Common knowledge in gaussian environments

Jayasri Dutta [1] Herakles Polemarchakis [2]

[1]University of Birmingham

[2]University of Warwick

The Problem

- We study the transmission of knowledge across players when all variables are jointly normally distributed.
- So far, most work on Common Knowledge (**(CK)** has been restricted to discrete and finite state spaces ([1], [4]) that facilitate analytics.
- But, it makes for difficulties in the application of the theory to market situations, where sources of information, as well as outcomes (such as prices) are neither discrete nor bounded.
- So, we follow [2] in setting up a model where all variables are jointly normally distributed.
- We have done this before in [3].

The Model

- There are two individuals, **A** and **B**, who both want to predict a variable y.
- They observe variables

$$X = (x_1, x_2, \cdots x_{K_A}) \quad \text{and} \quad Z = (z_1, z_2, \cdots z_{K_B}).$$

- Their objective is to predict another random variable, y, that is one-dimensional.

- **Gaussian:** (y, X, Z) is jointly normally distributed, each centered, with expectation 0, and variances and covariances

$$V(y) = \sigma_y^2; \quad V(X) = \Sigma_{xx}; \quad V(Z) = \Sigma_{zz};$$
$$cov(y, X) = \sigma_{yx}; \quad cov(y, Z) = \sigma_{yz}, \quad cov(X, Z) = \Sigma_{xz}.$$

- The variance-covariance matrix of (y, X, Z) is of dimension $1 + K_A + K_B$.
- The matrices $V(X), V(Z)$ are nonsingular.
- The realizations of the random variables X and of Z are **private information**.
- However both individuals are aware of the parameters of $\Sigma_{y,X,Z}$.

Predictions and Learning

- Individuals begin by making their own predictions; they use expectations of y conditional on their own information sets

$$\mathcal{I}_{\mathbf{A0}} = X; \quad \mathcal{I}_{\mathbf{B0}} = Z.$$

- So, the first step yields

$$\hat{y}_{A0} = \mathbf{E}y|\mathcal{I}_{A0} = X\beta; \quad \hat{y}_{B0} = \mathbf{E}y|\mathcal{I}_{B0} = Z\delta;$$

here,

$$\beta = \Sigma_{xx}^{-1}\sigma_{yx}'; \quad \delta = \Sigma_{zz}^{-1}\sigma_{yz}'.$$

- In the next step, individuals update their information as

$$\mathcal{I}_{A1} = \mathcal{I}_{A0} | \hat{y}_{B0}; \quad \mathcal{I}_{B1} = \mathcal{I}_{B0} | \hat{y}_{A0},$$

  and announce

$$\hat{y}_{A1} = \mathbf{E} y | \mathcal{I}_{A1}, \quad \hat{y}_{B1} = \mathbf{E} y | \mathcal{I}_{B1}.$$

- ....
- And this goes on until

$$\mathbf{CK(n)} : \hat{y}_{An} = \hat{y}_{Bn}$$

- In step n, the predictions coincide and we have **Common Knowledge.**

- Common Knowledge typically occurs in finitely many steps: in fact, in $\min\{K_A, K_B\} + 1$ steps if the first deduction is made by the individual with higher of K_A and K_B, followed by the other individual.
- In this paper/presentation, we want to consider shorter paths to CK, and so concentrate on models and examples displaying **CK(1)**.
- We distinguish between two types of phenomena:
- **(P)** for parametric ..... where **CK(1)** occurs for all sample paths X, Z because it is a property of the parameters;
- **(S)** for sample-dependent ... where **CK(1)** occurs only for designated samples (X^*, Z^*) and may disappear for ϵ-perturbations.

Parametric CK(1)

We first look at parametric properties that restrict $\Sigma_{xx}, \Sigma_{zz}, \Sigma_{xz}....$ to ensure that common knowledge occurs in one step: that is, the model displays **CK(1)** for *all* sample paths $X.Z$.

- **Few Variables** Suppose $K_A = 1$. Then

$$\hat{y}_{A0} = x\beta \quad \hat{y}_{B0} = Z'\delta.$$

We have, in consequence, that **B** can figure out the value of x from **A**'s prediction because she is assumed to know the value of the parameter $\beta = \frac{\sigma_{xy}}{\sigma_{xx}}$, and so ...

$$\hat{y}_{B1} = \mathbf{E}y|(x, Z),$$

and **A** simply waits until round 2 after which

$$\hat{y}_{A2} = \hat{y}_{B1} = \mathbf{E}]y|(x, Z).$$

Common Knowledge is achieved in two steps – at most.

This is *a bit easy* so to rule it out we assume that

$$k = \min\{K_A, K_B\} \geq 2.$$

NB All examples assume $K_A = K_B = 2$

Theorem 1.

CK(1) occurs parametrically if and only if

$$\mathbf{E}y|(X, Z) = \alpha_A \mathbf{E}y|X + \alpha_B \mathbf{E}y|Z, \quad \mathbf{CK(1)}$$

for some parameters α_A, α_B.

Necessity is obvious — if not true then **CK** cannot occur in one step because the predictions of both individuals will differ from the desired one, $\mathbf{E}y|(X, Z)$, and usually from one another's. It is sufficient as long as the parameters α_A, α_B are known.

- **Linear Dependence** First suppose that X and Z are fully correlated. eg $z_1 = x_1 + x_2$ and $z_2 = x_1 - x_2$. Then their predictions will be the same:

$$\mathbf{E}y|X = \mathbf{E}y|Z = \hat{y},$$

and, in consequence,

$$\mathbf{E}y|X, Z = \mathbf{E}y|X$$

or $C(1)$ holds with $\alpha_A = 1; \alpha_B = 0$ (or indeed the other way round).

- **Conditional Independence** Suppose A is better informed than B: in particular, has access to better quality of observation X.
  A is Goldman Sachs and **B** is a poor investor JD who tries to estimate **A**'s information:

$$z_i = x_i + \epsilon_i \quad i = 1, \cdots k.$$

  In this case, conditionally on X, y and Z are independent:

$$\mathbf{E}y|(X, Z) = \mathbf{E}y|X = \hat{y}_{0A}$$

  A ignores **B**'s announcement and **B** simply adopts **A**'s prediction.
  CK(1) holds with $\alpha_A = 1, \alpha_B = 0$

- **Uncorrelated observations** [2] Suppose

$$\Sigma_{XX} = I, \Sigma_{ZZ} = I \Sigma_{XZ} = 0 :$$

all the x and z variables are independent and identically distributed; further

$$\sigma_{xy} = [1, \cdots, 1]; \sigma_{zy} = [1, \cdots, 1].$$

Then

$$\mathbf{E}y|X = x_1 + x_2 + \cdots + x_{k_A}; \quad \mathbf{E}y|Z = z_1 + z_2 + \cdots + z_{k_B};$$

and further

$$\mathbf{E}y|(X, Z) = \sum_i x_i + \sum_j z_j = \mathbf{E}y|X + \mathbf{E}y|Z.$$

The fact that this leads to **CK(1)** is evident.

- **Independence** In the Bacharach example it is only the independence conditions that matter rather than the identical distribution: it holds for $\Sigma_{XX} = DIAG[\cdots, \sigma^2_{x_i}, \cdots]$, $\Sigma_{ZZ} = DIAG[\cdots, \sigma^2_{z_j}, \cdots]$ and $\Sigma_{XZ} = 0$. This leads to

$$\mathbf{E}y|(X, Z) = X\beta + Z\delta$$

where $\beta = \Sigma^{-1}_{XX}\sigma_{xy}$ $\delta = \Sigma^{-1}_{ZZ}\sigma_{zy}$. It follows that

$$\mathbf{E}y|(X, Z) = \mathbf{E}y|X + \mathbf{E}y|Z.$$

This implies **CK(1)**.

- **The Generalized Bacharach Condition** In fact we do not need the X variables to be independent of one another but only of the Z variables ... similarly the Z variables. So let us assume that $cov(X, Z) = \Sigma_{XZ} = 0$. This leads to

$$\mathbf{E}y|X, Z = X\beta + Z\delta = \mathbf{E}y|X + \mathbf{E}y|Z :$$

condition **(C1)** holds with $\alpha_A = \alpha_B = 1$

- We have seen that complete dependence, conditional independence, and complete independence lead to condition **(C1)** holding and then to $\alpha_A = 1; \alpha_B = 0$ or to $\alpha_A = \alpha_B = 1$. Is the property $\alpha_i \in \{0, 1\}$ universal?
- (Unfortunately perhaps) The answer is NO, as our next example demonstrates!

An Example of Parametric CK(1)

Let us suppose that

$$y = y_1 + y_2,$$

where $\mathrm{cov}(y_1, y_2) = 0$, $\mathbf{E}y_i = 0$; and suppose individuals **A** & **B** make observations of y_1 and y_2 with error. Thus

$$x_1 = y_1 + u_1; \quad x_2 = y_2 + u_2,$$

and

$$z_1 = y_1 + \epsilon_1; \quad z_2 = y_2 + \epsilon_2.$$

To keep things simple, assume that $(u_1, u_2, \epsilon_1, \epsilon_2)$ are uncorrelated with each other; and that $V(x_1) = V(x_2) = V(z_1) = V(z_2) = 1$, while $cov(x_1, z_1) = V(y_1) = \rho$ and $cov(x_2, z_2) = V(y_2) = \rho$. It is possible to show that

$$\mathbf{E}y|(X, Z) = \frac{1}{1+\rho}(\mathbf{E}y|X + \mathbf{E}y|Z):$$

we have a failure of the block-independence condition, but **CK(1)**!

Note that as ρ approaches zero, the condition above approaches the **CK(1)** condition. As it approaches one, and the two predictions come close to coinciding, the weights approach $\frac{1}{2}$

Sample Paths with $\mathbf{CK}_s(1)$

We start with the simplest possible example.

- Suppose $k_A = k_B = 2$ so individual **A** uses her observation on x_1, x_2 and individual **B** uses his observation of z_1, z_2. Suppose now they observe

$$X^* = [x_1^* = 0 x_2^* = 0], \quad \text{and} \quad Z^* = [z_1^* = 0 z_2^* = 0].$$

- Then their first round of announcements are

$$\hat{y}_{A0} = 0; \quad \hat{y}_{B0} = 0.$$

- It follows that

$$\hat{y}_{A1} = \hat{y}_{B1} = 0$$

- Hence, we have $\mathbf{CK}_s\mathbf{(1)}$.
- As it happens, for the sample $(0,0)(0,0)$ we have common knowledge in the first period and this is the optimal outcome because we have $\mathbf{E}y|(X^*, Z^*) = 0$.
- It is worth noting that the result holds irrespective of the parametric conditions.
- **But** it is truly sample dependent ..... in the sense that for any perturbation of X^*, Z^*

We saw that $\mathbf{CK}_S\mathbf{(1)}$ can yield efficient outcomes. But this is not necessarily true!

- Now suppose $\beta_1 = \beta_2 = 1$ and $\delta_1 = \delta_2 = 1$ for simplicity and

$$X^* = [10, -10]; \quad Z^* = [-5, 5].$$

- Once again, we have

$$\hat{y}_{A0} = 0; \quad \hat{y}_{B0} = 0.$$

- And exactly as in the previous example, we obtain

$$\hat{y}_{A1} = \hat{y}_{B1} = 0.$$

- So we have $\mathbf{CK}_S\mathbf{(1)}$.......
- But, this is inefficient because

$$\mathbf{E}y|X, Z \neq 0,$$

unless the parametric condition for $\mathbf{CK}_P\mathbf{(1)}$ holds.

- This example demonstrates that players may agree on something other than the truth!

Characterizing CK_s(n)

Suppose

$$\hat{y}_{A0} = X^*\beta; \hat{y}_{B0} = Z^*_\delta.$$

We have $\mathbf{CK}_s\mathbf{(1)}$ whenever $\hat{y}_{A0} = \hat{y}_{B0}$ because predictions agree: in other words whenever $X^*\beta = Z^*_\delta$.
Suppose $x_2^* = z_1^* = 0$. Then this amounts to saying that

$$\frac{x_1^*}{z_2^*} = s_1 \equiv \frac{\delta_2}{\beta_1} \qquad \mathbf{CK_s(1)}.$$

It is possible to characterize the "timing" of CK by a sequence $\{s_n\}$ such that

$$\frac{x_1^*}{z_2^*} = s_n \Leftrightarrow \mathbf{CK_s(n)}.$$

Conclusions

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