Secure implementation in allotment economies *

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Abstract

An allocation rule is *securely implementable* if it is strategy-proof and has no "bad" Nash equilibrium in its associated direct revelation game (Saijo, Sjöström, and Yamato, 2007). We study this implementability notion in allotment economies with single-peaked preferences (Sprumont, 1991). The equal division rule and priority rules are characterized on the basis of secure implementability, which underlines a strong trade-off between efficiency and symmetry. Though the uniform rule is not securely implementable, we show that, in its direct revelation game, any "bad" Nash equilibrium is blocked by a credible coalitional deviation, and any "good" Nash equilibrium is never blocked. Thus the impossibility of securely implementing the uniform rule can be resolved by allowing pre-play communication among players.

Keywords: Secure implementation, Strategy-proofness, Uniform rule, Priority rule, Nash implementation, Coalition-proof Nash equilibrium, Single-peaked preference, Fair allocation.

JEL codes: C72, D63, D61, C78, D71.

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1 Introduction

Strategy-proofness is a central condition in implementation and mechanism theory. It states that truthtelling is a dominant strategy for everyone in its associated direct revelation game. However, even if a rule is strategy-proof, agents may play some "bad" Nash equilibrium so that undesirable outcomes are realized.[1] To avoid this problem, a seminal work by Saijo, Sjöström, and Yamato (2007) (hereafter, SSY) suggests to impose the additional protective criterion of *secure implementability*. A rule is securely implementable if it is strategy-proof and its associated direct revelation game has no Nash equilibrium that realizes an outcome that differs from the outcome obtained under truthful revelation. Aside from the theoretical motivation for secure implementation, the concept is also motivated by experimental evidence. In laboratory experiments on Clarke-Groves mechanisms, Cason, Saijo, Sjöström and Yamato (2006) show that rules that are not secure do not perform well. Subjects do not necessarily play dominant strategies, and they moreover get stuck at bad Nash equilibria. In that sense, secure implementation draws an important behavioral segmentation among strategy-proof rules.

Our purpose is to examine the possibility of secure implementation in the problem of allocating a divisible resource to agents with single-peaked preferences over shares of this resource. The central allocation rule in this context is the uniform rule (Sprumont, 1991).[2] It has been well-known since Sprumont's (1991) and Ching (1994) that the uniform rule is the only symmetric, efficient, and strategy-proof rule. However, an earlier version of SSY points out that the direct revelation game of the uniform rule allows a continuum of inefficient Nash equilibria: the uniform rule is not securely implementable.

We first investigate which rules are securely implementable. Since secure implementability implies strategy-proofness, we investigate which rules can survive if we add each of symmetry and efficiency. In Theorem 1, we show that only constant rules can satisfy secure implementability and a mild independence condition. As a corollary, we obtain that the equal division rule is the only securely implementable and symmetric rule. We next turn our attention to efficiency. In Theorem 2, we show that, in the two person case, priority rules are the only securely implementable and efficient rules. This characterization does not hold when there are more than two agents. Using several examples, we give some hints about how large the class of se-

[1] This problem is observed in laboratory experiments of Clarke-Groves mechanisms by Cason, Saijo, Sjöström, Yamato (2006).

[2] See, Thomson (2005) for a survey. Recent studies by Fujinaka and Wakayama (2008, 2009) also analyze secure implementability in economies with indivisible goods.

curely implementable and efficient rules is: in contrast with the mere structure of the class of priority rules which contains only $n!$ rules, our examples show that the class of securely implementable and efficient rules contains a continuum of rules. While we do not have a full characterization, it is clear that many of these rules –as shown in our examples– are unappealing and cannot be part of recommendations made to the mechanism designer. We next investigate whether the result for the two-agent case can be generalized. In an extended model where the set of agents and the amount of the resource can vary, we show that the priority rules are the only securely implementable and efficient rules under the additional conditions of resource monotonicity and consistency (Theorem 3). A by-product of our result is that, in the rich family of fixed path rules (Moulin, 1999), the only securely implementable are the priority rules. Since constant rules have no flavor of efficiency and priority rules have no flavor of symmetry, these results suggest a strong trade-off between symmetry and efficiency under secure implementability.

Given the central position of the uniform rule in the literature, we next investigate the robustness of the uniform rule to "bad" Nash equilibria. In Theorem 4, we show that any "good" Nash equilibrium is robust against coalitional deviations in the sense of strong Nash equilibrium. Moreover, in Theorem 5, we clarify how vulnerable any bad Nash equilibrium is to coalitional deviations. In Theorem 6, we show that a Nash equilibrium is coalitionally stable if and only if it is "good". These results imply that the impossibility of securely implementing the uniform rule can be naturally resolved by allowing pre-play communication among agents. A by-product of our results is that the uniform rule is triply implemented by its associated direct revelation mechanism in dominant strategy, coalition-proof Nash, and strong Nash equilibria. The negative message of Theorems 1 and 2 is thus no longer an issue once pre-play communication among agents is possible.

The paper is organized as follows. In Section 2, we introduce the model, and we analyze secure implementation in Section 3. Next, we analyze the possibility of credible coalitional deviations in Section 4. Finally, we provide some concluding comments in Section 5. All the proofs are relegated to the Appendix.

2 Definitions

Let $I \equiv \{1, 2, \ldots, n\}$ be the finite set of *agents*. There is a fixed amount of an infinitely divisible resource $\Omega > 0$ to be allocated. An *allotment* for $i \in I$ is $x_i \in [0, \Omega]$. An *allocation* is a vector of allotments $x = (x_1, \ldots, x_n) \in [0, \Omega]^I$ such that $\sum_{i \in I} x_i = \Omega$. Let X be the set of allocations.

A *single-peaked preference* is a transitive, complete, and continuous binary relation R_i over $[0, \Omega]$ for which there exists a "peak" $r_i \in [0, \Omega]$ such that, for each $x_i, x_i' \in [0, \Omega]$,

$$x_i' < x_i \leq r_i \Longrightarrow x_i \; P_i \; x_i',$$
$$r_i \leq x_i < x_i' \Longrightarrow x_i \; P_i \; x_i'.$$

The symmetric and asymmetric parts of R_i are denoted by I_i and P_i, respectively. Let D be the set of single-peaked preferences. A preference profile is $R \equiv (R_i)_{i \in I} \in D^I$ and the *peak profile* of $R \in D^I$ is $r \equiv (r_i)_{i \in I} \in [0, \Omega]^I$.

A *rule* is a function $\psi : D^I \to X$, which maps each preference profile $R \in D^I$ to an allocation $\psi(R) \in X$. The rule that has played the prominent role in the literature is the *uniform rule* (Benassy, 1982; Sprumont, 1991):

Uniform rule, U: For each $R \in D^I$ and each $i \in I$,

$$U_i(R) = \begin{cases} \min\{r_i, \lambda\} & \text{if } \sum_{j \in I} r_j \geq \Omega, \\ \max\{r_i, \lambda\} & \text{if } \sum_{j \in I} r_j \leq \Omega, \end{cases}$$

where λ solves $\sum_{j \in I} U_j(R) = \Omega$.

The uniform rule satisfies the following two well-known axioms:

Efficiency: An allocation $x \in X$ is *(Pareto-)efficient* at R if $\Omega \leq \sum_{j \in I} r_j$ implies $x \leqq r$ and $\sum_{j \in I} r_j \leq \Omega$ implies $r \leqq x$.[3] A rule ψ is *efficient* if for each $R \in D^I$, $\psi(R)$ is efficient at R.

Symmetry: For each $R \in D^I$ and each $i, j \in I$ such that $R_i = R_j$, $\psi_i(R) \; I_i \; \psi_j(R)$.

3 Securely implementable rules

3.1 Basic notion

The notion of secure implementation is introduced in the seminal work of SSY. A rule is *strategy-proof* if any true preference profile constitutes a dominant strategy equilibrium in its associated direct revelation game. Furthermore, a strategy-proof rule is *secure* if every Nash equilibrium in the game realizes the same outcome as the outcome obtained under true preferences.

[3] It is easy to see that this "same-sidedness" definition is equivalent to the standard definition of efficiency based on dominance relations. On vector inequality, for $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I}$, "$a \leqq b$" denotes "$a_i \leq b_i$ for all $i \in I$".

Strategy-proofness: For each $R \in D^I$, each $i \in I$, and each $R'_i \in D$, $\psi_i(R)\ R_i\ \psi_i(R'_i, R_{-i})$.

Given a (true) preference profile $R^0 \in D^I$, a (reported) preference profile $R \in D^I$ is a *Nash equilibrium in the direct revelation game of* ψ *at* R^0 if for each $i \in I$ and each $R'_i \in D$, $\psi_i(R)\ R^0_i\ \psi_i(R'_i, R_{-i})$. Hereafter, the direct revelation game of ψ is simply referred to as the *game* of ψ. Let $N(\psi, R)$ be the set of Nash equilibria in the game of ψ at R.

Secure implementability: ψ is strategy-proof and for each $R^0 \in D^I$ and each $R \in N(\psi, R^0)$, $\psi(R) = \psi(R^0)$.[4]

An earlier version of SSY points out that the uniform rule is not securely implementable[5]. The underlying reason is that its direct revelation mechanism allows for some "bad" Nash equilibria, as described in the following example:

Example 1. *Bad Nash equilibria in the direct revelation game of* U
Let $I = \{1, 2\}$, $\Omega = 1$ and $R^0 \in D^I$ be such that $r^0_1 < 0.5 < r^0_2$, and $r^0_1 + r^0_2 > 1$. Notice that $U_1(R^0) < \frac{\Omega}{2}$ and $\frac{\Omega}{2} < U_2(R^0)$. Then, $R \in D^I$ with $r = (\frac{\Omega}{2}, \frac{\Omega}{2})$ is a bad Nash equilibrium since $U(R) \neq U(R^0)$. Note that $U(R)$ is not even efficient at R^0. $\diamond$

3.2 A trade-off between symmetry and efficiency

We introduce additional conditions that are useful to understand symmetric securely implementable rules.

Additional conditions:

- A rule ψ is *weakly symmetric* if for each $R \in D^I$ with $R_1 = R_2 = \cdots = R_n$, it holds that $\psi_i(R)\ I_i\ \psi_j(R)$ for all $i, j \in I$.
- A rule ψ satisfies *condition* α if, whenever $R, R' \in D^I$ are such that $r_i = \Omega$ and $r'_i = 0$ for all $i \in I$, it holds that $\psi(R) = \psi(R')$.
- A rule ψ is *constant* if $\psi(R) = \psi(R')$ for all $R, R' \in D^I$.

Weak symmetry and constancy are self-explanatory. Condition α is a newly introduced condition. It states that a rule identically deals with the situation in which all

[4]The original definition of secure implementability by SSY admits implementation by any indirect mechanism. However, the revelation principle of secure implementation (SSY, Theorem 1, Lemma 2) allows us to restrict our attention to the direct mechanism.

[5]The RIETI Discussion Paper (September 2003-E-019) version of SSY offers a discussion on the uniform rule. However, the version published in *Theoretical Economics* contains none of them.

agents want the full of the resource and the situation in which they need no resource. Weak symmetry implies condition α, since it requires the equal division be selected in the both situations. In this sense, condition α can be seen as a much weakened version of symmetry.[6] Furthermore, under the two extreme preference profiles considered in condition α, all allocations are efficient, and hence this condition itself does not bite efficiency. This mild condition, however, can be only satisfied by constant rules under secure implementability as show below.

Theorem 1. *A rule satisfies secure implementability and condition α if and only if it is constant.*

Proof. See, the Appendix. □

A rule ψ that trivially satisfies secure implementability and symmetry is the invariant rule that always recommends the point of equal division, i.e. for each $i \in I$ and each $R \in D^I$, $\psi_i(R) = \frac{\Omega}{n}$. We call this rule the *equal division rule*. Theorem 1 in fact implies that this rule is the only securely implementable rule satisfying weak symmetry:

Corollary 1. *A rule is securely implementable and weakly symmetric if and only if it is the equal division rule.*

Remark 1. In the problem of allocating a divisible resource where preferences are fixed to be strictly increasing and agents are characterized by certain non-preference parameters, Mizukami, Saijo, and Wakayama (2005) show that the equal division rule is the only symmetric and strategy-proof rule. In our problem, it is important to notice that the pair of symmetry and strategy-proofness does not characterize the equal division rule, since the uniform rule satisfies the two properties.[7] ◇

Theorem 1 is quite disappointing, since constant rules are not only inefficient but also reflect no information on reported preferences. This motivates us to analyze if there is any interesting efficient rule that is securely implementable. To study this issue, we first introduce a class of efficient rules:

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[6] Condition α is also satisfied by many asymmetric rules. For instance, in the two-agent case, consider the rule that allocates his peak amount to agent 1 and what is left to agent 2 when the sum of the peaks is less than Ω, and his peak amount to agent 2 and what is left to agent 1 when the sum of the peaks is less than or equal to Ω. Clearly this rule satisfies condition α but it is not symmetric.

[7] There are many other rules that satisfy symmetry and strategy-proofness. For instance, the rule which recommends the point of equal division whenever the peak of agent 1 is less than $\frac{\Omega}{n}$, and follows the uniform rule otherwise is symmetric and strategy-proof as well.

Priority rule: A *priority ordering* is an irreflexive, transitive, and complete binary relation $\succ$ on I.[8] A rule is a *priority rule* if there exists a priority ordering $\succ$ on I such that, for every $R \in D^I$ and every $i, j \in I$ with $i \succ j$, either $[\psi_i(R) = r_i]$ or $[\psi_i(R) < r_i$ and $\psi_j(R) = 0]$.

The next theorem shows that, in the two-person case, priority rules are the only securely implementable and efficient rules.

Theorem 2. *Assume $n = 2$. A rule is securely implementable and efficient if and only if it is a priority rule.*

Proof. See, the Appendix. □

Remark 2. Notice that for any priority rule, the set of Nash equilibrium is in general not a singleton. In fact, at some preference profiles, there can be a continuum of "untruthful" Nash equilibria. For instance suppose that the priority ordering is such that $1 \succ 2 \succ \cdots \succ n$. Then for each $R^0 \in D^I$ such that $r_3^0 > \Omega - r_1^0 - r_2^0$, any announcement $\widetilde{r} = (r_1^0, r_2^0, \widetilde{r}_3, \ldots, \widetilde{r}_n)$ with $\widetilde{r}_3 \geq r_3^0$ and $\widetilde{r}_i \in [0, \Omega]$ for each $i \in \{4, 5, \ldots, n\}$ is a Nash equilibrium. Obviously any such untruthful Nash equilibrium results in the good outcome. ◇

When there are more than two agents, Theorem 2 no longer holds. As it turns out, the full class of efficient and secure rules is much larger. To get an intuition of why this is the case, we provide below several examples which show that an increase in the number of agents gives a lot of flexibility in the way efficient and secure rules can be designed.

Example 2. *Priorities with flexible positioning*
Consider the following modified priority rule whose associated ordering is not a priori fixed, $1 \succ 2 \sim 3 \succ \cdots \succ n$. The priority between agent 2 and 3 is said to be *flexible* because the way the "tie" between agent 2 and 3 is broken depends on the peak profile.[9] For each $R \in D^I$, agent 1 always receives his peak amount $\psi_1(R) \equiv r_1$; When r_1 is a rational number, agent 2 receives $\psi_2(R) \equiv \min\{r_2, \Omega - r_1\}$, agent 3 receives $\psi_3(R) \equiv \min\{r_3, \max\{0, \Omega - \psi_1(R) - \psi_2(R)\}\}$ etc; On the other hand, when r_1 is an irrational number, agent 3 receives $\psi_3(R) \equiv \min\{r_3, \Omega - r_1\}$ and then agent 2 receives $\psi_2(R) \equiv \min\{r_2, \max\{0, \Omega - \psi_1(R) - \psi_3(R)\}\}$ etc. This rule is obviously securely implementable and efficient, but it is not a priority rule. ◇

[8] Irreflexibility means $[\nexists i \in I, i \succ i]$; Transitivity means $[\forall i, j, k \in I, i \succ j$ and $j \succ k$ imply $i \succ k]$; Completeness means $[\forall i, j \in I$, either $i \succ j$ or $j \succ i]$.

[9] Because secure implementability implies peak-onliness, the way a tie is broken can only be based on peak information.

Remark 3. Secure implementability imposes restrictions on the way ties in the priority ordering can be broken. In particular, the tie between agents 2 and 3 in Example 3 cannot be broken as a function of the peak of an agent whose priority is inferior to the one of agent 2 and 3. For that very reason, it is also not possible to have a tie between the first two agents in the priority ordering. To see this, suppose for instance that the role played by agent 1 in breaking the tie between agent 2 and 3 is instead played by agent 4. Let $R^0 \in D^I$ be such that $r_2^0 = r_3^0 = \Omega - r_1^0$, and r_4^0 is a rational number –giving the priority ordering $1 \succ 2 \succ 3 \succ \cdots \succ n$ at R^0. Then $\widetilde{r} = (r_1^0, r_2^0, r_3^0, \widetilde{r}_4, \ldots)$ is a Nash equilibrium for any $\widetilde{r}_4 \in [0, \Omega]$, in contradiction with secure implementability. $\diamond$

It is clear from Examples 2 that the class of secure and efficient rules contains an infinite number of rules, even in the three-agent case. However the class is even richer as shown below in the next example.

Example 3. *Priority rules with sequential rationing*
Consider the following modified priority which allocate resources in several steps. At step 1, half of the resource is distributed via the ordering $1 \succ_{\frac{\Omega}{2}} 2 \succ_{\frac{\Omega}{2}} \ldots \succ_{\frac{\Omega}{2}} n$. At step 2, the rest of the resources are distributed via the ordering $2 \succ_{\frac{\Omega}{2}} 1 \succ_{\frac{\Omega}{2}} 3 \succ_{\frac{\Omega}{2}} \ldots \succ_{\frac{\Omega}{2}} n$. This rule is obviously securely implementable and efficient, but it is not a priority rule. $\diamond$

It is possible to combine Example 2 and 3 in "interesting" ways to obtain rules that allow for a flexible positioning of agents as well as some sequential rationing. The examples above show that the class of efficient and secure rules share a non-empty intersection with both the sequential allotment rules and the fixed-path methods –e.g., the priority rules; while being neither a superset nor a subset of either of the two aforementioned classes –e.g., the uniform rule is not secure, among many others.

Although we do not have a full characterization of securely implementable and efficient rules when there are more than two agents, the examples above highlights the fact that not only the class is very large, but it also admits many questionable rules that would be hard to justify from a normative standpoint. One question we now ask is whether it is possible to generalize the result obtained for the two-agent case. It turns out that under the additional requirements of *resource monotonicity* and *consistency*, priority rules are again the only securely implementable and efficient rules. In order to study these two axioms, we need to extend our model so that the amount of the resource and the set of agents can vary. This is the object of our next subsection.

3.3 Variable factors model

In the previous subsection, both the set of agents and the amount of the resource were fixed. In this subsection, we introduce a model in which these two factors can vary (e.g., Thomson, 1994b). Let $\bar{I} \equiv \{1, 2, \ldots, \bar{n}\}$ be the set of potential agents and $\bar{\Omega} > 0$ be the potentially maximum amount of the resource.[10] Each $i \in \bar{I}$ has a single-peaked preference $R_i \in D$ over $[0, \bar{\Omega}]$.

Given any non-empty subset $I \subseteq \bar{I}$, an *I-problem* is a pair

$$(R_I, \Omega) \in D^I \times [0, \bar{\Omega}],$$

where $R_I \equiv (R_i)_{i \in I} \in D^I$ is a single-peaked preference profile of I and Ω is an amount of the resource to be divided. Let $\mathcal{E}^I$ be the set of I-problems. In this variable factors model, a *rule* is a function ψ that maps, for every non-empty $I \subseteq \bar{I}$, each I-problem $(R_I, \Omega) \in \mathcal{E}^I$ to a vector of allotments

$$\psi(R_I, \Omega) = (\psi_i(R_I, \Omega))_{i \in I} \in \mathbb{R}^I_+$$

that satisfies the feasibility constraint $\sum_{i \in I} \psi_i(R_I, \Omega) = \Omega$.

All the axioms defined in the previous "fixed factors model" can be defined in this "variable factors model" in the same way, so we omit their trivial definitions. We introduce two well-known axioms:

Resource monotonicity: For every non-empty $I \subseteq \bar{I}$, every $(R_I, \Omega) \in \mathcal{E}^I$, and every Ω' with $\Omega \leq \Omega' \leq \bar{\Omega}$,

$$\psi_i(R_I, \Omega) \leq \psi_i(R_I, \Omega') \;\; \forall i \in I.$$

This condition states that the allotment for each agent weakly increases according to any increase of the amount of the available resource.[11]

Consistency: For every non-empty $I \subseteq \bar{I}$, every $(R_I, \Omega) \in \mathcal{E}^I$, and every non-empty $I' \subseteq I$,

$$\psi_i(R_{I'}, \sum_{j \in I'} \psi_j(R_I, \Omega)) = \psi_i(R_I, \Omega) \;\; \forall i \in I'.$$

This condition states that the recommendation made by ψ in a larger problem should be still recommended by ψ in a smaller problem.

[10] Our result in this subsection holds even if the potential population and the potential amount of the resource are infinite.

[11] There are other definitions of resource monotonicity that have been used in this literature, but they are equivalent under efficiency. See, Ehlers (2002) on this point. Thomson (1994a) characterizes the uniform rule on the basis of resource monotonicity.

In the variable factors model, a *priority ordering* is an irreflexive, transitive, and complete binary relation $\succ$ on $\bar{I}$, and a rule ψ is a *priority rule* if there exists a priority ordering $\succ$ on $\bar{I}$ such that, for every non-empty $I \subseteq \bar{I}$, every $(R_I, \Omega) \in \mathcal{E}^I$, and every $i, j \in I$ with $i \succ j$,

$$\text{either } [\psi_i(R_I, \Omega) = \max\{r_i, \Omega\}] \text{ or } [\psi_i(R_I, \Omega) < \max\{r_i, \Omega\} \text{ and } \psi_j(R_I, \Omega) = 0].$$

As announced, we now offer a characterization of the family of priority rules.

Theorem 3. *In the variable factors model, a rule is securely implementable, efficient, resource monotonic, and consistent if and only if it is a priority rule.*

Proof. See, the Appendix. □

Moulin (1999) characterizes the class of fixed-path rules by strategy-proofness, efficiency, resource monotonicity, and consistency.[12] Since secure implementability is stronger than strategy-proofness, Theorem 3 and Moulin's result imply that priority rules are the only securely implementable fixed-path rules. However, notice that the proof of Theorem 3 is based on Theorem 2 and does not depend in any way on Moulin's result. It is easy to see that each of secure implementability and efficiency cannot be dispensed with. We show below that there are rules that differ from the priority rules and which satisfy all the axioms but consistency or resource monotonicity. The characterization statement offered in Theorem 3 is thus tight.

Example 4. *Dropping resource monotonicity*
Let $\bar{I} = \{1, 2, 3\}$ and $\bar{\Omega} = 6$. For $\Omega \leq 3$, let $\succ_\Omega$ be the priority ordering on $\bar{I}$ such that $1 \succ_\Omega 2 \succ_\Omega 3$. For $\Omega > 3$ let $\succ_\Omega$ be such that $2 \succ_\Omega 1 \succ_\Omega 3$. It is easy to see that this rule is efficient, securely implementable, and consistent but violates resource monotonicity. Let $R^0 \in D^I$ be such that $r^0 = (4, 3, 1)$. At $\Omega = 3$, the allocation is $(3, 0, 0)$. However, at $\Omega = 4$, it is $(1, 3, 0)$, a contradiction with resource monotonicity. ◇

Example 5. *Dropping consistency*
Let $\bar{I} = \{1, 2, 3\}$ and $\bar{\Omega} = 6$. For each $I \subset \bar{I}$ with $|I| = 2$, $i \succ j$ if $i < j$. For $\bar{I}$, let the priority be defined in two steps as follows. In the first step, half of the resource is distributed according to the priority ordering $1 \succ_{\frac{\Omega}{2}} 2 \succ_{\frac{\Omega}{2}} 3$. In the second step, the remaining half is distributed according to $2 \succ_{\frac{\Omega}{2}} 3 \succ_{\frac{\Omega}{2}} 1$. It is easy to see that this rule is efficient, securely implementable, and resource monotonic but not consistent. Let $R^0 \in D^I$ be such that $r^0 = (4, 3, 1)$. However, $\Omega = \bar{\Omega} = 6$, the chosen allocation is $(3, 3, 0)$, though $(4, 2)$is chosen when agent 3 leaves. This contradicts consistency. ◇

[12]We refer the reader to Moulin (1999) for details and definitions on the family of fixed path rules.

4 Robustness of the uniform rule

4.1 Motivation and notation

The difficulties of secure implementation observed in Theorems 1 and 2 motivates us to analyze whether bad Nash can be naturally eliminated by certain plausible criteria of equilibrium selection. In particular, given that the uniform rule has occupied a prominent role in the literature, we shall show a way out of the negative result. Our approach is to distinguish good and bad Nash equilibria in views of coalitional stability.

We need some additional notation. Given $T \subseteq I$, $R_T \in D^T$ and allocations $x, y \in X$, we write x wdom$[R_T]$ y if,

$$x_i \; R_i \; y_i \quad \text{for each } i \in T,$$
$$x_j \; P_j \; y_j \quad \text{for some } j \in T.$$

Then x is said to *weakly dominate* y at R_T. Similarly, we write x sdom$[R_T]$ y if,

$$x_i \; P_i \; y_i \quad \text{for each } i \in T.$$

Then x is said to *strictly dominate* y at R_T. Obviously, strict domination implies weak domination.

4.2 Coalitional stability of Nash equilibria

A profile $R \in D^I$ is a *strong Nash equilibrium* (Aumann, 1959) in the game of ψ at R^0 if there exist no $T \subseteq I$ and $R'_T \in D^T$ such that $\psi(R'_T, R_{I \setminus T})$ wdom$[R^0_T]$ $\psi(R)$. [13] Let $SN(\psi, R^0)$ be the set of strong Nash equilibria in the game of ψ at R^0. The next theorem ensures strong coalitional stability of any good Nash equilibrium.

Theorem 4. *For each $R^0 \in D^I$ and each $R \in N(U, R^0)$, if $U(R) = U(R^0)$, then $R \in SN(U, R^0)$.*

Proof. See, the Appendix. □

We remark that Theorem 4 generalizes the known fact that the uniform rule is "coalition strategy-proof", which states that the profile of true preferences constitutes a strong Nash equilibrium in its associated direct revelation game (e.g., Serizawa, 2006).

[13] We are defining strong Nash by weak domination, but all of our results hold if it is defined by strong domination.

Given that any good Nash equilibrium is coalitionally stable, if any bad Nash equilibrium is shown to be unstable in some sense, then we can distinguish good and bad Nash equilibria in views of coalitional stability and can conclude that only good Nash equilibria survive the stability test. Before establishing this implication, we first introduce a notion of credible coalitional deviations.

Given $R^0 \in D^I$ and $R \in N(\psi, R^0)$ such that $U(R) \neq U(R^0)$, a coalition $T \subseteq I$ *can robustly deviate from* R *via* $R'_T \in D^T$ *at* R^0 if "the deviation is strongly profitable":

$$U(R'_T, R_{I \setminus T}) \text{ sdom}[R^0_T]\, U(R),$$

and "no weakly profitable counter deviation is possible": there exist no $S \subseteq I$ and $R''_S \in D^S$ such that

$$U(R''_S, R'_{T \setminus S}, R_{I \setminus (T \cup S)}) \text{ wdom}[R^0_S]\, U(R'_T, R_{I \setminus T}).$$

Strong deviation is quite a demanding condition on credible deviations, and hence we can conclude that any Nash equilibrium blocked by such a deviation is unrealizable.

Our next result shows that for every bad Nash equilibrium in the game of the uniform rule, (i) the uniform allocation Pareto dominates the bad Nash equilibrium allocation, (ii) agents who prefer the uniform allocation to the bad Nash equilibrium allocation are "misreporting" agents, (iii) the coalition of such misreporting agents can robustly deviate from the bad Nash equilibrium via their true preferences, (iv) the robust deviation leads to the uniform allocation, and (v) the robust deviation does not change the allotment of anyone who does not belong to the coalition. The following example illustrates this result for a three-person case.

Example 6. *Structure and coalitional instability of bad Nash equilibria*
Let $I = \{1, 2, 3\}$, $\Omega = 6$, and $R^0 \in D^I$ be such that $r^0 = (1, 2, 4)$. Observe that R with $r = (2, 2, 2)$ and $U(R) = (2, 2, 2)$ is a Nash equilibrium in the game of the uniform rule at R^0. By definition of the uniform rule, no one can change this allocation by any unilateral deviation from r. It is crucial here that the misreport is such that $r_1 + r_2 + r_3 = \Omega$. Notice that $U(R^0) = (1, 2, 3)$ Pareto dominates $U(R)$ at R^0. Agents 1 and 3 have the joint profitable deviation of simply reporting their true preferences so that the true uniform allocation is obtained. By the same token, observe that the report R' with $r' = (1.5, 2, 2.5)$ and $U(R') = (1.5, 2, 2.5)$ is also a Nash equilibrium at R^0. One can verify that there is in fact an infinity of bad Nash equilibria in this example described by the following set, $\{R \in D^I : 1 < r_1 \leq 2, r_2 = 2, 2 \leq r_3 < 4 \text{ such that } r_1 + r_2 + r_3 = 6\}$. This turns out to be a general observation. Bad Nash equilibria occur only when the sum of the reported peaks is equal to Ω and the obtained allocation is Pareto dominated by the true uniform allocation. ◇

We are now ready to proceed to our next result.

Theorem 5. *For each $R^0 \in D^I$ and each $R \in N(U, R^0)$, if $U(R) \neq U(R^0)$, then the following statements hold:*
(i) $U(R^0)$ wdom$[R^0]$ $U(R)$, and so

$$T \equiv \{i \in I : U_i(R^0)\ P_i^0\ U_i(R)\} \neq \emptyset;$$

(ii) $r_i \neq r_i^0 \quad \forall i \in T$;
(iii) T can robustly deviate from R via R_T^0 at R^0;
(iv) $U(R_T^0, R_{I \setminus T}) = U(R^0)$;
(v) $U_i(R_T^0, R_{I \setminus T}) = U_i(R) \quad \forall i \in I \setminus T$.

Proof. See, the Appendix. □

In a companion paper, Bochet and Sakai (2007) show that in the game of the uniform rule, any efficient Nash equilibrium allocation is the uniform allocation. Theorem 5 (i) generalizes this result, since the fact that the uniform allocation Pareto dominates a bad Nash equilibrium allocation implies that the bad Nash equilibrium allocation is not efficient.

Note that, in the robust deviation in Theorem 5, finding deviation strategies and calculating the final outcome are quite easy: deviating agents only need to report their true preferences so as to realize the true uniform allocation. Now the implication of Theorem 5 is clear: in the direct revelation mechanism associated with the uniform rule, all bad Nash equilibria can be naturally eliminated by allowing pre-play communication among players.

4.3 Coalition-proof Nash equilibrium

Bernheim, Peleg, and Winston (1987) introduce the notion of credibility of coalitional deviations and define an equilibrium as a position at which no credible coalitional deviation occurs. We define this notion based on two credibility conditions.

***Coalition-proof Nash equilibrium under weak domination*:** We first inductively define the notion of *credible deviation under weak domination* in the game of ψ at R^0: (1) *Singleton coalitions: $T = \{i\}$ for some $i \in I$.* In this case, $\{i\}$ *can credibly deviate from R via R_i'* if $\psi_i(R_i', R_{-i})\ P_i^0\ \psi_i(R)$; (2) *Other coalitions: any T with $|T| \geq 2$.* In this case, T *can credibly deviate from R via R_T' under weak domination* if $\psi(R_T', R_{I \setminus T})$ wdom$[R_T]$ $\psi(R)$ and there exist no $S \subseteq T$ and $R_S'' \in D^S$ such that S can credibly deviate from $(R_T', R_{I \setminus T})$ via R_S'' under weak domination.

Then $R \in D^I$ is a *coalition-proof Nash equilibrium* if there exist no $T \subseteq I$ and $R'_T \in D^T$ such that T can credibly deviate from R via R'_T under weak domination in the game of ψ at R^0. Let $CN_w(\psi, R^0)$ be the set of coalition-proof Nash equilibria under weak domination in the game of ψ at R^0.

***Coalition-proof Nash equilibrium under strong domination*:** This notion is defined by replacing "weak domination" with "strong domination" in the definitions of credible deviation under weak domination and coalition-proof Nash equilibrium under weak domination. Let $CN_s(\psi, R^0)$ be the set of coalition-proof Nash equilibria under strong domination in the game of ψ at R^0.

By definition, any strong Nash equilibrium is a coalition-proof Nash equilibrium under weak domination and under strong deviation, but the converse does not always hold. Indeed, coalition-proof Nash equilibria can be even inefficient. Furthermore, Konishi, Le Breton, and Weber (1999) point out that the set of coalition-proof Nash equilibria under weak domination and the set under strong domination may have an empty intersection. However, in the game of the uniform rule, both sets coincide and each element of the sets realizes the uniform allocation. This is because every bad Nash equilibrium can be blocked by a coalitional deviation that meets both credibility conditions (Theorem 5), while every good Nash equilibrium is a strong Nash equilibrium (Theorem 4). The next theorem summarizes this fact and other characteristics of "good" Nash equilibria:

Theorem 6. *For each $R^0 \in D^I$ and each $R \in N(U, R^0)$, the following statements are equivalent:*
(i) $U(R) = U(R^0)$;
(ii) $U(R)$ is efficient at R^0;
(iii) There exists no $R' \in N(U, R^0)$ such that $U(R')$ wdom$[R^0]$ $U(R)$;
(iv) $R \in CN_w(U, R^0)$;
(v) $R \in CN_s(U, R^0)$;
(vi) $R \in SN(U, R^0)$.

Proof. By Theorem 4, (i) implies (vi). By definition, (vi) implies (ii, iii, iv, v). By Theorem 5, each one of (ii, iii, iv, v) implies (i). □

A few remarks are in order. Shinohara (2005) shows that, in a class of games in which each player's payoff only depends on his own strategy and the sum of other players' strategies, the set of coalition-proof Nash equilibria under weak domination is contained by the set under strict domination. Obviously, the game of the uniform rule does not belong to the class, since each player's payoff depends on the entire distributions of peaks. Hence our equivalence between (iv) and (v) is independent of Shinohara's observation.

Note that (iii) implies the equivalence between the Pareto frontier of the set of Nash equilibria and the set of coalition-proof Nash equilibria (under any dominance relation). Yi (1999) establishes the same equivalence in games with strategic substitutes where each player's payoff depends only on his own strategy and the sum of others' strategies. However, the game of the uniform rule apparently differs from those games, so our equivalence between the two sets is independent of Yi's result.

In terms of implementation theory, (iv, v, vi) imply that the direct revelation mechanism of the uniform rule implements the uniform rule in coalition-proof Nash (under both dominance relations) and strong Nash equilibria.

4.4 General notion of secure implementation

Secure implementability states that a strategy-proof rule admits no bad Nash equilibrium in its direct revelation game. This ingenious idea by SSY can be applied not only to Nash equilibrium but also to any other solution concept. Let us say that a strategy-proof rule is *securely implementable with respect to an equilibrium concept* if it admits no equilibrium defined by the equilibrium concept in its direct revelation game. This general notion would be helpful to understand how a rule is coalitionally secure. Indeed, we observed that, though the uniform rule is not securely implementable with respect to Nash equilibrium, it is securely implementable with respect to coalition-proof Nash (under both dominance relations) and strong Nash equilibria. Moreover, our results show that the violation of the secure criterion by the uniform rule is not "robust". As seen in Example 6, bad Nash equilibrium allocation are Pareto dominated by the true uniform allocation, and coalitional deviations simply involve lying agents to revert to truthful reports. We consider that this approach is useful to obtain more positive results in any economic environment where no reasonable strategy-proof can be securely implementable with respect to Nash equilibrium.

5 Concluding comments

We studied secure implementation in the problem of fairly allocating a divisible resource when agents have single-peaked preferences. We characterized constant rules and priority rules on the basis of secure implementability. Not surprisingly, this condition turns out to be quite strong and considerably restricts the class of rules satisfying it. Since the uniform rule is not securely implementable, we then analyzed the structure of bad Nash equilibria in the game of the uniform rule and showed that they can be naturally eliminated by many types of credible coalitional deviations. This significantly contrasts with the robustness of good Nash equilibria to arbitrary coalitional deviations. Thus the impossibility of securely implementing the uniform rule can be resolved when pre-play communication among players is possible.

6 Appendix

6.1 Basic results and notation

Characterizations of secure implementability by SSY imply that the pair of strategy-proofness and the following independence condition is necessary and sufficient for secure implementation:

Rectangular property: For each $R, R' \in D^I$, if for each $i \in I$, $\psi_i(R')\ I_i\ \ \psi_i(R_i, R'_{-i})$, then $\psi(R) = \psi(R')$.

Lemma 1. *A rule is securely implementable if and only if it satisfies strategy-proofness and the rectangular property.*

Proof. See, Theorem 1 in SSY. □

The next condition is an informational restriction on the choice of allocations. It states that a rule determines an allocation only depending on the peaks of preferences:

Peak-only: For each $R, R' \in D^I$, if $r = r'$, then $\psi(R) = \psi(R')$.

Lemma 2. *If a rule is securely implementable, then it satisfies peak-only.*

Proof. Let ψ be a securely implementable rule. By Lemma 1, ψ satisfies strategy-proofness and the rectangular property. It suffices to show that, for each $R \in D^I$, $i \in I$, and $R'_i \in D$ with $r_i = r'_i$, $\psi(R) = \psi(R'_i, R_{-i})$.

Let $x \equiv \psi(R)$ and $y \equiv \psi(R'_i, R_{-i})$. Without loss of generality, assume that $x_i \leq y_i$. By strategy-proofness at R, it is not possible that $x_i < y_i \leq r_i$. By strategy-proofness at (R'_i, R_{-i}), it is not possible that $r'_i \leq x_i < y_i$. Hence, $x_i \leq r_i \leq y_i$. Let $R''_i \in D$ be such that $x_i\ I''_i\ y_i$ and $r''_i = r_i$.

Let $z \equiv \psi(R''_i, R_{-i})$. By strategy-proofness, either $x_i = z_i$ or $y_i = z_i$. Consider the case $x_i = z_i$. By the rectangular property, $x = z$. Note that

$$\psi_i(R''_i, R_{-i}) = z_i\ I''_i\ y_i = \psi_i(R'_i, R_{-i}).$$

By the rectangular property, $y = z$. Hence $x = y$. The same logic applies to the case $y_i = z_i$. □

Lemma 2 implies that any securely implementable rule ψ is peak-only. This means that ψ can be seen as a function that maps each peak profile $r \in [0, \Omega]^I$ to an allocation $\psi(r) \in X$. Therefore, when there is no danger of confusion, we may write $\psi(r)$ instead of $\psi(R)$. Also, $\Omega_i, 0_i$ mean that i's peak is $\Omega, 0$, respectively, and $\mathbf{\Omega}, \mathbf{0}$ denote the peak profiles that only consist of $\Omega, 0$, respectively. These notational rules are particularly used in the proofs of Theorems 1 and 3.

Before going to formal proofs, it is useful to note some basic observations on any strategy-proof and peak-only rule ψ:

- If $\psi_i(r) < r_i$ and $\psi_i(r) \leq r'_i$, then $\psi_i(r'_i, r_i) = \psi_i(r)$. This is because (i) $\psi_i(r'_i, r_i) < \psi_i(r)$ contradicts i's manipulation at (r'_i, r_{-i}) via r_i when i's true peak is r'_i; (ii) $\psi_i(r) < \psi_i(r'_i, r_i)$ contradicts i's manipulation at r via r'_i when his true preference is R_i such that $\psi_i(r'_i, r_{-i}) \; P_i \; \psi_i(r)$ with peak r_i.
- If $r_i < \psi_i(r)$ and $r'_i \leq \psi_i(r)$, then $\psi_i(r'_i, r_i) = \psi_i(r)$. This can be shown in the same fashion.

These observations will be implicitly used in our proofs.

6.2 Proof of Theorem 1

Proof of Theorem 1. Since the "only if" part is trivial, we show the "if" part. Let ψ be a rule satisfying the two conditions and $x \equiv \psi(\mathbf{\Omega}) = \psi(\mathbf{0})$. We shall show that, for every $R \in D^I$, $\psi(R) = x$.

By *strategy-proofness*, for every $i \in I$, $\psi_i(x_i, \mathbf{\Omega}_{-i}) = x_i$. Hence by the *rectangular property* at $\psi(\mathbf{\Omega}) = x$,

$$\psi(x_{N \setminus S}, \mathbf{\Omega}_S) = x \;\; \forall S \subseteq I. \tag{1}$$

Let $A \equiv \{i \in I : x_i < r_i\}$. By (1), $\psi(x_{N \setminus A}, \mathbf{\Omega}_A) = x$. For every $i \in A$, since $x_i < r_i \leq \Omega$, *strategy-proofness* implies $\psi_i(x_{N \setminus A}, r_i, \mathbf{\Omega}_{A \setminus \{i\}}) = x_i$. Hence by the *rectangular property* at $\psi(x_{N \setminus A}, \mathbf{\Omega}_A)$,

$$\psi(x_{N \setminus A}, r_A) = x.$$

Again, for every $i \in A$, since $x_i < r_i$, *strategy-proofness* implies $\psi_i(x_{N \setminus A}, x_i, r_{A \setminus \{i\}}) = x_i$. Hence by the *rectangular property* at $\psi(x_{N \setminus A}, r_A)$,

$$\psi(x_{N \setminus A}, x_{A \setminus \{j\}}, r_j) = x \;\; \forall j \in A. \tag{2}$$

Let $B \equiv \{i \in I : r_i < x_i\}$. Following the same logic, we have

$$\psi(x_{N \setminus B}, x_{B \setminus \{j\}}, r_j) = x \;\; \forall j \in B. \tag{3}$$

Since (1) implies $\psi(x) = x$, by (2), (3), and the *rectangular property*, we have

$$\psi(r) = x,$$

which completes the proof. $\square$

6.3 Proof of Theorem 2

In this proof, we only deal with the fixed set $I = \{1, 2\}$. Therefore, we often avoid subscripts of agents when there is no danger of confusion. For example, we simply write $\psi(0, \Omega)$ instead of $\psi(0_1, \Omega_2)$.

Proof of Theorem 2. We only show that any *securely implementable* and *efficient* rule ψ is a priority rule, since the other direction is trivial.

Let $x \equiv \psi(\Omega, \Omega)$ and $y \equiv \psi(0, 0)$. If $x = y$, then ψ satisfies *condition* α, so ψ is constant by Theorem 1, a contradiction to *efficiency*. Therefore, $x \neq y$. Hence, without loss of generality, we can assume $x_1 < y_1$.

Step 1. Either $\boldsymbol{x_1 = \Omega}$ or $\boldsymbol{x_2 = \Omega}$. Suppose, by contradiction, that $0 < x_1 < \Omega$ and $0 < x_2 < \Omega$.

By *strategy-proofness* at $\psi(\Omega, \Omega) = x$, we have $\psi(x_1, \Omega) = \psi(\Omega, x_2) = x$. By the *rectangular property*, $\psi(x_1, x_2) = x$.

Since $\psi(x_1, \Omega) = x$ and $\psi(x_1, x_2) = x$, if $\psi(0, x_2) = x$, then by the *rectangular property*, $\psi(0, \Omega) = x$, a contradiction to *efficiency*. Therefore, $\psi_1(0, x_2) \neq x_1$, and so by $\psi_1(x_1, x_2) = x_1$ and *strategy-proofness*, $\psi_1(0, x_2) < x_1$. Let $r_1 \equiv \psi_1(0, x_2)$. By *strategy-proofness*,

$$\psi(r_1, x_2) = (r_1, \Omega - r_1). \tag{4}$$

Since

$$0 \leq r_1 < x_1 < y_1 \leq \Omega,$$

we have

$$0 \leq \Omega - y_1 < \Omega - x_1 < \Omega - r_1,$$

meaning that

$$0 \leq y_2 < x_2 < \Omega - r_1. \tag{5}$$

By (4), (5), and *strategy-proofness*, we have

$$\psi(r_1, y_2) = (r_1, \Omega - r_1). \tag{6}$$

On the other hand, since $\psi(0,0) = y$ and $r_1 < y_1$, by *strategy-proofness*, $\psi(r_1, 0) = y$. Also, since $\psi(0,0) = y$, by *strategy-proofness*, $\psi(0, y_2) = y$. Thus by the *rectangular property*, $\psi(r_1, y_2) = y$, a contradiction to (6).

Step 2. Either $y_1 = 0$ or $y_2 = 0$. This can be shown in a way similar to Step 1.

Step 3. f is serially dictatorial. By Step 1, without loss of generality, assume $x_1 = \Omega$. If $y_2 = 0$, then $x = y$ and ψ satisfies condition α, a contradiction with secure implementability. Hence, by Step 2, $y_1 = 0$. Overall, $\psi(\Omega, \Omega) = (\Omega, 0)$ and $\psi(0,0) = (0, \Omega)$.

Let $R \in D^N$ be any preference profile. By *strategy-proofness*, $\psi(\Omega, r_2) = (\Omega, 0)$ and $\psi(0, r_2) = (0, \Omega)$. Hence by *strategy-proofness*, $\psi(r_1, r_2) = (r_1, \Omega - r_1)$. Thus agent 1 is the dictator . □

6.4 Proof of Theorem 3

Given any rule ψ, any population $I \subseteq \bar{I}$ with $|I| = 2$, and any amount of the resource $\Omega \in [0, \bar{\Omega}]$, we say that $i \in I$ is the (ψ, I, Ω)-*dictator* if for every $R_I \in D^I$, $\psi_i(R_I, \Omega) = \max\{r_i, \Omega\}$.

Lemma 3. *Let ψ be any securely implementable, efficient, and resource monotonic rule. Then for every $I \subseteq \bar{I}$ with $|I| = 2$, there exists $i \in I$ such that for each $\Omega \in [0, \bar{\Omega}]$, i is the (ψ, I, Ω)-dictator.*

Proof. By Theorem 2, for every $\Omega, \Omega' \in [0, \bar{\Omega}]$ with $\Omega < \Omega'$, there exist $i, j \in I$ who are the (ψ, I, Ω)-dictator, (ψ, I, Ω')-dictator, respectively. We claim that $i = j$. Suppose, not, $i \neq j$. Then $\psi_j(0_i, 0_j; \Omega) = \Omega$ and $\psi_j(0_i, 0_j; \Omega') = 0$, a contradiction to *resource monotonicity.* □

Proof of Theorem 3. We omit the easy proof that any priority rule satisfies the four axioms. Conversely, let ψ be any rule satisfying these axioms. We shall show that ψ is a priority rule.

Define the binary relation $\succ$ on $\bar{I}$ as follows: for every $i, j \in \bar{I}$ with $i \neq j$, $i \succ j$ if i is the $(\psi, \{i, j\}, \Omega)$-dictator for all $\Omega \in [0, \bar{\Omega}]$; $j \succ i$ if j is the $(\psi, \{i, j\}, \Omega)$-dictator for all $\Omega \in [0, \bar{\Omega}]$. Lemma 3 implies that $\succ$ is well-defined. Irreflexibility and completeness of $\succ$ are obvious. We shall show that $\succ$ is transitive. Assume $i \succ j$ and $j \succ k$. Let $\Omega \in (0, \bar{\Omega}]$. Let $x \equiv \psi(\Omega_i, \Omega_j, \Omega_k; \Omega)$. Since $i \succ j$, by *consistency*, $x_j = 0$. Since $j \succ k$, by *consistency*, $x_k = 0$. Therefore, $x_i = \Omega$, so by *consistency*, $\psi_i(\Omega_i, \Omega_k; \Omega) = \Omega$. Thus by Theorem 2, $i \succ k$. Hence, $\succ$ is a priority ordering.

We shall show that ψ allocates the resource on the basis of $\succ$. Let $I \subseteq \bar{I}$, $(R_I, \Omega) \in \mathcal{E}^I$, and $i, j \in I$ with $i \succ j$. Let $x \equiv \psi(R_I, \Omega)$. We need to prove that

$$\text{either } [x_i = \max\{r_i, \Omega\}] \text{ or } [x_i < \max\{r_i, \Omega\} \text{ and } x_j = 0].$$

Suppose, not,

$$[x_i \neq \max\{r_i, \Omega\}] \text{ and } [\max\{r_i, \Omega\} \leq x_i \text{ or } x_j \neq 0].$$

Then, there are two cases.

Case 1. Consider the case $[x_i \neq \max\{r_i, \Omega\}$ and $\max\{r_i, \Omega\} \leq x_i]$. Then $r_i < x_i$, so by *consistency*, $r_i < x_i = \psi_i(R_i, R_j; x_i + x_j)$, a contradiction to $i \succ j$.

Case 2. Consider the case $[x_i \neq \max\{r_i, \Omega\}$ and $x_j \neq 0]$. By *consistency*, $(x_i, x_j) = \psi(R_i, R_j; x_i + x_j)$. Since $i \succ j$, $x_i \neq \max\{r_i, \Omega\}$ is possible only if $x_j = 0$, a contradiction to $x_j \neq 0$. □

6.5 Proof of Theorem 4

We fix the true preference profile R^0 throughout the proof of Theorem 4. We only deal with the case $\Omega \leq \sum_{i \in I} r_i^0$. The opposite case can be handled in a similar way.

Lemma 4. *For each $R \in N(U, R^0)$, if $\sum_{i \in I} r_i < \Omega$, then $U(R^0) = U(R) = r^0$.*

Proof. Pick any $R \in N(U, R^0)$ and assume $\sum_{i \in I} r_i < \Omega$. Since $\sum_{i \in I} r_i < \Omega$, every $i \in I$ could get more by reporting a preference whose peak is more than $U_i(R)$. Therefore,

$$r_i^0 \leq U_i(R) \quad \forall i \in I. \tag{7}$$

By assumption, $\Omega \leq \sum_{i \in I} r_i^0$, hence (7) implies $r^0 = U(R)$. Therefore, $\sum_{i \in I} r_i^0 = \Omega$, so that $U(R^0) = U(R) = r^0$. □

In Lemmas 5 and 6, we write

$$I_1 \equiv \{i \in I : U_i(R^0) = r_i^0 < \lambda^0\}, \tag{8}$$

$$I_2 \equiv \{i \in I : U_i(R^0) = \lambda^0 \leq r_i^0\}, \tag{9}$$

where R^0 is a given preference profile and λ^0 is such that $U(R^0) = (\min\{r_i^0, \lambda^0\})_{i \in I}$. Note that I_1, I_2, λ^0 depend on R^0, but we use these notations for simplicity.

Lemma 5. *For each $R \in N(U, R^0)$, if $U(R) = U(R^0) \neq r^0$ and $\Omega \leq \sum_{i \in I} r_i$, then*

$$r_i = r_i^0 \quad \forall i \in I_1,$$

$$\lambda^0 \leq r_i \quad \forall i \in I_2.$$

Proof. Pick any $R \in N(U, R^0)$ and assume $U(R) = U(R^0)$ and $\Omega \leq \sum_{i \in I} r_i$.

Consider the case $\Omega = \sum_{i \in I} r_i$. Then $U(R) = U(R^0) = r$. Hence, $r_i = U_i(R) = U_i(R^0) = r_i^0$ for all $i \in I_1$, and $r_i = U_i(R) = U_i(R^0) = \lambda^0$ for all $i \in I_2$.

Consider the case $\Omega < \sum_{i \in I} r_i$. Let $\lambda > \Omega$ be such that $U(R) = (\min\{r_i, \lambda\})_{i \in I}$. Note that $U(R^0) \neq r^0$ implies the non-emptyness of I_2. Therefore, $\lambda < \lambda^0$ contradicts $U_i(R) = U_i(R^0)$ for $i \in I_2$, so that $\lambda^0 \leq \lambda$ holds. Hence, for every $i \in I_1$, since $U_i(R) = U_i(R^0) = r_i^0 < \lambda^0 \leq \lambda$, we have $r_i = r_i^0$. Furthermore, for every $i \in I_2$, since $U_i(R) = U_i(R^0) = \lambda^0 \leq \lambda$, we have $\lambda^0 \leq r_i$. □

Lemma 6. *For each $R \in N(U, R^0)$, if $U(R) = U(R^0) \neq r^0$, $\Omega \leq \sum_{i \in I} r_i$, and*

$$r_i = r_i^0 \quad \forall i \in I_1,$$
$$\lambda^0 \leq r_i \quad \forall i \in I_2,$$

then there exist no $T \subseteq I$ and $R'_T \in D^T$ such that

$$U(R'_T, R_{I \setminus T}) \; \textit{wdom}[R_T^0] \; U(R).$$

Proof. Suppose, by contradiction, that there exist $T \subseteq I$ and $R'_T \in D^T$ such that

$$U(R'_T, R_{I \setminus T}) \; \mathrm{wdom}[R_T^0] \; U(R).$$

Let $y \equiv U(R'_T, R_{I \setminus T})$ and $j \in T$ be such that $y_j \; P_j^0 \; x_j$. Then $j \in I_2$ and $x_j < y_j$.

We first consider the case $\sum_{i \in T} r'_i + \sum_{i \in I \setminus T} r_i \geq \Omega$. Let $\mu \in [\frac{\Omega}{n}, \Omega]$ be such that $y = ((\min\{r'_i, \mu\})_{i \in T}, (\min\{r_i, \mu\})_{i \in I \setminus T})$. Note that $x_j < y_j$ implies $\lambda^0 < \mu$. Then for each $i \in I \setminus T$, $x_i \leq y_i$. Thus there exists $k \in T$ such that $y_k < x_k$, a contradiction.

We next consider the case $\sum_{i \in T} r'_i + \sum_{i \in I \setminus T} r_i < \Omega$. Let $\mu \in [0, \frac{\Omega}{n}]$ be such that $y = ((\max\{r'_i, \mu\})_{i \in T}, (\max\{r_i, \mu\})_{i \in I \setminus T})$. For each $i \in I \setminus T$, since $y_i = \max\{r_i, \mu\}$ and $x_i = r_i$, we have $x_i \leq y_i$. Hence there exists $k \in T$ such that $y_k < x_k$, a contradiction. □

Lemma 7. *For each $R \in N(U, R^0)$, if $U(R) = U(R^0) \neq r^0$ and $\Omega \leq \sum_{i \in I} r_i$, then there exist no $T \subseteq I$ and $R'_T \in D^T$ such that*

$$U(R'_T, R_{I \setminus T}) \; \textit{wdom}[R_T^0] \; U(R).$$

Proof. Follows from Lemmas 5 and 6. □

Proof of Theorem 4. Let $R \in N(U, R^0)$ be such that $U(R) = U(R^0)$. If $U(R) = r^0$, then obviously no coalition can be weakly made better off by any deviation from R. Hence consider the case $U(R^0) \neq r^0$. By Lemma 4, $\Omega \leq \sum_{i \in I} r_i$. Thus by Lemma 7, $R \in SN(U, R^0)$. □

6.6 Proof of Theorem 5

Again, we fix the true preference profile R^0 throughout the proof of Theorem 5. We only deal with the case $\Omega \leq \sum_{i \in I} r_i^0$. The opposite case can be handled in a similar way.

Lemma 8. *For each $R \in N(U, R^0)$ such that $U(R) \neq U(R^0)$, if $U(R) = (\frac{\Omega}{n}, \ldots, \frac{\Omega}{n})$, then the conclusion of Theorem 5 holds.*

Proof. Let $R \in N(U, R^0)$ be such that $U(R) \neq U(R^0)$. Assume that $x \equiv U(R) = (\frac{\Omega}{n}, \ldots, \frac{\Omega}{n})$.

Case 1: $\boldsymbol{\Omega < \sum_{i \in I} r_i}$. Let $\lambda > \frac{\Omega}{n}$ be such that $x = (\min\{\lambda, r_i\})_{i \in I}$. Since $R \in N(U, R^0)$, it is easy to see that for each $i \in I$, $x_i \leq r_i^0$. Then one can easily verify that

$$x_i = \min\{\lambda, r_i^0\} \ \ \forall i \in I.$$

Hence $x = U(R^0)$, which contradicts $U(R) \neq U(R^0)$. Thus this case does not occur.

Case 2: $\boldsymbol{\sum_{i \in I} r_i < \Omega}$. Since $R \in N(U, R^0)$, it is easy to see that for each $i \in I$, $r_i^0 \leq x_i$. This is possible only if for each $i \in I$, $r_i^0 = x_i$. Hence $x = U(R^0)$, which contradicts $U(R) \neq U(R^0)$. Thus this case does not occur, too.

Case 3: $\boldsymbol{\Omega = \sum_{i \in I} r_i}$. By Cases 1 and 2, only this case can occur. Let $\lambda \geq \frac{\Omega}{n}$ be such that $U(R^0) = (\min\{r_i^0, \lambda\})_{i \in I}$. Since $U(R)$ is the equal division but not the uniform allocation, $\lambda > \frac{\Omega}{n}$ and $U(R^0)$ Pareto dominates $U(R)$ at R^0.

Let $T \equiv \{i \in I : r_i \neq r_i^0\}$ be the set of agents who are not reporting their true peaks. Because $U(R)$ is the equal division by assumption, we have that,

$$r_i^0 = \frac{\Omega}{n} \ \ \forall i \in I \setminus T, \tag{10}$$

$$r_i^0 \neq \frac{\Omega}{n} \ \ \forall i \in T. \tag{11}$$

(10) and (11) imply that, when coalition T deviates from R via R_T^0, then the true uniform allocation $U(R^0)$ is obtained. Hence

$$U_i(R) = \frac{\Omega}{n} < \lambda = U_i(R^0) \leq r_i^0 \ \ \forall i \in T \text{ with } \lambda < r_i^0,$$

$$U_i(R^0) = r_i^0 \ \ \forall i \in T \text{ with } r_i^0 \leq \lambda.$$

Therefore $U(R_T^0, R_{I \setminus T})$ sdom$[R_T^0]$ $U(R)$. Robustness of this deviation is ensured by Lemma 5. Thus we verified all the properties of Theorem 5. □

Lemma 9. *For each $R \in N(U, R^0)$, if $\Omega < \sum_{i \in I} r_i$ and $U(R) \neq (\frac{\Omega}{n}, \ldots, \frac{\Omega}{n})$, then $U(R) = U(R^0)$.*

Proof. Let $R \in N(U, R^0)$ be such that $\Omega < \sum_{i \in I} r_i$ and $x \equiv U(R) \neq (\frac{\Omega}{n}, \ldots, \frac{\Omega}{n})$.

Then there exists a number $\lambda \in (\frac{\Omega}{n}, \Omega)$ such that

$$x_i = \min\{r_i, \lambda\} \quad \forall i \in I.$$

Note that

$$\begin{aligned} I_1 &\equiv \{i \in I : x_i = r_i < \lambda\} \neq \emptyset, \\ I_2 &\equiv \{i \in I : x_i = \lambda \leq r_i\} \neq \emptyset, \\ I_1 \cap I_2 &= \emptyset \text{ and } I_1 \cup I_2 = I. \end{aligned}$$

Every $i \in I$ with $x_i > 0$ could get less by reporting a preference whose peak is less than $\min\{r_i, \lambda\}$. Therefore,

$$x_i \leq r_i^0 \quad \forall i \in I \text{ with } 0 < x_i. \tag{12}$$

In particular,

$$x_i \leq r_i^0 \quad \forall i \in I_2. \tag{13}$$

Every $i \in I_1$ with $x_i = 0$ could get more by reporting a preference whose peak is positive. Therefore,

$$x_i = r_i^0 \quad \forall i \in I_1 \text{ with } x_i = 0. \tag{14}$$

Furthermore, every $i \in I_1$ with $x_i > 0$ could get more by reporting a preference whose peak is more than r_i. This and (12) together imply that,

$$x_i = r_i^0 \quad \forall i \in I_1 \text{ with } x_i > 0. \tag{15}$$

By (13), (14), and (15),

$$x_i = \min\{r_i^0, \lambda\} \quad \forall i \in I.$$

Thus $x = U(R^0)$. □

Lemma 10. *For each $R \in N(U, R^0)$ such that $U(R) \neq U(R^0)$, if $U(R) \neq (\frac{\Omega}{n}, \ldots, \frac{\Omega}{n})$, then the conclusion of Theorem 5 holds.*

Proof. Let $R \in N(U, R^0)$ be such that $U(R) \neq U(R^0)$. Assume that $U(R) \neq (\frac{\Omega}{n}, \ldots, \frac{\Omega}{n})$. By $U(R) \neq U(R^0)$ and Lemma 4, $\Omega \leq \sum_{i \in I} r_i$. Therefore by $U(R) \neq (\frac{\Omega}{n}, \ldots, \frac{\Omega}{n})$ and Lemma 9, $\sum_{i \in I} r_i = \Omega$.

Without loss of generality, assume that

$$x_1 \leq x_2 \leq \cdots \leq x_n.$$

Since $\Omega = \sum_{i \in I} r_i$,

$$x_i = r_i \quad \forall i \in I. \tag{16}$$

Let $I_1 \equiv \{i \in I : x_i = x_1\}$ and $I_n \equiv \{i \in I : x_i = x_n\}$. Since $U(R) \neq (\frac{\Omega}{n}, \ldots, \frac{\Omega}{n})$, we have $x_1 < x_n$, and so $I_1 \cap I_n = \emptyset$.

Step 1: For each $i \in I$, if $x_1 < x_i$, then $r_i = x_i \leq r_i^0$. Then by definition of the uniform rule and $\Omega = \sum_{i \in I} r_i$, i could get less by reporting a preference whose peak is less than x_i. Hence $r_i = x_i \leq r_i^0$.

Step 2: For each $i \in I$, if $x_i < x_n$, then $r_i^0 \leq x_i = r_i$. Then by definition of the uniform rule and $\Omega = \sum_{i \in I} r_i$, i could get more by reporting a preference whose peak is more than x_i. Hence $r_i^0 \leq x_i = r_i$.

Step 3: For each $i \in I$, if $x_1 < x_i < x_n$, then $r_i = r_i^0 = x_i$. This is implied by Steps 1 and 2.

Step 4: Finding a credible deviation. By Steps 1 and 2,

$$\max_{i \in I_1} r_i^0 \leq x_1 \leq r_2^0 = x_2 \leq r_3^0 = x_3 \leq \cdots \leq r_{n-1}^0 = x_{n-1} \leq x_n \leq \min_{i \in I_n} r_i^0. \tag{17}$$

If for each $i \in I_1$, $r_i^0 = x_i$, then $x = (\min\{r_i^0, x_n\})_{i \in I} = U(R^0)$, a contradiction. Hence $I_1' \equiv \{i \in I_1 : r_i^0 < x_i\} \neq \emptyset$. This fact, the assumption $\Omega \leq \sum_{i \in I} r_i^0$, and (17) in turn imply that $I_n' \equiv \{i \in I_n : x_i < r_i^0\} \neq \emptyset$.

Let $T \equiv \{i \in I : r_i \neq r_i^0\}$ be the set of agents who are not reporting their true peaks. By the above discussion, $T = I_1' \cup I_n'$. We consider the coalitional deviation of T at R via R_T^0, which realizes the true uniform allocation $U(R_T^0, R_{I \setminus T}) = U(R^0)$.

We shall show that

$$U(R^0) \text{ sdom}[R_T^0] \, U(R).$$

Let $y \equiv U(R^0)$ and $\eta > \frac{\Omega}{n}$ be such that

$$y_i = \min\{r_i^0, \eta\} \quad \forall i \in I.$$

For each $i \in I'_1$, since $r_i^0 \leq \frac{\Omega}{n}$, $y_i = r_i^0 < x_i$. Thus

$$U(R^0) \text{ sdom}[R^0_{I'_1}]\, U(R). \tag{18}$$

We observed that, after the deviation by T, the allotment of every $i \in I'_1$ decreased from x_i to r_i^0. This fact, (17), and $\sum_{i\in I} y_i = \Omega$ together imply that $x_n < \eta$. Also, $\Omega \leq \sum_{i\in I} r_i^0$ implies that $\eta \leq \max_{i\in I} r_i^0$. Overall, $x_n < \eta \leq \max_{i\in I} r_i^0$. Therefore,

$$x_i < y_i = r_i^0 \quad \forall i \in I'_n \text{ with } r_i^0 \leq \eta,$$
$$x_i < \eta = y_i < r_i^0 \quad \forall i \in I'_n \text{ with } \eta < r_i^0.$$

Thus

$$U(R^0) \text{ sdom}[R^0_{I'_n}]\, U(R). \tag{19}$$

By (18, 19),

$$U(R^0) \text{ sdom}[R^0_T]\, U(R).$$

Robustness of this deviation is ensured by Lemma 5. Thus we verified all the properties of Theorem 5. □

Proof of Theorem 5. Follows from Lemmas 8 and 10. □

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