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Short-Term Momentum and Long-Term Reversal of Returns under Limited Enforceability and Belief Heterogeneity*

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Abstract

We evaluate the ability of the Lucas [27] tree and the Alvarez-Jermann [3] models, both with homogeneous as well as heterogeneous beliefs, to generate a time series of excess returns that displays both short-term momentum and long-term reversal at quarterly frequency. We calibrate the model to U.S. data as in Alvarez and Jermann [4]. We find that only the Alvarez-Jermann model with heterogeneous beliefs delivers autocorrelations that not only have the correct sign but are also of magnitude similar to the US data.

Keywords: Heterogeneous beliefs, Endogenously Incomplete Markets, Financial Markets Anomalies, Limited Enforceability, Constrained Pareto Optimality, Recursive Methods

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1 Introduction

Over the last several years, a large volume of empirical work has documented that excess returns in the stock market appear to exhibit *short-term momentum*, that is positive autocorrelation, in the short to medium run and *long-term reversal*, that is negative autocorrelation, in the long run (see Moskowitz et al [28], Poterba and Summers [32] and Lo and MacKinlay [26]).

In Table 1 we report the annual averages of the risk-free interest rate and equity-premium and also the empirical quarterly autocorrelations up to order 8 (two-years) for the US stock market using Shiller's dataset.

Table 1: (*Shiller's Quarterly Data 1871-2012*)

Risk-Free Rate	*Equity Premium*	*1st*	*2nd*	*3rd*	*4th*	*5th*	*6th*	*7th*	*8th*
2.36%	*5.91%*	*0.823*	*0.559*	*0.278*	*-0.009*	*-0.132*	*-0.177*	*-0.188*	*-0.142*

As can be seen in Figure 1, where we include the 95% confidence intervals, the first three autocorrelations are significantly positive while the fifth to eighth autocorrelations are significantly negative.

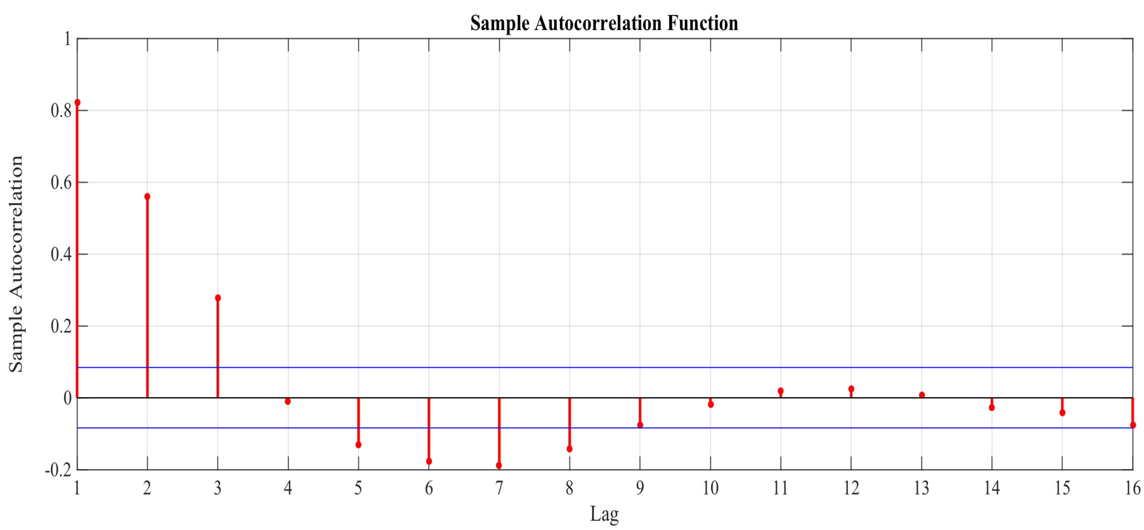

Figure 1: Autocorrelation Coefficients in Shiller's Quarterly Data $(1871 - 2012)$

There is a tendency to interpret these properties of excess returns as a rejection of standard models of asset pricing and so they are known as *financial markets anomalies.*[1] Many have argued that these anomalies are evidence that investors behave in an irrational way. Although this interpretation might be correct, it could instead be evidence of market frictions such as credit constraints.

In this paper we evaluate the ability of two standard general equilibrium asset pricing models to generate a time series of excess returns that display both short-term momentum and long-term

[1] Fama and French [19] suggest this interpretation as a logical possibility, while Poterba and Summers [32] argue that these properties of excess returns should be attributed to "price fads."

reversal. We consider both an economy without frictions, the Lucas [27] tree model adapted to allow for stochastic growth as in Mehra and Prescott [29], as well as an economy where credit frictions arise due to limited enforceability, the Alvarez-Jermann [3] model.

If one defines equity return as the sum of quarterly dividends plus capital gains, the autocorrelations of excess returns in these models are zero when expectations are computed using the so-called equivalent martingale measure or, as we call it, the market belief. Yet, as it has been noticed long time ago, the *empirical* excess returns could be autocorrelated.[2] This is because (i) the empirical autocorrelations converge to the autocorrelations computed with respect to the true probability measure and (ii) the market belief, typically, differs from the true probability measure.[3]

Loosely speaking, short-term momentum and long-term reversal occurs if the short-run equity premium conditional on an innovation has the sign of the innovation and the long-run equity premium conditional on an innovation has the oposite sign. In the language of Ottaviani and Sorensen [30], short-term momentum occurs if returns underreact in the short-run and overreact in the long run. They show that if agents have concordant heterogeneous beliefs and all signals are public, then underreaction occurs whenever the econometrician computes the autocorrelations using a neutral belief. They argue that in complete markets underreaction occurs because the market belief becomes more pessimistic than the neutral belief after a positive innovation and more optimistic than the neutral belief after a negative innovation.

We underscore that if one construct equity returns as they are computed in the data, the empirical autocorrelations are typically different from zero when computed with respect to the market belief. Indeed, the (annualised) quarterly equity returns are computed by Shiller as the return of buying equity in a given quarter and held it for one year. Therefore, some of the dividends that enter into the computation of one quarter equity return enter also in the succesive quarter equity return. We argue that this feature of the data is key as it changes the reasons why underreaction and overreaction could occur.

We consider a pure exchange economy where the state of nature follows a finite first-order time homogeneous Markov process. There is a finite number of infinitely-lived agents who are subjective utility maximisers and have arbitrary heterogeneous beliefs regarding the transition probability matrix. In particular, agents might have non-concordant beliefs.[4]

We calibrate the stochastic process of individual income and aggregate growth rates of a two-agent economy to aggregate and US household data as in Alvarez and Jermann [4] but we assume growth rates are serially uncorrelated. For each economy we analyse both the case of homogeneous and heterogeneous beliefs. We say that a model's predictions are *qualitatively* accurate if the sign of the predicted autocorrelations coincide with that of their empirical counterparts for some preferences parameters.

[2]See Leroy [25] or Lucas [27], for example.

[3]Note that (ii) is true even if some agents have correct beliefs because the market belief adjusts the true probability to take into account the effect of time and risk on the marginal valuation of future consumption.

[4]Our framework is general enough to accommodate bounded or unbounded aggregate growth and priors with and without the true transition matrix in its support. If some agent has correct beliefs but others do not, belief heterogeneity does not vanish.

We say that a model's predictions are *quantitatively* accurate if its predicted autocorrelations are both of the same sign and order of magnitude as in the data when one sets the discount rate and coefficient of risk aversion to match the average annual risk-free rate of 2.36% and equity premium of 5.91%. For each case, we first ask whether its predictions are qualitatively accurate. If the answer is positive, we study whether the model's predictions are quantitatively accurate as well. We first consider the Lucas tree model and its competitive equilibria (CE), i.e. full risk sharing equilibria. Under some mild assumptions, CE prices and excess returns converge to those of an economy where only agents with correct beliefs have positive wealth (see Sandroni [34] and Blume and Easley [10]). Thus, we restrict attention to the case where everybody has correct beliefs.[5] We find that the predictions of the Lucas tree model are quantitatively correct in the short-term as the model generates significant short-term momentum. However, the predictions are both quantitatively as well as qualitatively inaccurate as the Lucas tree model is unable to generate long-term reversals when the growth rates are uncorrelated.

Next we consider competitive equilibria with solvency constraints (CESC) that prevent agents to attain full risk sharing, an equilibrium concept very close to the one used in Alvarez and Jermann [3]. Following Alvarez and Jermann [3] and Kehoe and Levine [22], we say that an allocation is enforceable if agents would at no time be better off reverting permanently to autarky. We say that an allocation is constrained Pareto optimal (CPO) if it is optimal within the set of enforceable allocations. Our analysis of CESC allocations is based on a methodological contribution that complements the techniques developed by Spear and Srivastava [35] and Abreu et al [2]. Indeed, we first provide a complete recursive characterisation of the set of constrained Pareto optimal allocations and a version of the principle of optimality for these economies. We also show how to decentralise a CPO allocation as a CESC using a suitable adaptation of the methodology that Beker and Espino [5] develop to decentralise a Pareto optimal allocation of an economy with belief heterogeneity as a CE.

To assess the impact of belief heterogeneity on CESC, we assume agent 1 has correct beliefs and agent 2 has dogmatic beliefs that are (possibly) incorrect about the persistency of the expansion state and correct otherwise. The presence of solvency constraints ensures that the consumption of every agent is bounded away from zero, i.e. both agents survive.

The main lesson is that if one insists in that some agents must eventually have correct beliefs, then perpetual optimism, belief heterogeneity and limited enforceability are three ingredients that together give a qualitative explanation for short-term momentum and long-term reversal in a general equilibrium setting. However, we find that its predictions are never quantitatively accurate. Indeed, when on chooses the preferences paramters to match the risk-free rate and equity premium, the model generates short-term momentum but it fails to generate long-term reversal both for optimistic and pesimistic beliefs.

We are not the first to use incorrect beliefs to explain asset pricing puzzles in general equilibrium. However, most of the previous papers are representative agent models. Abel [1], assumes the belief of the representative agent is characterised by pessimism and doubt and he shows that these effects reduce the risk-free rate and increase the equity premium. Cogley and Sargent [14] focus on the quantitative

[5] Note that since we are interested in asymptotic results, this restriction is without loss in generality

effects of pessimism on the equity premium. However, neither of these authors tackle the effect of pessimism on the autocorrelations of excess returns. Cecchetti et al [13] explain several anomalies, including long-term reversal, but they are silent about short-term momentum. Our approach differs from theirs in one important aspect. In their model the stochastic discount factor has a non stationary behaviour because they assume not only that the representative agent is pessimist but also that she believes the endowment growth follows a peculiar non-stationary process. In our model, instead, the agents correctly believe the true process is stationary while the non stationary behaviour of the stochastic discount factor arises endogenously due to changes in the wealth distribution.

Cogley and Sargent [15] combine both pessimism and belief heterogeneity but they focus only on their effect on the market price of risk on a finite sample. Although the pessimistic agent ends up learning, they show that, for a plausible calibration of their model, it takes a long time for the effect of large pessimism on CE asset prices to be erased unless the agents with correct beliefs own a large fraction of the initial wealth.

Finally, Cao [12] and Cogley et al [16] also combine the same three ingredients to study the dynamics of asset prices. Cao focuses on survival and excess volatility of asset prices. Cogley et al focus on the wealth dynamics of a bond economy when solvency constraints are exogenously given and proportional to the agents' income.

This paper is organised as follows. Section 2, describes the model. Our methodological contribution is introduced in sections 3 and 4. Section 5 provides a statistical and economic characterisation of short-term momentum and long-term reversal. In section 6, we evaluate the ability of CE and CESC allocations to generate short-term momentum and long-term reversal. Section **??** provides a final discussion. Proofs are gathered in the Appendix.

2 The Model

We consider a one-good infinite horizon pure exchange stochastic economy. In this section we establish the basic notation and describe the main assumptions.

2.1 The Environment

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. The set of possible states of nature is $S \equiv \{1, \ldots, K\}$. The state of nature at date zero is known and denoted by $s_0 \in S$. The set of partial histories up to date $t \geq 1$, S^t, is the $t-$Cartesian product of S with typical element $s^t = (s_1, \ldots, s_t)$. S^∞ is the set of infinite sequences of the states of nature and $s = (s_1, s_2, \cdots)$, called a path, is a typical element. For every partial history s^t, $t \geq 1$, a *cylinder* with base on s^t is the set $C(s^t) \equiv \{\tilde{s} \in S^\infty : \tilde{s} = (s^t, \tilde{s}_{t+1}, \cdots)\}$ of all paths whose t initial elements coincide with s^t. Let $\mathcal{F}_t$ be the σ-algebra that consists of all finite unions of the sets $C(s^t)$. The σ-algebras $\mathcal{F}_t$ define a filtration $\mathcal{F}_0 \subset \ldots \subset \mathcal{F}_t \subset \ldots \subset \mathcal{F}$ where $\mathcal{F}_0 \equiv \{\emptyset, S^\infty\}$ is the trivial $\sigma-$algebra and $\mathcal{F}$ is the σ-algebra generated by the algebra $\bigcup_{t=1}^\infty \mathcal{F}_t$.

Let Δ^{K-1} be the $K-1$ dimensional unit simplex in $\Re^K$. We say that $\pi : S \times S \to [0, 1]$ is a transition probability matrix if $\pi(\cdot\,|\xi) \in \Delta^{K-1}$ for all $\xi \in S$. If $\{s_t\}$ follows a first-order time-homogeneous Markov process with a $K \times K$ transition probability matrix π, then P^{π} denotes the probability measure on $(S^{\infty}, \mathcal{F})$ uniquely induced by π. Let Π^{K} denote the set of $K \times K$ transition probability matrices and Π^{K}_{++} be the subset consisting of all transitions matrices with strictly positive entries. The following assumption will be used for the characterisation of the dynamics in Sections 5-6 where we need to be explicit about the true data generating process (henceforth, dgp).

A.0 The true dgp is given by P^{π^*} for some $\pi^* \in \Pi^{K}_{++}$.

2.2 The Economy

There is a single perishable consumption good every period. The economy is populated by I (types of) infinitely-lived agents where $i \in \mathcal{I} = \{1, ..., I\}$ denotes an agent's name. A consumption plan is a sequence $\{c_t\}_{t=0}^{\infty}$ such that $c_0 \in \mathbb{R}_+$, $c_t : S^{\infty} \to \mathbb{R}_+$ is $\mathcal{F}_t-$measurable for all $t \geq 1$ and $\sup_{(t,s)} c_t(s) < \infty$. Given s_0, the agent's consumption set, $\mathbb{C}(s_0)$, is the set of all consumption plans.

2.2.1 Beliefs

P_i is the probability measure on $(S^{\infty}, \mathcal{F})$ that represents agent i's prior. We assume agent i's beliefs are *dogmatic* in the sense that there is $\pi_i \in \Pi^{K}$ such that for every event $A \in \mathcal{F}$.[6]

$$P_i\left(A\right) = P^{\pi_i}\left(A\right)$$

The following assumption defines a large class of beliefs that we use In Proposition 4.

A1 There exists $\xi^* \in S$ such that $\frac{\pi_1(\xi^*|\xi^{**})}{\pi_2(\xi^*|\xi^{**})}\frac{\pi_1(\xi^{**}|\xi^{*})}{\pi_2(\xi^{**}|\xi^{*})} \neq 1$ for some $\xi^{**} \in S$.

2.2.2 Preferences

Agents' preferences over consumption plans have a subjective expected utility representation that is time separable, i.e., for every $c_i \in \mathbb{C}(s_0)$ her preferences are represented by

$$U_i^{P_i}(c_i) = E^{P_i}\left(\sum_{t=0}^{\infty} \rho_{i,t}\ u_i(c_{i,t})\right),$$

where $u_i : \mathbb{R}_+ \to \{-\infty\} \cup \mathbb{R}$ is continuously differentiable, strictly increasing, strictly concave and $\lim_{x \to 0} \frac{\partial u_i(x)}{\partial x} = +\infty$ and $\rho_{i,t}$ is agent i's multi-period stochastic discount factor recursively defined by

$$\rho_{i,t+1}(s) = \beta(s_t, \pi_i)\rho_{i,t}(s) \text{ for all } t \text{ and } s,$$

where $\rho_{i,0}(s_0) \in (0,1)$ is given and $\beta(\xi, \cdot) : \Pi^{K} \to (0,1)$ is continuous for all ξ and uniformly bounded above by $\overline{\beta} \in (0,1)$.[7,8] Sometimes we write $\beta_i(\xi) \equiv \beta(\xi, \pi_i)$ for all ξ.

[6]Since $\pi_i \in \Pi^{K}_{++}$ for all i, it follows that each agent assigns positive probability to every partial history s^t, i.e., $P_i(C(s^t)) > 0$ for all s^t.

[7]We allow for utility functions unbounded from below.

[8]In the standard case where $\beta(\xi, \pi_i) = \beta$ for all ξ, $\rho_{i,t}(s) = \beta^t$ for all $t \geq 1$ and s.

2.3 Feasibility, Enforceability and Constrained Optimality

Agent i's endowment at date t is a time-homogeneous function of the current state of nature and we denote it by $y_i(\xi) > 0$ for all ξ. The aggregate endowment is denoted by $y(\xi) \equiv \sum_{i=1}^{I} y_i(\xi) \leq \overline{y} < \infty$. Let $y_{i,t}(s) \equiv y_i(s_t)$ and $y_t(s) \equiv y(s_t)$.

Given a consumption plan $c_i \in \mathbb{C}(s_0)$, the agent's utility from the consumption plan can be recursively defined as

$$U_i(c_i)(s^t) = u_i(c_i(s^t)) + \beta(s_t, \pi_i) \sum\nolimits_{\xi'} \pi_i(\xi' \,|s_t) U_i(c_i)(s^t, \xi') \text{ for all } t \text{ and } s^t.$$

When c_i is the endowment of agent i, we simply write $U_i(s_t, \pi_i)$ to make clear that the utility attained from consuming the individual endowment forever can be expressed as a function only of s_t and π_i.

Let $Y(s_0)$ be the set of feasible allocations. Given s_0, a feasible allocation $\{c_i\}_{i=1}^{I}$ is *enforceable* if $U_i(c_i)(s^t) \geq U_i(s_t, \pi_i)$ for all t, s^t and i. Let $Y^E(s_0) \subset Y(s_0)$ be the set of enforceable allocations. A feasible allocation $\{c_i\}_{i=1}^{I}$ is *Pareto optimal (PO)* if there is no alternative feasible allocation $\{\tilde{c}_i\}_{i=1}^{I} \in Y(s_0)$ such that $U_i^{P_i}(\tilde{c}_i) > U_i^{P_i}(c_i)$ for all i. An enforceable allocation $\{c_i\}_{i=1}^{I}$ is *constrained Pareto optimal (CPO) given* s_0 if there is no other enforceable allocation $\{\tilde{c}_i\}_{i=1}^{I} \in Y^E(s_0)$ such that $U_i^{P_i}(\tilde{c}_i) > U_i^{P_i}(c_i)$ for all i.

Given s_0, define the *utility possibility correspondence* by

$$\mathcal{U}(s_0) = \{\tilde{u} \in \mathbb{R}^I : \exists \{c_i\}_{i=1}^{I} \in Y(s_0), \; \tilde{u}_i \leq U_i^{P_i}(c_i) \quad \forall i\},$$

and the *enforceable utility possibility correspondence* by

$$\mathcal{U}^E(s_0) = \{\tilde{u} \in \mathbb{R}^I : \exists \{c_i\}_{i=1}^{I} \in Y^E(s_0), \; U_i(s_0, \pi_i) \leq \tilde{u}_i \leq U_i^{P_i}(c_i) \quad \forall i\}.$$

Given s_0, the set of CPO allocations can be characterised as the solution to the following planner's problem with welfare weights $\alpha \in \mathbb{R}_+^I$:

$$v^*(s_0, \alpha) \equiv \sup_{\{c_i\}_{i=1}^{I} \in Y^E(s_0)} \sum_{i=1}^{I} \alpha_i \; E^{P_i}\left(\sum_t \rho_{i,t} \; u_i(c_{i,t})\right). \tag{1}$$

It is straightforward to prove that (1) can be rewritten as

$$v^*(s_0, \alpha) = \sup_{\tilde{u} \in \mathcal{U}^E(s_0)} \sum_{i=1}^{I} \alpha_i \; \tilde{u}_i. \tag{2}$$

The maximum is attained since the objective function is continuous and the constraint set is compact.

2.3.1 An Economy with Aggregate Growth

Let $g : S \to \mathbb{R}_+$ and $\epsilon_i : S \to (0, 1)$ denote the (stochastic) growth rate and income share of agent i, respectively. Then,

$$y_t(s) = g(s_t) \; y_{t-1}(s) \textit{ and } y_{i,t}(s) = \epsilon_i(s_t) \; y_t(s) \textit{ for all } i, \, t \textit{ and } s. \tag{3}$$

Definition. *An economy where the aggregate endowment satisfies (3), the discount factor is non-stochastic and preferences display constant relative risk aversion is called a growth economy.*

Our specification of the discount factor let us accommodate growth as in Alvarez and Jermann [3]. Indeed, we now argue that the set of enforceable allocations of a growth economy can be characterised by studying the set of enforceable allocations of an economy with constant aggregate endowment and an stochastic discount factor.

Let $\widehat{c}_{i,t}(s) = c_{i,t}(s)/y_t(s)$, $\widehat{y}_{i,t}(s) = y_{i,t}(s)/y_t(s) = \epsilon_i(s_t)$ for all i, s and t. Notice that $\widehat{y}_t(s) = \sum_{i=1}^{I} \widehat{y}_{i,t}(s) = 1$ for all s and t. Then,

$$\widehat{U}_i(\widehat{c}_i)(s^t) = u_i(\widehat{c}_{i,t}(s)) + \widehat{\beta}(s_t, \pi_i) \sum\nolimits_{\xi'} \widehat{\pi}_i(\xi' \,|s_t) \widehat{U}_i(\widehat{c}_i)(s^t, \xi') \text{ for all } t \text{ and } s,$$

where

$$\widehat{\pi}_i(\xi' \,|s_t) = \frac{\pi_i(\xi' \,|s_t)\, g(\xi')^{1-\sigma}}{\sum_{\tilde{\xi}} \pi_i(\tilde{\xi} \,|s_t) g(\tilde{\xi})^{1-\sigma}} \quad \text{and} \quad \widehat{\beta}(s_t, \pi_i) = \beta \sum_{\xi'} \pi_i(\xi' \,|s_t)\, g(\xi')^{1-\sigma}.$$

As in Mehra and Prescott [29], expected utility is well defined if

$$\sup_{\xi} \left\{ \beta \sum_{\xi'} \pi_i(\xi' \,|\xi) g(\xi')^{1-\sigma} \right\} < 1. \tag{4}$$

Let $\widehat{c}_i \equiv \{\widehat{c}_{i,t}\}_{t=0}^{\infty}$ for all i and $\widehat{y} \equiv \{\widehat{y}_t\}_{t=0}^{\infty}$. We define the *normalised stationary economy* associated to the growth economy by $\left(\widehat{y}, \left\{\widehat{c}_i, \widehat{U}_i\right\}_{i \in \mathcal{I}}\right)$.

Finally, $\{c_i\}_{i=1}^{I}$ is an enforceable allocation in the growth economy iff $\{\widehat{c}_i\}_{i=1}^{I}$ is an enforceable allocation in the normalised stationary economy. Also, the preference orderings are identical in the two corresponding economies and the discount factor is stochastic if and only if the growth rate is.

3 A Recursive Approach to CPO

In this section, we provide the recursive characterisation of the set of CPO allocations and a version of the Principle of Optimality for economies with heterogeneous beliefs and limited enforceability.

3.1 The Recursive Planner's Problem

In Appendix A we show that $v^* : S \times \mathbb{R}_+^I \times \mathcal{P}(\Pi) \to \mathbb{R}$ solves the functional equation[9]

$$v^*(\xi, \alpha) = \max_{(c, w'(\xi'))} \sum_{i=1}^{I} \alpha_i \left\{ u_i(c_i) + \beta(\xi, \pi_i) \sum_{\xi'} \pi_i(\xi' \,|\xi)\; w_i'(\xi') \right\}, \tag{5}$$

subject to

$$c_i \geq 0, \qquad \sum\nolimits_{i=1}^{I} c_i = y(\xi), \tag{6}$$

$$u_i(c_i) + \beta(\xi, \pi_i) \sum\nolimits_{\xi'} \pi_i(\xi' \,|\xi)\; w_i'(\xi') \geq U_i(\xi, \pi_i), \tag{7}$$

$$w_i'(\xi') \geq U_i(\xi', \pi_i) \qquad \text{for all } \xi', \tag{8}$$

$$\min_{\tilde{\alpha} \in \mathbb{R}_+^I} \left[v^*(\xi', \tilde{\alpha}) - \sum\nolimits_{i=1}^{I} \tilde{\alpha}\; w_i'(\xi') \right] \geq 0 \text{ for all } \xi', \tag{9}$$

[9] In section 3.1 we abuse notation and let c to be a non-negative vector and c_i its i^{th} component.

and $\alpha'(\xi')$ is the solution to problem (9) for state of nature ξ'.

In the recursive dynamic program defined by (5) - (9), the current state of nature, ξ, captures the impact of changes in aggregate output while α summarises and isolates the history dependence introduced by the $\mathcal{B}$-margin of heterogeneity, $\frac{\pi_i(\xi'|\xi)}{\pi_j(\xi'|\xi)}$, introduced by Beker and Espino [5] and limited enforceability.[10] The planner takes as given (ξ, α) and allocates current consumption and continuation utility levels among agents. The optimisation problem defined in condition (9) characterises the set of continuation utility levels attainable at ξ' (see Lemma A.1 in Appendix A).[11] The weights $\alpha'(\xi, \alpha)(\xi')$ that attain the minimum in (9) are the weights that support next period allocation.

Any (c, w', α') that satisfies (6) - (9) will be referred as a *set of policy functions.* Given (s_0, α_0), we say the policy functions (c, α') generate an allocation $\{c_t\}_{t=0}^{\infty} \in \mathbb{C}(s_0)^I$ if

$$\begin{aligned} c_{i,t}(s) &= c_i(s_t, \alpha_t(s)), \\ \alpha_{t+1}(s) &= \alpha'(s_t, \alpha_t(s), \end{aligned}$$

for all i, $t \geq 0$ and $s \in S^{\infty}$, where $\alpha_0(s) = \alpha_0$.

It follows by standard arguments that the corresponding optimal consumption policy function, $c_i(\xi, \alpha)$, is the unique solution to

$$c_i(\xi, \alpha) + \sum_{h \neq i} \left(\frac{\partial u_h}{\partial c_h} \right)^{-1} \left(\frac{\alpha_i}{\alpha_h} \frac{\partial u_i(c_i(\xi, \alpha))}{\partial c_i} \right) = y(\xi).$$

where $\left(\frac{\partial u_h}{\partial c_h} \right)^{-1}$ denotes the inverse of the function $\frac{\partial u_h}{\partial c_h}$.

The following Theorem states our version of the Principle of Optimality. It shows that there is a one-to-one mapping between the set of CPO allocations and the allocations generated by the optimal policy functions solving (5) - (9).

Theorem 1. *An allocation $(c_i^*)_{i=1}^{I}$ is CPO given (ξ, α) if and only if it is generated by the set of policy functions solving (5) - (9).*

Given $\alpha_{-i} \in \mathbb{R}_+^{I-1}$, define

$$\underline{\alpha}_i(\xi)(\alpha_{-i}) = \min_{\alpha_i} \left\{ \alpha_i \in \mathbb{R}_+ : u_i(c_i(\xi, (\alpha_i, \alpha_{-i}))) + \beta(\xi, \pi_i) \sum_{\xi'} \pi_i(\xi' \,|\xi)\ w_i'(\xi, (\alpha_i, \alpha_{-i}))(\xi') = U_i(\xi, \pi_i) \right\}$$

where $c_i(\xi, \alpha)$ and $w_i'(\xi, (\alpha_i, \alpha_{-i}))(\xi^{'})$ are the maximisers in problem (5) - (9). For $I = 2$, we simply write $\underline{\alpha}_1(\xi)$ and $\underline{\alpha}_2(\xi)$.

[10] To be more precise, Beker and Espino define the $\mathcal{B}$-margin as the ratio of the priors about the states of nature in the following t periods while here it is the ratio of the priors about the realisations of next period state of nature.

[11] To understand condition (9) notice that the utility possibility correspondence is convex, compact and contains its corresponding frontier. The frontier of a convex set can always be parameterised by supporting hyperplanes. Thus, a utility level vector w is in the utility possibility correspondence if and only if for every welfare weight α the hyperplane parameterised by α and passing through w, αw, lies below the hyperplane generated by the utility levels attained by the CPO allocation corresponding to that welfare weight α, attaining the value $v(\xi, \alpha)$. This is why we must have $\alpha w \leq v(\xi, \alpha)$ for all α or, equivalently, $\min_{\tilde{\alpha}} [v(\xi, \tilde{\alpha}) - \tilde{\alpha} w] \geq 0$.

The following Proposition shows that constraint (7) can be ignored by restricting the welfare weights to lie in $\Delta(\xi) \equiv \{\alpha \in \Delta^{I-1} : \alpha_i \geq \underline{\alpha}_i(\xi)(\alpha_{-i}) \text{ for all } i\}$.[12]

Proposition 2. *Let $\xi \in S$. (i) If $\alpha \in \Delta(\xi)$, then constraint (7) does not bind at any solution to (5) - (9). (ii) If $\alpha \notin \Delta(\xi)$, then there exists some $\tilde{\alpha} \in \Delta(\xi)$ such that $c(\xi, \alpha) = c(\xi, \tilde{\alpha})$.*

The (normalised optimal) law of motion for the welfare weights, $\alpha'_{i,cpo}(\xi, \alpha)(\xi')$, follows from the first order conditions with respect to the continuation utility levels for each individual. In the two-agent case, the CPO law of motion for agent 1's welfare weight is

$$\alpha'_{1,cpo}(\xi,\alpha)(\xi') = \begin{cases} \underline{\alpha}_1(\xi') & \text{if } \alpha'_{1,po}(\xi,\alpha)(\xi') < \underline{\alpha}_1(\xi') \\ 1 - \underline{\alpha}_2(\xi') & \text{if } \alpha'_{1,po}(\xi,\alpha)(\xi') > 1 - \underline{\alpha}_2(\xi') \\ \alpha'_{1,po}(\xi,\alpha)(\xi') & \text{otherwise} \end{cases}$$

where

$$\alpha'_{1,po}(\xi,\alpha)(\xi') = \frac{\alpha_1 \beta_1(\xi) \pi_1(\xi' \,|\, \xi)}{\alpha_1 \beta_1(\xi) \pi_1(\xi' \,|\, \xi) + \alpha_2 \beta_2(\xi) \pi_2(\xi' \,|\, \xi)}$$

is the PO law of motion for agent 1's welfare weight that depends only on the extent to which beliefs are heterogeneous as captured by the $\mathcal{B}$-margin. The CPO law of motion for agent 1's welfare weight, instead, combines two effects: belief heterogeneity and limited enforceability. To understand the impact of each effect we discuss them in isolation. If beliefs are heterogeneous but enforceability is perfect, the CPO law of motion for agent 1's welfare weight becomes the PO law of motion. Therefore, the changes in agent 1's welfare weight are purely driven by changes in the $\mathcal{B}$-margin. If beliefs are homogeneous and enforceability is imperfect, the case analysed by Alvarez and Jermann [3], the CPO law of motion requires the next period welfare weight to be equal to the current one unless that conflicts with the need to provide incentives to avoid the agent to revert to autarky, i.e. there is some state of nature for which the current welfare weight does not lie in the interval defined by the corresponding minimum enforceable weights. Therefore, the changes in agent 1's welfare weights are purely driven by the need to satisfy enforceability. If agents have heterogeneous beliefs and enforceability is limited, both effects might interact. Consequently, changes in agent 1's welfare weight are driven by the $\mathcal{B}$-margin unless that conflicts with enforceability.

3.2 Computation

For many purposes it is important to have an algorithm capable of finding the value function v^*. Let $\tilde{v}$ be the value function solving the recursive problem when the enforceability constraints are ignored (see Beker and Espino[5]). Evidently, $v^*(\xi, \alpha) \leq \tilde{v}(\xi, \alpha)$ for all (ξ, α).

Proposition 3. *Let $v_0 = \tilde{v}$ and $v_n = T(v_{n-1})$ for all $n \geq 1$. Then, $\{v_n\}$ is a monotone decreasing sequence and $\lim_{n\to\infty} v_n = v^*$.*

[12] The proof of Proposition 2 is included in the supplementary material.

3.3 The Welfare Weights Dynamic with Dogmatic Beliefs

In this section we assume there are two agents. $\Omega \equiv \{(\xi, \alpha) \in S \times \Delta^1 : \alpha \in \Delta(\xi)\}$ is the state space and $\mathcal{G}$ its $\sigma-$algebra. For $t \geq 0$, Ω^t is the $t-$cartesian product of Ω with typical element $\omega^t = (\xi_0, \alpha_0, ..., \xi_t, \alpha_t)$ and $\Omega^\infty = \Omega \times \Omega \times ...$ is the infinite product of the state space with typical element $\omega = (\omega_0, \omega_1, ...)$. $\mathcal{G}_{-1} \equiv \{\varnothing, \Omega^\infty\}$ is the trivial $\sigma-$algebra, $\mathcal{G}_t$ is the $\sigma-$algebra that consists of all the cylinder sets of length t. The $\sigma-$algebras $\mathcal{G}_t$ define a filtration $\mathcal{G}_{-1} \subset \mathcal{G}_0 \subset ... \subset \mathcal{G}_t \subset ... \subset \mathcal{G}^\infty$, where $\mathcal{G}^\infty \equiv \mathcal{G} \times \mathcal{G} \times ...$ is the $\sigma-$algebra on Ω^∞.

The law of motion for the welfare weights, α'_{cpo}, coupled with π^* define a time-homogeneous transition function on the states of nature and the welfare weights, $F_{cpo} : \Omega \times \mathcal{G} \to [0, 1]$, given by

$$F_{cpo}\left[(\xi, \alpha), \mathcal{S} \times \mathcal{A}\right] = \sum_{\xi' \in \mathcal{S},\ \alpha'_{cpo}(\xi,\alpha)(\xi') \in \mathcal{A}} \pi^*\left(\xi' \,|\xi\right) \text{ for all } (\mathcal{S} \times \mathcal{A}) \in \mathcal{G}$$

The transition function F_{cpo} together with a probability measure ψ on $(\Omega, \mathcal{G})$ induces a unique probability measure $P^{F_{cpo}}(\psi, \cdot)$ on $(\Omega^\infty, \mathcal{G}^\infty)$. We define the operator T^* on the space of probability measures on $(\Omega, \mathcal{G})$ as

$$T^*\psi\left(\mathcal{S}, \mathcal{A}\right) = \int F_{cpo}\left((\xi, \alpha), \mathcal{S} \times \mathcal{A}\right) d\psi \text{ for all } (\mathcal{S} \times \mathcal{A}) \in \mathcal{G}$$

We use standard arguments to show that T^* has a unique invariant measure on $(\Omega, \mathcal{G})$ and that the distribution of states converges weakly to that measure.

Proposition 4. *Suppose $I = 2$, A.0 and A1 for both agents. Then there exists a unique invariant measure $\psi_{cpo} : \mathcal{G} \to [0, 1]$. Moreover, ψ_{cpo} is globally stable and non-degenerate.*

Actually, Beker and Espino [6] show that CPO allocations are *never* PO for a large class of heterogeneous priors in any two-agent economy. Moreover, our numerical simulations led us to conjecture that, typically, the support of the invariant distribution has a finite number of points.

4 Competitive Equilibrium with Solvency Constraints

In this section we define a competitive equilibrium with solvency constraints (CESC). In Section 4.1 we show that CPO allocations can be decentralised as CESC and study the determinants of the financial wealth distribution. In section 4.2 we study the limit distribution of wealth and consumption in a CESC.

Every period t, after observing s^t, agents trade both the consumption good and a complete set of Arrow securities in competitive markets. Security ξ' issued at date t pays one unit of consumption if next period's state of nature is ξ' and 0 otherwise. We denote by $q_t^{\xi'}(s)$ and $a_{i,t}^{\xi'}(s)$ the price of Arrow security ξ' and agent i's asset holdings, respectively, at date t on path s. Let $a_{i,-1}^{\xi'} = 0$ for all ξ', $a_{i,t} = \left(a_{i,t}^1, ..., a_{i,t}^K\right)$ and $a_i \equiv \{a_{i,t-1}\}_{t=0}^\infty$ for all i. Prices are in units of the date–t consumption good and a price system is given by $q \equiv \left\{q_t^1, ..., q_t^K\right\}_{t=0}^\infty$. Agent i faces a state contingent solvency constraint, $B_{i,t}^{\xi'}(s)$, that limits security ξ' holdings at date t and $B_i \equiv \left\{B_{i,t}^1, ..., B_{i,t}^K\right\}_{t=0}^\infty$ for all i.

Given q and B_i, *agent i's problem* is

$$\max_{(c_i,a_i)} \quad E^{P_i}\left(\sum_{t=0}^{\infty} \rho_{i,t}\ u_i(c_{i,t})\right)$$

$$\text{s.t.} \quad \begin{cases} c_{i,t}(s) + \sum_{\xi'} q_t^{\xi'}(s)\ a_{i,t}^{\xi'}(s) = y_i(s_t) + a_{i,t-1}^{s_t}(s) & \text{for all s and t.} \\ c_{i,t}(s) \geq 0,\ a_{i,-1} = 0,\ a_{i,t}^{\xi'}(s) \geq B_{i,t}^{\xi'}(s) & \text{for all } \xi'\text{, s and t.} \end{cases}$$

Markets clear if

$$\begin{array}{lll} \sum_{i=1}^{I} c_{i,t}(s) & = y(s_t) & \text{for all s and t.} \\ \sum_{i=1}^{I} a_{i,t}^{\xi'}(s) & = 0 & \text{for all } \xi'\text{, s and t.} \end{array}$$

Definition. *A competitive equilibrium with solvency constraints (CESC) is an allocation* $\{c_i\}_{i\in\mathcal{I}}$, portfolios $\{a_i\}_{i\in\mathcal{I}}$*, a price system* q *and solvency constraints* $\{B_i\}_{i\in\mathcal{I}}$ *such that:*
(CESC 1) Given q *and* B_i*,* (c_i, a_i) *solves agent i's problem for all i.*
(CESC 2) Markets clear.

Of course, a CESC need not be CPO (see Bloise et al [9]). In what follows, however, when we refer to CESC we always mean a CESC that is CPO. A Competitive Equilibrium (CE, hereafter) is a CESC in which the corresponding allocation is PO.

4.1 Decentralisation

Now we study the determinants of the financial wealth distribution that supports a $CESC$ allocation. First, we construct recursively the date zero-transfers needed to decentralise a CPO allocation as a time invariant function of the (ξ, α). Later, we employ a properly adapted version of the Negishi's approach to pin down the CPO allocation that can be decentralised as a CESC with zero transfers.

We begin defining $A_{i,e}(\xi, \alpha)$, for $e \in \{po, cpo\}$, as the solution to the functional equation

$$A_{i,e}(\xi,\alpha) = c_{i,e}(\xi,\alpha) - y_i(\xi) + \sum_{\xi'} Q_e(\xi,\alpha)(\xi')\ A_{i,e}(\xi',\alpha'), \tag{10}$$

where

$$Q_e(\xi,\alpha)(\xi') = \max_h \left\{ \beta(\xi,\pi_h)\ \pi_i(\xi'\,|\xi)\ \frac{\partial u_h(c_{h,e}(\xi',\alpha'(\xi,\alpha)(\xi')))\,/\partial c_{h,e}}{\partial u_h(c_{h,e}(\xi,\alpha))\,/\partial c_{h,e}} \right\}.$$

Expression (10) computes recursively the present discounted value of agent i's excess demand at the CPO allocation priced by the implicit state price $Q_e(\xi,\alpha)(\xi')$. Let $R_e^F(\xi,\alpha) = \left(\sum_{\xi'} Q_e(\xi,\alpha)(\xi')\right)^{-1}$ be the (implicit) risk-free interest rate

Definition. *A CPO allocation generates positive risk-free interest rates if* $R_{cpo}^F(\xi,\alpha) > 1$ *for all* (ξ,α).

Proposition 5 shows that positive risk-free interest rates guarantees that A_i is well-defined and there exist a welfare weight α_0 such that A_i is zero for every i. The allocation parameterised by α_0 is the natural candidate to be decentralised as a $CESC$.[13]

[13]In the literature studying competitive decentralisation of PO allocations in growth economies with homogeneous beliefs, the positive risk-free interest rate condition is ubiquitous to make utility levels bounded and, thus, to establish the existence of a competitive equilibrium. Since $\widehat{Q}(\xi,\alpha)(\xi') = \beta(\xi)\ \widehat{\pi}(\xi'\,|\xi) = \beta\ \pi(\xi'\,|\xi)\ g(\xi')^{1-\sigma}$ is the state price of the normalised stationary economy, the positive risk-free interest rate condition is equivalent to condition (4).

Proposition 5. *If the CPO allocation generates positive risk-free interest rates, there is a unique continuous function $A_{i,e}$ solving (10). Moreover, for each (s_0) there exists $\alpha_0 = \alpha(s_0) \in \mathbb{R}^I_+$ such that $A_{i,e}(s_0, \alpha_0) = 0$ for all i.*

We follow the Negishi's approach to decentralise the CPO allocation parameterised by α_0 as a $CESC$. For each s, t and ξ', we define recursively

$$a^{\xi'}_{i,t}(s) = A_{i,cpo}\left(\xi', \alpha'_{cpo}(s_t, \alpha_t(s))\right) \tag{11}$$

$$q^{\xi'}_t(s) = Q_{cpo}(s_t, \alpha_t(s))(\xi') \tag{12}$$

$$B^{\xi'}_{i,t}(s) = A_{i,cpo}\left(\xi', \alpha'_{cpo}(s_t, \alpha_t(s))(\xi')\right), \tag{13}$$

where α_t for $t \geq 1$ is generated by α'_{cpo} and $\alpha_0 = \alpha(s_0)$.

In a decentralised competitive setting with sequential trading, $A_{i,cpo}(s_t, \alpha_t(s))$ is the financial wealth that agent i needs at date t to afford the consumption bundle corresponding to the CPO allocation parameterised by $\alpha_t(s)$ given s_t (see Espino and Hintermaier [18] for further discussion).[14]

Theorem 6. *If the CPO allocation parameterised by $\alpha_0 = \alpha(s_0)$ generates positive risk-free interest rates, then it can be decentralised as a $CESC$ with portfolios $\{a_i\}_{i\in I}$, price system q and solvency constraints $\{B_i\}_{i\in I}$ defined by (11)-(13).*

4.2 The Limiting Distribution of Wealth and Consumption

Theorem 6 shows that the dynamics of the individuals' wealth and consumption in a CESC allocation is driven by the dynamic of the welfare weights. Proposition 4 shows that welfare weights have a non-degenerate limiting distribution. The following Proposition couples these two results.

Proposition 7. *Suppose $I = 2$, A.0 and A1. The limiting distribution of wealth and consumption in a CESC is non-degenerate.*

An important implication of this result is that every agents' consumption is bounded away from zero regardless of whether her beliefs are correct or not (see Cao [12] for an alternative discussion). Therefore, the so-called Market Selection Hypothesis does not hold in this setting.

5 Short-Term Momentum and Long-Term Reversal

In Section 5.1 we introduce a formal definition of short-term momentum and long-term reversal in terms of the empirical autocorrelations of the equity excess returns. In Section 5.2, we argue that in any CE or CESC, the empirical autocorrelations can be approximated using the population autocorrelations. In Section 5.3 we provide a statistical characterisation of the population autocorrelations

[14]Our equilibrium concept does not rely on solvency constraints that are not too tight, see Alvarez and Jermann [3] and [4]. In our decentralisation, individual asset holdings are always at the solvency constraints by construction. However, as discussed in Alvarez and Jermann [4, pp 1131], some of these are "false corners", i.e., if the solvency constraints were marginally relaxed, the agent would not change the optimal choice of consumption and asset holdings.

in terms of the reaction of the conditional equity-premium to the realisation of the excess returns. Finally, in Section 5.4 we reinterpret the equivalent martingale measure as a market belief.

5.1 Definitions

We are interested in the asset that Mehra and Prescott [29] study, that is, a claim to the aggregate endowment. Let $y_t(s)$, $p_t(s)$ and $r_t^f(s)$ be the dividend of the asset, its ex-dividend price and the (gross) risk-free interest rate, respectively, at date t on path s. Its theoretical return is given by:

$$r_{t+1}(s) = \frac{p_{t+1}(s) + d_{t+1}(s)}{p_t(s)}$$

However, in order to assess the quantitative performance of the model, it is important to construct quarterly returns as they are in the data. Shiller constructs quarterly (annualised) excess returns (the return hereafter) by computing the excess return of investing one unit in the Mehra-Prescott asset and holding it four periods (quarters). Consequently, the return in the data is given by

$$r_{t,t+4}(s) = \frac{p_{t+4}(s) + y_t(s) + \ldots + y_{t+4}(s)}{p_t(s)} - r_{t,t+4}^f(s)$$

where for each $t \geq 0$ and s, $r_{t,t+4}^f(s)$ is the return from investing one unit in the risk-free bond in period t and holding the investment for 4 periods.

We imagine an econometrician who observes data on returns for T consecutive periods. Let

$$\overline{r}_T(s) \equiv \frac{1}{T}\sum_{t=1}^{T} r_{t,t+4}(s) \text{ and } \sigma_T^2 \equiv \frac{1}{T}\sum_{t=1}^{T}\left(r_{t,t+4}(s) - \overline{r}_T(s)\right)^2$$

be the empirical average and variance of the returns. Let

$$cov_{k,T}(s) \equiv \frac{1}{T}\sum_{t=1}^{T}\left(r_{t,t+4}(s) - \overline{r}_T(s)\right)\left(r_{t+k,t+k+4}(s) - \overline{r}_T(s)\right) \text{ and } \rho_{k,T}(s) \equiv \frac{cov_{k,T}(s)}{\sigma_T(s)\ \sigma_T(s)}$$

be the empirical autocovariance and autocorrelation coefficient of order $k \geq 1$.

Now we give a formal definition of the so-called *financial markets anomalies* that we explain.

Definition. *The asset displays short-term momentum on a path s if $\lim_{T\to\infty} \rho_{\tau,T}(s) > 0$ for $\tau \leq 3$. The asset displays long-term reversal on a path s if $\lim_{T\to\infty} \rho_{\tau,T}(s) < 0$ for $\tau \geq 4$.*

5.2 Asymptotic Approximation

The autocorrelations are continuous functions of the returns and (CE or $CESC$) equilibrium returns are continuous functions of a Markov process with transition function F_e on $(\Omega, \mathcal{G})$, where $e \in \{po, cpo\}$.[15] That is, there exists a $\mathcal{G}_\tau-$ measurable function $R_{\tau,\tau+4,e} : \Omega^\infty \to \Re$ and a function $R_e : S \times \Delta^1 \times S^4 \mapsto \Re$ such that

$$R_{t,t+4,e}(\omega) \equiv R_e\left(\xi_t(\omega), \alpha_t(\omega)\right)\left(\xi_{t+1}(\omega), \ldots, \xi_{t+4}(\omega)\right) = r_{t,t+4}(s),$$

where ω and s are related by $\omega_t = (s_t, \alpha_t(s))$.

[15] When allocations are PO, with some abuse of notation, we define $\Omega \equiv S \times \Delta^1$ and $\mathcal{G}$ its $\sigma-$algebra.

If one argues that the Markov process is ergodic with invariant distribution ψ_e, then standard arguments show that the following asymptotic approximation holds for any $\tau \geq 1$

$$\lim_{T\to\infty} cov_{\tau,T}(s) = cov^{P_e}(R_4, R_{\tau+4}) \qquad and \qquad \lim_{T\to\infty} \sigma_T(s) = \sigma^{P_e}(R_{0,4}), \qquad P^{\pi^*} - a.s., \tag{14}$$

where $P_e \equiv P^{F_e}(\psi_e, \cdot)$ and $R_{t+4,e} = R_{t,t+4,e}$ for all $t \geq 0$.

Theorem 8. *Assume A.0, A.1 and that some agent has correct beliefs. Then the asymptotic approximation (14) holds if*
(a) Allocations are PO or
(b) Allocations are CPO, $I = 2$ and both agents have dogmatic beliefs satisfying A3.

Theorem 8 can be intuitively explained as follows. For the case in which allocations are PO and the dgp is iid, Beker and Espino [5] show that if some agent has correct beliefs, then the vector of welfare weights associated with a PO allocation converges to a fixed vector almost surely. An analogous result can be proved in the case that the dgp is generated by draws from a time-homogeneous transition matrix as in this paper. This result implies the ergodicity of the Markov process with transition F_{po}. For the case in which allocations are CPO, the result follows directly from Proposition 4.

REMARK: The well-known result on convergence of posteriors implies that if every agent satisfies A1, there exists $\pi = (\pi_1, ..., \pi_I)$ such that the agent's posteriors converges weakly to a point mass on π_i for $P^{\pi^*}-$almost all $s \in S^\infty$.[16]

5.3 Statistical Characterisation

For $\tau \geq 2$, the law of iterated expectations implies that

$$cov^{P_e}(R_{4,e}, R_{\tau+4,e}) = E^{P_e}\left[\overline{R}_{4,e}\ E^{P_e}(R_{\tau+4,e}|\,\mathcal{G}_4)\right], \tag{15}$$

where $\overline{R}_{4,e}(\omega) \equiv R_{4,e}(\omega) - E^{P_e}(R_{4,e})$ is the *abnormal return* and $E^{P_e}(R_{\tau+4,e}|\,\mathcal{G}_4)(\omega)$ denotes the $\tau-$period ahead conditional equity premium. We refer to $R_{\tau+4,e}$ as the short-run return if $\tau \leq 3$ and as the long-run return if $\tau \geq 4$. Likewise, $E^{P_e}(R_{\tau+4,e}|\,\mathcal{G}_4)(\omega)$ is the *conditional short-run equity premium if* $\tau \leq 3$ *and* the *conditional long-run equity premium if* $\tau \geq 4$.

Condition (15) makes clear that the sign of the autocovariance of order τ depends on how the conditional equity premium reacts to abnormal returns at date 4. The important question is what kind of reaction of the conditional equity premium leads to short-term momentum and long-term reversal. The following definitions will be used in Proposition 9 to provide an answer to that question.

Definition. *Consider $\omega^+, \omega^- \in \Omega$ such that $\overline{R}_{4,e}(\omega^+) > 0$ and $\overline{R}_{4,e}(\omega^-) < 0$. For $\tau \geq 1$, the return underreacts by date $\tau + 4$ if $E^{P_e}(R_{\tau+4,e}|\,\mathcal{G}_4)(\omega^+) > E^{P_e}(R_{\tau+4,e}|\,\mathcal{G}_4)(\omega^-)$. The return overreacts by date $\tau + 4$ if $E^{P_e}(R_{\tau+4,e}|\,\mathcal{G}_4)(\omega^+) < E^{P_e}(R_{\tau+4,e}|\,\mathcal{G}_4)(\omega^-)$.*

This Proposition provides a sufficient condition for both short-term momentum and long-term reversal that follows immediately from (15) and the definition above.

[16] Where $\pi_i \in \Pi^K$ is the point in the support of agent i's prior which minimises the Kullback-Leibler divergence with respect to π^*. See the seminal work of Berk [8] for the i.i.d. case and Yamada [40] for the Markov extension.

Proposition 9. *If the return underreacts by date τ, then the $\tau-$order autocorrelation is positive. If the return overreacts by date τ, then the $\tau-$order autocorrelation is negative. That is, (i) if the conditional short-run equity premium is positive, then the asset displays short-term momentum and (ii) if the conditional long-run equity premium is negative, then the asset displays long-term reversal.*

5.4 The Economics of Predictable Returns

To explain when the hypothesis of Proposition 9 are met we have to understand the behaviour of $E^{P_e}\left(R_{\tau+4,e}|\,\mathcal{G}_4\right)(\omega)$.

Definition. *Returns are unpredictable if $E^{P_e}\left(R_{\tau+4,e}|\,\mathcal{G}_4\right)(\omega)$ is $\mathcal{G}_0-$measurable for all τ.*

Returns are unpredictable if $E^{P_e}\left(R_{\tau+4,e}|\,\mathcal{G}_4\right)(\omega)$ does not change with the information released at date 4, i.e. the conditional equity premium coincides with the (unconditional) equity premium. Our next result follows immediately from (15) and the definition of unpredictable return.

Proposition 10. *If returns are unpredictable, the asset does not display financial markets anomalies.*

The case we are interested is when returns are predictable, that is $E^{P_e}\left(R_{\tau,e}|\,\mathcal{G}_4\right)(\omega)$ varies with the information released at date 1. Unfortunately, this case is more complex because returns can be predictable in many different ways.

For $e \in \{po, cpo\}$, let $M_e : \mathcal{G}^\infty \to [0,1]$, be the equivalent martingale measure on $(\Omega^\infty, \mathcal{G}^\infty)$ and let $m_e : \mathcal{G} \to [0,1]$ be given by

$$m_e\left(\xi' \,|\xi, \alpha\right) \equiv \frac{Q_e(\xi,\alpha)(\xi')}{\sum_{\tilde{\xi}\in S} Q_e(\xi,\alpha)(\tilde{\xi})} = R_e^F\left(\xi,\alpha\right) Q_e(\xi,\alpha)(\xi') > 0.$$

Then $M_e(\,C\left(\omega^\tau, \xi'\right)|\,\mathcal{G}_\tau)\left(\omega\right) = m_e\left(\xi' \,|\xi_\tau\left(\omega\right), \alpha_\tau\left(\omega\right)\right)$ and so m_e can be reinterpreted as the *market belief* about the states of nature next period.

6 Financial Market Anomalies?

In this section we evaluate qualitatively and quantitatively the ability of CE and CESC allocations to generate short-term momentum and long-term reversal.

6.1 Asset returns with Aggregate Growth

To facilitate quantitative analysis, in this section we define asset returns for a growth economy. The particular form of the growth process we assume in (3) makes the state prices in the growth economy, Q_e, independent of current output. Indeed,[17]

$$Q_e(\xi,\alpha)(\xi') = \beta\ \max_h \left\{\pi_h(\xi' \,|\xi)\ \frac{(\widehat{c}_h(\xi',\alpha_e'(\xi,\alpha)(\xi'))^{-\sigma}}{(\widehat{c}_h(\xi,\alpha))^{-\sigma}}\right\}\ g(\xi')^{-1} = \widehat{Q}_e(\xi,\alpha)(\xi')\ g(\xi')^{-1}$$

[17]Note that (4) reduces to $\max_{\xi'}\left\{\beta\sum_{\xi'}\pi_i(\xi' \,|\xi)g(\xi')^{1-\sigma}\right\} < 1$.

The price-dividend ratio in the growth economy, P_e^D, is the solution to

$$P_e^D(\xi,\alpha) = \sum_{\xi'} Q_e(\xi,\alpha)(\xi')g(\xi')\ \left(1+P_e^D(\xi',\alpha_e'(\xi,\alpha)(\xi'))\right) = \sum_{\xi'} \widehat{Q}_e(\xi,\alpha)(\xi')\ \left(1+P_e^D(\xi',\alpha_e'(\xi,\alpha)(\xi'))\right)$$

Therefore, the price of the Mehra-Prescott asset can be written as $P_e^E(\xi,\alpha) = P_e^D(\xi,\alpha)\xi$

6.2 Returns

There is a simple relationship between $r^E_{t,t+4}$ and $r^E_{t+1,t+5}$.

$$\begin{aligned} r^E_{t+1,t+5} &= \frac{p_{t+5}+y_{t+2}+\ldots+y_{t+5}}{p_{t+1}} \\ &= \frac{p_{t+5}-p_{t+4}+y_{t+5}-y_{t+1}}{p_{t+1}} + \frac{p_t}{p_{t+1}} r^E_{t,t+4} \\ &= \frac{p_t}{p_{t+1}} r^E_{t,t+4} + \frac{p_{t+5}+y_{t+5}-p_{t+4}}{p_{t+1}} - \frac{y_{t+1}}{p_{t+1}} \end{aligned}$$

This simple formula makes evident that typically there is some correlation in equity returns. It also highlights that expected returns of $r^E_{t+1,t+5}$ conditional on $r^E_{t,t+4}$ are typically not zero according to the market belief. This is a key difference with the case analised by Ottaviani and Sorensen [30] and is due to the way Shiller constructs returns.

6.3 Calibration

We set $S = 4$ to allow for both aggregate and idiosyncratic risk while respecting symmetry across agents. We specify the endowment process with four values for the income of each agent and two values for the growth rate. Even and odd states correspond to high and low, respectively, growth rates. Agent 1's income share is high in state 1 and 2 and low otherwise. Because of symmetry, there are 10 parameters to be selected: six for π^*, two for $y_1(\cdot)$ and two for $g(\cdot)$. We calibrate these 10 free parameters using 10 moments describing the US aggregate and household income data. The first moment is the autocorrelation of the growth rates. There is no unanimity in the profession as to what is the most reasonable value for this moment. On the one hand, Mehra and Prescott use Shiller's annual data to argue that it is negative. On the other hand, recen work using quarterly data for the post-war period suggests it is positive (see Campbell [11]). However, this has been criticised because the methodology to construct the consumption series induces spurious correlations (see Ferson and Harvey [20]). In this work we assume growth rates are uncorrelated. The advantage is that any correlation in returns we obtain cannot be attributed to correlation of dividend growth. The remaining nine moments are given by M2-M10 of Alvarez and Jermann (see Appendix C for the calibrated parameters.)

We say that a model's predictions are *qualitatively* accurate if the model generates a time series of returns that displays both short-term momentum and long-term reversal for some values of β and σ. We say that a model's predictions are *quantitatively* accurate if the model generates a time series of returns that displays both short-term momentum as well as long-term reversal of an order of magnitude as those in Table 1 when β and σ are set to minimise the distance between the model predictions to the average annual risk-free rate of 2.36% and equity premium of 5.91%.

6.4 CE Allocations

In this section we evaluate the case of PO allocations. It is well known that CE prices and excess returns converge to those of an economy where only agents with correct beliefs have positive wealth (see Sandroni [34] and Blume and Easley [10]). Thus, without loss in generality, we restrict attention to the case where everybody has correct beliefs, i.e. the Lucas tree model.[18] We find that the predictions of the Lucas tree model are quantitatively correct in the short-term as the model generates significant short-term momentum. However, the predictions are both quantitatively and qualitatively inaccurate as the Lucas tree model is unable to generate long-term reversals when the growth rates are uncorrelated.

Since we assume growth rates are uncorrelated, the risk-free rate and the price dividend ratio are constant. Indeed, for any (ξ, α)

$$
\begin{aligned}
R^F_{po}(\xi,\alpha) &= \frac{1}{\beta E(g^{-\sigma})} \\
P^D_{po}(\xi,\alpha) &= \frac{\beta E(g^{1-\sigma})}{1-\beta E(g^{1-\sigma})} \equiv \phi(\beta,\sigma)
\end{aligned}
$$

Thus, equity returns can be written as:

$$
R^E_{po}(\xi_0,\alpha)(\xi_1,...,\xi_4) = \frac{\phi\xi_4+\xi_1+...+\xi_4}{\phi\xi_0} = \frac{1+\phi}{\phi}\prod_{i=1}^{4} g_{t+i} + \frac{1}{\phi}\left(g_{t+1} + \prod_{i=1}^{2} g_{t+i} ... + \prod_{i=1}^{3} g_{t+i}\right)
$$

Since the risk-free rate is history independent, the correlation of excess returns equals that of the equity returns. Thus,

$$
\begin{aligned}
cov^{P_{po}}(R_{4,po}, R_{\tau+4,po}) &= cov^{P^{\pi^*}}\left(\frac{1+\phi}{\phi}\prod_{i=1}^{4} g_i + \frac{1}{\phi}\sum_{j=1}^{3}\prod_{i=1}^{j} g_i, \frac{1+\phi}{\phi}\prod_{i=1}^{4} g_{\tau+i} + \frac{1}{\phi}\sum_{j=1}^{3}\prod_{i=1}^{j} g_{\tau+i}\right) \\
var^{P_{po}}(R_{4,po}) &= var^{P^{\pi^*}}\left(\frac{1+\phi}{\phi}\prod_{i=1}^{4} g_i + \frac{1}{\phi}\sum_{j=1}^{3}\prod_{i=1}^{j} g_i\right)
\end{aligned}
$$

6.4.1 Qualitative Predictions

To evaluate the model qualitatively, we first obtain the closed form solutions for the correlations. It is straightforward to prove that

Proposition 11. *For any β and σ, the autocorrelations of order 1 to 3 are positive and every autocorrelation of order 4 or higher is zero. Moreover,*

$$
\lim_{\phi\to\infty} \frac{cov^{P_{po}}(R_{4,po}, R_{\tau+4,po})}{var^{P_{po}}(R_{4,po})} = \frac{cov^{P^{\pi^*}}\left(\prod_{i=1}^{4} g_i, \prod_{i=1}^{4} g_{\tau+i}\right)}{var^{P^{\pi^*}}\left(\prod_{i=1}^{4} g_i\right)} = \bar{g}^{2\tau}\frac{\left(var(g)+\bar{g}^2\right)^{4-\tau} - \left(\bar{g}^2\right)^{4-\tau}}{\left(var(g)+\bar{g}^2\right)^{4} - \left(\bar{g}^2\right)^{4}}
$$

[18] Note that since we are interested in asymptotic results, this restriction is without loss in generality

where $\overline{g}$ and $var(g)$ are the mean and variance of the growth rates, respectively.

The first part of Proposition 11 imply that CE allocations always display short-term momentum but they never display long-term reversal. Consequently, the short-term predictions of the Lucas tree model are qualitatively accurate but the long-term predictions are not. The second part of Proposition 11 shows that the limit of the correlation coefficient, as the price dividend ratio grows to infinity, is independent of β and σ. This result would prove useful to understand the quantitative predictions in the next section.

6.4.2 Quantitative Predictions

To evaluate the model quantitatively, we set β so that the model matches the historical average of the risk-free rate and we let σ between 1 and 500. We denote by $\beta(\sigma)$ the calibrated value of β.

Figure 2 plots the equity premium and the implied β for values of σ between 1 and 500.

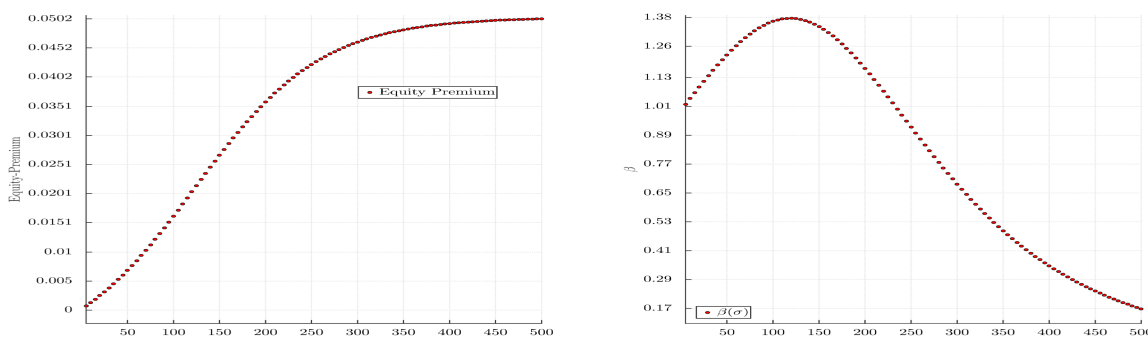

Figure 2: Equity-Premium and Calibrated β in CE.

The plot on the left-hand side of Figure 2 shows that the model generates an equity premium very close to the historical data for values of σ above 250. This is in line with previous results in the literature (see Mehra and Prescott [29]). Moreover, the plot on the right-hand side shows that the implied $\beta(\sigma)$ is smaller than one for values of σ larger than 250.

The left and righ-hand side of Figure 3 plots the functions $\phi(\beta(\sigma), \sigma)$ and $\frac{cov^{P_{po}}(R_{4,po}, R_{\tau+4,po})}{var^{P_{po}}(R_{4,po})}$ as a function of σ, respectively.

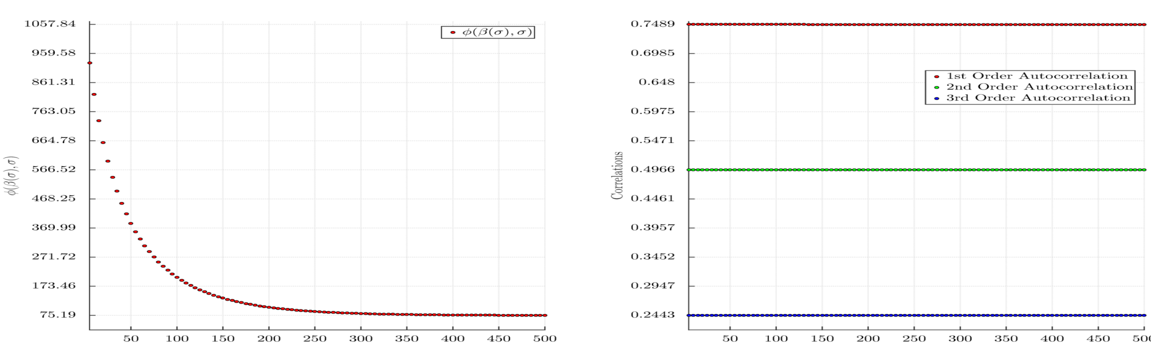

Figure 3: Price-Dividend Ratios and Correlations in CE.

Remarkably, the autocorrelations of order 1-3 are approximately $0.7489, 0.4966$ and 0.2443, respectively, and so they are very close to the targets in Table 1. One can easily see that the price-dividend ratio is larger than 75 and that correlations seem to be constant with respect to σ. The latter can be understood using the second part of Proposition 11. Indeed, in our calibration $\overline{g} = 1.0050$ and $var(g) = 0.0051$ and so the limit values of the correlation coefficients of order 1 to 3 are given by $0.7481, 0.4975$ and 0.2481, respectively. The following Theorem summarises our finding for Competitive Equilibria.

Theorem 12. *Competitive equilibrium allocations generate quantitatively significant short-term momentum but cannot generate long-term reversal.*

6.5 CESC Allocations: Homogeneous and Heterogeneous Beliefs

In this section we report the results of our numerical simulations for CESC. We first report the autocorrelations for the calibrated economy and then we explain the role played by belief heterogeneity.

Agent 1 has correct beliefs and agent 2 believes the transition matrix belongs to the a family parameterised by $\varepsilon_R \in (-\pi^*(2|\,1), \pi^*(1|\,1))$ and $\varepsilon_E \in (-\pi^*(1|\,2), \pi^*(2|\,2))$ given by:

$$\pi^* + \begin{bmatrix} -\varepsilon_R & \varepsilon_R & 0 & 0 \\ \varepsilon_E & -\varepsilon_E & 0 & 0 \\ 0 & 0 & -\varepsilon_R & \varepsilon_R \\ 0 & 0 & \varepsilon_E & -\varepsilon_E \end{bmatrix}$$

Note that agent 2 has (possibly) incorrect beliefs regarding the persistency of recessions and expansions, i.e. $\pi_2(1|\,1) = \pi_2(3|\,3) = 0.1146 - \varepsilon_R$ and $\pi_2(2|\,2) = \pi_2(4|\,4) = 0.7831 - \varepsilon_E$, and correct beliefs otherwise. In particular, he has correct beliefs regarding the idiosyncratic state.[19]

6.5.1 Qualitative Predictions

We first argue that optimism can generate autocorrelations of the same sign than those in Table 1. We set $\sigma = 0.8020$ and $\beta = 0.9520$ and we restrict $|\varepsilon_R|$ and $|\varepsilon_E|$ to take values in the sets $\{0, 0.02, 0.4\}$ and $\{0, 0.06, 0.10\}$ representing correct, moderate and highly optimistic beliefs, respectively. Figures 4 - 6 plot the autocorrelations of order 1-8 in the data (full red circles), in a CESC with correct beliefs[20] (blue and white circles) and in a CESC with heterogeneous beliefs given by one combination of $(\varepsilon_R, \varepsilon_E)$ in the set described above (black squares). The first column correspond to moderate optimism while the second colum correspond to high optimism.

[19] Clearly, this parameterisation satisfies A3.
[20] That is, the Alvarez-Jermann model.

In Figure 4 we assume agent 2 is optimistic only during recessions (states 1 and 3). We show the asset displays short-term momentum for highly optimistic beliefs but it does not display long-term reversal.

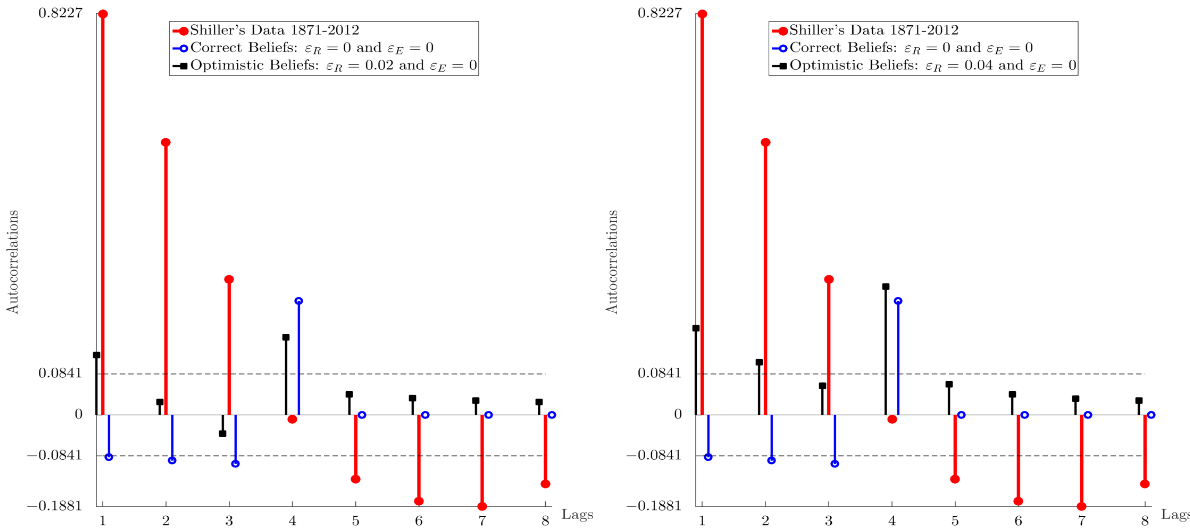

Figure 4: Autocorrelations in CESC for $\sigma = 0.8020$, $\beta = 0.9520$ – Optimism during recessions

In Figure 5 we assume agent 2 is optimistic only during expansions (states 2 and 4). We show the asset displays both short-term momentum and long-term reversal for highly optimistic beliefs.

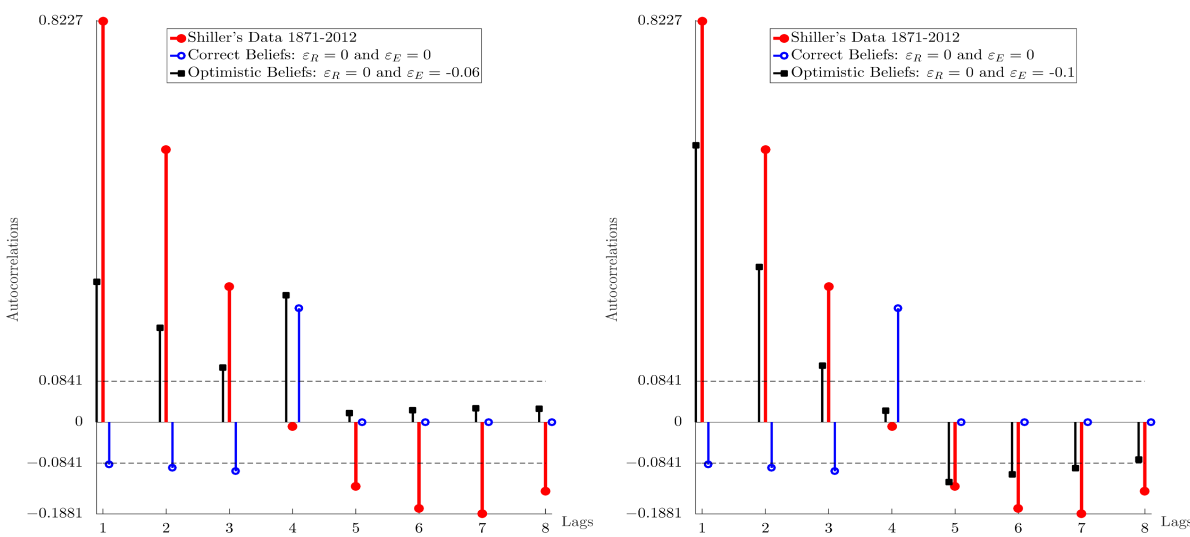

Figure 5: Autocorrelations in CESC for $\sigma = 0.8020$, $\beta = 0.9520$ – Optimism during expansions

In Figure 6 we assume agent 2 is optimistic during both states. Again, the asset displays both short-term momentum and long-term reversal when agent 2 displays high optimism.

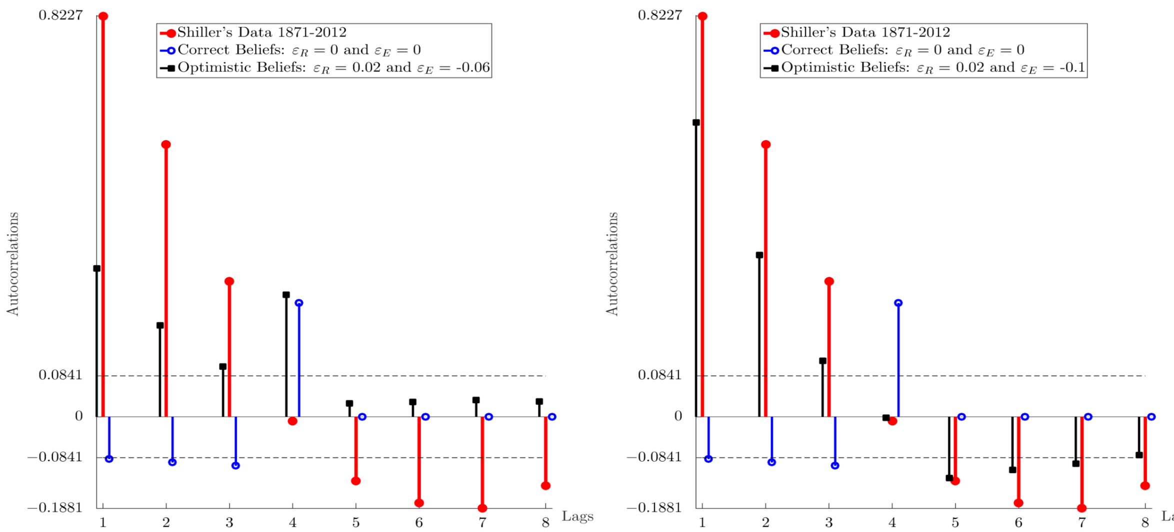

Figure 6: Autocorrelations in CESC for $\sigma = 0.8020$, $\beta = 0.9520$ - Optimism

Interestingly, CESC allocations with correct beliefs can neither generate short-term momentum nor long-term reversals. Since CE allocations could not generate long-term reversals either, we conclude that it is the interaction of belief heterogeneity, optimism and limited enforceability that generates short-term momentum and long-term reversal when $\sigma = 0.8020$, and $\beta = 0.9520$.

To understand how short-term momentum arises, in Figure 7 we plot the cumulative distribution of the conditional equity premium after a negative (the dashed blue line) and a positive abnormal return (the full black line), respectively. For completeness we also plot the cumulative distribution of the conditional equity-premium (the full red line).

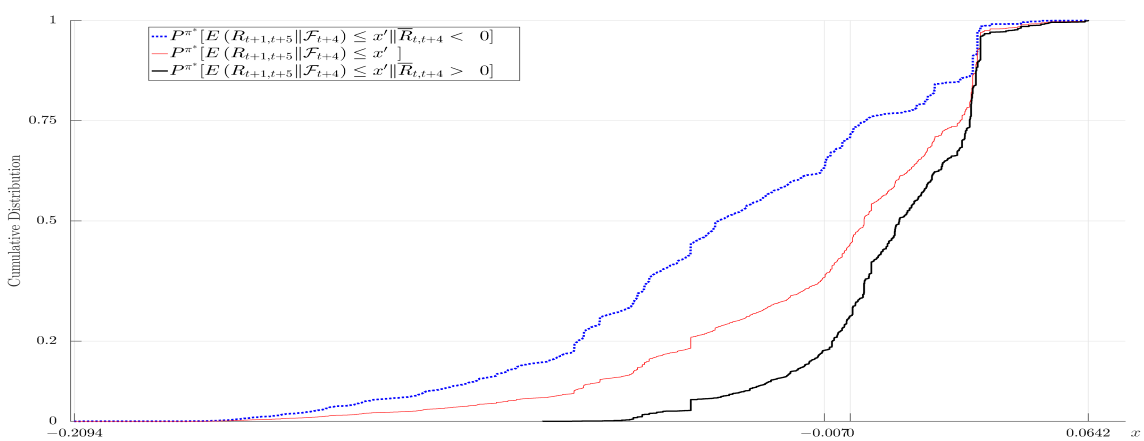

Figure 7: Conditional equity-premium cumulative distribution for $(\sigma, \beta, \varepsilon_R, \varepsilon_E) = (0.8020, 0.9520, 0.02, -0.06)$

We see that the conditional equity premium is negative with very large probability (above 70%) after a negative abnormal return (bad news) and positive with very large probability (around 70%) after a positive abnormal return (good news). That is, the return underreacts by date 5 with very high probability. In the light of Proposition 9, it is not surprising that short-term momentum occurs.

Next we allow $|\varepsilon_R|$ and $|\varepsilon_E|$ to take values in the set $\{0, 0.02, 0.04, 0.06, 0.08, 0.10\}$ and we ask what is the subset of heterogeneous beliefs for which the sign of the autocorrelations coincide with the sign observed in the data for some values of σ and β. For each element in that subset of heterogeneous beliefs, we choose σ and β to minimise the distance to the observed risk-free rate and equity-premium.

Figures 8 - 10 plots our results. Each plot displays the autocorrelations of order 1-8 for three cases: (a) in the data (the full red circles), (b) CESC with correct beliefs (the blue and white circles) and CESC for one combination of $(\varepsilon_R, \varepsilon_E)$ (the black squares) for which the signs of the autocorrelations are as in the data. Our numerical simulations show that for values of $|\varepsilon_E| < 0.06$, the signs are not as expected. The model, however, does predict the right signs if agent 2 is highly optimistic during expansions, that is for $\varepsilon_E \geq -0.06$. This is the reason why we only report simulations where $\varepsilon_E \in \{-0.08, -0.1\}$.

Figure 8 plots the case where the agent has ambiguous beliefs as it is pessimistic during recessions and optimistic during expansions.

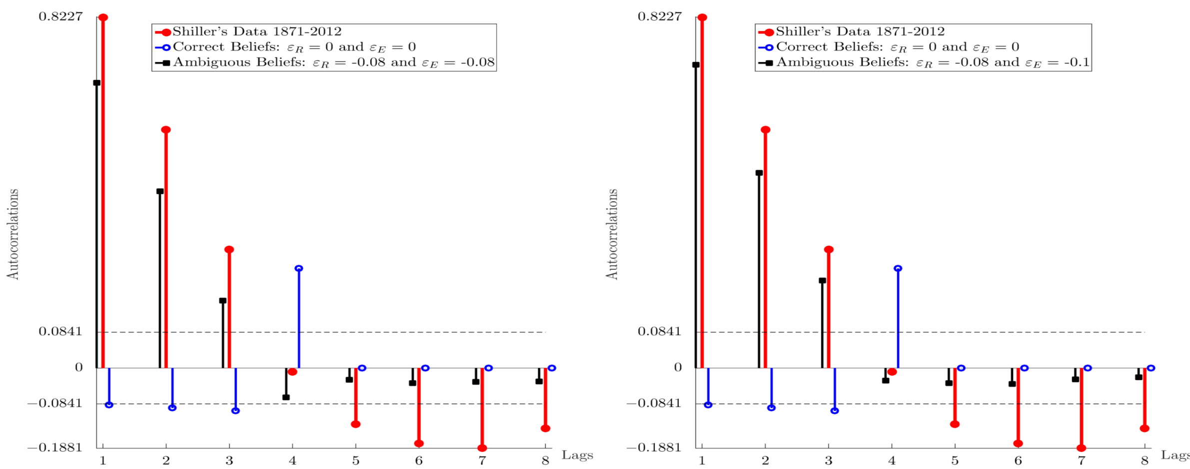

Figure 8: Autocorrelations in CESC for calibrated σ and β

Figure 9 plots the case where the agent is optimistic only during expansions.

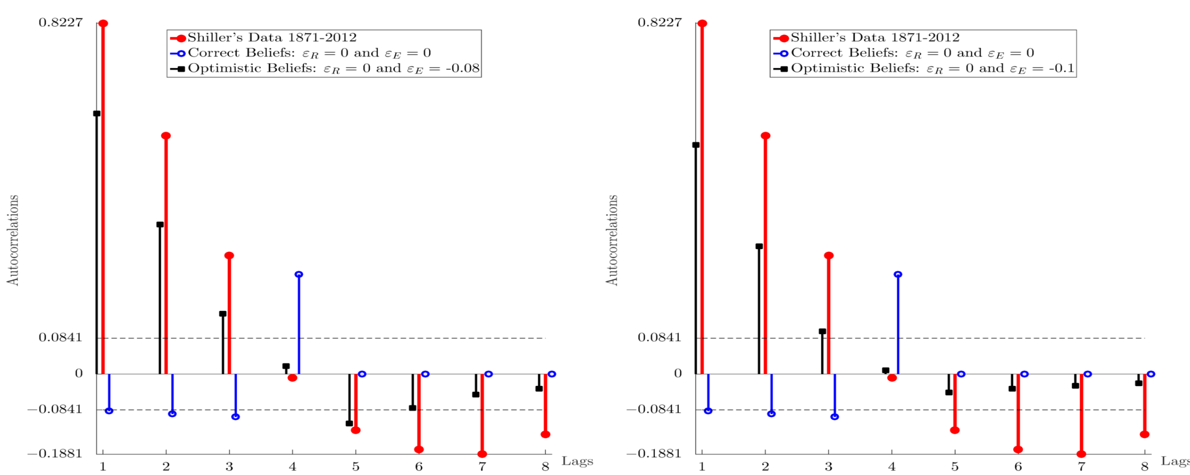

Figure 9: Autocorrelations in CESC for calibrated σ and β

Figure 10 plots the case where the agent is optimistic during both recessions and expansions.

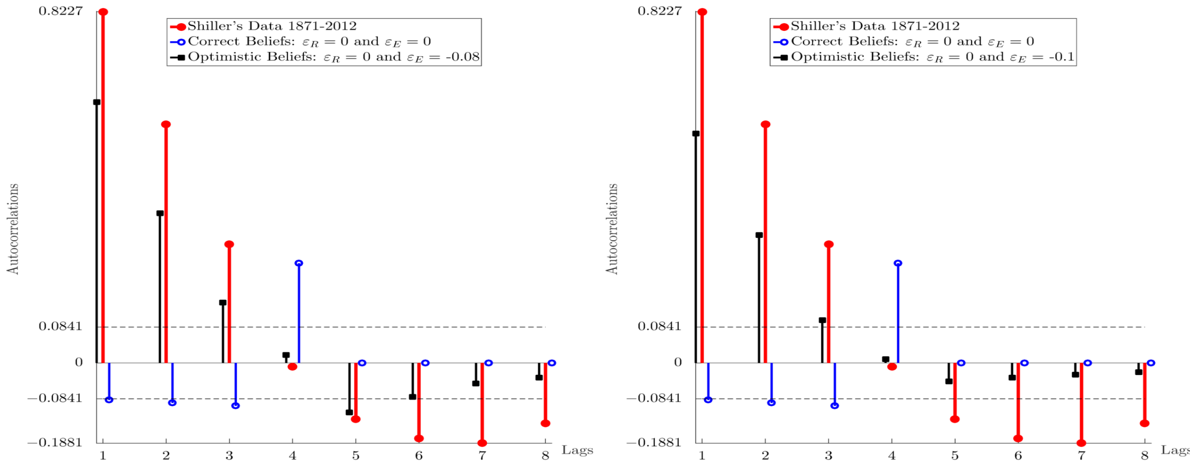

Figure 10: Autocorrelations in CESC for calibrated σ and β

We underscore that neither the model with correct beliefs nor with pessimistic can generate both short-term momentum and long-term reversal. We conclude that the predictions of CESC are qualitatively accurate only if agents are sufficiently optimistic during expansions. Pessimism during recessions tend to make the 4th autocorrelation too negative.

Figure 11 plots the values of the risk-free rate and the equity-premium for the calibrated parameters. The model predicts a high interest rate and a negative equity premium.[21]

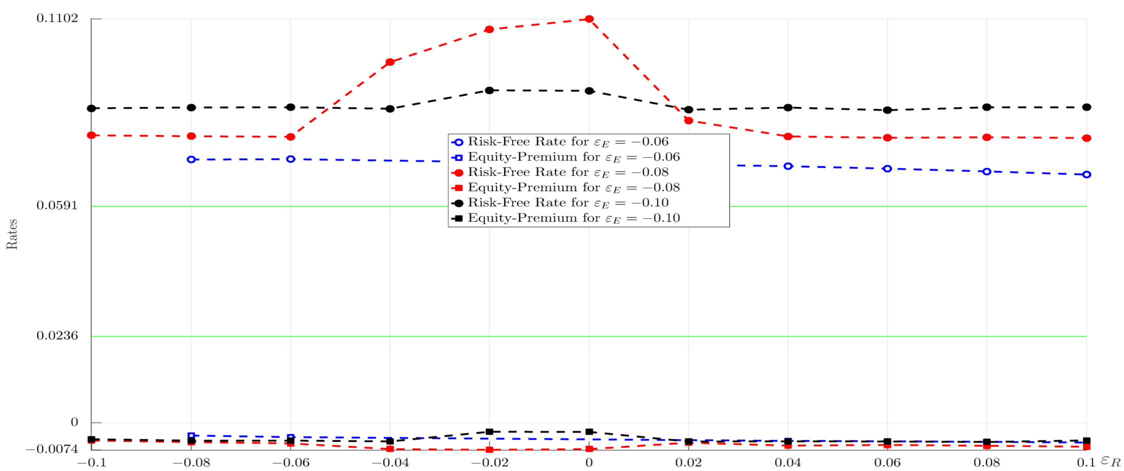

Figure 11: Average Risk-Free Rate and Equity-Premium in CESC for calibrated σ and β

6.5.2 Quantitative Predictions

In this section we ask whether CESC allocations can account for the risk-free rate and the equity-premium in Table 1. To address this, we consider a range of values for $(\varepsilon_E, \varepsilon_R)$ and we choose σ and

[21]Figure 16 in Appendix C plots the calibrated values of σ and β as a function of $(\varepsilon_E, \varepsilon_R)$.

β to match the observed risk-free rate and equity-premium. We plot our results in Figure 12

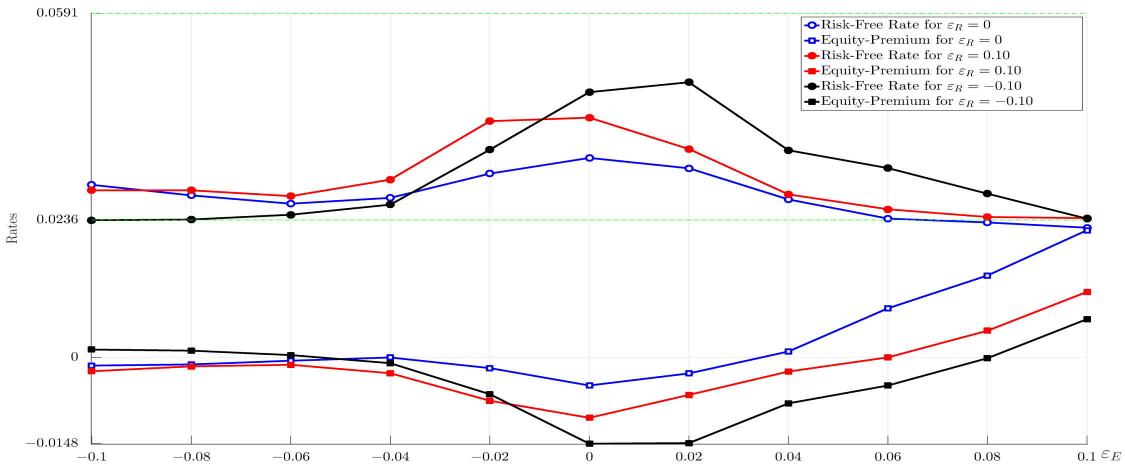

Figure 12: Average Risk-Free Rate and Equity-Premium in CESC for calibrated σ and β

We can see the model does a good job in matching the risk-free rate but it needs substancial pessimism during expansions (positive and large ε_E) to generate a significant equity premium.[22]

Figures 13 - 15 evaluate the ability of the model to generate short-term momentum and long-term reversal. The plot on the left and right-hand sides corresponds to pessimistic and optimistic beliefs, respectively.

Figure 13 plots our results for small pessimism and optimism.

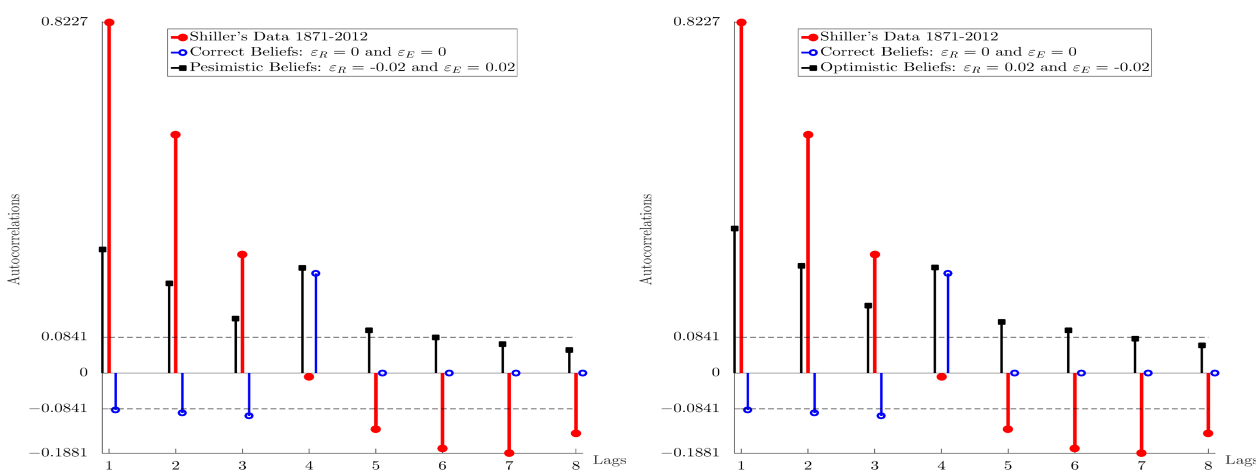

Figure 13: Autocorrelations in CESC for calibrated σ and β – Small pessimism and optimism

[22]Figure 17 in Appendix C plots the calibrated values of σ and β as a function of $(\varepsilon_E, \varepsilon_R)$.

Figure 14 plots our results for moderate pessimism and optimism.

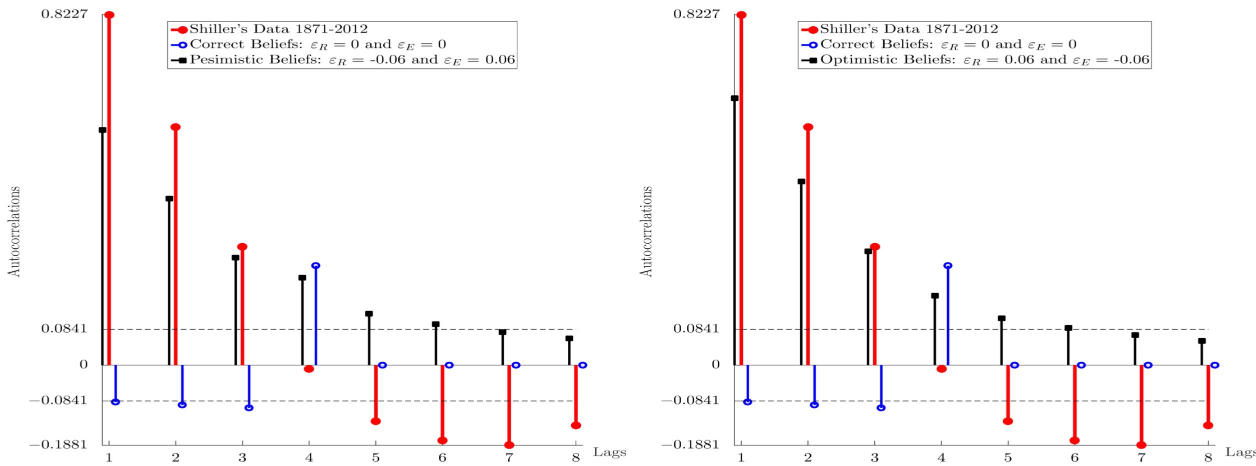

Figure 14: Autocorrelations in CESC for calibrated σ and β – Moderate pessimism and optimism

Finally, Figure 15 plots our results for large pessimism and optimism.

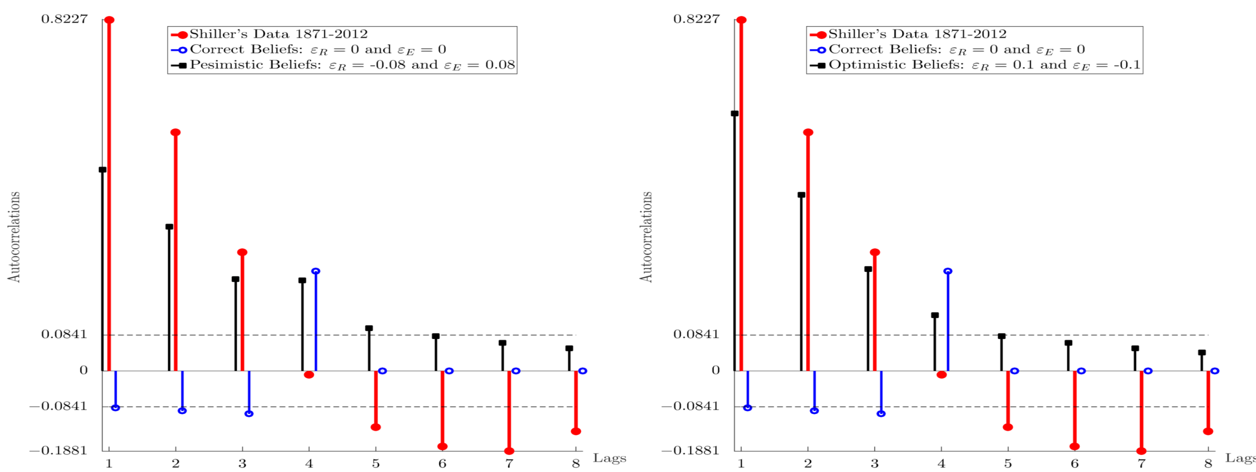

Figure 15: Autocorrelations in CESC for calibrated σ and β – Large pessimism and optimism

Although the model generates short-term momentum, it fails to generate long-term reversal both for optimistic and pesimistic beliefs. We conclude its predictions are never quantitatively accurate.

Theorem 13. *CESC allocations predictions are qualitatively accurate if agents are highly optimistic during expansions but they are never quantitatively accurate.*

Appendix A

In this Appendix we prove the results in Section 3. We begin with some definitions.

Let $f : S \times \mathbb{R}^I_+ \times \mathcal{P}(\Pi) \to \mathbb{R}_+$, $\|f\| \equiv \sup_{(\xi,\alpha)} \mid f(\xi,\alpha,\mu) : \alpha \in \Delta^{I-1} \mid$ and

$$
\begin{aligned}
F &\equiv \{f : S \times \mathbb{R}^I_+ \times \mathcal{P}(\Pi) \to \mathbb{R}_+ : \ f \text{ is continuous and } \|f\| < \infty\}. \\
F_H &\equiv \left\{ f \in F : f(\xi,\alpha) - \sum_{i=1}^{I} \alpha_i U_i(\xi,\mu_i) \geq 0 \text{ for all } (\xi,\alpha,\mu), \text{ HOD 1 w.r.t. } \alpha \right\}
\end{aligned}
$$

where HOD 1 stands for *homogeneous of degree one*. F_H is a closed subset of the Banach space F and thus a Banach space itself. Continuity is with respect to the weak topology and thus the metric on F is induced by $\|.\|$.

Given $(\xi,\alpha) \in S \times \mathbb{R}^I_+ \times \mathcal{P}(\Pi)$, we define the operator T on F_H as follows

$$
(Tf)(\xi,\alpha) = \max_{(c,w'(\xi'))} \sum_{i=1}^{I} \alpha_i \left\{ u_i(c_i) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)\ w'_i(\xi') \right\}, \tag{16}
$$

subject to

$$
c_i \geq 0, \qquad \sum_{i=1}^{I} c_i = y(\xi), \tag{17}
$$

$$
u_i(c_i) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)\ w'_i(\xi') \geq U_i(\xi,\mu_i), \tag{18}
$$

$$
w'_i(\xi') \geq U_i(\xi', \mu'_i(\xi,\mu)(\xi')) \qquad \text{for all } \xi', \tag{19}
$$

$$
\left[f(\xi', \tilde{\alpha}, \mu'(\xi,\mu)(\xi')) - \sum_{i=1}^{I} \tilde{\alpha}_i\ w'_i(\xi') \right] \geq 0 \tag{20}
$$

for all $\tilde{\alpha} \in \Delta^{I-1}$ and all ξ'.

Now define

$$
\begin{aligned}
\mathcal{U}^E(\xi,\mu)(f) \equiv \big\{ & w \in \mathbf{R}^I_+ : \exists (c,w') \text{ such that } (18) - (20) \text{ are satisfied} \\
& \text{and } w_i = u_i(c_i) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)\ w'_i(\xi') \big\}.
\end{aligned}
$$

The following Lemma characterises the utility possibility set and follows from a reasoning analogous to the one in Lemma 14 in Beker and Espino [5].

Lemma A.1. *$u \in \mathcal{U}^E(\xi,\mu)(f)$ if and only if $u_i \geq U_i(\xi,\mu_i)$ for all i and*

$$
\left[Tf(\xi,\alpha,\mu) - \sum_{i=1}^{I} \alpha_i\ u_i \right] \geq 0.
$$

for all $\alpha \in \Delta^{I-1}$.

It is easy to see that condition $(A.1)$ is satisfied if and only if

$$\min_{\tilde{\alpha}\in\Delta^{I-1}}\left[Tf(\xi,\tilde{\alpha},\mu)-\sum_{i=1}^{I}\alpha_i\ u_i\right]\geq 0.$$

Theorem 1 follows from Propositions A.2 and A.3. We say that $f\in F_H$ is *preserved under* T if $f(\xi,\alpha,\mu)\leq (Tf)(\xi,\alpha)$ for all (ξ,α,μ).

Proposition A.2. *If $f\in F_H$ is preserved under T, then $(Tf)(\xi,\alpha)\leq v^*(\xi,\alpha)$ for all (ξ,α).*

Proof. Let $f\in F_H$ and define $\mathcal{W}(\xi,\mu)(f)$ as the constraint correspondence defined by (17)-(20) evaluated at f and (ξ,μ).

Take any arbitrary $\left(\tilde{c}_0,(\tilde{w}_1'(\xi'),\tilde{\alpha}_1'(\xi'))_{\xi'}\right)\in\mathcal{W}(\xi,\mu)(f)$ and notice that this implies, by (20) and Lemma $A.1$, that

$$\sum_{i=1}^{I}\alpha'\tilde{w}_{i,1}'(s_1)\leq f(s_1,\alpha',\mu_{s^1}) \tag{21}$$

for all $\alpha'\in\Delta^{I-1}$. On the other hand, since f is preserved under T, it follows from (20) that

$$f(s_1,\alpha_1',\mu_{s^1})\leq (Tf)(s_1,\alpha_1',\mu_{s^1}) \tag{22}$$

for all $\alpha'\in\Delta^{I-1}$.

Hence, as we couple conditions 21 and (22), we conclude that

$$\sum_{i=1}^{I}\alpha'\tilde{w}_{i,1}'(s_1)\leq (Tf)(s_1,\alpha_1',\mu_{s^1}) \tag{23}$$

for all $\alpha'\in\Delta^{I-1}$ and therefore $\tilde{w}_1'(s_1)\in\mathcal{U}^E(s_1,\mu_{s^1})(f)$ as a direct implication of Lemma $A.1$. Therefore, there exists some $\left(\tilde{c}_1(s_1),(\tilde{w}_2'(s_1,\xi'),\tilde{\alpha}_2'(s_1,\xi'))_{\xi'}\right)\in\mathcal{W}(s_1,\mu_{s^1})(f)$ such that

$$\tilde{w}_{i,1}'(s_1)=u_i(\tilde{c}_{i,1}(s_1))+\beta(s_1,\mu_{s^1})\sum_{\xi'}\pi_{\mu_{i,s^1}}(\xi'\,|s_1)\ \tilde{w}_{i,2}'(s_1,\xi')\text{ for all } i$$

Following this strategy, one can construct a collection of functions $\{\tilde{c}_t(s^t),\tilde{w}_t(s^t)\}$ for all s^t and $t\geq 1$. Define, $\{c_t\}_{t=0}^{\infty}\in\mathbb{C}(\xi)$ as follows:

$$\begin{aligned} &c_0=\tilde{c}_0,\quad c_t(s)=\tilde{c}_t\left(s^t\right) && \text{for all } s \text{ and } t\geq 1,\\ &w_t(s)=\tilde{w}_t\left(s^t\right) && \text{for all } s \text{ and } t\geq 1.\end{aligned}$$

Since $\{c_t\}_{t=0}^{\infty}$ is feasible by construction, we show it is enforceable. By construction, we have that

$$\begin{aligned}\left|U_i(c_i)(s^t)-\tilde{w}_{i,t}'(s^t)\right| &\leq \overline{\beta}\left|\sum_{\xi'}\pi_{\pi_i}(\xi'\,|s_t)\left(U_i(c_i)(s^t,\xi')-\tilde{w}_{i,t}'(s^t,\xi')\right)\right|\\ &\leq \overline{\beta}\sup_{\xi'}\left|U_i(c_i)(s^t,\xi')-\tilde{w}_{i,t}'(s^t,\xi')\right|\\ &\leq \overline{\beta}^k\sup_{(\xi_1',\ldots\xi_k')}\left|U_i(c_i)(s^t,\xi_1',\ldots\xi_k')-\tilde{w}_{i,t}'(s^t,\xi_1',\ldots\xi_k')\right|.\end{aligned}$$

Since $U_i(s_t, \pi_i) \leq \tilde{w}'_{i,t} \leq \|f\| < \infty$ for all i and t and $U_i(\cdot)$ is uniformly bounded, it follows that

$$\begin{aligned} \left|U_i(c_i)(s^t) - \tilde{w}'_{i,t}(s^t)\right| &\leq \limsup_{k\to\infty} \left\{ \overline{\beta}^k \sup_{(\xi'_1,\ldots\xi'_k)} \left|U_i(c_i)(s^t, \xi'_1, \ldots\xi'_k) - \tilde{w}'_{i,t}(s^t, \xi'_1, \ldots\xi'_k)\right| \right\} \\ &= 0. \end{aligned}$$

and consequently $U_i(c_i)(s^t) = \tilde{w}'_{i,t}(s^t)$ for all i and all s^t. Finally, since by construction $\tilde{w}'_{i,t}(s^t) \geq U_i(s_t, \pi_i)$ for all i, we can conclude that $\{c_t\}_{t=0}^{\infty}$ is enforceable.

We conclude that for any arbitrary $\alpha \in \mathbb{R}^I_+$

$$\begin{aligned} & \sum_{i=1}^{I} \alpha_i \left(u_i(\tilde{c}_{i,0}) + \beta(\xi, \mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)\ \tilde{w}'_{i,1}(\xi') \right) \\ = & \sum_{i=1}^{I} \alpha_i u_i(c_{i,0}) + \sum_{i=1}^{I} \alpha_i E^{P_i} \left(\rho_{i,0}\ w'_{i,1}\right) \\ = & \sum_{i=1}^{I} \alpha_i E^{P_i} \left(\sum_{t=0}^{T} \rho_{i,t}\ u_i(c_{i,t}) \right) + \sum_{i=1}^{I} \alpha_i E^{P_i} \left(\rho_{i,T+1}\ w'_{i,T+1}\right) \\ \leq & \sum_{i=1}^{I} \alpha_i E^{P_i} \left(\sum_{t=0}^{T} \rho_{i,t}\ u_i(c_{i,t}) \right) + \overline{\beta}^{T+1} \|f\|. \end{aligned}$$

where the inequality follows from the first inequality in (22). Taking limits, we obtain

$$\sum_{i=1}^{I} \alpha_i \left(u_i(\tilde{c}_{i,0}) + \beta(\xi, \mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)\ \tilde{w}'_{i,1}(\xi') \right) \leq \sum_{i=1}^{I} \alpha_i E^{P_i} \left(\sum_{t=0}^{\infty} \rho_{i,t}\ u_i(c_{i,t}) \right) \leq v^*(\xi, \alpha).$$

where the first inequality follows because weak inequalities are preserved under limits and $\overline{\beta} \in (0,1)$ and the last one because $\{c_t\}_{t=0}^{\infty}$ is enforceable.

Since $\left(\tilde{c}_0, (\tilde{w}'_1(\xi'), \tilde{\alpha}'_1(\xi'))_{\xi'}\right) \in \mathcal{W}(\xi, \mu)(f)$ is arbitrary,

$$\sum_{i=1}^{I} \alpha_i \left(u_i(\tilde{c}_{i,0}) + \beta(\xi, \mu_i) \sum_{\xi'} \pi_{\mu}(\xi' \,|\xi)\ \tilde{w}'_{i,1}(\xi') \right) \leq v^*(\xi, \alpha),$$

for all $\left(\tilde{c}_0, (\tilde{w}'_1(\xi'), \tilde{\alpha}'_1(\xi'))_{\xi'}\right) \in \mathcal{W}(\xi, \mu)(f)$. Therefore,

$$\begin{aligned} Tf(\xi, \alpha) &= \max_{(c,w')\in\mathcal{W}(\xi,\mu)} \sum_{i=1}^{I} \alpha_i \left(u_i(c_i) + \beta(\xi, \mu_i) \sum_{\xi'} \pi_{\mu}(\xi' \,|\xi)\ w'_i(\xi') \right) \\ &\leq v^*(\xi, \alpha). \end{aligned}$$

as desired. □

Proposition A.3. *$v^* \in F_H$ is preserved under T and $v^*(\xi, \alpha) = (Tv^*)(\xi, \alpha, \mu)$ for all (ξ, α).*

Proof. Given (ξ, α), take any $u \in \mathcal{U}^E(\xi, \mu)$ and let $c \in \mathbb{C}(\xi)$ denote the corresponding enforceable feasible allocation. For each ξ', ${}_{\xi'}c_i \in \mathbb{C}(\xi')$ given by

$${}_{\xi'}c_i(s^t) = c_i(\xi', s^t) \text{ for all } t \geq 1,$$

denotes the ξ'−continuation of c_i. For every $t \geq 1$, let

$$P_{i,\xi'}(s^t) = \frac{P_i(C(s^t))}{\pi_{\pi_i}(\xi' \,|s_t)},$$

and note that

$$\sum_{i=1}^{I} \alpha_i U_i^{P_i}(c_i) = \sum_{i=1}^{I} \alpha_i \left[u_i(c_{i,0}) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_{\mu_i}(\xi' \,|\xi)) U_i^{P_{i,\xi'}}({}_{\xi'}c_i) \right].$$

Since $\left(U_i^{P_{i,\xi'}}({}_{\xi'}c_i)\right)_{i=1}^{I} \in \mathcal{U}(\xi', \mu'(\xi,\mu)(\xi'))$ for all ξ', it follows by Lemma A.1 that

$$\sum_{i=1}^{I} \alpha_i' U_i^{P_{i,\xi'}}({}_{\xi'}c_i) \leq v^*(\xi', \mu'(\xi,\mu)(\xi'), \alpha') \quad \text{for all } \xi' \text{ and } \alpha' \in \Delta^{I-1}$$

and so

$$\sum_{i=1}^{I} \alpha_i U_i^{P_i}(c_i) \leq (Tv^*)(\xi,\alpha) \quad \text{for all } \xi' \text{ and } \alpha' \in \Delta^{I-1}.$$

We conclude that v^* is preserved under T since

$$v^*(\xi,\alpha) = \sup_{c \in Y^\infty} \sum_{i=1}^{I} \alpha_i U_i^{P_i}(c_i) \leq (Tv^*)(\xi,\alpha) \quad \text{for all } (\xi,\alpha).$$

It follows from Proposition A.2 that $(Tv^*)(\xi,\alpha) \leq v^*(\xi,\alpha)$ and so $v^*(\xi,\alpha) = (Tv^*)(\xi,\alpha)$ for all (ξ,α). □

Proof of Theorem 1. Since Proposition A.3 shows that v^* is a fixed point of T, the rest of the proof is analogous to that of Theorem 2 in Beker and Espino [5]. □

Proof of Proposition 3. Note that T^n is a monotone operator for all $n \geq 1$ (i.e., if $f \geq g$ then $T^n f \geq T^n g$.) Let $\tilde{T}$ be the operator when the enforceability constraints are ignored. Theorem 2 in Beker and Espino [5] show that $\tilde{T}$ has a unique fixed point, say $\tilde{v}$.

Let $\{v_n\}_{n=0}^{\infty}$ be the sequence of functions defined by $v_0 = \tilde{v}$ and $v_n = T(v_{n-1})$ for all $n \geq 1$. Next we show that $v_n \geq v_{n+1} \geq v^*$ for all n. Indeed, since $T(\tilde{v}) \leq \tilde{T}(\tilde{v}) = \tilde{v}$, it follows that $v_1 \leq v_0$ and

$$\begin{aligned} v_{n+1} &= T^{n+1}(\tilde{v}) = T^n(T\tilde{v}) = T^n(v_1) \leq T^n(v_0) = v_n && \text{for all } n, \\ v_{n+1} &= T^{n+1}(\tilde{v}) \geq T^{n+1}(v^*) = v^* && \text{for all } n. \end{aligned}$$

Since $\{v_n\}_{n=0}^{\infty}$ is a monotone decreasing sequence of uniformly bounded functions bounded below by v^*, there exists a uniformly bounded function $v_\infty \geq v^*$ such that $\lim_{n\to\infty} v_n = v_\infty$. To show that $v_\infty \leq v^*$ we argue that v_∞ is preserved under T and apply Proposition A.2.

Given (ξ,α), $v_\infty(\xi,\alpha) \leq v_n(\xi,\alpha,\mu)$ for all n and so there is $\left(\tilde{c}_n, (\tilde{w}_n'(\xi'), \tilde{\alpha}_n'(\xi'))_{\xi'}\right) \in W(\xi,\mu)(v_n)$ such that

$$v_n(\xi,\alpha) = \sum_{i=1}^{I} \alpha_i \left(u_i(\tilde{c}_{i,n}) + \beta(\xi,\mu_i) \sum_{\xi'} \pi_\mu(\xi' \,|\xi)\ \tilde{w}_{i,n}'(\xi') \right), \quad \text{for all } n. \tag{24}$$

Since $\left(\tilde{c}_n, (\tilde{w}'_n(\xi'), \tilde{\alpha}'_n(\xi'))_{\xi'}\right)$ lies in a compact set, it has a convergent subsequence with limit point $\left(\tilde{c}, (\tilde{w}'(\xi'), \tilde{\alpha}'(\xi'))_{\xi'}\right)$.

Note that

$$\begin{aligned} v_n(\xi', \alpha'(\xi,\mu)(\xi'), \mu'(\xi,\mu)(\xi')) - \sum_{i=1}^{I} \alpha'_i \ \tilde{w}'_{i,n}(\xi') &\geq 0, \\ \tilde{w}'_{i,n}(\xi') - U_i(\xi, \mu_i) &\geq 0, \end{aligned}$$

for all n and all ξ'. Since weak inequalities are preserved in the limit

$$\begin{aligned} v_\infty(\xi', \alpha'(\xi,\mu)(\xi'), \mu'(\xi,\mu)(\xi')) - \sum_{i=1}^{I} \alpha'_i \ \tilde{w}'_i(\xi') &\geq 0, \\ \tilde{w}'_i(\xi') - U_i(\xi, \mu_i) &\geq 0, \end{aligned}$$

and, therefore, $\left(\tilde{c}, (\tilde{w}'(\xi'), \tilde{\alpha}'(\xi'))_{\xi'}\right) \in W(\xi,\mu)(v_\infty)$. Consequently,

$$\begin{aligned} \left(T v_\infty\right)(\xi, \alpha) &\geq \sum_{i=1}^{I} \alpha_i \left(u_i(\tilde{c}_i(\xi)) + \beta(\xi, \mu_i) \sum_{\xi_2} \pi_{\mu_i}(\xi' \,|\xi) \ \tilde{w}'_i(\xi') \right) \\ &= v_\infty(\xi, \alpha). \end{aligned}$$

where the equality follows by (24) and continuity. □

The following Lemma will be used in the proof of Proposition 4

Lemma A.4. *If A1 and A3 holds, there exists N and α^* such that $(\xi_N(\omega), \alpha_N(\omega)) = (\xi^*, \alpha^*)$ for all $\omega \in \{\tilde{\omega} : \xi_t(\tilde{\omega}) = \xi^*$ for t even, $\xi_t(\tilde{\omega}) = \xi^{**}$ for t odd, $1 \leq t \leq N\}$.*

Proof. Let $\delta \equiv \frac{\pi_1(\xi^*|\xi^{**})}{\pi_2(\xi^*|\xi^{**})} \frac{\pi_1(\xi^{**}|\xi^*)}{\pi_2(\xi^{**}|\xi^*)}$. Consider the case in which $\delta > 1$. Note that without loss of generality we can assume $\frac{\pi_1(\xi^{**}|\xi^*)}{\pi_2(\xi^{**}|\xi^*)} > 1$. Let $\alpha_1^* \equiv \max_{\alpha \in \Delta(\xi^{**}, \mu^\pi)} \alpha'_{1,cpo}(\xi^{**}, \alpha, \mu^\pi)(\xi^*)$. Let N^* be the smallest $n \in \mathbb{N} \cup \{0\}$ satisfying

$$\delta^n \frac{\underline{\alpha}_1(\xi^*, \mu^\pi)}{1 - \underline{\alpha}_1(\xi^*, \mu^\pi)} \geq \frac{\alpha_1^*}{1 - \alpha_1^*} > \delta^{n-1} \frac{\underline{\alpha}_1(\xi^*, \mu^\pi)}{1 - \underline{\alpha}_1(\xi^*, \mu^\pi)}.$$

For any $\alpha \in \Delta(\xi^*, \mu^\pi)$ such that $\alpha_1 \leq \alpha_1^*$

$$\frac{\alpha'_{1,cpo}(\xi^*, \alpha, \mu^\pi)(\xi^{**})}{1 - \alpha'_{1,cpo}(\xi^*, \alpha, \mu^\pi)(\xi^{**})} = \max \left\{ \frac{\alpha'_{1,po}(\xi^*, \alpha, \mu^\pi)(\xi^{**})}{1 - \alpha'_{1,po}(\xi^*, \alpha, \mu^\pi)(\xi^{**})}, \frac{\underline{\alpha}_1(\xi^{**}, \mu^\pi)}{1 - \underline{\alpha}_1(\xi^{**}, \mu^\pi)} \right\}$$

and so $\alpha'_{1,cpo}(\xi^*, \alpha, \mu^\pi)(\xi^{**}) \geq \alpha'_{1,po}(\xi^*, \alpha, \mu^\pi)(\xi^{**})$. It follows that

$$\begin{aligned} \frac{\alpha'_{1,cpo}(\xi^{**}, \alpha'_{cpo}(\xi^*, \alpha, \mu^\pi)(\xi^{**}), \mu^\pi)(\xi^*)}{1 - \alpha'_{1,cpo}(\xi^{**}, \alpha'_{cpo}(\xi^*, \alpha, \mu^\pi)(\xi^{**}), \mu^\pi)(\xi^*)} &\geq \frac{\alpha'_{1,cpo}(\xi^{**}, \alpha'_{po}(\xi^*, \alpha, \mu^\pi)(\xi^{**}), \mu^\pi)(\xi^*)}{1 - \alpha'_{1,cpo}(\xi^{**}, \alpha'_{po}(\xi^*, \alpha, \mu^\pi)(\xi^{**}), \mu^\pi)(\xi^*)} \\ &\geq \frac{\pi_1(\xi^*|\xi^{**})}{\pi_2(\xi^*|\xi^{**})} \frac{\alpha'_{1,po}(\xi^*, \alpha, \mu^\pi)(\xi^{**})}{1 - \alpha'_{1,po}(\xi^*, \alpha, \mu^\pi)(\xi^{**})} \\ &= \frac{\pi_1(\xi^*|\xi^{**})}{\pi_2(\xi^*|\xi^{**})} \frac{\pi_1(\xi^{**}|\xi^*)}{\pi_2(\xi^{**}|\xi^*)} \frac{\alpha_1}{1 - \alpha_1} \\ &= \delta \frac{\alpha_1}{1 - \alpha_1}. \end{aligned}$$

Let $N \equiv 2(N^* + 1)$. Consider $\omega \in \Omega^* \equiv \{\tilde{\omega} : \xi_t(\tilde{\omega}) = \xi^*$ for t even, $\xi_t(\tilde{\omega}) = \xi^{**}$ for t odd, $1 \leq t \leq N\}$. The sequence $\{\alpha_{1,t}(\omega)\}$ generated by α'_{cpo} satisfies $\underline{\alpha}_1(\xi^*, \mu^\pi) \leq \alpha_{1,t}(\omega) \leq \alpha_1^*$ and, therefore, $\alpha_{1,t+2}(\omega) \geq \delta\alpha_{1,t}(\omega)$ for all even t such that $2 \leq t \leq N$. Thus, for any even t such that $2 \leq t \leq N$

$$\alpha_{1,t}(\omega) \geq \min\left\{\delta^{\frac{t-2}{2}}\frac{\alpha_{1,2}(\omega)}{1-\alpha_{1,2}(\omega)}, \frac{\alpha_1^*}{1-\alpha_1^*}\right\}$$

and so it follows by the definition of N^* that $\alpha_N(\omega) = \alpha^*$.

If $\delta < 1$, we define $\alpha_1^* \equiv \min_{\alpha\in\Delta(\xi^{**},\mu^\pi)} \alpha'_{1,cpo}(\xi^{**}, \alpha, \mu^\pi)(\xi^*)$ and the proof is analogous to the case $\delta > 1$. $\square$

Proof of Proposition 4. The existence of a unique invariant distribution that is globally stable follows by Theorem 11.12 in Stokey and Lucas [36]. It suffices to show that F_{cpo} satisfies the following condition:

Condition M: There exists $\epsilon > 0$ and an integer $N \geq 1$ such that for any $A \in \mathscr{S}$, either $P^N(s, A) \geq \epsilon$, all $s \in S$, or $P^N(s, A^c) \geq \epsilon$, all $s \in S$.

Define N and α^* as in Lemma A.4 and $\epsilon \equiv (\min_\xi \pi^*(\xi^{**}|\xi))\left(\pi^*(\xi^*|\xi^{**})\pi^*(\xi^{**}|\xi^*)\right)^N > 0$. Let $A \in \mathscr{S}$ and $(\xi, \alpha) \in \Omega$ be arbitrary. If $\alpha^* \in A$, then $P^N((\xi, \alpha), A) \geq P^N((\xi, \alpha), \alpha^*) \geq \epsilon$ by Lemma A.4. If $\alpha^* \in A^c$, then $P^N((\xi, \alpha), A^c) \geq P^N((\xi, \alpha), \alpha^*) \geq \epsilon$ by Lemma A.4 . To show the invariant distribution is not degenerate note that α_1^* must be part of the support. If $\alpha_1^* \notin \Delta(\xi^{**}, \mu^\pi)$, the result follows trivially. If $\alpha_1^* \in \Delta(\xi^{**}, \mu^\pi)$, either $\alpha'_{1,cpo}(\xi^{**}, \alpha^*, \mu^\pi)(\xi^*) \neq \alpha^*$ or $\alpha'_{1,cpo}(\xi^*, \alpha^*, \mu^\pi)(\xi^{**}) \neq \alpha^*$. $\square$

Proof of Proposition 5. Since risk-free rates are assumed to be positive and (ξ, α) lies in a compact set, it follows by continuity of R^F that $R^F_{\min} \equiv \min_{(\xi,\alpha)} R^F(\xi, \alpha) > 1$. Let $m(\xi'|\xi, \alpha) \equiv Q(\xi, \alpha)(\xi')R^F(\xi, \alpha)$ be the equivalent martingale measure.

Let $f \in F$ and consider the operator T_i defined by

$$(T_i f)(\xi, \alpha) = c_i(\xi, \alpha) - y_i(\xi) + \frac{\sum_{\xi'} m(\xi'|\xi, \alpha) f(\xi', \alpha'_{cpo}(\xi, \alpha)(\xi'), \mu'(\xi, \mu)(\xi'))}{R^F(\xi, \alpha)}.$$

Step 1: We check that $T_i : F \to F$.

Suppose that $f \in F$. Since α' and μ' are both continuous, then

$$\frac{\sum_{\xi'} m(\xi'|\xi, \alpha) f(\xi', \alpha'_{cpo}(\xi, \alpha)(\xi'), \mu'(\xi, \mu))}{R^F(\xi, \alpha)}, \tag{25}$$

is continuous in (ξ, α). Also, (25) is bounded because f and R^F are both bounded. Since $|c_i(\xi, \alpha) - y_i(\xi)|$ is uniformly bounded, we can conclude that $(T_i f) \in F$.

Step 2: We check that T_i satisfies Blackwell's sufficient conditions for a contraction mapping.

Discounting. Consider any $a > 0$ and note that

$$\begin{aligned} T_i(f + a)(\xi, \alpha) &= c_i(\xi, \alpha) - y_i(\xi) + \frac{\sum_{\xi'} m(\xi'|\xi, \alpha) f(\xi', \alpha'_{cpo}(\xi, \alpha)(\xi'), \mu'(\xi, \mu)(\xi'))}{R^F(\xi, \alpha)} + \frac{a}{R^F(\xi, \alpha)} \\ &\leq (T_i f)(\xi, \alpha) + \left(R^F_{\min}\right)^{-1} a. \end{aligned}$$

Monotonicity. If $f(\xi,\alpha) \geq g(\xi,\alpha)$ for all (ξ,α), it is immediate that $(T_i f)(\xi,\alpha) \geq (T_i g)(\xi,\alpha)$ for all (ξ,α).

Thus, we can apply the Contraction Mapping Theorem to conclude that T_i is a contraction with a unique fixed point $A_i \in F$ and that the fixed point is the unique solution to (10) for each i. Finally, the same arguments used in Espino and Hintermaier [18] show that there exists $\alpha_0 = \alpha(s_0) \in \mathbb{R}^I_+$ such that $A_i(s_0, \alpha_0, \mu_0) = 0$ for all i. □

Proof of Theorem 6. Given q and B_i, we argue that (c_i, a_i) satisfies *(CESC 1)*.

First, we argue that (c_i, a_i) is in agent i's budget set. Note that the solvency constrains are satisfied by construction. Since $a^{s_0}_{i,-1} = 0$ for all i, it follows by construction of (c_i, a_i) and the definition of A_i that the sequential budget constraint is satisfied.

Next, we argue that (c_i, a_i) is optimal given q and B_i. Notice that (12) implies that

$$\begin{aligned} q^{\xi'}_t(s) &= \max_h \left\{ \beta\left(s_t, \mu_{h,s^t}\right)\ \pi_{\mu_{h,s^t}}(\xi'\,|s_t)\frac{u'_h(c_{h,t+1}(s'))}{u'_h(c_{h,t}(s))} \right\} \text{ where } s' \in C\left(s^t, \xi'\right), \qquad (26)\\ &\geq\ \beta\left(s_t, \pi_i\right) \pi_{\pi_i}(\xi'\,|s_t)\frac{u'_i(c_{i,t+1}(s'))}{u'_i(c_{i,t}(s))}. \end{aligned}$$

for all i (with equality if $U_i(c_i)(s^t) > U_i(s_t, \mu_{i,s^{t-1}})$). Consider any alternative plan $(\tilde{c}_i, \tilde{a}_i)$ in agent i's budget set. It follows by concavity that

$$u_i(c_{i,t}) - u_i(\tilde{c}_{i,t}) \geq u'_i(c_{i,t})\left(c_{i,t} - \tilde{c}_{i,t}\right) \qquad (27)$$

while

$$c_{i,t}(s) - \tilde{c}_{i,t}(s) = a^{s_t}_{i,t-1}(s) - \tilde{a}^{s_t}_{i,t-1}(s) + \sum_{\xi'} q^{\xi'}_t(s)\left(\tilde{a}^{\xi'}_{i,t}(s) - a^{\xi'}_{i,t}(s)\right) = -b_{i,t}(s) + b^*_{i,t}(s),$$

where $b_{i,0} \equiv 0$ and

$$\begin{aligned} b_{i,t}(s) &\equiv \tilde{a}^{s_t}_{i,t-1}(s) - a^{s_t}_{i,t-1}(s) = \tilde{a}^{s_t}_{i,t-1}(s) - A(s_t, \alpha_t(s), \mu_{s^t}) = \tilde{a}^{s_t}_{i,t-1}(s) - B^{s_t}_{i,t-1}(s) \quad &\text{for } t \geq 1,\\ b^*_{i,t}(s) &\equiv \sum_{\xi'} q^{\xi'}_t(s)\left(\tilde{a}^{\xi'}_{i,t}(s) - a^{\xi'}_{i,t}(s)\right) &\text{for } t \geq 0. \end{aligned}$$

Note that

$$\begin{aligned} u'_i(c_{i,t}(s))b^*_{i,t} &= u'_i(c_{i,t}(s))\sum_{\xi'} q^{\xi'}_t(s)\left(\tilde{a}^{\xi'}_{i,t}(s) - a^{\xi'}_{i,t}(s)\right)\\ &= \sum_{\xi'} u'_i(c_{i,t}(s))q^{\xi'}_t(s)\left(\tilde{a}^{\xi'}_{i,t}(s) - a^{\xi'}_{i,t}(s)\right)\\ &\geq E^{P_i}\left[\beta u'_i(c_{i,t+1})b_{i,t+1}\right|\mathcal{F}_t]. \qquad (28) \end{aligned}$$

where the inequality follows from (26). For $T < \infty$, let $\Delta \equiv E^{P_i}\left(\sum_{t=0}^{T} \rho_t\ \left(u_i(c_{i,t}) - u_i(\tilde{c}_{i,t})\right)\right)$. Then,

$$
\begin{aligned}
\Delta &\geq E^{P_i}\left(\sum_{t=0}^{T}\rho_t u_i'(c_{i,t})\left(c_{i,t}-\tilde{c}_{i,t}\right)\right)\\
&= E^{P_i}\left(\sum_{t=0}^{T}\rho_t u_i'(c_{i,t})\left(-b_{i,t}+b^*_{i,t}\right)\right)\\
&= -E^{P_i}\left[\sum_{t=0}^{T}\rho_t u_i'(c_{i,t})b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T}\rho_t u_i'(c_{i,t})b^*_{i,t}\right]\\
&= -E^{P_i}\left[\sum_{t=0}^{T}\rho_t u_i'(c_{i,t})b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T}\rho_t E^{P_i}\left[\left. u_i'(c_{i,t})b^*_{i,t}\right| \mathcal{F}_t\right]\right]\\
&\geq -E^{P_i}\left[\sum_{t=0}^{T}\rho_t u_i'(c_{i,t})b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T} E^{P_i}\left[\left.\rho_{t+1}\ u_i'(c_{i,t+1})b_{i,t+1}\right| \mathcal{F}_t\right]\right]\\
&= -E^{P_i}\left[\sum_{t=0}^{T}\rho_t u_i'(c_{i,t})b_{i,t}\right] + E^{P_i}\left[\sum_{t=0}^{T}\rho_{t+1} u_i'(c_{i,t+1})b_{i,t+1}\right],
\end{aligned}
$$

where the first line uses (27), the fourth and last lines follows from the law of iterated expectations and the inequality in the fifth line follows from (28). Since $b_{i,0} = 0$,

$$
\Delta \equiv E^{P_i}\left[\sum_{t=0}^{T}\rho_t\ \left(u_i(c_{i,t})-u_i(\tilde{c}_{i,t})\right)\right] \geq E^{P_i}\left[\rho_{T+1}u_i'(c_{i,T+1}(s))b_{i,T+1}\right].
$$

Now we argue that $b_{i,t}$ is uniformly bounded. Since $\mathcal{P}(\Pi^K)$ is compact (in the weak topology), the continuous functions $\underline{\alpha}_i(\xi,\mu)$ and $A_i(\xi,\alpha,\mu)$ and, therefore, $B^{\xi'}_{i,t}$ is uniformly bounded for all ξ'. So it suffices to show $\tilde{a}^{\xi'}_{i,t}$ is uniformly bounded for all ξ'. Note that $\tilde{a}^{\xi'}_{i,t}(s)$ is bounded below by $B^{\xi'}_{i,t}(s)$ and market clearing implies it is bounded above by $-B^{\xi'}_{j,t}(s)$ for $j \neq i$.

It follows from the Dominated Convergence Theorem that

$$
\begin{aligned}
E^{P_i}\left[\sum_{t=0}^{\infty}\rho_t\ \left(u_i(c_{i,t})-u_i(\tilde{c}_{i,t})\right)\right] &= \lim_{T\to\infty} E^{P_i}\left[\sum_{t=0}^{T}\rho_t\ \left(u_i(c_{i,t})-u_i(\tilde{c}_{i,t})\right)\right]\\
&\geq \lim_{T\to\infty} E^{P_i}\left[\rho_{T+1}u_i'(c_{i,T+1})b_{i,T+1}\right]\\
&= 0.
\end{aligned}
$$

since $\beta(\xi,\mu) \leq \overline{\beta} \in (0,1)$ for all (ξ,μ). Consequently, given q and B_i, (c_i, a_i) solves agent i's problem.

Finally, note that $\{a_i\}$ satisfies *(CESC 2)* since $\sum_{i=1}^{I} A_i(\xi,\alpha) = 0$ for all (ξ,α) (see (11)). $\square$

Appendix B

In this Appendix we prove the results of Section 5.

Theorem B.1 (Stout [37] and Jensen and Rahbek [21]). *Assume* $\{z_t\}_{t=0}^{\infty}$ *is a time homogeneous Markov process with transition function* F *on* $(Z, \mathcal{Z})$. *If there exists a unique invariant distribution* $\psi : \mathcal{Z} \to [0,1]$, *then for any* $z_0 \in Z$, *any integer* k *and any continuous function* $f : Z^k \to \Re$,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} f\left(z_t, ..., z_{t+k}\right) = E^{P^F(\psi,\cdot)}\left(f\left(\tilde{z}_0, ..., \tilde{z}_k\right)\right), \qquad P^F\left(z_0, \cdot\right) - a.s.$$

Proof of Theorem 8. For the case of CPO allocations when agents have dogmatic priors, the result follows directly from Proposition 4. So we only deal here with the case of PO. Under our assumptions *PO* allocations can be parameterized by welfare weights. Let agent h be some agent whose prior satisfies A2. A straightforward extension of Beker and Espino [5] to handle Markov uncertainty can be used to show that the welfare weights associated with a *PO* allocation satisfy that for every agent i and every path $s \in S^{\infty}$

$$\alpha_{i,t}\left(s\right) = \frac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\sum_{j=1}^{I} \alpha_{j,0}\ P_{j,t}\left(s\right)} = \frac{\dfrac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)}}{\sum\limits_{i=1}^{I} \dfrac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)}}$$

and so the limit behaviour of the welfare weights depends on the limit behaviour of the likelihood ratio

$$\frac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)}.$$

If h's prior satisfies A1.a then one can use Sandroni's results to show that, $P^{\pi^*} - a.s.$,

$$\frac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)} \to \frac{\alpha_{i,0}}{\alpha_{h,0}} \frac{\mu_i\left(\pi^*\right)}{\mu_h\left(\pi^*\right)}$$

while if h's prior satisfies A1.b then one can use Phillip and Ploberger's [31, Theorem 4.1] results to show that, $P^{\pi^*} - a.s.$,

$$\frac{\alpha_{i,0}\ P_{i,t}\left(s\right)}{\alpha_{h,0}\ P_{h,t}\left(s\right)} \to \frac{\alpha_{i,0}}{\alpha_{h,0}} \frac{f_i\left(\pi^*\right)}{f_h\left(\pi^*\right)}.$$

It follows that $\alpha_{i,t}\left(s\right) \to \alpha_{\infty}$, $P^{\pi^*} - a.s.$

Since every agent's prior satisfies A1, it is well known that there exists some $\pi = (\pi_1, ..., \pi_I)$ where $\pi_i \in \Pi^K$ such that π_i converges weakly to μ^{π_i} for P^{π_i}–almost all $s \in S^{\infty}$ and π_i is the element of i's support which is closer to π^* in terms of entropy. By assumption A.2, $\pi_h = \pi^*$.

Since convergence almost surely implies convergence in distribution, we conclude that, $P^{\pi^*} - a.s.$, the marginal distribution over welfare weights and beliefs converges to a mass point on $(\alpha_{\infty}, \mu^{\pi})$. $\square$

Proof of Proposition 9. We need to show that

$$E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)\right](\omega) > 0 \qquad \text{if the return underreacts by date } \tau, \tag{29}$$

$$E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)\right](\omega) < 0 \qquad \text{if the return overreacts by date } \tau. \tag{30}$$

Let $\Omega^+ \equiv \left\{\tilde{\omega} : \overline{R}_{1,e}(\tilde{\omega}) \geq 0\right\}$ and $\Omega^- \equiv \left\{\tilde{\omega} : \overline{R}_{1,e}(\tilde{\omega}) < 0\right\}$. Note that

$$\begin{aligned} E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)\right](\omega) &= P_e\left(\Omega^+\right)\ E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)\middle|\,\Omega^+\right] + \\ &\quad\ P_e\left(\Omega^-\right)\ E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\,\overline{R}_{1,e}\right)\middle|\,\Omega^-\right] \end{aligned}$$

and so $E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)\right](\omega)$ is bounded below by

$$P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\,\Omega^+\right) \inf_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\tilde{\omega}) + P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\,\Omega^-\right) \sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\tilde{\omega})$$

and above by

$$P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\,\Omega^+\right) \sup_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\tilde{\omega}) + P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\,\Omega^-\right) \inf_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\tilde{\omega})$$

If $E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\omega)$ underreacts by date τ, then

$$\inf_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\tilde{\omega}) > \sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\tilde{\omega})$$

and so (29) holds because

$$\begin{aligned} E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)\right](\omega) &> \left(P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\,\Omega^+\right)\right) \sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\overline{R}_{1,e}\right)(\tilde{\omega}) + \\ &\quad\ \left(P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\,\Omega^-\right)\right) \sup_{\omega\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\overline{R}_{1,e}\right)(\tilde{\omega}) \\ &= E^{P_e}\left(\overline{R}_{1,e}\right) \sup_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\overline{R}_{1,e}\right)(\tilde{\omega}) \\ &= 0. \end{aligned}$$

If $E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\omega)$ underreacts by date τ, then

$$\sup_{\tilde{\omega}\in\Omega^+} E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\tilde{\omega}) < \inf_{\omega\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)(\omega)$$

and so (30) holds because

$$\begin{aligned} E^{P_e}\left[\overline{R}_{1,e}\ E^{P_e}\left(R_{\tau,e}\middle|\,\mathcal{G}_1\right)\right](\omega) &< \left(P_e\left(\Omega^+\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\,\Omega^+\right)\right) \inf_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\overline{R}_{1,e}\right)(\tilde{\omega}) + \\ &\quad\ \left(P_e\left(\Omega^-\right)\ E^{P_e}\left(\overline{R}_{1,e}\middle|\,\Omega^-\right)\right) \inf_{\omega\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\overline{R}_{1,e}\right)(\tilde{\omega}) \\ &= E^{P_e}\left(\overline{R}_{1,e}\right) \inf_{\tilde{\omega}\in\Omega^-} E^{P_e}\left(R_{\tau,e}\middle|\,\overline{R}_{1,e}\right)(\tilde{\omega}) \\ &= 0. \end{aligned}$$

$\square$

Appendix C

Calibrated parameters:

$$\pi^* = \begin{bmatrix} 0.1146 & 0.7150 & 0.0318 & 0.1386 \\ 0.1334 & 0.7831 & 0.0130 & 0.0705 \\ 0.0318 & 0.1386 & 0.1146 & 0.7150 \\ 0.0130 & 0.0705 & 0.1334 & 0.7831 \end{bmatrix}$$

s_t	Endowments	Growth Rates
1	0.8178	0.9926
2	0.6298	1.0071
3	0.1822	0.9926
4	0.3702	1.0071

C.1 Calibrated β and σ

Figure 16 plots the calibrated values of σ and β as a function of $(\varepsilon_E, \varepsilon_R)$.

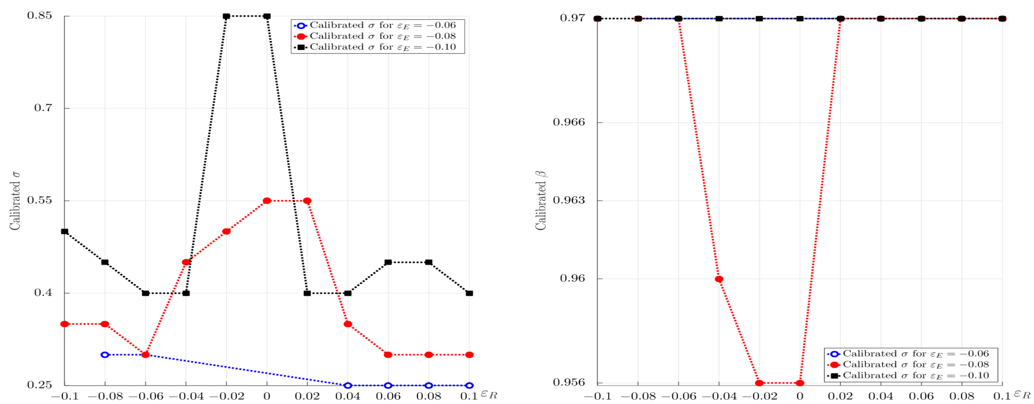

Figure 16: Calibrated values of σ and β in CESC

Figure 17 plots the calibrated values of σ and β as a function of $(\varepsilon_E, \varepsilon_R)$.

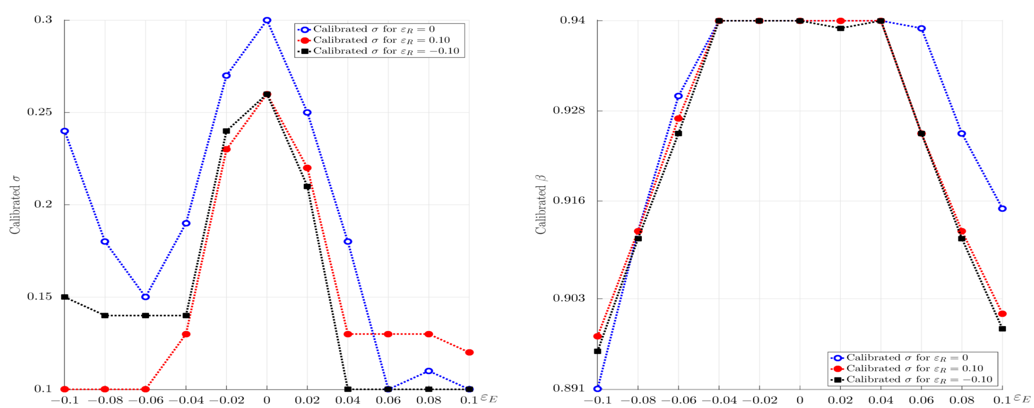

Figure 17: Calibrated values of σ and β in CESC

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Supplementary Material

Let $c_i(\xi,\alpha)$ and $w_i'(\xi,\alpha,\mu)(\xi^{'})$ be the maximisers in problem (5) - (9) and let $\lambda_i(\xi,\alpha,\mu)$ be the Lagrange multiplier associated to constraint (7). Let

$$\tilde{u}_i(\xi,\alpha,\mu) = u_i(c_i(\xi,\alpha)) + \beta(\xi,\mu_i)\sum_{\xi'} \pi_{\mu_i}(\xi'\,|\xi\,)\; w_i'(\xi,\alpha,\mu)(\xi').$$

Claim 1. *$\tilde{u}_i(\xi,\alpha,\mu)$ is nondecreasing in α_i for all $\alpha \in \mathbb{R}^I_+$.*

Proof. Let $\tilde{\alpha},\alpha \in \mathbb{R}^I_+$ be such that $\tilde{\alpha}_i > \alpha_i$ and $\tilde{\alpha}_j = \alpha_j$ for every $j \neq i$. To get a contradiction, suppose $\tilde{u}_i(\xi,\tilde{\alpha},\mu) < \tilde{u}_i\,(\xi,\alpha,\mu)$. Since the constrained set is independent of the welfare weights, then

$$\sum_h \tilde{\alpha}_h\left(\tilde{u}_h\left(\xi,\tilde{\alpha},\mu\right) - \tilde{u}_h\left(\xi,\alpha,\mu\right)\right) \geq 0 \text{ and } \sum_h \alpha_h\left(\tilde{u}_h\left(\xi,\alpha,\mu\right) - \tilde{u}_h\left(\xi,\tilde{\alpha},\mu\right)\right) \geq 0$$

and so, on the one hand,

$$\sum_h (\tilde{\alpha}_h - \alpha_h)\left(\tilde{u}_h\left(\xi,\tilde{\alpha},\mu\right) - \tilde{u}_h\left(\xi,\alpha,\mu\right)\right) \geq 0$$

But, on the other hand,

$$\sum_h (\tilde{\alpha}_h - \alpha_h)\left(\tilde{u}_h\left(\xi,\tilde{\alpha},\mu\right) - \tilde{u}_h\left(\xi,\alpha,\mu\right)\right) = (\tilde{\alpha}_i - \alpha_i)\left(\tilde{u}_i\left(\xi,\tilde{\alpha},\mu\right) - \tilde{u}_i\left(\xi,\alpha,\mu\right)\right) < 0$$

a contradiction. $\square$

Let $\overline{c}_i(\xi,\alpha)$ and $\overline{w}_i'(\xi,\alpha,\mu)(\xi^{'})$ be the maximisers of the relaxed problem where (7) is ignored. Let

$$\overline{u}(\xi,\alpha,\mu) = u_i(\overline{c}_i(\xi,\alpha)) + \beta(\xi,\mu_i)\sum_{\xi'} \pi_{\mu_i}(\xi'\,|\xi\,)\; \overline{w}_i'(\xi,\alpha,\mu)(\xi').$$

Claim 2. *Let $\alpha \in \mathbb{R}^I_+$. If $\alpha_i < \tilde{\alpha}_i$ and $\alpha_h = \tilde{\alpha}_h$ for all $h \neq i$, then $\overline{u}_i\left(\xi,\alpha,\mu\right) < \overline{u}_i\left(\xi,\tilde{\alpha},\mu\right)$.*

Proof. Note that $\overline{c}_i(\xi,\alpha)$ is the unique solution to

$$c_i + \sum_{h\neq i}\left(\frac{\partial u_h}{\partial c_h}\right)^{-1}\left(\frac{\alpha_i}{\alpha_h}\;\frac{\partial u_i(c_i)}{\partial c_i}\right) = y(\xi).$$

and so it is strictly increasing in α_i. Therefore, $\overline{c}_i(\xi,\tilde{\alpha}) > \overline{c}_i(\xi,\alpha)$. Note that

$$\overline{\alpha}_i'(\xi,\alpha)(\xi') = \frac{\alpha_i \int \pi(\xi'\,|\xi)\mu_i'(\xi,\mu)\,(\xi')\,(d\pi)}{\sum_h \alpha_h \int \pi(\xi'\,|\xi)\mu_h'(\xi,\mu)\,(\xi')\,(d\pi)}$$

Thus, $\overline{\alpha}_i'(\xi,\alpha,\mu)(\xi')$ is nondecreasing in α_i. Since $\overline{w}_i'(\xi,\alpha,\mu)(\xi^{'})$ satisfies (8) and (9), it follows by Lemma A.1 and Theorem 1 that $\overline{w}_i'(\xi,\alpha,\mu)(\xi^{'}) = \overline{u}_i(\xi',\overline{\alpha}'(\xi,\alpha,\mu)(\xi'),\mu'(\xi,\mu)(\xi'))$. Thus, Claim 1 implies that $\overline{w}_i'(\xi,\tilde{\alpha},\mu)(\xi^{'}) \geq \overline{w}_i'(\xi,\alpha,\mu)(\xi^{'})$ for all ξ'. We conclude that $\overline{u}_i\left(\xi,\alpha,\mu\right) < \overline{u}_i\left(\xi,\tilde{\alpha},\mu\right)$, as desired. $\square$

Proof of Proposition 2 . (i) Suppose $\alpha \in \Delta(\xi,\mu)$. Consider first the case where $\alpha_i > \underline{\alpha}_i(\xi,\mu)$ for all i. By the definition of $\tilde{u}_i(\xi,\alpha,\mu)$, we have that $\tilde{u}_i(\xi,\alpha,\mu) \geq U_i(\xi,\mu)$ and $\sum_i^I \alpha_i \tilde{u}_i(\xi,\alpha,\mu) = v^*(\xi,\alpha,\mu)$. It follows by Lemma A.1, that $(\tilde{u}_1(\xi,\alpha,\mu)...\tilde{u}_I(\xi,\alpha,\mu)) \in \mathcal{U}^{\mathrm{E}}(\xi,\mu)$. Since $\sum_i^I \alpha_i \tilde{u}_i(\xi,\alpha,\mu) = v^*(\xi,\alpha,\mu)$, it is easy to see that $(u_1(\xi,\alpha,\mu)...u_I(\xi,\alpha,\mu)) \in \overline{\mathcal{U}}^E(\xi,\mu)$. Then, $\tilde{u}_i(\xi,\alpha,\mu) > U_i(\xi,\mu_i)$ for all i by definition of $\underline{\alpha}_i(\xi,\mu)$. Thus, $\lambda_i(\xi,\alpha,\mu) = 0$. Let $\alpha \in \Delta(\xi,\mu)$ be such that $\alpha_i = \underline{\alpha}_i(\xi,\mu)$ for some i. Then there is a sequence $\{\alpha^n\}_{n=1}^{\infty}$ such that $\alpha_i^n > \underline{\alpha}_i(\xi,\mu)$ for all i and n and $\alpha^n \to \alpha$. It follows that

$$\lambda_i(\xi,\alpha,\mu) = \lambda_i(\xi, \lim_{n\to\infty} \alpha^n, \mu) = \lim_{n\to\infty} \lambda_i(\xi,\alpha^n,\mu) = 0,$$

where the second equality follows by continuity of $\lambda_i(\xi,\alpha,\mu)$ in α and the last one because weak inequalities are preserved under limits. It follows that, $\tilde{u}_i(\xi,\alpha,\mu) = \overline{u}_i(\xi,\alpha,\mu)$ and so $c_i(\xi,\alpha) = \overline{c}_i(\xi,\alpha)$, i.e. $c_i(\xi,\alpha)$ solves the relaxed problem.

(ii) Let $\alpha \in \mathbb{R}_+^I$ and $\alpha^* \equiv \left(\frac{\alpha_1}{\sum_{i=1}^I \alpha_i}...\frac{\alpha_I}{\sum_{i=1}^I \alpha_i}\right)$. If $\alpha^* \in \Delta(\xi,\mu)$, then $c_i(\xi,\alpha) = c_i(\xi,\alpha^*)$ because $\tilde{u}_i(\xi,\alpha,\mu)$ is homogeneous of degree zero in α. If $\alpha^* \notin \Delta(\xi,\mu)$, there is i such that $\alpha_i^* < \underline{\alpha}_i(\xi,\mu)\left(\alpha_{-i}^*\right)$.

- First, we show that $\lambda_i(\xi,\alpha,\mu) > 0$. To get a contradiction, suppose $\lambda_i(\xi,\alpha,\mu) = 0$. It follows that

$$\begin{aligned}
\tilde{u}_i\left(\xi,\left(\alpha_i,\alpha_{-i}\right),\mu\right) &= \tilde{u}_i\left(\xi,\left(\alpha_i^*,\alpha_{-i}^*\right),\mu\right)\\
&= \overline{u}_i\left(\xi,\left(\alpha_i^*,\alpha_{-i}^*\right),\mu\right)\\
&= \overline{u}_i\left(\xi,\left(\frac{\alpha_i^*}{\underline{\alpha}_i(\xi,\mu)\left(\alpha_{-i}^*\right)},\frac{\alpha_{-i}^*}{\underline{\alpha}_i(\xi,\mu)\left(\alpha_{-i}^*\right)}\right),\mu\right)\\
&< \overline{u}_i\left(\xi,\left(1,\frac{\alpha_{-i}^*}{\underline{\alpha}_i(\xi,\mu)\left(\alpha_{-i}^*\right)}\right),\mu\right)\\
&= \overline{u}_i\left(\xi,\left(\underline{\alpha}_i(\xi,\mu)\left(\alpha_{-i}^*\right),\alpha_{-i}^*\right),\mu\right)\\
&= U_i(\xi,\mu),
\end{aligned}$$

  where the first equality follows because $\tilde{u}_i$ is homogeneous of degree zero in α, the second one is due to the assumption that $\lambda_i(\xi,\alpha,\mu) = 0$ and the homogeneity of degree zero of $\lambda_i(\xi,\alpha,\mu)$ in α, the third and fifth follows by homogeneity of degree zero of $\overline{u}_i(\cdot)$ in α, the inequality follows by Claim 2 and the last equality follows by definition of the minimum enforceable weights. But then, $\tilde{u}_i\left(\xi,\left(\alpha_i,\alpha_{-i}\right),\mu\right) < U_i(\xi,\mu)$ which contradicts constraint (7).

- Second, note that problem (5) - (9) is equivalent to maximising

$$\sum_{i=1}^I (\alpha_i + \lambda_i)\left\{u_i(c_i) + \beta(\xi,\mu_i)\sum_{\xi'} \pi_{\mu_i}(\xi'\,|\xi)\; w_i'(\xi')\right\},$$

  subject to constraints (6), (8) and (9).

- Finally, the latter is equivalent to the relaxed problem with welfare weights $\tilde{\alpha}$ given by

$$\tilde{\alpha}_i = \frac{\alpha_i + \lambda_i(\xi,\alpha,\mu)}{\sum_{h=1}^I \left(\alpha_h + \lambda_h(\xi,\alpha,\mu)\right)},$$

Thus, $\overline{u}_i\left(\xi,\tilde{\alpha},\mu\right) = \tilde{u}_i\left(\xi,\alpha,\mu\right) \geq U_i(\xi,\mu) = \overline{u}_i\left(\xi,\underline{\alpha}_i,\mu\right)$. It follows by Claim 2 that $\tilde{\alpha}_i \geq \underline{\alpha}_i$. Therefore, $\tilde{\alpha} \in \Delta(\xi,\mu)$ and $c_i(\xi,\alpha) = \overline{c}_i(\xi,\tilde{\alpha}) = c_i(\xi,\tilde{\alpha})$ as desired. □