Time-varying Consumption Tax, Productive Government Spending, and Aggregate Instability.

Literature

- Schmitt-Grohe and Uribe (JPE 1997): Ramsey model with endogenous labor income tax + balanced budget (fiscal) policy rule. Also (Constant) Government spending is pure waste. Endogenous labor supply and additively-separable preferences. Local indeterminacy
- Giannitsaru (EJ 2007): Endogenous (time-varying) consumption tax never leads to local indeterminacy.
- Nourry et al. (JET 2011): Endogenous (time-varying) consumption tax may imply multiple steady states and local indeterminacy under GHH utility function. Result is robust to government spending in the utility function.

Aggregate instability means possibility of sunspot equilibria. Woodford (1984): local indeterminacy implies sunspot equilibria.

Motivation

No contribution on endogenous time-varying consumption tax financing productive government spending.

No contribution on time-varying endogenous taxes in an endogenous growth model a la Barro (JEP 1990).

Aim of the paper is to fill this gap.

Model Setup

The representative household solves the following problem taking as given the time-varying paths of $\mathcal{G}$ and τ:

$$
\begin{aligned}
\max & \int_0^\infty \frac{c^{1-\sigma}-1}{1-\sigma} e^{-\rho t} dt \\
s.t. \;\; & \dot{k} = Ak^\alpha \mathcal{G}^{1-\alpha} - \delta k - (1+\tau)c \\
& k \geq 0, \;\; c \geq 0 \\
& k(0) = k_0 > 0 \;\; given
\end{aligned}
$$

where

$$
\Theta \stackrel{def}{=} \{(\alpha, \rho, \delta, \sigma, A) : \; \alpha \in (0,1), \; \rho > 0, \; \delta > 0, \; \sigma > 0 \; and \; A > 0\}.
$$

Given an initial capital stock k_0 and the path $(\tau(t), \mathcal{G}(t))_{t\geq 0}$, the representative household maximizes his/her utility by choosing any path $(c(t), k(t))_{t\geq 0}$ which solves the system of ODEs

$$\begin{aligned} \dot{k} &= Ak^{\alpha}\mathcal{G}^{1-\alpha} - \delta k - (1+\tau)c && (1) \\ \frac{\dot{c}}{c} &= \frac{1}{\sigma}\left[\alpha A \left(\frac{k}{\mathcal{G}}\right)^{\alpha-1} - \delta - \rho - \frac{\dot{\tau}}{1+\tau}\right] && (2) \end{aligned}$$

respects the inequality constraints and the transversality condition

$$\lim_{t\to+\infty} \frac{k}{c^{\sigma}(1+\tau)} e^{-\rho t} = 0 \qquad (3)$$

The government balances its budget in every period:

$$\mathcal{G} = \tau(\tilde{c}) \cdot c \tag{4}$$

where $\tilde{c} = ce^{-\gamma t}$, with γ the asymptotic (endogenous) growth rate and $\tau(.)$ continuous and differentiable.

- Barro: $\tau = \tau_0$ for all t.
- Giannitsaru: $\mathcal{G}$ constant and therefore τ time-varying.
- Nourry et al.: $\tau = \tau(c)$ since no growth.

with positive growth we need to “de-trend” consumption to avoid an explosive (counterfactual) consumption tax.

We also assume the following:

Assumption

The elasticity of τ with respect to $\tilde{c}$ is constant:

$$\phi \stackrel{def}{=} \frac{d\tau}{d\tilde{c}} \cdot \frac{\tilde{c}}{\tau} = \text{constant} \tag{5}$$

Therefore by differentiating $\tau(\tilde{c})$ w.r.t. t

$$\frac{\dot{\tau}}{\tau} = \phi \left(\frac{\dot{c}}{c} - \gamma \right) \tag{6}$$

Observe that integrating (6) leads to a menu of fiscal policies:

$$\tau(t) = B\left(c(t)e^{-\gamma t}\right)^{\phi} \tag{7}$$

with B a generic (and not exogenously given) constant.

To avoid to introduce a trivial form of indeterminacy we need to select just one of them. To do so τ_0 has to be exogenously given.

Intertemporal equilibrium

Definition

Given $k_0 > 0$ and $\tau_0 > 0$, an intertemporal equilibrium is any path $(c(t), k(t), \tau(t), \mathcal{G}(t))_{t \geq 0}$ which satisfies the system of equations (1), (2), (4) and (6), respects the inequality constraints and the transversality condition (3).

In particular we may define the control-like variable $x \overset{def}{=} \frac{c}{k}$ and observe that the intertemporal equilibrium can be derived studying the following system of nonlinear ODEs:

$$
\begin{aligned}
\frac{\dot{x}}{x} &= \frac{[(1+\tau)(1-\sigma)-\phi\tau]\left[\alpha A(x\tau)^{1-\alpha}-\delta-\rho-\sigma\gamma\right]}{\sigma(1+\tau)+\phi\tau} \\
&+ \gamma(1-\sigma)+(1+\tau)x-(1-\alpha)A(x\tau)^{1-\alpha}-\rho \qquad (8) \\
\dot{\tau} &= \frac{\phi\tau(1+\tau)}{\sigma(1+\tau)+\phi\tau}\left[\alpha A(x\tau)^{1-\alpha}-\delta-\rho-\sigma\gamma\right] \qquad (9)
\end{aligned}
$$

Balanced Growth Paths

A balanced growth path (BGP) is an intertemporal equilibrium where consumption, and capital are purely exponential functions from t greater than zero. Formally:

$$k(t) = k_0 e^{\gamma t} \qquad \textit{and} \qquad c(t) = c_0 e^{\gamma t} \qquad \forall t \geq 0. \tag{10}$$

From equation (6) it follows immediately that along a BGP

$$\tau(t) = \hat{\tau} = \tau_0 \qquad \forall t \geq 0$$

with the hat symbol indicating, from now on, the value of a variable on a BGP. Also purely exponential government spending.

Proposition (Existence and Uniqueness of a BGP)

Given any initial condition of capital $k_0 > 0$ and the tax rate $\tau_0 > 0$, there exist $\underline{A} > 0$, $\underline{\tau} > 0$ and $\bar{\tau}(\sigma) \in (0, +\infty]$ with $\bar{\tau}(\sigma) > \underline{\tau}$ such that when $A > \underline{A}$ and one of the following conditions hold
i) $\sigma \geq 1$ *and* $\tau_0 > \underline{\tau}$,
ii) $\sigma \in (0, 1)$ *and* $\tau_0 \in (\underline{\tau}, \bar{\tau}(\sigma))$,
there is a unique balanced growth path where the consumption over capital ratio is constant and equal to $\hat{x}$ *and the growth rate of the economy is*

$$\hat{\gamma} = \alpha A(\hat{x}\hat{\tau})^{1-\alpha} - \delta - \rho > 0, \tag{11}$$

with $\hat{\tau} = \tau_0$.

Remark

$(\hat{x}, \hat{\tau}) = \left(\frac{\hat{c}_0}{k}_0, \tau_0\right)$ *is a steady state of the system (8), (9).*

Local determinacy of the steady state $(\hat{x}, \hat{\tau})$

Proposition

Consider the steady state $(\hat{x}, \hat{\tau})$ *(with* $\hat{\tau} = \tau_0$*) which characterizes the unique BGP of our economy. For any given initial conditions* (k_0, τ_0)*, there is no transitional dynamics, i.e. there exists a unique* $c_0 = k_0 \hat{x}$ *such that the economy directly jumps on the BGP from the initial date* $t = 0$*.*

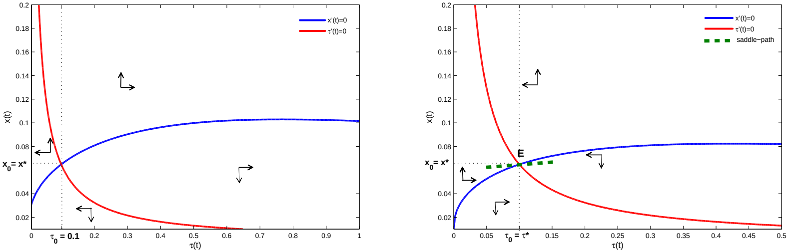

Figure: Phase Diagrams when $\phi = 0.5$ (left) and $\phi = -0.01$ (right) and $\gamma = \hat{\gamma}^0$

Existence of other equilibria

Differently from Barro, some transitional dynamics may occur in our model. Indeed, the BGP as defined by $(\hat{x}, \hat{\tau})$ and $\hat{\gamma}$ is not the unique possible equilibrium of our economy.

Look for the existence of equilibrium path $(x_t, \tau_t)_{t\geq 0}$ which may eventually converge to an asymptotic BGP.

Definition

An ABGP is any path $(x(t), \tau(t))_{t \geq 0} = (x^*, \tau^*)$ such that:

a) τ^* is a positive arbitrary constant sufficiently close to (but different from) τ_0;

b) (x^*, τ^*) is a steady state of (8)-(9) with $x^* > 0$ and $\gamma^* > 0$ solution of

$$0 = (\alpha - 1)A(x\tau^*)^{1-\alpha} + (1 + \tau^*)x - \rho \quad (12)$$

$$\gamma = \alpha A(x\tau^*)^{1-\alpha} - \delta - \rho. \quad (13)$$

c) (x^*, τ^*) satisfies the transversality condition.

Remark

An ABGP is not an equilibrium since it does not satisfy the initial condition $\tau(0) = \tau_0$.

The existence of an equilibrium path converging to an ABGP is associated to the existence of consumers' beliefs that are different from those associated to the BGP.

Indeed, they may believe that the consumption tax profile will not remain constant but rather change over time and eventually converge to a positive value $\tau^* \neq \tau_0$ where the consumption over capital ratio and the growth rate are x^* and γ^* respectively.

Based on that we will prove that under some conditions the consumers may indeed decide a consumption path which makes this belief self-fulfilled.

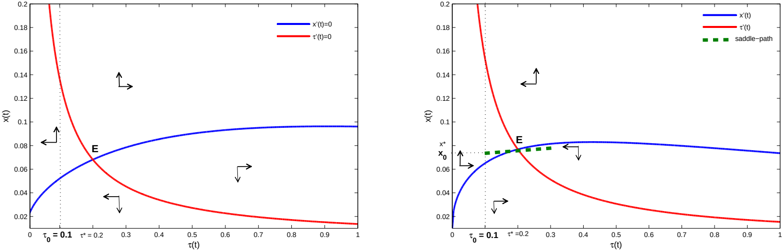

Figure: Phase Diagrams when $\phi = 0.5$ (left) and $\phi = -0.01$ (right) and $\gamma = \hat{\gamma}$[1]

Proposition

Given any initial condition $k_0 > 0$ *and* $\tau_0 > 0$, *consider* $\underline{\tau}$ *and* $\bar{\tau}(\sigma)$ *as defined by Proposition 1. There exist* $\underline{A} > 0$ *such that when* $A > \underline{A}$, *there is a unique equilibrium path* $(x_t, \tau_t)_{t\geq 0}$ *converging over time to the ABGP* (x^*, τ^*) *if and only if* $\phi \in (-\sigma(1+\tau^*)/\tau^*, 0)$ *and one of the following conditions hold:*
i) $\sigma \geq 1$ *and* $\tau^* > \underline{\tau}$,
ii) $\sigma \in (0, 1)$ *and* $\tau^* \in (\underline{\tau}, \bar{\tau}(\sigma))$.

The last proposition actually proves that there exists a continuum of equilibria each of them converging to a different ABGP.

Any value of τ^* in a neighborhood of the given initial value τ_0 is a belief that the consumers' may self-fulfilled if the conditions of the Proposition are met.

Of course this implies a form of global indeterminacy since from a given τ_0, one can select the unique BGP by jumping on it from the initial date or any other equilibrium converging to an ABGP.

Theorem

Given the initial conditions k_0 and τ_0 let $\tau_{inf} = \tau_0 - \epsilon > 0$ and $\tau_{sup} = \tau_0 + \varepsilon$ with $\varepsilon > 0$ small enough. Assume that one of the following conditions holds

i) $\sigma < 1$ *and* $A \in (\underline{A}_2(\tau_{inf}), \bar{A}(\tau_{sup}))$*;*

ii) $1 < \sigma < 1 + \frac{\rho}{\delta}$ *and* $A > \underline{A}_2(\tau_{inf})$*;*

iii) $\sigma > 1 + \frac{\rho}{\delta}$ *and* $A > \max\{\underline{A}_1(\tau_{inf}), \underline{A}_2(\tau_{inf})\}$*;*

Then if $\phi \in (-\frac{\sigma(1+\tau_{sup})}{\tau_{sup}}, 0)$, there is a continuum of equilibrium paths indexed by the letter j, departing from (τ_0, x_0^j), each of them converging to a different ABGP (τ^{*j}, x^{*j}) with $\tau^{*j} \in (\tau_{inf}, \tau_{sup})$, i.e. the dynamics of the economy is globally, but not locally, indeterminate. On the other hand, the dynamics of the economy is globally and locally determinate if $\phi \in (0, +\infty)$.

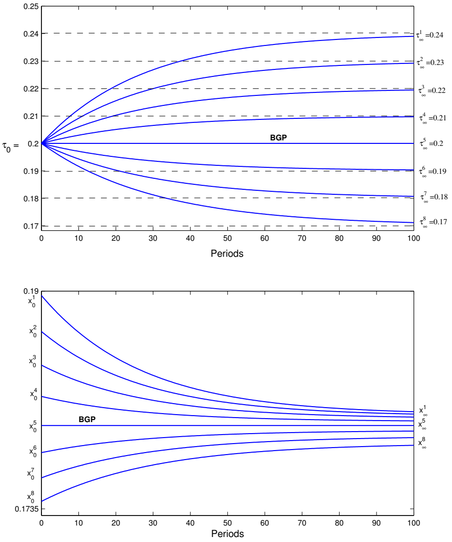

Figure: Dynamics of the Tax Rate and of the Consumption-capital Ratio

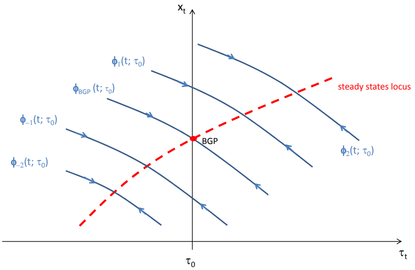

Figure: Examples of Saddle-Paths

Sunspot Equilibria

Sunspot equilibria, and therefore aggregate instability, may emerge in our framework once extrinsic uncertainty is introduced.

We do that introducing extrinsic uncertainty in the discrete time counterpart of the dynamical system $F(x, \tau)$.

An illustrative example

- Discrete time with period length h;
- Dynamics is then described by

$$\begin{pmatrix} \mathbb{E}_t(\Delta x_{t+h}) \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h \qquad \textit{with } (x_0, \tau_0) \textit{ given}$$

  where $\Delta \chi_{t+h} \equiv \chi_{t+h} - \chi_t$ with $\chi = x, \tau$. Equivalently

$$\begin{pmatrix} \Delta x_{t+h} \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h + s \begin{pmatrix} \Delta \varepsilon_{t+h} \\ 0 \end{pmatrix} \qquad \textit{with } (x_0, \tau_0) \textit{ given} \tag{14}$$

  where $\mathbb{E}_t(\Delta \varepsilon_{t+h}) = 0$ with ε_t the sunspot variable.
- ε_t takes the values $(0, z_1, z_2)$ at dates $(0, t_1 + h, t_2 + h)$ respectively – (deterministic) sunspots

Dynamics in $t \in [0, t_1 - h]$ will be described by the initial value problem (from now on IVP)

$$\begin{pmatrix} \Delta x_{t+h} \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h \qquad \textit{with } (x_0, \tau_0) \textit{ given}$$

Unique equilibrium path

$$\{x_t, \tau_t\}_{t=0}^{t_1} = \{\phi_{1x}(t), \phi_{1\tau}(t)\}_{t=0}^{t_1}.$$

Then at date $t = t_1$ due to the arrival of the second sunspot at date $t = t_1 + h$ we have that

$$\begin{pmatrix} \Delta x_{t_1+h} \\ \Delta \tau_{t_1+h} \end{pmatrix} = F(x_{t_1}, \tau_{t_1})h + s \begin{pmatrix} z_1 - 0 \\ 0 \end{pmatrix}$$

which clearly implies that

$$\begin{pmatrix} x_{t_1+h} \\ \tau_{t_1+h} \end{pmatrix} = \begin{pmatrix} \phi_{1x}(t_1) \\ \phi_{1\tau}(t_1) \end{pmatrix} + F(\phi_{1x}(t_1), \phi_{1\tau}(t_1))h + s \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \qquad (15)$$

Dynamics in $t \in [t_1 + h, t_2 - h]$ will be given by the IVP

$$\begin{pmatrix} \Delta x_{t+h} \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h \qquad \textit{with } (x_{t_1+h}, \tau_{t_1+h}) \textit{ given by } (15). \tag{16}$$

Unique equilibrium path

$$\{x_t, \tau_t\}_{t=t_1+h}^{t_2} = \{\phi_{2x}(t), \phi_{2\tau}(t)\}_{t=t_1+h}^{t_2}.$$

The presence of the sunspot variable has just modified the deterministic framework by allowing a "jump" at date $t_1 + h$ of size sz_1 in the no-predetermined variable while the dynamics of the economy is still described by $F(.)$ since the uncertainty is extrinsic and does not affect the fundamentals.

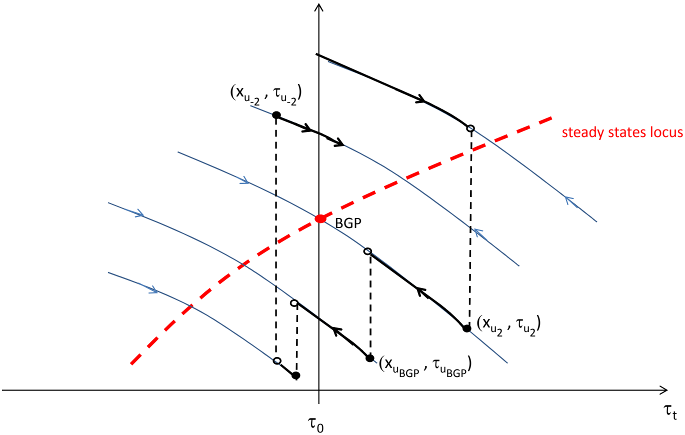

Figure: Example of a Sunspot Equilibrium

A similar argument can be applied for the arrival of the third sunspot.

A sunspot equilibrium can be obtained through a randomization over the deterministic equilibrium paths.

Continuous time case can be naturally derived by considering the limit $h \to 0$.

In the paper, we prove the existence of sunspot equilibria by explicitly describing the stochastic process (continuous time Markov chain) governing the sunspot variable.