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Extreme Idealism and Equilibrium in the Hotelling-Downs Model of Political Competition

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Abstract

In the classic Hotelling-Downs model of political competition there is (almost always) no pure strategy equilibrium with three or more potential strategic candidates where the distribution of voters' preferred policies are single-peaked. I study the effect of introducing two *idealist* candidates who are non-strategic (i.e., fixed to their policy platform), to an unlimited number of potential strategic entrants. I present results that hold for a non-degenerate class of cases: (i) For any equilibrium, it must be that the left-most and right-most candidates (i.e., extremists) are idealists; (ii) Hotelling's Law fails: in any equilibrium, candidates do not share their policy platforms, which instead are spread out across the policy space; (iii) Characterizations for symmetric and asymmetric single-peaked distributions of voters' ideal policy preferences. Equilibria where many strategic candidates enter exist only if the distribution of voter preferences is asymmetric. (*JEL: C72; D72*)

Keywords: Hotelling; political competition; equilibrium existence; idealism

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1 Introduction

The Hotelling-Downs model of political competition is the workhorse of political scientists and political economists. It is well known that the unique equilibrium when there are $N = 2$ strategic candidates is that they both locate at the median voter's ideal policy. However, in his seminal work Osborne (1993) shows the negative result that this model, adapted for endogenous entry with $N \geq 3$ strategic candidates who maximize their plurality, fails to admit an equilibrium in pure strategies for all but pathological cases of single-peaked distributions of voter ideal points.

Equilibrium existence fails generically when $N \geq 3$ because there must be two candidates located at the left-most and right-most policy platforms, which requires the mass of voters to the left and right of these positions to be equal: an onerous requirement of the distribution of voter ideal points. These conditions are only met by a degenerate class of distributions including the uniform, which significantly weakened the results of previous studies that employed such distributions e.g., Cox (1987, Theorem 2). In this article I consider a slight variant of the classical model, show when this can restore the existence of pure strategy equilibria and offer a characterization thereof.[1]

I suppose that in addition to strategic candidates, there are two *idealist* candidates, who are fixed to their policy platforms. My first result establishes that for equilibria to generically exist (within the class of single-peaked distributions), these idealists must be extremists i.e., occupy left-most and right-most positions. This appeals to the notion that in reality, candidates with the most extreme positions may often be those unwilling to compromise on their positions.

The classic lesson from Hotelling's result under $N = 2$ is that candidates' incentives to maximize their vote share will lead them converge on the same (median) platform in equilibrium, a prediction referred to as Hotelling's Law or the principle of minimum differentiation. An initial prediction for the nature of equilibrium under larger N was made by Chamberlain (1933, Appendix C) who conjectured (in relation to firms and candidates) that "they may group in twos". The second result I present says that in almost any equilibrium, it must be that there

[1] I study pure equilibria in this article and hereon refer to these simply as 'equilibria'.

is exactly *one* candidate at every policy platform. Combined with the fact that strategic candidates who are willing to enter tie, this implies that their positions are spaced evenly throughout the distribution of voter preferences. This *maximal* differentiation of candidate positions in equilibrium shows that in this setup, Hotelling's Law fails.

Other researchers have also proposed variations of the canonical model in which a pure strategy equilibrium obtains. Osborne (1993) defines a dynamic version of the model, and offers results for $N = 3$ (and partial results for $N = 4, 5$) showing, among other findings, that there is always an equilibrium in which $N - 2$ candidates enter and locate at the median. Xefteris (2016) shows that when one allows for each voter to cast $k \geq 2$ votes each instead of just $k = 1$, then equilibrium exists for a non-degenerate class of distributions where there are at least $k + 1$ candidates at every location. In contrast, I offer results in a plurality voting system (common to many countries e.g., United States, Canada, India and United Kingdom) where the number of potential strategic candidates, who may choose whether to enter in equilibrium, is unlimited i.e., $N = \infty$.

I characterize the equilibria for a non-degenerate set of single-peaked distributions of voter ideal points. For symmetric distributions, there is a unique equilibrium in which one strategic candidate enters and wins the election outright when the idealists are not too extreme or too moderate, relative to the distribution of voter preferences. I then show that equilibria with multiple strategic entrants exist only if the distribution of voter ideal points is asymmetric, but that the converse is not true. I then provide a characterization for equilibria under asymmetric single-peaked distributions. I also give examples of these equilibria for various symmetric and asymmetric distributions.

2 Model

The model setup is the same as that of the canonical Hotelling-Downs model, but allows for an endogenous entry, an unlimited number of candidates, and a reasonable objective function for candidates. The policy space is represented by some interval $X \subseteq \mathbb{R}$. The ideal policies of voters are spread out along X by an atomless distribution function F which is assumed continuous, guaranteeing it has a density, f. Voters are assumed to be sincere and to have symmetric preferences around their ideal points, meaning that they vote for the candidate positioned closest to that point. There is an unlimited number of *strategic* candidates (i.e., $N = \infty$) and two *idealist* candidates. Idealist candidates always enter and occupy positions denoted $z_1, z_2 \in X$ where $z_1 < z_2$.[2] Strategic candidates each have the action set $X \cup \{\text{out}\}$ i.e., they either enter and choose a policy platform denoted x_i, or they choose not to enter the race. The number of strategic candidates choosing to enter the race is denoted n and the vector of positions x. Candidates who do not enter are *inactive*. The utility of a candidate who stays out is normalized to zero. The functions $v_i : X^n \to [0, 1]$ denote the share of votes obtained by each candidate i given a vector of positions x.

Strategic candidates maximize their plurality i.e., their margin of victory. Their preferences are represented by the following utility function:

$$u_i(x) = v_i(x) - \max_{l \neq i}\{v_l(x)\}$$

An oft-used objective function for candidates is that of vote maximization. However, vote maximization is not a reasonable objective function for candidates when $N > 2$, as it is incompatible with preferences in which winning an election is preferred to losing it (Osborne, 1995, p.280). To illustrate, consider the following example: $X = [0, 1]$, f uniform and position vector $x_A = (0, 0.5, 0.8)$ which gives $v_1(x_A) = 0.25, v_2(x_A) = 0.4, v_3(x_A) = 0.35$ and a victory for candidate 2. Now consider $x_B = (0, 0.2, 0.8)$ i.e., candidate 2 moves left, which gives $v_1(x_B) = 0.1, v_2(x_B) = 0.4, v_3(x_B) = 0.5$ and a victory for candidate 3. Under vote-maximization, candidate 2 should be indifferent between x_A and x_B yet wins the election under x_A and loses under x_B. Plurality maximization does not suffer this criticism, saying that candidates prefer to: (1) stay out rather than enter and lose; (2) win (or tie for the win) than to lose; and (3) win outright by wider margins.

There are $r + 1 \leq n + 2$ occupied positions denoted $y_0, \ldots, y_r$ indexed without loss of generality such that $y_0 < \cdots < y_r$. The midpoints of two adjacent locations is denoted $m_j = \frac{1}{2}(y_j + y_{j+1})$. The number of candidates located at y_j is denoted k_j. The *constituency* of a position y_j is the share of voters that vote for one of the candidates at y_j. The *left (right)*

[2]Although z_1 and z_2 refer to locations, sometimes I also call the idealists z_1 and z_2.

constituency of y_j denotes the mass of voters voting for a candidate at y_j who have ideal points to the left (right) of y_j, denoted L_j, R_j i.e., $L_j = F(y_j) - F(m_{j-1})$ and $R_j = F(m_j) - F(y_j)$ for $j = 0, \ldots, r$ but with $F(m_{-1}) \equiv 0$ and $F(m_r) \equiv 1$.

3 Results

I first present some necessary conditions for an equilibrium to exist generically within the class of single-peaked densities f. These include the results that idealists must be the extreme candidates and that Hotelling's Law fails. Proofs and intermediate Lemmas are found in the Appendix. I then add sufficient conditions in order to characterize equilibria for symmetric and asymmetric single-peaked distributions.

Proposition 1 (Extreme idealism). *For almost any single-peaked f: $y_0 = z_1$, $y_r = z_2$ and $k_1 = k_r = 1$ in equilibrium.*

Proposition 1 reveals that the left-most and right-most (extreme) positions must be occupied by idealists for an equilibrium to exist for almost any single-peaked f. With only strategic candidates, it must be that $k_0 = k_r = 2$ and therefore that $L_0 = R_0$ and $L_r = R_r$ (Lemma A1, b and c) which are so restrictive that they preclude equilibrium in all but a degenerate class of distributions (Osborne, 1993). In contrast, when extreme positions are occupied by candidates who are void of strategic concerns these requirements do not arise. This gives rise to Proposition 1: in any equilibrium, extremists must be idealists.

Proposition 2 (Hotelling's Law fails). *For almost any single-peaked f, $k_j = 1$ for all j when $n \geq 2$.*

Proposition 1 dealt with the extreme locations. Proposition 2 deals with the intermediate positions and shows that these also cannot hold two strategic candidates in equilibrium, except for very special cases of F. Due to the endogenous entry decision all strategic candidates who enter, tie in equilibrium (Lemma A1, d). Together with Proposition 2 this implies that a necessary condition of equilibrium is that the strategic candidates are spaced evenly throughout the distribution of voter preferences. However, Hotelling's Law stipulates that candidates are incentivized to converge upon shared locations. The separation of candidates' equilibrium positions here shows this law can fail, irrespective of the number of strategic entrants.

The results of Propositions 1 and 2 lay the ground work for the equilibrium characterizations. Firstly, Proposition 3 provides the conditions for which an equilibrium exists for a non-degenerate class of symmetric single-peaked distributions of voter preferences.

Proposition 3 (Symmetric distributions). *For almost any symmetric, single-peaked f, there is a unique equilibrium where $n = 1$ strategic candidate enters at location y_1, where y_1 solves (1):*

$$F(m_0) = 1 - F(m_1) \tag{1}$$

whenever the positions of the idealists (z_1, z_2) satisfy (2) and (3):

(2) *Not too moderate:* $m_0 < F^{-1}\left(\frac{1}{3}\right) \iff m_1 > F^{-1}\left(\frac{2}{3}\right)$

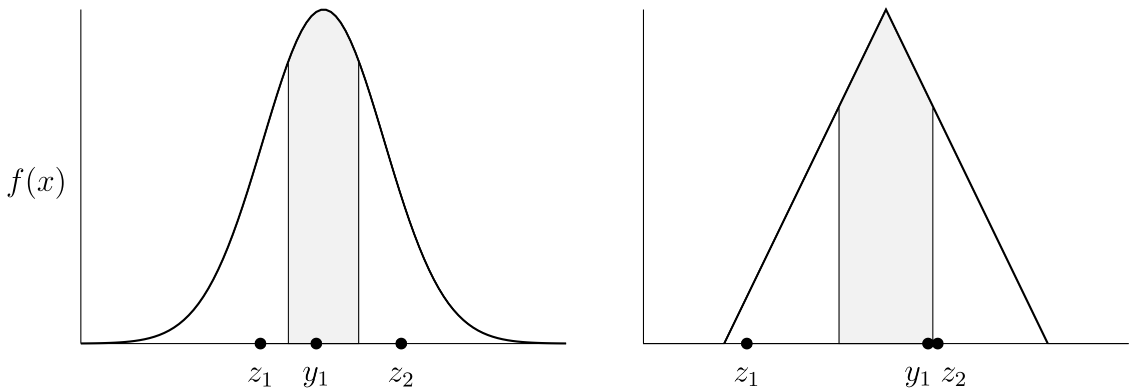
(3) *Not too extreme: If z_1 is closer to the peak of f than z_2, $F(y_1) \geq 1 - 2F(m_0)$.*
If z_2 is closer to the peak of f than z_1, $F(y_1) \leq 2F(m_0)$.

Except for single-peakedness, the conditions of Proposition 3 deliver equilibrium existence without other restrictions on the shape of f. Condition (1) is implied by the requirement that the idealists' vote-shares must be equal in equilibrium (if not, then due to symmetry the strategic candidate could profitably deviate by moving slightly towards the idealist with the higher vote share). Conditions (2)-(3) state that relative to the distribution of voter preferences, the idealists cannot be too moderate or too extreme. They cannot be too moderate because a strategic candidate must win. They cannot be too extreme else there is room for an entrant to deviate in and win. I now illustrate the characterization with two examples.

Example 1: Let F be the standard Normal Distribution and the idealists be located at the 15th and 90th percentiles: $(z_1, z_2) = (F^{-1}(0.15), F^{-1}(0.90)) = (-1.04, 1.28)$. Condition (1) then gives $y_1 = -0.12$. The remaining conditions are also satisfied: (2) becomes $m_0 = -0.58 < -0.43 = F^{-1}\left(\frac{1}{3}\right)$ and the first statement of (3) becomes $F(y_1) = 0.45 \geq 0.44 = 1 - 2F(m_0)$. The left panel of Figure 1 shows this equilibrium.

Example 2: Let F be the triangular distribution with the density $f(x) = 1 - |x|$ for $x \in [-1, 1]$ and the idealists be located at the 1st and 77th percentiles: $(z_1, z_2) = (F^{-1}(0.01), F^{-1}(0.77)) = (-0.86, 0.32)$. Condition (1) then gives $y_1 = 0.26$. The remaining conditions are also satisfied: (2) becomes $m_0 = -0.30 < -0.18 = F^{-1}\left(\frac{1}{3}\right)$ and the second statement of (3) becomes $F(y_1) = 0.73 \geq 0.51 = 1 - 2F(m_0)$. The right panel of Figure 1 shows this equilibrium.

Figure 1: Equilibrium for symmetric single-peaked distributions



Left panel: Differentiable f with unbounded support (the standard Normal).
Right panel: Non-differentiable f with bounded support (a triangular distribution).
The shaded area is the constituency of the winning candidate.

A feature of Proposition 3 is that with symmetric single-peaked densities, only one strategic candidate enters in equilibrium. In Corollary 1, I show that this feature is not special to symmetry per se: it will hold generically in equilibrium for any single-peaked distribution where the mode (*Mo*) equals the median (*Md*).

Corollary 1. *For almost any single-peaked f where Mo$(f) =$ Md(f), $n = 1$.*

To understand the result, suppose instead that $n > 1$. I show in the Appendix that exactly one idealist loses (Lemma A4). Further, there cannot be more than one strategic candidate with any of their constitution on the same side of the mode as the losing idealist (else candidate closest to the losing idealist could profitably deviate by moving slightly towards the mode). With $n > 1$, there would then be at least one strategic candidate with their whole constituency on the same side of the mode as the idealist who ties for the win. However, for these candidates to win, there must be more than half the probability density on that side of the mode, contradicting $\text{Mo}(f) = \text{Md}(f)$.

I now characterize equilibria where $n > 1$ strategic candidates enter. By Corollary 1 we know that distributions of voter preferences that support such equilibria are such that $\text{Mo}(f) \neq$

Md(f), and hence are necessarily asymmetric. Furthermore, the simple fact of whether the median or the mode of f is greater will play a role in determining equilibria. Proposition 4 provides conditions for an equilibrium to exist for asymmetric single-peaked distributions of voter preferences where Mo$(f) \neq$ Md(f): Equilibria in which multiple strategic candidates enter exist for a non-degenerate set of distributions and idealist positions. Figure 2 gives two examples.

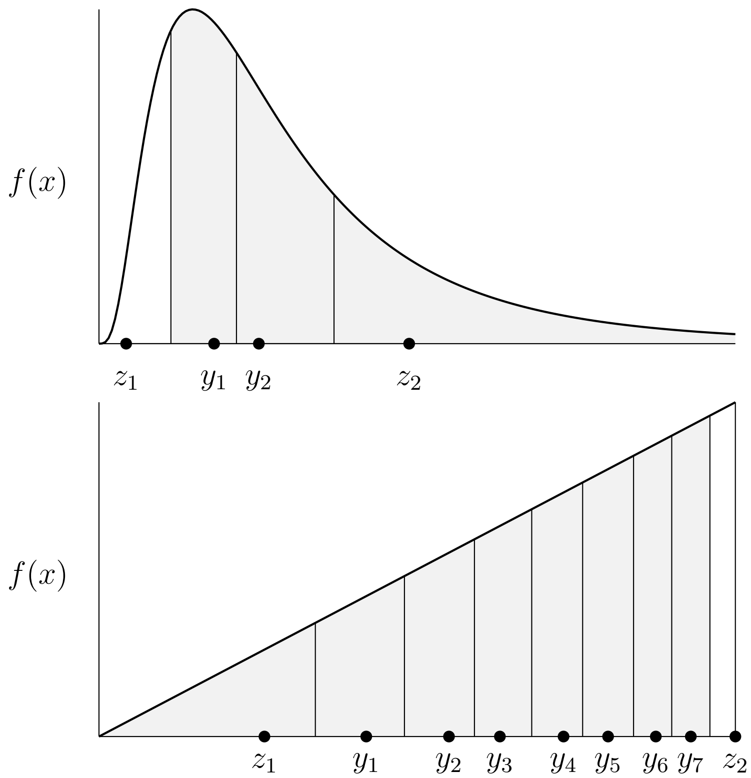
Proposition 4 (Asymmetric distributions). *For almost any asymmetric, single-peaked f satisfying (4) - (6) where Mo$(f) \neq$ Md(f), there is an equilibrium with $n > 1$ strategic candidates where locations and vote-shares are given by Lemma A7.*

	If Mo$(f) <$ Md(f)	*If Mo$(f) >$ Md(f)*
(4)	$f(m_0) \in [f(m_1), 2f(m_1)]$	$f(m_n) \in [f(m_{n-1}), 2f(m_{n-1})]$
(5)	$f(m_j) \leq 2f(m_{j+1}) \quad j = 1, \dots, n$	$f(m_{j-1}) \leq 2f(m_j) \quad j = 1, \dots, n$
(6)	$f(m_0) \leq \max\{f(y_1), f(z_1)\}$	$f(m_n) \leq \max\{f(y_n), f(z_2)\}$

Compared to the symmetric case, there are more equilibrium conditions when F is asymmetric and $n > 1$. Lemma A7 provides conditions (A6) and (A9) which are analogous to condition (2) of Proposition 3, saying that the losing idealist must be extreme enough to lose. The Lemma also provides the exact equilibrium location of strategic candidates (conditions A4, A5, A7, A8) which as Proposition 2 revealed, are spaced out evenly through the distribution of voter ideal points. Specifically, the locations of the idealist candidates pin down the vote share, s^*, enjoyed by each of the strategic candidates in equilibrium. The strategic candidates' locations are then determined by the following 'spacing procedure' (detailed in precisely in Lemma A5): To illustrate, suppose that z_1 ties for the win (which is the case if Mo$(f) <$ Md(f)); then place the first strategic candidate at y_1, such that z_1 has a vote share of s^*; then place the second strategic candidate at y_2, such that the candidate at y_1 has a vote share of s^*, and so on; the losing idealist, in this case z_2, will then be left with the residual vote share of $1 - s(n+1)$.

For asymmetric distributions, there are also conditions concerning the shape of f, given by (4)-(6). The requirements of (4) and (5) that $f(m_j) \leq 2f(m_{j+1})$ for $j = 0, \dots, n$ are driven by the fact that strategic candidates are plurality maximizers. To see this, consider the candidate at y_2 in the upper panel of Figure 2, and a deviation slightly to the left. This reduces the vote share of the candidate at y_1, but raises that of z_2. The marginal gain in plurality is $f(m_1)$, but the marginal loss is $2f(m_2)$: $f(m_2)$ for the loss in vote share and another $f(m_2)$ for the gain in vote share of z_2. Therefore, if $f(m_1) > 2f(m_2)$ there would be such a deviation. Also, (6) requires that the density of the midpoint between the losing extremist and their neighboring strategic candidate not be higher than the density of either of those candidate's locations. The condition precludes the possibility that there could be a profitable deviation for an inactive candidate to enter and locate in such a way that they win the election with the peak of f in their constituency. All conditions are met by the examples in Figure 2 which are therefore equilibria with asymmetric, single-peaked distributions of voter preferences.

Figure 2:
Equilibrium for asymmetric single-peaked distributions with $n > 1$ strategic candidates



Top panel: $n = 2$; Log-Normal distribution $\ln \mathcal{N}(0, 0.5)$; idealists at 1st and 83rd percentiles.
Bottom panel: $n = 7$; Linear distribution; idealists at the 7th and 100th percentiles.
The shaded areas are the constituencies of the winning candidates.

More generally, the results of this article do not change qualitatively if other extreme idealist candidates are added appropriately. For example, more idealists can be added at any positions to the left (right) of z_1 (z_2) when z_1 (z_2) loses.[3] Rather than having two idealists, one can contemplate potentially many fringe candidates on the extremes of the political spectrum.

I established a simple relationship between the mode and median of f as a determinant of the number of candidates entering in equilibrium. Such a measure cannot of course hope to capture every way in which distributions can be asymmetric, but nevertheless acts as a succinct predictive measure in plurality voting systems with idealist candidates. With multiple strategic candidates in such an election, it must be that the distribution of voter preferences is such that the mode and median are distinct. Equally, if the distribution of voter preferences is symmetric and there are fringe idealists on each extreme, one strategic candidate will run and win.

References

Chamberlain, Edward, 1933, The Theory of Monopolistic Competition (Harvard University Press, Cambridge, MA).

Cox, Gary W, 1987, Electoral equilibrium under alternative voting institutions, American Journal of Political Science 31(1), 82–108.

Downs, Anthony, 1957, An Economic Theory of Democracy (Harper & Row, New York).

Hotelling, Harold, 1929, Stability in competition, Economic Journal 3, 41–57.

Osborne, Martin J, 1993, Candidate positioning and entry in a political competition, Games and Economic Behavior 5(1), 133–151.

_ 1995, Spatial models of political competition under plurality rule: A survey of some explanations of the number of candidates and the positions they take, The Canadian Journal of Economics 28(2), 261–301.

Xefteris, Dimitrios, 2016, Stability in electoral competition: A case for multiple votes, Journal of Economic Theory, forthcoming.

[3]Idealists can also be added to the left (right) of z_1 (z_2) when z_1 (z_2) wins, but it requires a minor reworking to the expressions in the results presented here to reflect the fact the strategic candidate's equilibrium locations need to be shifted to ensure z_1 (z_2) still ties for the win.

Appendix

Lemma A1. *When at least one strategic type is at y_j:*

(a) $k_j \leq 2$.

(b) $k_j = 2$ *for* $j = 0, r$.

(c) *If* $k_j = 2$, $L_j = R_j$.

(d) *All strategic candidates who enter, tie and win.*

Proof: See Cox (1987) and Osborne (1993) Lemma 1 who prove these when all candidates are strategic. If none of the candidates at the positions in question are strategic, then the proofs do not apply, hence the qualified version of their Lemmas here. ■

Lemma A2. *For almost any distribution F, not all candidates tie.*

Proof: Suppose not. Firstly, consider the case where there are two candidates at an extreme location and without loss of generality, suppose this is on the left i.e., $k_0 = 2$. By Lemma A1 (c), $y_0 = F^{-1}\left(\frac{1}{n+2}\right)$ and $m_0 = F^{-1}\left(\frac{2}{n+2}\right)$. If $k_1 = 2$, then $y_1 = F^{-1}\left(\frac{3}{n+2}\right)$ which implies $F^{-1}\left(\frac{1}{n+2}\right) + F^{-1}\left(\frac{3}{n+2}\right) = 2F^{-1}\left(\frac{2}{n+2}\right)$, which is not satisfied for almost any distribution. Continuing similarly, one shows that generically, $k_j = 1$ for all $j > 1$ (see the proof of Lemma 2 in Osborne, 1993 which I have presented an adapted version of, up to this point). It must be therefore that $r = n$ and $y_r = z_2$. For all candidates to tie, $m_j = F^{-1}\left(\frac{j+2}{n+2}\right)$ for $j = 0, \ldots, n-1$. Solving recursively yields $y_0 = (-1)^n z_2 + 2\sum_{j=0}^{n-1}(-1)^j F^{-1}\left(\frac{j+2}{n+2}\right)$. However, we also required $y_0 = F^{-1}\left(\frac{1}{n+2}\right)$. These two expressions are not satisfied for almost any distribution.

Now consider the case where there is one candidate at each extreme location $k_0 = k_r = 1$, which by Lemma A1 implies $y_0 = z_1$ and $y_r = z_2$. For all to tie, $F(m_j) = F(m_{j-1}) + s_j$ for all $j = 0, \ldots, r-1$ where $s_j = \frac{k_j}{n+2}$ and $F(m_{-1}) \equiv 0$. Solving recursively yields $z_1 = (-1)^r z_2 + 2\sum_{j=0}^{r-1}(-1)^j F^{-1}(S_j)$, where $S_j = \sum_{i=1}^{j} s_i$ which is not true for almost any F. ■

Proposition 1 (Extreme idealism). *For almost any single-peaked f: $y_0 = z_1$, $y_r = z_2$ and $k_1 = k_r = 1$ in equilibrium.*

Proof: Suppose not. Either $k_0 = 2$ or $k_r = 2$ by Lemma A1 (b). Without loss of generality say $k_0 = 2$, which implies $L_0 = R_0$ by Lemma A1 (c). Denote the equilibrium vote share of the winning candidates by s.

If $n = 1$ this imposes $F(z_1) = F(\frac{1}{2}(z_1 + z_2)) - F(z_1)$, which is not true for almost any F. If $n = 2$, $s \geq \frac{1}{4}$. If $s = \frac{1}{4}$, all candidates tie, which is ruled out by Lemma A2. If $s > \frac{1}{4}$, then by Lemma A1 (d), z_2 is the sole loser. It must be that the strategic candidate is located at $y_1 < z_2$: if they were located at z_2, then they would tie with z_2; if they were located right of z_2, they could profitably deviate slightly to the left. If $f(m_0) > f(m_1)$, then the candidate at y_1 can profitably deviate by moving slightly to the left (they increase their share, and decrease the winning candidates' shares). If $f(m_0) \leq f(m_1)$, $R_0 < L_1$ because f is single-peaked. But $L_0 = R_0 = s$, hence the candidate at y_1 must get strictly more than s votes and wins outright, a contradiction.

For $n \geq 3$ strategic candidates, $y_0 = F^{-1}(s)$ and $m_0 = F^{-1}(2s)$. If there is a strategic candidate at y_1 and $k_1 = 2$, then $y_1 = F^{-1}(3s)$ which implies $\frac{1}{2}(F^{-1}(s) + F^{-1}(3s)) = F^{-1}(2s)$, which is true for almost no distribution F. Hence $k_1 = 1$ and $m_1 = F^{-1}(3s)$. Similarly, if there is a strategic candidate at y_2, $k_2 = 1$ for almost any F, and so on. Denote y_i as the left-most position after y_0 where there is an idealist. What I have shown so far is that for almost any F, $k_i = 1$. Notice that by Lemma A1 (d) and Lemma A2, z_2 must lose for almost all F. Now I consider two cases, both of which end in a contradiction.

(i) If there are no strategic candidates to the right of y_i, then for the single-peaked density f: if $f(m_{i-2}) \leq f(m_{i-1})$, then $L_1 > s$ because $R_0 = s$, which contradicts Lemma A1 (d); if $f(m_{i-2}) > f(m_{i-1})$, then the candidate at y_i has a profitable deviation slightly to the left (by increasing their own vote share and decreasing that of the winning candidates).

(ii) If there is a strategic candidate to the right of y_i, let the right-most such candidate be at y_j. If $f(m_{i-1}) \leq f(m_i)$, $L_1 > s$. If $f(m_{i-1}) > f(m_i)$, I consider two sub-cases: If $j = r$, then $k_r = 2$ and $R_r = L_r = s < R_{r-1}$. If $j < r$, then $j = r - 1$ and there is a lone idealist at y_r, in which case y_j can deviate profitably by moving slightly to the left (by increasing their own vote share and decreasing that of the winning candidates). ■

Lemma A3. *For almost any distribution F, $k_j = 1$ for all j when $n = 2$.*

Proof: Suppose not. Then by Proposition 1 and Lemma A1 (c), $k_1 = 2$ and $L_1 = R_1$. If z_1 gets a strictly lower (higher) vote share than z_2, an entrant can locate slightly to the right (left) of the strategic candidates at y_1 and win outright. Thus all candidates tie, contradicting Lemma A2. ■

Lemma A4. *For almost any single-peaked f, exactly one idealist must tie with the strategic candidates when $n \geq 2$.*

Proof: First, I show that both idealists cannot lose. Suppose they do and consider first $n = 2$. By Lemma A3, $k_1 = k_2 = 1$. If $f(m_0) < f(m_1)$, the candidate at y_1 can move slightly to the right, increasing their vote-share and decreasing that of the other strategic candidate; if $f(m_0) \geq f(m_1)$ then because f is single-peaked, the maximizer of f must lie to the left of m_1, which implies $f(m_1) > f(m_2)$ and hence that the candidate at y_2 can profitably deviate by moving slightly to the left, increasing their vote-share and decreasing that of the other strategic candidate.

Now consider $n \geq 3$. Denote the equilibrium vote share of strategic candidates as s. If y_2 is to the left (weakly) of the maximizer of f, then $k_1 = 1$ because if $k_1 = 2$, $s = R_1 < L_2$, contradicting Lemma A1 (d). As $k_1 = 1$ and z_1 loses, the candidate at y_1 can profitably deviate slightly to the right. Now consider the case where y_2 is to the right of the maximizer. There can be no more strategic candidates to the right of y_2. If there were, then $k_j = 1$, $j > 2$ because if $k_j = 2$ for one such j, then $R_{j-1} > L_j = s$. Note now that the candidate at y_{r-1} has a profitable deviation to the left because z_2 loses. Next I show that it must be that $k_1 = k_2 = 2$ and hence that $n = 4$. If $k_2 = 1$ and $f(m_1) > f(m_2)$, the candidate at y_2 can profitably deviate to the left; if $k_2 = 1$ and $f(m_1) \leq f(m_2)$, $k_1 = 1$ (else $s = R_1 < L_2$) and the candidate at y_1 can profitably deviate right. Hence $k_2 = 2$. If $k_1 = 1$ and $f(m_0) < f(m_1)$, the candidate at y_1 can profitably deviate right; if $k_1 = 1$ and $f(m_0) \geq f(m_1)$ then $f(y_1) > f(m_1)$ implying $R_1 > L_2 = s$ as $k_2 = 2$. As $k_1 = k_2 = 2$, by Lemma A1 (c) and (d), $L_1 = R_1 = L_2 = R_2$. But with only two free variables (y_1 and y_2) these three conditions will not be satisfied for almost any F.

Hence, for almost all single-peaked distributions at least one idealist must tie, but by Lemma A2, exactly one idealist must tie. ■

Lemma A5. *For almost any single-peaked f, $k_j = 1$ for all j when $n \geq 3$.*

Proof: By Proposition 1, $y_0 = z_1$ and $y_r = z_2$ while by Lemma A1 (d) all strategic entrants tie for the win. This implies $F(z_1) < \frac{1}{n+2}$ and $F(z_2) > \frac{n+1}{n+2}$ in any equilibrium. By Lemma A4, exactly one idealist ties with the strategic types and without loss of generality, let this be z_1. Now consider the following *spacing procedure* which spaces candidate locations throughout the distribution F for some arbitrary number of candidates n, where $k_0 = k_r = 2$, $k_j = 1, 2$ for $j = 1, \ldots, r-1$ and strategic types tie with the idealist z_1.

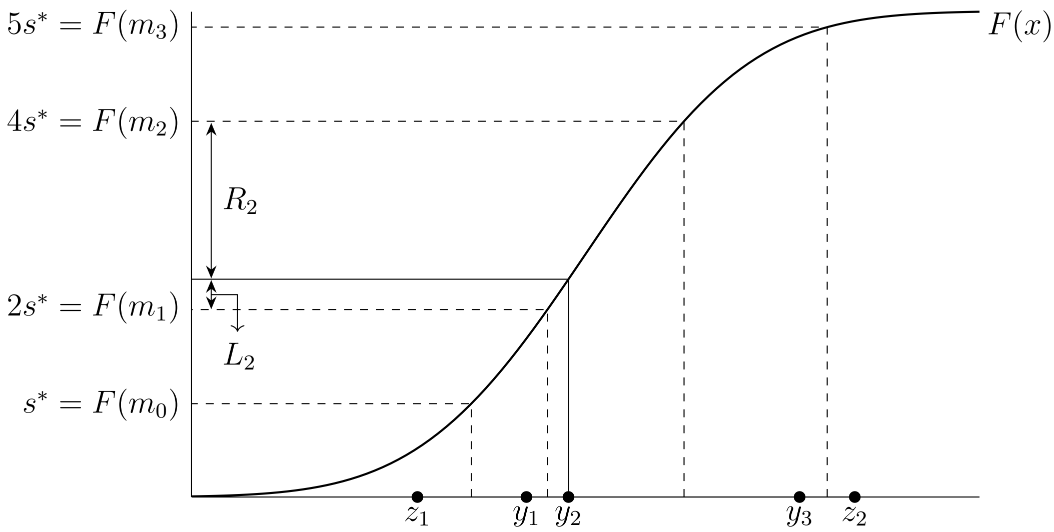
Spacing Procedure:

1. Choose y_1 such that $s \equiv F(m_0) \in \left(F(z_1), \frac{1}{n+2}\right)$.
2. Place the remaining $r-2$ candidate locations at y_j for $j = 2, ..., r-1$ in turn, such that $F(m_{j-1}) = F(m_{j-2}) + k_{j-1}s$.
3. Observe whether $\frac{1}{2}(y_{r-1} + z_2) = m_{r-1}$. If yes, stop and denote s as s^*; if $m_{r-1} < (>)$ $\frac{1}{2}(y_{r-1} + z_2)$ return to step 1 and choose a higher (lower) value of s.

Iterating on this procedure, the value of s will converge to s^*. As F is continuous, s^* exists, and as F is strictly increasing, s^* is unique. An example result of the procedure is illustrated below in Figure A1. The points $y_1, \ldots, y_{r-1}$ associated with s^* pin-down the necessary locations of the strategic candidates in equilibrium.[4]

[4]Notice that although s^* is necessarily the equilibrium share of the vote for the winning candidates, this procedure is not sufficient to define an equilibrium as for example, it may not be that $y_j > y_{j-1}$ for all $j = 1, ..., r-1$.

Figure A1: An example result of the spacing procedure



The example shown has $n = 4$ and $r = 4$ where $k_i = 1$ for all i except $k_2 = 2$. F is the standard Normal distribution and $z_1 = F^{-1}(0.10)$, $z_2 = F^{-1}(0.98)$. Solving the procedure yields $s^* = 0.19$ (2 d.p.) with candidate positions as shown.

It is now straightforward to see that for almost any distribution F, $k_i = 1$ for all i. Suppose instead that $k_j = 2$ for some $j = 2, ..., r$. By Lemma A1 (c) we must have that $L_j = R_j$. However, as is illustrated in Figure A1 for the example of $j = 2$, this extra condition will not be satisfied for all except very particular distributions. ■

Proposition 2 (Hotelling's Law fails). *For almost any single-peaked f, $k_j = 1$ for all j when $n \geq 2$.*

Proof: Immediate from Lemmas A3 and A5. ■

Lemma A6. *For any symmetric, single-peaked f, when there is $n = 1$ strategic entrant, the idealists' vote shares are equal.*

Proof: Suppose not. Without loss of generality, suppose that the idealist z_1 has a higher vote share than z_2 which implies that $f(m_0) > f(m_1)$. The strategic candidate at y_1 can move slightly to the left, simultaneously increasing their own vote share and reducing the vote share of z_2, giving strictly higher utility. ■

Proposition 3 (Symmetric distributions). *For almost any symmetric, single-peaked f, there is a unique equilibrium where $n = 1$ strategic candidate enters at location y_1, where y_1 solves (1):*

$$(1) \qquad F(m_0) = 1 - F(m_1)$$

whenever the positions of the idealists (z_1, z_2) satisfy (2) and (3):

(2) *Not too moderate:* $m_0 < F^{-1}\left(\frac{1}{3}\right) \iff m_1 > F^{-1}\left(\frac{2}{3}\right)$

(3) *Not too extreme: If z_1 is closer to the peak of f than z_2, $F(y_1) \geq 1 - 2F(m_0)$.*
If z_2 is closer to the peak of f than z_1, $F(y_1) \leq 2F(m_0)$.

Proof: Firstly I show that $n = 1$ in equilibrium. Suppose instead $n > 1$. By Lemmas 1, A3 and A5, for almost all single-peaked f, the strategic candidates occupy the non-extreme locations and $k_j = 1$ for all j. As f is symmetric, there must be at least one strategic candidate on either side of the maximizer of f, else Lemma A1 (d) is violated. I now show this implies that both idealists tie with the strategic candidates. Suppose not and without loss of generality that z_1 loses. As f is symmetric, this implies $f(m_0) < f(m_1)$ (if not, z_1 gets at least as many votes as the candidate at y_2). The candidate at y_1 then can profitably deviate slightly to the right. But by Lemma A2 for almost all distributions F, not all candidates can tie.

I now characterize the equilibrium. By Lemma A6, the idealists' vote shares must be equal, meaning that the strategic candidate's position y_1 must solve (1). To be an equilibrium, the strategic candidate must win, which implies $F(m_1) - F(m_0) > \frac{1}{3}$. Using (1), this becomes (2).

In equilibrium, the strategic candidate must not want to deviate to the left of z_1 or the right of z_2. Note that (2) implies that $z_1 < F^{-1}(\frac{1}{3})$ and $z_2 > F^{-1}(\frac{2}{3})$. As the strategic candidate gets at least $\frac{1}{3}$ of the vote share in order to win, there is no such profitable deviation. The strategic candidate would also lose if they deviated to an idealist's location as the other idealist would win outright. Finally, the strategic candidate does not have incentive to deviate to another location in (z_1, z_2): Without loss of generality, consider such a deviation to the left. By Lemma A6 this increases z_2's vote share (and z_2 now beats rather than ties with z_1). However, as f symmetric, this deviation also decreases the strategic candidate's vote share and hence also their plurality.

In equilibrium, inactive strategic candidates must not wish to enter. Notice that an inactive candidate could only profitably locate in (z_1, z_2). Assume first that z_1 is closer to the maximizer of f than z_2, so that y_1 is to the left of the maximizer. Notice that the payoff of the entrant is increasing as their location approaches y_1 from the right. Hence, entry is not profitable if the right constituency of y_1 is less than the vote share of the idealists $F(m_1) - F(y_1) \leq F(m_0)$ which gives (3). Similarly, the case of z_2 being closer to the peak gives the other expression in (3). ■

Corollary 1. *For almost any single-peaked f where Mo$(f) =$ Md(f), $n = 1$.*

Proof: Suppose instead $n > 1$. By Lemma A4 exactly one idealist loses and without loss of generality assume this is z_2. This implies that $f(m_{r-2}) \leq f(m_{r-1})$ else the candidate at y_{r-1} deviates left. This implies that m_{r-2} is strictly to the left of the maximizer of f. For the candidate at y_{r-2} and z_1 to tie (along with any number of others on the left of the maximizer), there must be strictly more than half the density to the left of the maximizer, contradicting Mo$(f) =$ Md(f). ■

Lemma A7. *For almost any single-peaked f, when $n \geq 2$, strategic candidates and one idealist tie for the win with vote share s^*, where:*

If Mo$(f) <$ Md(f), then s^* solves (A4), locations are given by (A5) and the left extremist loses (A6);

(A4) $$z_1 = (-1)^{n+1} z_2 - 2 \sum_{i=1}^{n+1} (-1)^{n+i} F^{-1}(1 - is)$$

(A5) $$y_j = (-1)^{n+1-j} z_2 - 2 \sum_{i=1}^{n+1-j} (-1)^{n-j+i} F^{-1}(1 - is^*), \quad \textit{s.t. } z_1 < y_j < y_{j+1}, \quad j = 1, ..., n$$

(A6) $$z_1 < 2F^{-1}(s^*) - y_1.$$

If Mo$(f) >$ Md(f), s^* solves (A7), locations are given by (A8) and the right extremist loses (A9);

(A7) $$z_1 = (-1)^{n+1} z_2 + 2 \sum_{i=1}^{n+1} (-1)^{n+i} F^{-1}(is)$$

(A8) $$y_j = (-1)^{j} z_1 + 2 \sum_{i=1}^{j} (-1)^{j+i} F^{-1}(is^*), \quad \textit{s.t. } z_1 < y_j < y_{j+1}, \quad j = 1, ..., n$$

(A9) $$z_2 > 2F^{-1}(1 - s^*) - y_n.$$

Proof: I first show that if Mo$(f) <$ Md(f) and $n > 1$, z_1 loses: If not, by Lemma A4 z_2 loses and one can then then follow the proof of Corollary 1 to show that there must be strictly more than half the density to the left of the maximizer, contradicting $\text{Mo}(f) < \text{Md}(f)$. Given z_1 loses, z_2 must tie with the strategic candidates by Lemma A4 and $k_j = 1$ for all j by Lemmas A3 and A5. This implies that $r = n + 1$ and that $F(m_j) = F(m_{j-1}) + s$ for $j = 1, \ldots, n + 1$ where s is the equilibrium vote share and $F(m_{n+1}) \equiv 1$. Solving recursively yields (A4) which the equilibrium s solves, giving equilibrium locations as (A5) where (A6) is the requirement for z_1 to lose: $F(m_0) < s^*$. Similarly, one finds (A7)-(A9) in the case of $\text{Mo}(f) > \text{Md}(f)$. ■

Proposition 4 (Asymmetric distributions). *For almost any asymmetric, single-peaked f satisfying (4) - (6) where Mo$(f) \neq$ Md(f), there is an equilibrium with $n > 1$ strategic candidates where locations and vote-shares are given by Lemma A7.*

	If Mo$(f) <$ Md(f)	*If Mo$(f) >$ Md(f)*
(4)	$f(m_0) \in [f(m_1), 2f(m_1)]$	$f(m_n) \in [f(m_{n-1}), 2f(m_{n-1})]$
(5)	$f(m_j) \leq 2f(m_{j+1}) \quad j = 1, \ldots, n$	$f(m_{j-1}) \leq 2f(m_j) \quad j = 1, \ldots, n$
(6)	$f(m_0) \leq \max\{f(y_1), f(z_1)\}$	$f(m_n) \leq \max\{f(y_n), f(z_2)\}$

Proof: I show that conditions (4) - (6) are sufficient for an equilibrium by considering all possible deviations in the case of $\text{Mo}(f) < \text{Md}(f)$; those for $\text{Mo}(f) > \text{Md}(f)$ follow similarly.

Consider deviations of the candidate at y_1 within (z_1, y_2) (the candidate at y_1 is the only strategic candidate who could have a constituency boundary to the left of the maximizer of f) (i) to the left: the candidate at y_2 then becomes the candidate with the highest vote-share of all other candidates, hence if $f(m_0) \leq 2f(m_1)$ there is no profitable deviation within (z_1, y_1); (ii) to the right: for a small move, z_1 remains a loser and the candidate at y_2 becomes a loser. It must be that $f(m_0) \geq f(m_1)$ else the candidate at y_1 could profit from such a move. This implies that any deviation within (y_1, y_2) reduces this candidate's vote share, hence there is no such profitable deviation. This gives (4).

Next consider deviations for the candidate at y_j, $j > 1$ within (y_{j-1}, y_{j+1}) (i) to the left: their vote share would increase, but so will that of the candidate at y_{j+1} who then becomes the candidate with the highest share of all the others, but the plurality of the deviating candidate decreases if $f(m_{j-1}) \leq 2f(m_j)$ which gives (5); (ii) to the right: their own vote share would decrease while increasing that of the candidate at y_{j-1}.

Next consider an inactive candidate entering (i) at an occupied location: this is not profitable as it results in an outright loss; (ii) left of z_1 or right of z_2: this results in an outright loss; (iii) between two strategic candidates y_j and y_{j+1}, $j > 1$: such an interval does not contain the maximizer of f, hence the optimal such deviation is as close as possible to the candidate whose position is has higher density, y_j. But this cannot be profitable because the maximum vote share is bounded from above by $\max\{L_j, R_j\} < s^*$; (iv) between z_1 and y_1, which contains the maximizer of f: under (6), the optimal such deviation is to locate arbitrarily close to z_1 or y_1 (whichever has the higher density), but as in case (iii) this is unprofitable because $\max\{R_0, L_1\} < s^*$.

Finally, for deviations of the candidate at y_j to locations outside the interval (y_{j-1}, y_{j+1}), $j = 1, \ldots, n$, it suffices to follow the steps above relating to an inactive candidate. ■