gap between the theoretical literature on efficiency characterizations available for OLG models with two-period lived agents and the applied literature on stochastic life-cycle economies with many-period-lived agents. Our framework is close to Ríos-Rull (1996) who considers a life-cycle model with many-period-lived agents with stochastic lifetimes and productivity shocks. Our main result is that the price characterizations in the Cass (1972) tradition carry over from stochastic two-generation models to stochastic OLG models with quite general demographic structure.

In particular, the efficiency characterization of Chattopadhyay and Gottardi (1999) that was developed in a two-generation pure exchange stochastic OLG model holds under mild conditions in a multi-generation OLG model with capital, productivity shocks and stochastic lifetime. This implies that existing Cass-type tests [see e.g. Chattopadhyay (2006) and Barbie et al. (2007)] carry over to stochastic life-cycle models with arbitrary period length and household lifetimes. Our results therefore allow to assess Pareto efficiency empirically by making use of return data with observed maturities, e.g. information incorporated in the yield curve.

There are some other papers that deal with optimality in multigeneration overlapping generations models. Molina-Abraldes and Pintos-Clapes (2004) deal with a characterization of efficiency in general overlapping generations economies without uncertainty. Henriksen and Spear (2008) analyze how sequentially incomplete markets influence risk sharing among individuals in a three-generation stochastic overlapping generations economy. Departing from the canonical notion of interim Pareto optimality that we employ, Bloise and Calziano (2007) characterize *robust* inefficiency in the sense of Debreu (1951) in a stochastic $N-$generation overlapping generations model. Demange (2002) provides a sufficient condition for interim optimality in a stochastic $N-$generation model with sequentially complete markets.

Section 2 introduces the model and key definitions. Section 3 presents our results. Section 4 contains proofs.

2 The Model

We consider an infinite-horizon production economy which is populated by overlapping generations at every point of time. Time is discrete, starts at 0 and extends infinitely into the future. There is a single perishable good produced with a production technology at each point of time. The good is used for production and consumption. At every point of time, there is production and a consumption-savings decision. The sources of uncertainty are stochastic shocks to the production technology and shocks to the lifetimes of (representative) generations. Essentially, we consider a stochastic multi-generations version of the Diamond (1965) model, similar to Ríos-Rull (1996).

Uncertainty The shocks affect both the production technology and the lifetime of representative generations. W.l.o.g, for all $t \geq 1$, the set of shocks, $Z_t = Z$, has finite cardinality where $|Z|$ is the cardinality of Z and Z_0 is single-valued, i.e. $Z_0 = \{z_0\}$. We denote by $z^t = (z_0, z_1, ..., z_t)$ an arbitrary history of shocks. We denote by $\mathcal{H}$ the set of all (finite) histories, i.e. $z^t \in \mathcal{H}$. For any histories z^s and z^t with $s > t$, we write $z^s \succ z^t$ if $z^s = (z^t, z_{t+1}, ..., z_s)$ for $z_{t+i} \in Z$ and we write $z^s \succeq z^t$ if either $z^s \succ z^t$ or $z^s = z^t$. For every $z^t \in \mathcal{H}$, $(z^t)^+$ denotes the set of immediate successor histories of z^t, i.e. $(z^t)^+ = \{z^{t+1} \in \mathcal{H} \,|\, z^{t+1} \succ z^t\}$. In order to capture generations born before period 0, we extend $\mathcal{H}$ to $\widetilde{\mathcal{H}} := \left\{z^{-(N-1)}, z^{-(N-2)}, ..., z^{-1}\right\} \cup \mathcal{H}$. We define that $z^t \succ z^{-i}$ for all $t = 0, 1, ...$ and $i = 1, ..., N-1$. Further, $z^{-j} \succ z^{-i}$ if and only if $j < i$.

Production The production technology at time t is described by a neoclassical constant-returns-to-scale production function $F : \mathbb{R}^2_+ \times Z_t \rightarrow \mathbb{R}_+$, where $F(k_t, l_t, z_t)$ is the output produced at time t, k_t is the capital stock, l_t is the labor input and z_t is the current stochastic shock drawn from Z_t. The perishable good produced with the technology is the only good in the economy and is used for production and consumption. For simplicity, the depreciation rate is assumed to be 1.

Households For simplicity, we assume that there is no population growth. At each node in the tree, one representative household per generation is born who lives a maximum of N periods and at least for two periods. The history of stochastic shocks z^t determines which of the households alive at $z^{t-1} \prec z^t$ will not survive in z^t. Households are identified with the date and the state of nature in which they are born. Since time starts at $t = 0$, we start with $N - 1$ initially old households (born in periods $-1, ..., -(N-1)$).

Labor Supply In each of the first $N - 1$ periods of his life (in the case he reaches this age), every household born in $t \geq 0$ inelastically supplies one unit of labor.[1] Similarly, the households born before period 0 inelastically supply one unit of labor at each date-event, except for the period that defines their maximum lifetime (conditional on living that long). Thus the aggregate amount of labor offered is L_t for each date-event z^t will depend on the date and is written as $l(z^t)$. For convenience we work with the production function $f(k_t, z^t) = F(k_t, l(z^t), z_t)$ with the following standard properties: $f(0, z^t) = 0$, $f' = \frac{\partial f}{\partial k} > 0$, $f'' < 0$, $f'(0, z^t) = \infty$, $f'(\infty, z^t) = 0$.

Preferences Except for the households born in period $-(N-1)$, who face no uncertainty, all households face uncertainty starting with their second period of life due to the stochastic shocks. Consider the family of functions $U_I : \mathbb{R}^I_+ \times \Theta \to \mathbb{R}$, where $1 \leq I \leq 1 + |Z| + ... + |Z|^{N-1}$ and $\Theta \subseteq \mathbb{R}^q$ (for some $q \in \mathbb{N}$) is compact. Here, $|.|$ denotes the cardinality of a (finite) set. For fixed I and θ, $U_I(., \theta)$ is twice continuously differentiable (in the interior of its domain), strictly increasing in each argument and with negative definite Hessian matrix. Further, for fixed I, the first and second partial derivatives are continuous functions of θ. I is the number of date-events a certain individual is alive, and $\theta \in \Theta$ is a preference parameter. To the household born in $z^t \in \widetilde{\mathcal{H}}$, we assign a set of histories $I(z^t) \subseteq \mathcal{H}$ at which the household is alive. Formally, we define $I(z^t)$ as a subset of $\left\{z^s \in \mathcal{H} \left| z^{\max\{0,t\}} \preceq z^s \preceq z^{t+N-1} \text{ for some } z^{t+N-1}\right.\right\}$, which satisfies (1) $z^{\max\{0,t\}} \in I(z^t)$ and (2) $z^t \preceq z^{s'} \preceq z^s$ and $z^s \in I(z^t)$ implies $z^{s'} \in I(z^t)$. For every z^t, define $N(z^t) = \{i \in \{1, ..., N\} | z^t \in I(z^{t-i+1})\}$, the generations alive in z^t. Note that $N(z_0) = \{1, ..., N\}$. Further, we assign to each individual z^t via a function $\theta : \widetilde{\mathcal{H}} \to \Theta$ a parameter value θ. The consumption set of a household born in some date-event z^t is $\mathbb{R}^{|I(z^t)|}_+$ and the preferences are given by $U_{|I(z^t)|}(c(z^t), \theta(z^t))$. We denote by $c(z^t)$ the consumption vector of individual z^t and it is given by $c(z^t) = (c_{s-t+1}(z^s))_{z^s \in I(z^t)}$. Here we denote by $c_i(z^s)$ the consumption of a person with age $i = 1, ..., N-1$ in history z^s, i.e. a person born in t in some z^t with $z^s \succeq z^t$.

Households' Problem Conditional on surviving, a household born in z^t receives a wage income $w(z^{t-1+i})$ for $i = 1, ..., N-1$ in the first $N-1$ periods of his life from inelastically supplying one unit of labor. Let $W(z^t)$ denote the date-events in which individual z^t supplies labor. i.e. $W(z^t) = I(z^t) \cap \{z^s \in \mathcal{H} | z^s \succeq z^t \text{ and } s \leq t + N - 2\}$. We assume here that a complete set of Arrow securities exists at each history z^t, so that markets are sequentially complete. Facing state contingent prices $p\left(z^{\max\{0,t+i\}}\right)$ for $i = 0, ..., N-1$,

[1] Endogenous labor supply as in Ríos-Rull (1996) is a straightforward extension.

the household's problem for all households except the household born in period $-(N-1)$ is to maximize utility subject to his intertemporal budget constraint:

$$\max_{c(z^t)} U_{|I(z^t)|}\left(c\left(z^t\right), \theta\left(z^t\right)\right) \tag{1}$$

$$s.t. \qquad \sum_{z^s \in I(z^t)} p\left(z^s\right) c_{s-t+1}\left(z^s\right) \leq \sum_{z^s \in W(z^t)} p\left(z^s\right) w\left(z^s\right)$$

The consumer born in period $-(N-1)$ maximizes the consumption of the single good $c_N\left(z_0\right)$. He is endowed with $f'\left(k_{-1}, z_0\right) \cdot k_{-1}$,where k_{-1} is the initial stock of capital. In a competitive equilibrium, as defined below, this simply means that he is the initial owner of the capital stock. In the following, we write $U_{z^t}\left(c\left(z^t\right)\right)$ for $U_{|I(z^t)|}\left(c\left(z^t\right), \theta\left(z^t\right)\right)$ for simplicity.

Firm's Problem The firm's problem is to decide after the shock realization at every history z^t how much capital to invest. This capital is then used to produce output at the immediate successor histories of z^t. The firm maximizes profits given state contingent prices $p\left(z^t\right)$ and $p\left(z^{t+1}\right)$ for $z^{t+1} \succ z^t$. Let $k\left(z^t\right)$ be the firm's investment in z^t. The firm's problem in z^t is:[2]

$$\max_{k(z^t)\geq 0} \sum_{z^{t+1} \succ z^t} \left[p\left(z^{t+1}\right) \cdot f\left(k\left(z^t\right), z^{t+1}\right)\right] - p\left(z^t\right) \cdot k\left(z^t\right). \tag{2}$$

Key Definitions Next, we define feasible allocations, the notion of optimality and a competitive equilibrium. Our definition of Pareto efficiency is - in the terminology of Demange and Laroque (1999) - *interim Pareto efficiency.* It is also used in Chattopadhyay and Gottardi (1999). Households born in different *date-events* are considered distinct households. Then, the usual concept of Pareto efficiency is applied to this set of households. This feature of our model distinguishes our analysis e.g. from Gottardi and Kubler (2008) who adopt ex ante Pareto efficiency as a welfare criterion.

Definition 1 *Let the initial capital stock k_{-1} be given. A feasible allocation is a tuple $(c, k) = \left(\left(c_i\left(z^t\right)\right)_{i \in N(z^t)}\right)_{z^t \in \mathcal{H}}, \left(\left(k\left(z^t\right)\right)_{z^t \in \mathcal{H}}, k_{-1}\right)\right)$ such that*

1. $\sum_{i=1}^N c_i\left(z_0\right) + k\left(z_0\right) = f\left(k_{-1}, z_0\right)$

[2]Problem (2) only describes finding the optimal capital stock for labor supply fixed at $l(z^{t+1})$. Since $l(z^{t+1})$ is in inelastic supply, in equilibrium it suffices to find the optimal capital stock given $l(z^{t+1})$. For simplicity and without loss of generality we therefore abstract from considering optimal labor demand of the firm.

2. For $z^t \in \mathcal{H}$ with $t \geq 1 : \sum_{i \in N(z^t)} c_i(z^t) + k(z^t) = f(k(z^{t-1}), z^t)$.

Definition 2 *A feasible allocation (c, k) is called interim Pareto efficient if there exists no other feasible allocation $\left(\widehat{c}, \widehat{k}\right)$ such that $U_{z^t}(\widehat{c}(z^t)) \geq U_{z^t}(c(z^t))$ for all $z^t \in \widetilde{\mathcal{H}}$, with at least one strict inequality.*

Definition 3 *(c^*, k^*, p^*, w^*) is a competitive equilibrium if (a) c^* solves the household's problem (1) for every household born after $-(N-1)$ given competitive wages w^* and state contingent prices p^*, (b) the household born in $-(N-1)$ consumes $f'\left(k^*_{-1}, z_0\right) \cdot k^*_{-1}$, (c) firms maximize profits given p^*, and (d) (c^*, k^*) is a feasible allocation, i.e. markets clear.*

Throughout the rest of the paper, we say that a competitive equilibrium (c, k) or more generally an allocation (c, k) is *interior* if the capital stock k and consumption c is uniformly bounded away from 0 (given the initial capital stock k_{-1}).

3 Characterization of Efficiency

In this section, we show that the efficiency characterization derived by Chattopadhyay and Gottardi (1999) as well as its extension to production economies by Barbie et al. (2007) for a two-generation stochastic OLG model remains valid for a multi-generation economy. First, since all the assumptions of Theorem 2 in Chattopadhyay and Gottardi (1999) hold and since we can w.l.o.g. restrict transfer payments to persons of age 1 and age 2 at each date-event, the condition given in their Theorem 2 remains valid as a sufficient condition for interim Pareto inefficiency (see Lemma 1 in section 4.1).

To show that the condition is also necessary for inefficiency, we proceed in two steps. First, in subsection 4.2, we construct an artificial two-period OLG economy with aggregated date-events formed from the date-events of our original economy with the property that every individual only lives two periods in the artificial economy. We derive a Chattopadhyay-Gottardi-like necessary condition for efficiency expressed in terms of the date-event tree and equilibrium price vectors of this artificial economy. Second, we show that this condition can be equivalently expressed as the sufficient condition for efficiency from Lemma 1 in section 4.1, see the proof of Theorem 1. This condition coincides with the characterization of efficient equilibrium allocations in the two-generation model.

The main difficulties in proving the necessity of the condition for inefficiency arise at three points: (1) The appropriate definition of the date events for the artificial two period OLG economy. This is done in subsection 4.2. (2) The incorporation of joint changes in production and consumption This is carried out in subsection 4.3 in step 2 of the proof of Proposition 1. This step is considerably more involved than in Barbie et al. (2007). A firm invests in some date event z^t and produces at sucessor date events $(z^t)^+$ of the *original* economy. By our construction of the artificial date events from the original economy in subsection 4.2, it may happen that z^t and some elements of $(z^t)^+$ belong to the same artificial date event σ_s, whereas other elements of $(z^t)^+$ belong to σ_{s+1}. For a useful definition of a roll-over scheme (equation (10)), it is necessary to split the change in the capital stock in an appropriate way. This is done by introducing a new weight function for production (see equation (9)). This is not necessary in Barbie et al. (2007), where investment and production always take place in different date events. (3) The construction of the transfer pattern for the original economy from the transfer pattern of the artificial economy. This is carried out in section 4.4 in the proof of Theorem 1.

To start, we recall the definition of a transfer pattern, which plays a crucial role in determining efficiency of a competitive allocation in stochastic OLG models.[3]

Definition 4 *A transfer pattern* λ *is a function* $\lambda : \mathcal{H} \rightarrow [0,1]$ *with the following properties. The set of histories* $\mathcal{H}$ *can be partitioned,* $\mathcal{H} = \mathcal{H}_+ \dot{\bigcup} \mathcal{H}_0 \dot{\bigcup} \{z_0\}$ *with* $\mathcal{H}_+ \neq \emptyset$*, so that the following holds:*

1. $\lambda(z_0) = 1$.

2. $z^t \in \mathcal{H}_0$ *if and only if* $\lambda(z^t) = 0$ *and* $z^t \in \mathcal{H}_0$ *implies for all* $z^{t+1} \succ z^t$ *that* $z^{t+1} \in \mathcal{H}_0$.

3. *If* $z^t \in \mathcal{H}_+$ *then* $\sum_{z^{t+1} \succ z^t} \lambda(z^{t+1}) = 1$.

The definition says that $\mathcal{H}$ can be divided into three subsets, one set of nodes $\mathcal{H}_0$ where no transfers are assigned, one set of nodes $\mathcal{H}_+$ associated with positive transfers, and the root σ_0 (whose weight is normalized to one). If no transfers are assigned at one node then no transfers are assigned to any successor nodes of this node. Most importantly, once transfers are assigned at one node then they are assigned at some direct successor node(s). Furthermore, transfers are normalized so that their weights sum up to one.

Every sequence $(z_0, z_1, z_2, ...)$ with $z_i \in Z_i$ for every $i = 0, 1, 2, ...$ defines a path z^∞, i.e. a path is an infinite history. We denote the set of all paths by $\mathcal{H}^\infty$. We write z_t^∞ for the $t-$th coordinate of the path z^∞. $(z^\infty)^t$ denotes the history consisting of the first $t+1$ coordinates, $(z_0^\infty, z_1^\infty, ..., z_t^\infty)$, of the path z^∞. We now have the following characterization:

[3]The concept was introduced by Chattopadhyay and Gottardi (1999) who called it "weight function". We use the version of Barbie et al. (2007) which is at first glance slightly different but in fact equivalent to the Chattopadhyay and Gottardi (1999) definition.

Theorem 1 *Consider an interior competitive equilibrium allocation* (c^*, k^*, p^*, w^*)*. The allocation is interim Pareto inefficient if and only if there exists a transfer pattern* λ *and a constant* $M > 0$ *such that for every path* z^∞

$$\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \lambda\left((z^\infty)^s \right) \right) \cdot \frac{1}{p^*\left((z^\infty)^t \right)} \leq M. \tag{3}$$

An important feature of theorem 1 is that given the set of histories and the set of competitive equilibrium prices, the characterization for interim Pareto efficiency is independent of the number periods an individual lives. In particular, the condition is the same as for a two-generation stochastic OLG model, and as a consequence, all tests of efficiency and inefficiency in terms of rates of return of certain assets as developed e.g. in Chattopadhyay (2006) and Barbie et al. (2007) apply without any change also to multi-generation stochastic OLG models. This holds because the tests are all based on condition (3) which remains unchanged compared to the two-period case. The fact that one can use asset returns that correspond to one time period in the original time scale (e.g. annual data) is important. To see this, consider the special case of a stochastic $N-$generation economy. The construction of artificial date events outlined in section 4.2 would imply that an artificial date event has a length of $N-1$ periods in the original time scale. Thus a test that would e.g. use the artificial date events would practically be impossible to implement for values of N typically used in applied work (e.g. 60 to 80 years) due to the lack of appropriate asset market data. Likewise, modelling the economy as a two-period OLG model with one period lasting $N/2$ periods in the original time scale causes the well-known problem of one model period being too long to draw any conclusions from the usually available data sets. For example, US government bonds are not available with a maturity of more than 30 years so that yield curve information could not be used. Thus, it is not only important that we have *some* characterization of efficiency in a general stochastic OLG model, but that this characterization works with the prices from the original date-event tree.

It may be desirable to have a condition for efficiency/inefficiency that does not require the specification of a transfer pattern λ as in (3), but only uses competitive equilibrium prices. In Theorem 2 of Demange (2002) a sufficient condition for efficiency is given which states that if $\sum_{z^t \in \mathcal{H}_t} p^*\left(z^t\right) \leq C$ for all t, the allocation is Pareto efficient. A stronger result is obtained in a two-period model by simply doing a traditional Cass type argument that sums over all date events at one point of time (see Theorem 3 and Corollary 1 in Chattopadhyay and Gottardi (1999)). It follows from our Theorem 1 and the results of

Chattopadhyay and Gottardi (1999) that this result also holds for our general stochastic economy.

Corollary 1 *Consider an interior competitive equilibrium allocation* (c^*, k^*, p^*, w^*)*. If*

$$\sum_{t=0}^{\infty} \frac{1}{\sum_{z^t \in \mathcal{H}_t} p^*\left(z^t\right)} = \infty, \tag{4}$$

then the allocation is interim Pareto efficient.

4 Proofs

4.1 A Sufficient Condition for Inefficiency

We first show that condition (3) is sufficient for inefficiency:

Lemma 1 *Consider an interior competitive equilibrium allocation* (c^*, k^*, p^*, w^*)*. If there exists a transfer pattern* λ *and a constant* $A > 0$ *such that for every path* z^∞

$$\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \lambda\left(\left(z^\infty\right)^s\right) \right) \cdot \frac{1}{p^*\left(\left(z^\infty\right)^t\right)} \leq A, \tag{5}$$

then the allocation is interim Pareto inefficient.

Proof. We fix everything except for consumption of households of age 1 and 2 at the competitive equilibrium for every history z^t. We can interpret the equilibrium as the equilibrium of a two-generation pure exchange stochastic OLG model with consumption dates 1 and 2 with appropriately chosen endowments. Further, one can show that the bounded Gaussian curvature condition of Chattopadhyay and Gottardi (1999) holds in our setup. Then by Theorem 2 in Chattopadhyay and Gottardi (1999), an improvement exists for the two-generation pure-exchange economy and thus also for our multi-generation production economy. ■

4.2 Construction of an Artificial Two-Period OLG Economy

We now show that (5) is not only sufficient but also necessary for interim Pareto inefficiency. We do this by constructing an artificial economy in which certain histories z^t are summarized to new histories. The recursive construction is as follows. For any $z^t \in \mathcal{H}$ we define:

$$\mathcal{H}^\infty\left(z^t\right) = \left\{ z^\infty \in \mathcal{H}^\infty \,\middle|\, \left(z^\infty\right)^t = z^t \right\},$$

$$t^{\max}\left(z^{\infty}\left|z^{t}\right.\right)=\max\left\{s\left|(z^{\infty})^{s}\in I\left(z^{t-i}\right)\right.\text{ for }i=0,...,N-1\right\}\text{ for }z^{\infty}\in\mathcal{H}^{\infty}\left(z^{t}\right),$$

and

$$\sigma\left(z^{t}\right)=\left\{z^{s}\in\mathcal{H}\left|z^{t}\preceq z^{s}\prec(z^{\infty})^{t^{\max}\left(z^{\infty}\left|z^{t}\right.\right)}\text{ for all }z^{\infty}\in\mathcal{H}^{\infty}\left(z^{t}\right)\right.\right\}.$$

Further

$$\left(\sigma\left(z^{t}\right)\right)^{+}=\left\{z^{s+1}\in\mathcal{H}\left|z^{s+1}\in(z^{s})^{+}\text{ for some }z^{s}\in\sigma\left(z^{t}\right)\right.\right\}.$$

Starting with $\mathcal{F}^0 := \{z^0\}$, we define recursively:

$$\mathcal{F}^{i}=\bigcup_{z^{s}\in\mathcal{F}^{i-1}}\left(\sigma\left(z^{s}\right)\right)^{+}\backslash\sigma\left(z^{s}\right)$$

for $i \geq 1$. We obtain a new set of artificial date-events by considering the set $\Sigma := \left\{\sigma\left(z^{s}\right)\left|z^{s}\in\cup_{i\in\{0,1,...\}}\mathcal{F}^{i}\right.\right\}$. We say that date-event $\sigma(z^t)$ is at time i in the new timeline if $z^t \in \mathcal{F}^i$.

We often simply write σ_t if the particular $z^s \in \mathcal{F}^t$ is not important. We write $\sigma_t \succ \sigma_{t-1}$ if $z^u \succ z^v$ for $z^u \in \sigma_t$ and $z^v \in \sigma_{t-1}$. We define a path σ^∞ as a sequence of date-events $(\sigma_0, \sigma_1, ...)$ such that $\sigma_t \succ \sigma_{t-1}$ for $t = 1, 2, ...$ σ_t^∞ denotes the $t-$th coordinate of the path σ^∞. Note that any consumer born in some history $z^s \in \sigma_t$ lives on the new timeline in date-events σ_t and all $\sigma_{t+1} \succ \sigma_t$. So we have transformed the multi-generation stochastic OLG model into a two-generation stochastic OLG model in which the consumption sets of agents born in the same (artificial) date-event σ_t differ. For an agent born in $z^s \in \sigma_t$, we denote by $c^y_{z^s}(\sigma_t)$ the young age consumption vector $\left(c_{i+1}\left(z^{s+i}\right)\right)_{z^{s+i}\in I(z^S)\cap\sigma_t}$. For any $\sigma_{t+1} \succ \sigma_t$, we obtain an old-age consumption vector $c^o_{z^s}(\sigma_{t+1}) = \left(c_{i+1}\left(z^{s+i}\right)\right)_{z^{s+i}\in I(z^s)\cap\sigma_{t+1}}$. In the context of our artificial two-generation economy, we denote the preferences of a consumer born in history $z^s \in \sigma_t$ by $U_{z^s}\left(c^y_{z^s}(\sigma_t), \left(c^o_{z^s}(\sigma_{t+1})\right)_{\sigma_{t+1}\succ\sigma_t}\right)$. For a date-event $\sigma(z^t)$ we define with a slight abuse of notation $p\left(\sigma\left(z^t\right)\right) = \left(p\left(z^s\right)\right)_{z^s\in\sigma(z^t)}$.

In analogy to our original multi-generation economy, we define a transfer pattern μ for the artificial economy. The definition is exactly the same as *Definition 4* with the set of histories $\mathcal{H}$ replaced by Σ.

4.3 A Necessary Condition for Inefficiency

We now state a necessary condition for interim Pareto inefficiency for the *original* economy in terms of a Cass-Chattopadhyay-Gottardi sum test for the *artificial* two-period OLG economy. Let $\|\cdot\|$ denote the euclidian norm:

Proposition 1 *Consider an interior competitive equilibrium allocation* (c^*, k^*, p^*, w^*). *If* *the allocation is interim Pareto inefficient, there exist a transfer pattern μ (on Σ) and a constant $B>0$ such that for every path σ^{∞}*

$$\sum_{t=0}^{\infty}\left(\prod_{s=0}^{t}\mu\left(\sigma_{s}^{\infty}\right)\right)\cdot\frac{1}{\left\|p^{*}\left(\sigma_{t}^{\infty}\right)\right\|}\leq B. \tag{6}$$

Proof. *Step 1:* Let (c^{*},k^{*}) denote the competitive equilibrium allocation and $\left(\widetilde{c},\widetilde{k}\right)$ an improving allocation. By convexity of preferences and technology we can w.l.o.g. assume that $\left(\widetilde{c},\widetilde{k}\right)$ is contained in some compact set $K\subseteq\prod_{z^{t}\in\mathcal{H}}\mathbb{R}_{++}^{N\left(z^{t}\right)+1}$. Consider now the artificial date events. $U_{z^{s}}\left(\widetilde{c}_{z^{s}}^{y}\left(\sigma_{t}\right),\left(\widetilde{c}_{z^{s}}^{o}\left(\sigma_{t+1}\right)\right)_{\sigma_{t+1}\succ\sigma_{t}}\right)\geq U_{z^{s}}\left(c_{z^{s}}^{*y}\left(\sigma_{t}\right),\left(c_{z^{s}}^{*o}\left(\sigma_{t+1}\right)\right)_{\sigma_{t+1}\succ\sigma_{t}}\right)$ holds by construction for any σ_{t} and any $z^{s}\in\sigma_{t}$. If we define $\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)=\widetilde{c}_{z^{s}}^{y}\left(\sigma_{t}\right)-c_{z^{s}}^{*y}\left(\sigma_{t}\right)$ and $\Delta c_{z^{s}}^{o}\left(\sigma_{t}\right)=\widetilde{c}_{z^{s}}^{o}\left(\sigma_{t}\right)-c_{z^{s}}^{*o}\left(\sigma_{t}\right)$ for any σ_{t} and any $z^{s}\in\sigma_{t},\sigma_{t-1}$ there exists[4] a $\underline{\rho}>0$ such that $DU_{z^{s}}\left(c^{*}\left(z^{s}\right)\right)\cdot\Delta c\left(z^{s}\right)\geq\underline{\rho}\left\|\Delta c\left(z^{s}\right)\right\|^{2}$., where $\Delta c\left(z^{s}\right):=\widetilde{c}\left(z^{s}\right)-c^{*}\left(z^{s}\right)$. At an interior competitive equilibrium we have $DU_{z^{s}}\left(c^{*}\left(z^{s}\right)\right)=\lambda_{z^{s}}p_{z^{s}}^{*}$ for some $\lambda_{z^{s}}>0$, where $p_{z^{s}}^{*}=\left(p^{*}\left(z^{t}\right)\right)_{z^{t}\in I\left(z^{s}\right)}$. Thus we have:

$$p_{z^{s}}^{*}\cdot\Delta c\left(z^{s}\right)\geq\frac{\underline{\rho}}{\lambda_{z^{s}}}\left\|\Delta c\left(z^{s}\right)\right\|^{2}\geq\frac{\underline{\rho}}{\lambda_{z^{s}}}\left\|\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)\right\|^{2}\geq\frac{\underline{\rho}}{\lambda_{z^{s}}}\frac{1}{\left\|p_{z^{s}}^{*}\left(\sigma_{t}\right)\right\|^{2}}\left(p_{z^{s}}^{*}\left(\sigma_{t}\right)\cdot\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)\right)^{2},$$

with $p_{z^{s}}^{*}\left(\sigma_{t}\right)=\left(p^{*}\left(z^{t}\right)\right)_{z^{t}\in I\left(z^{s}\right)\cap\sigma_{t}}$. The last inequality follows from $\left\|p_{z^{s}}^{*}\left(\sigma_{t}\right)\right\|\cdot\left\|\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)\right\|\geq p_{z^{s}}^{*}\left(\sigma_{t}\right)\cdot\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)$. Thus

$$\sum_{\sigma_{t+1}\succ\sigma_{t}}p^{*}\left(\sigma_{t+1}\right)\cdot\Delta c_{z^{s}}^{o}\left(\sigma_{t+1}\right)\geq-\left(p^{*}\left(\sigma_{t}\right)\cdot\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)\right)+\frac{\underline{\rho}}{\left\|p\left(\sigma_{t}\right)\right\|}\frac{\left(p^{*}\left(\sigma_{t}\right)\cdot\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)\right)^{2}}{\lambda_{z^{s}}\left\|p_{z^{s}}^{*}\left(\sigma_{t}\right)\right\|}.$$

$\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)$ is here in slight abuse of notation interpreted as vector from $\mathbb{R}^{\left|\sigma_{t}\right|}$ which is zero in all entries $z^{t}$for which $z^{t}\notin I\left(z^{s}\right)$. Now $\left\|\left(DU_{z^{s}}\left(c_{z^{s}}^{y*}\left(\sigma_{t}\right)\right)\right)\right\|=\lambda_{z^{s}}\left\|p_{z^{s}}^{*}\left(\sigma_{t}\right)\right\|$ implies that

$$\sum_{\sigma_{t+1}\succ\sigma_{t}}p^{*}\left(\sigma_{t+1}\right)\Delta c_{z^{s}}^{o}\left(\sigma_{t+1}\right)\geq-\left(p^{*}\left(\sigma_{t}\right)\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)\right)+\frac{\underline{\rho}}{\left\|p^{*}\left(\sigma_{t}\right)\right\|}\frac{\left(p^{*}\left(\sigma_{t}\right)\Delta c_{z^{s}}^{y}\left(\sigma_{t}\right)\right)^{2}}{\left\|\left(DU_{z^{s}}\left(c_{z^{s}}^{y*}\left(\sigma_{t}\right)\right)\right)\right\|}.$$

Given that the allocation (c^{*},k^{*}) is interior and the properties of the utility functions there exists some constant $b>0$ such that $\frac{\underline{\rho}}{\left\|\left(DU_{z^{s}}\left(c_{z^{s}}^{y*}\left(\sigma_{t}\right)\right)\right)\right\|}>b$ for any σ_{t} and any $z^{s}\in\sigma_{t}$.

[4] This follows from a second order Taylor approximation by using that allocation are from a compact set and our assumptions on differentiability and continuity of the second partial derivatives with respect to parameter θ from the compact set Θ.

Thus for σ_t and any $z^s \in \sigma_t$:

$$\sum_{\sigma_{t+1}\succ\sigma_t} p^*(\sigma_{t+1})\cdot\Delta c^o_{z^s}(\sigma_{t+1}) \geq -\left(p^*(\sigma_t)\cdot\Delta c^y_{z^s}(\sigma_t)\right) + \frac{b}{\|p^*(\sigma_t)\|}\left(p^*(\sigma_t)\cdot\Delta c^y_{z^s}(\sigma_t)\right)^2. \quad (7)$$

Note that the number of consumers born in an artificial date event is equal to the number of histories it contains, which is equal to $|\sigma_t| \leq 1 + |Z| + ... + |Z|^{N-1} =: I_{\max}$. Summing (7) over all agents born in σ_t (which we identify with the particular history z^s in which they are born) and dividing by $|\sigma_t|$ we obtain:

$$\begin{aligned} &\sum_{\sigma_{t+1}\succ\sigma_t} p^*(\sigma_{t+1})\cdot\frac{1}{|\sigma_t|}\cdot\left(\sum_{z^s\in\sigma_t}\Delta c^o_{z^s}(\sigma_{t+1})\right) \\ \geq\ & -\left(p^*(\sigma_t)\cdot\frac{1}{|\sigma_t|}\cdot\sum_{z^s\in\sigma_t}\Delta c^y_{z^s}(\sigma_t)\right) + \frac{b}{\|p^*(\sigma_t)\|}\cdot\frac{1}{|\sigma_t|}\cdot\sum_{z^s\in\sigma_t}\left(p^*(\sigma_t)\cdot\Delta c^y_{z^s}(\sigma_t)\right)^2 \\ \geq\ & -\left(p^*(\sigma_t)\cdot\frac{1}{|\sigma_t|}\cdot\sum_{z^s\in\sigma_t}\Delta c^y_{z^s}(\sigma_t)\right) + \frac{b}{\|p^*(\sigma_t)\|}\cdot\left(p^*(\sigma_t)\cdot\frac{1}{|\sigma_t|}\cdot\sum_{z^s\in\sigma_t}\Delta c^y_{z^s}(\sigma_t)\right)^2, \end{aligned}$$

so that $\Delta c^o(\sigma_{t+1}) := \sum_{z^s\in\sigma_t}\Delta c^o_{z^s}(\sigma_{t+1})$ and $\Delta c^y(\sigma_t) := \sum_{z^s\in\sigma_t}\Delta c^y_{z^s}(\sigma_t)$ satisfy

$$\sum_{\sigma_{t+1}\succ\sigma_t} p^*(\sigma_{t+1})\cdot\Delta c^o(\sigma_{t+1}) \geq -\left(p^*(\sigma_t)\cdot\Delta c^y(\sigma_t)\right) + \frac{b}{I_{\max}\cdot\|p^*(\sigma_t)\|}\cdot\left(p^*(\sigma_t)\cdot\Delta c^y(\sigma_t)\right)^2. \quad (8)$$

This is the usual form of the so-called non-vanishing Gaussian curvature condition for a two-generation stochastic OLG model that we derived for our artificial two-period date-event setting from our primitives.

Step 2: We now incorporate the possible joint changes in production and consumption into this estimate. The estimates are considerably more involved than in Barbie et al. (2007). For each $z^s \in \sigma_t$, we define the weight

$$\alpha(z^s) = \frac{\sum_{z^{s+1}\notin\left((z^s)^+\cap\sigma_t\right)} p^*(z^{s+1})\, f'(k^*(z^s), z^{s+1})}{\sum_{z^{s+1}\in(z^s)^+} p^*(z^{s+1})\, f'(k^*(z^s), z^{s+1})}. \quad (9)$$

Further, we denote by $\alpha(\sigma_t) = (\alpha(z^s))_{z^s\in\sigma_t}$ and define the Hadamard product $x\otimes y := (x_i\cdot y_i)_{i=1,\dots,n}$ for two vectors of dimension n. For any given σ_t, define:

$$\begin{aligned} T(\sigma_t) &:= -\Delta c^y(\sigma_t) - \alpha(\sigma_t)\otimes\Delta k(\sigma_t) \quad (10)\\ \widehat{T}(\sigma_t) &:= \Delta c^o(\sigma_t) - \Delta f(\sigma_t) + (\mathbf{1}-\alpha(\sigma_t))\otimes\Delta k(\sigma_t), \end{aligned}$$

where $\Delta f(\sigma_t) := (f(\widetilde{k}(z^s), z^{s+1}) - f(k^*(z^s), z^{s+1}))_{z^{s+1}\in\sigma_t}$ and $\mathbf{1}$ is the unit vector $(1, 1, ..., 1)$. Note that from the resource constraint $T(\sigma_t) = \widehat{T}(\sigma_t)$ for every σ_t. Since $\Delta c^o(\sigma_{t+1}) = \widehat{T}(\sigma_{t+1}) + \Delta f(\sigma_{t+1}) - (\mathbf{1} - \alpha(\sigma_{t+1})) \otimes \Delta k(\sigma_{t+1})$ and $\Delta c^y(\sigma_t) = -\alpha(\sigma_t) \otimes \Delta k(\sigma_t) - T(\sigma_t)$, we obtain from (8):

$$\begin{aligned} &\sum_{\sigma_{t+1}\succ\sigma_t} p^*(\sigma_{t+1}) \cdot (\widehat{T}(\sigma_{t+1}) + \Delta f(\sigma_{t+1}) \\ -(\mathbf{1} - \alpha(\sigma_{t+1})) \otimes \Delta k(\sigma_{t+1})) \geq\ & p^*(\sigma_t) \cdot (\alpha(\sigma_t) \otimes \Delta k(\sigma_t) + T(\sigma_t)) \\ &+\frac{b}{I_{\max} \cdot \|p^*(\sigma_t)\|} \cdot (p^*(\sigma_t) \cdot \Delta c^y(\sigma_t))^2 . \end{aligned}$$

We define $\underline{\sigma}_t := \left\{ z^s \in \sigma_t \middle| z^s \in (z^{s-1})^+ \text{ for some } z^{s-1} \in \sigma_{t-1} \right\}$ and $\overline{\sigma}^t := \sigma_t \backslash \underline{\sigma}_t$. From the concavity of f and the interiority of the allocations considered, we have $\Delta f(z^{s+1}) \leq f'(k^*(z^s), z^{s+1}) \Delta k(z^s) - d(\Delta k(z^s))^2 \leq f'(k^*(z^s), z^{s+1}) \Delta k(z^s)$ for some constant $d > 0$ for all histories z^s. Thus

$$\sum_{\sigma_{t+1}\succ\sigma_t} p^*(\sigma_{t+1}) \Delta f(\sigma_{t+1}) = \sum_{\sigma_{t+1}\succ\sigma_t} \sum_{z^{s+1}\in\overline{\sigma}_{t+1}} p^*(z^{s+1}) \Delta f(z^{s+1}) + \sum_{\sigma_{t+1}\succ\sigma_t} \sum_{z^{s+1}\in\underline{\sigma}_{t+1}} p^*(z^{s+1}) \Delta f(z^{s+1})$$

$$\begin{aligned} \leq\ & \sum_{\sigma_{t+1}\succ\sigma_t} \sum_{z^{s+1}\in\underline{\sigma}_{t+1}} p^*(z^{s+1}) f'(k^*(z^s), z^{s+1}) \Delta k(z^s) - \sum_{\sigma_{t+1}\succ\sigma_t} \sum_{z^{s+1}\in\underline{\sigma}_{t+1}} p^*(z^{s+1}) d(\Delta k(z^s))^2 \\ & + \sum_{\sigma_{t+1}\succ\sigma_t} \sum_{z^{s+1}\in\overline{\sigma}_{t+1}} p^*(z^{s+1}) f'(k^*(z^s), z^{s+1}) \Delta k(z^s) \end{aligned}$$

$$\begin{aligned} =\ & \sum_{z^s\in\sigma_t} \alpha(z^s) \left(\sum_{z^{s+1}\in(z^s)^+} p^*(z^{s+1}) f'(k^*(z^s), z^{s+1}) \right) \Delta k(z^s) \\ & -\frac{d}{\overline{b}} \sum_{z^s\in\sigma_t} \left(\sum_{z^{s+1}\in(z^s)^+} p^*(z^{s+1}) f'(k^*(z^s), z^{s+1}) \right) \alpha(z^s) (\Delta k(z^s))^2 \\ & + \sum_{\sigma_{t+1}\succ\sigma_t} \sum_{z^s\in\sigma_{t+1}} (1 - \alpha(z^s)) \left(\sum_{z^{s+1}\in(z^s)^+} p^*(z^{s+1}) f'(k^*(z^s), z^{s+1}) \right) \Delta k(z^s) . \end{aligned}$$

Here $f'(k^*(z^s), z^{s+1}) \leq \overline{b}$ for some strictly positive constant $\overline{b}$ by the interiority of the competitive equilibrium allocation. The equality follows from the definition of $\underline{\sigma}_{t+1}$, $\overline{\sigma}_{t+1}$ and $\alpha(z^s)$. At an interior competitive equilibrium the first order condition of (2) requires that $\sum_{z^s \in (z^{s-1})^+} p^* (z^s) f' (k^* (z^{s-1}), z_s) = p^* (z^{s-1})$. Thus

$$\sum_{\sigma_{t+1} \succ \sigma_t} p^* (\sigma_{t+1}) \Delta f (\sigma_{t+1}) \leq p^* (\sigma_t) \cdot (\alpha (\sigma_t) \otimes (\Delta k (\sigma_t) - \frac{d}{\bar{b}} (\Delta k)^2 (\sigma_t))) + \sum_{\sigma_{t+1} \succ \sigma_t} p^* (\sigma_{t+1}) \cdot ((\mathbf{1} - \alpha (\sigma_{t+1})) \otimes \Delta k (\sigma_{t+1})).$$

Here, $(\Delta k)^2 (\sigma_t)$ denotes the vector $((\Delta k(z_s))^2)_{z_s \in \sigma_t}$. This yields:

$$\sum_{\sigma_{t+1} \succ \sigma_t} p^* (\sigma_{t+1}) \cdot \widehat{T} (\sigma_{t+1}) \geq p^* (\sigma_t) \cdot T (\sigma_t) + \frac{b}{I_{\max} \cdot \|p^* (\sigma_t)\|} \cdot (p^* (\sigma_t) \cdot \Delta c^y (\sigma_t))^2 + \frac{d}{\bar{b}} . p^* (\sigma_t) \cdot \left(\alpha (\sigma_t) \otimes (\Delta k)^2 (\sigma_t) \right).$$

$p^* (\sigma_t) (\alpha (\sigma_t) \otimes (\Delta k)^2 (\sigma_t)) \geq \sum_{z^s \in \sigma_t} \frac{(p^*(z^s) \alpha(z^s) \Delta k(z^s))^2}{p^*(z^s)} \geq \frac{1}{\|p^*(\sigma_t)\|} \frac{1}{I_{\max}} (p^* (\sigma_t) (\alpha (\sigma_t) \otimes \Delta k (\sigma_t)))^2$ holds, where the last relation uses Jensen's inequality. We thus have for $\underline{c} := \min \left\{ \frac{c}{I_{\max}}, \frac{d/\bar{b}}{I_{\max}} \right\}$:

$$\sum_{\sigma_{t+1} \succ \sigma_t} p^* (\sigma_{t+1}) \cdot \widehat{T} (\sigma_{t+1}) \geq p^* (\sigma_t) \cdot T (\sigma_t) + \frac{\underline{c} \cdot \left[(p^* (\sigma_t) \cdot \Delta c^y (\sigma_t))^2 + (p^* (\sigma_t) \cdot (\alpha (\sigma_t) \otimes \Delta k (\sigma_t)))^2 \right]}{\|p^* (\sigma_t)\|}.$$

Now recall the elementary inequality $\alpha^2 + \beta^2 \geq \frac{1}{2} (\alpha + \beta)^2$. Using that $\widehat{T} (\sigma_{t+1}) = T (\sigma_{t+1})$ and $T (\sigma_t) = -\Delta c^y (\sigma_t) - \alpha (\sigma_t) \otimes \Delta k (\sigma_t)$, we finally obtain:

$$\sum_{\sigma_{t+1} \succ \sigma_t} p^* (\sigma_{t+1}) \cdot T (\sigma_{t+1}) \geq p^* (\sigma_t) \cdot T (\sigma_t) + \frac{\underline{c}/2}{\|p^* (\sigma_t)\|} (p^* (\sigma_t) \Delta T (\sigma_t))^2 . \qquad (11)$$

It is easy to show (available on request) that there exists some σ_t with $p^* (\sigma_t) \cdot T (\sigma_t) > 0$. From here on the proof follows the arguments in Chattopadhyay and Gottardi (1999), proof of Theorem 1. ■

4.4 Proof of Theorem 1

Proof. Suppose that a competitive equilibrium allocation is interim Pareto inefficient. By proposition 1 there exists a transfer pattern μ on Σ such that (6) holds. Then also:

$$\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \mu (\sigma_s^\infty) \right) \cdot \frac{N - 1}{\|p^* (\sigma_t^\infty)\|} \leq B (N - 1) \qquad (12)$$

for every path σ^{∞}. Consider an arbitrary path z^{∞} in the original economy. There exists a unique path σ^{∞} in the artificial economy such that every history $(z^{\infty})^{s}$ is contained in some σ_t^{∞}. Every σ_t^{∞} contains exactly $1 \leq N(\sigma_t^{\infty}, z^{\infty}) \leq N-1$ histories belonging to path z^{∞}. We can thus assign to every history $(z^{\infty})^{s}$ at least one summand in the series (12) by assigning $\frac{1}{\|p^*(\sigma_0^{\infty})\|}$ to histories $(z^{\infty})^{0}, (z^{\infty})^{1}, ..., (z^{\infty})^{N(\sigma_0^{\infty}, z^{\infty})-1}$, $\left(\prod_{s=0}^{1} \mu(\sigma_s^{\infty})\right) \cdot \frac{1}{\|p^*(\sigma_1^{\infty})\|}$ to histories $(z^{\infty})^{N(\sigma_0^{\infty}, z^{\infty})}, ..., (z^{\infty})^{N(\sigma_0^{\infty}, z^{\infty})+N(\sigma_1^{\infty}, z^{\infty})-1}$ and generally $\left(\prod_{s=0}^{t} \mu(\sigma_s^{\infty})\right) \cdot \frac{1}{\|p^*(\sigma_t^{\infty})\|}$ to histories $(z^{\infty})^{\sum_{i=0}^{t-1} N(\sigma_i^{\infty}, z^{\infty})}, ..., (z^{\infty})^{\sum_{i=0}^{t} N(\sigma_i^{\infty}, z^{\infty})-1}$. We construct a transfer pattern on $\mathcal{H}$ by setting $\lambda(z^0) := 1$, $\prod_{s=0}^{N(\sigma_0^{\infty}, z^{\infty})} \lambda((z^{\infty})^{s}) := \prod_{s=0}^{1} \mu(\sigma_s^{\infty}), ...,$ $\prod_{s=0}^{\sum_{i=0}^{t-1} N(\sigma_i^{\infty}, z^{\infty})} \lambda((z^{\infty})^{s}) := \prod_{s=0}^{t} \mu(\sigma_s^{\infty}), ...$ for every path z^{∞}, where σ^{∞} is the unique path in the artificial economy such that every history $(z^{\infty})^{s}$ is contained in some σ_t^{∞}. We define recursively:

$$\prod_{s=0}^{\sum_{i=0}^{t-1} N(\sigma_i^{\infty}, z^{\infty})+j} \lambda((z^{\infty})^{s})$$

$$: = \sum_{z^{\sum_{i=0}^{t-1} N(\sigma_i^{\infty}, z^{\infty})+j+1} \succ (z^{\infty})^{\sum_{i=0}^{t-1} N(\sigma_i^{\infty}, z^{\infty})+j}} \prod_{s=0}^{\sum_{i=0}^{t-1} N(\sigma_i^{\infty}, z^{\infty})+j+1} \lambda(z^{s})$$

for every $j = 1, ..., N(\sigma_t^{\infty}, z^{\infty}) - 1$ and every z^{∞}. Note that this construction gives a well defined transfer pattern λ and that by construction:

$$\sum_{j=0}^{N(\sigma_t^{\infty}, z^{\infty})-1} \prod_{s=0}^{\sum_{i=0}^{t-1} N(\sigma_i^{\infty}, z^{\infty})+j} \lambda((z^{\infty})^{s}) \cdot \frac{1}{\|p^*(\sigma_t^{\infty})\|} \leq \left(\prod_{s=0}^{t} \mu(\sigma_s^{\infty})\right) \cdot \frac{N-1}{\|p^*(\sigma_t^{\infty})\|}.$$

Consider $\sigma_t^{\infty} = \sigma(z^v)$ for $z^v \in \mathcal{F}^t$. For every $z^s \in \sigma_t^{\infty}$ there exists some $z^u \preceq z^v$ with $z^s \in I(z^u)$. At an interior competitive equilibrium we have

$$\frac{p^*(z^v)}{p^*(z^s)} = \frac{\frac{\partial U_{z^u}(c^*(z^u))}{\partial c_{v-u+1}(z^v)}}{\frac{\partial U_{z^u}(c^*(z^u))}{\partial c_{s-u+1}(z^s)}}.$$

Given that the allocation is interior and the properties of the utility functions, there exist constants $0 < \underline{m} < \overline{m}$ (independent of σ_t^{∞}) such that $\underline{m} \leq \frac{p^*(z^s)}{p^*(z^{\tilde{s}})} \leq \overline{m}$ for every $z^s, z^{\tilde{s}} \in \sigma_t^{\infty}$ for every t and every σ^{∞}. Thus from $\frac{1}{\|p^*(\sigma_t^{\infty})\|} \geq \frac{1}{\sqrt{I_{\max}} p^*_{\max}(\sigma_t^{\infty})}$ with $p^*_{\max}(\sigma_t^{\infty}) :=$

$\max_{z^s \in \sigma_t^\infty} p^*(z^s)$ and from $\frac{1}{p^*_{\max}(\sigma_t^\infty)} \geq \frac{m}{p^*(z^s)}$ for every $z^s \in \sigma_t^\infty$ we obtain

$$\sum_{j=0}^{N(\sigma_t^\infty, z^\infty)-1} \prod_{s=0}^{\sum_{i=0}^{t-1} N\left(\sigma_i^\infty, z^\infty\right)+j} \lambda\left((z^\infty)^s\right) \cdot \frac{m}{\sqrt{I_{\max}}} \cdot \frac{1}{p^*\left((z^\infty)^{t(N-1)+i}\right)}$$

$$\leq \sum_{j=0}^{N(\sigma_t^\infty, z^\infty)-1} \prod_{s=0}^{\sum_{i=0}^{t-1} N\left(\sigma_i^\infty, z^\infty\right)+j} \lambda\left((z^\infty)^s\right) \cdot \frac{1}{\|p^*(\sigma_t^\infty)\|},$$

and thus

$$\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \lambda\left((z^\infty)^s\right) \right) \cdot \frac{1}{p^*\left((z^\infty)^t\right)} \leq \frac{\sqrt{I_{\max}}}{m} B(N-1)$$

for every path z^∞, showing that condition (5) is also necessary for inefficiency. ■

4.5 Proof of Corollary 1

Proof. We prove by contraposition. Consider an equilibrium allocation that is interim inefficient. Then by Theorem 1, condition (3) holds. Given the competitive equilibrium prices p^* it is not difficult to construct a two-generation stochastic pure-exchange economy with two-period lifetimes that satisfies the assumptions of Theorem 2 in Chattopadhyay and Gottardi (1999). For this economy, condition (3) remains true (since the condition only depends on the competitive prices), and so by Theorem 2 in Chattopadhyay and Gottardi (1999), this two-period economy is also interim Pareto inefficient. By Corollary 1 in Chattopadhyay and Gottardi (1999), $\sum_{t=0}^{\infty} \frac{1}{\sum_{z^t \in \mathcal{H}_t} p^*(z^t)} < \infty$. ■

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