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The One-Way Fubini Property and Conditional Independence: An Equivalence Result

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Abstract

A process defined by a continuum of random variables with non-degenerate idiosyncratic risk is not jointly measurable with respect to the usual product σ-algebra. We show that the process is jointly measurable in a one-way Fubini extension of the product space if and only if there is a countably generated σ-algebra given which the random variables are essentially pairwise conditionally independent, while their conditional distributions also satisfy a suitable joint measurability condition. Applications include: (i) new characterizations of essential pairwise independence and essential pairwise exchangeability; (ii) when a one-way Fubini extension exists, the need for the sample space to be saturated if there is an essentially random regular conditional distribution with respect to the usual product σ-algebra.

Keywords: Continuum of random variables, joint measurability problem, one-way Fubini property, conditional distributions, characterizations of conditional independence.

Contents

1 Introduction

In this paper, a continuous parameter random process (or simply a process) is formalized as a continuum of random variables — i.e., a collection of random variables indexed by points in an atomless measure space. When the index space is the time line, such a process is usually assumed to be jointly measurable with respect to time and the random state in the usual product σ-algebra of probability theory.

Following the work on oceanic games by Milnor and Shapley [37] and on large economies by Aumann [3], [4] and Hildenbrand [22], economists and game theorists have long been interested in the "continuum limit" of an economic model or game as the number of agents or players tends to infinity — see also the surveys in [29] for games with many players, and in the latter part of [16] for economies with many traders. Agents or players in such a limit are indexed by points in an atomless probability space. Such "continuum" models of random processes involving many agents work well for systemic risks taking the form of common random shocks that influence a non-negligible set of agents. Yet reality suggests that these systemic risks are supplemented by risks at the individual level in the form of idiosyncratic micro shocks that, as in [5], influence a negligible set of agents. As shown in Corollary 1 below, however, a process that generates a continuum of random shocks satisfies a standard joint measurability condition only if there is essentially no idiosyncratic risk at all.

In terms of the Kolmogorov construction for a continuum of independent random variables, the first references to this non-measurability issue are by Doob [8, Theorem 2.2, p. 113] and [9, p. 67]. In general, [43, Proposition 1.1] shows that independence and joint measurability with respect to the usual product σ-algebra are never compatible with each other except for the trivial case. Indeed, the failure of joint measurability led Doob to claim in [9, p. 102] that processes with mutually independent random variables are only useful in the discrete parameter case. Nevertheless we note that studying a continuum of (conditionally) independent random variables within an appropriate analytic framework has allowed the discovery of several new connections between some basic concepts in probability theory. For example, Theorem 1 in both [43] and [20] shows the essential equivalence of pairwise and mutual (conditional) independence, which also implies the essential equivalence of pairwise and multiple versions of exchangeability.[1]

The papers [42] and [43] adopted the framework of Loeb product measure spaces in nonstandard analysis, which extends the usual measure-theoretic product, as noted by Anderson in [1], while retaining the common Fubini property (as shown by Keisler in [25] — see also [26], [33] and [36]). Any atomless Loeb product measure space always has an abundance of nontrivial independent processes (Theorem 6.2 in [42]). In particular, Keisler's Fubini theorem implies that the joint measurability problem is automatically solved for independent processes that are Loeb product measurable.

On the other hand, for an arbitrarily given process with a continuum of independent random variables, a two-way Fubini extension as in [42] (and more generally in [44]) may not be possible in general (see Remark 3.2 in [19]). Indeed, it will be impossible if the random variables are indexed by points in the unit interval with Lebesgue measure. The approach that we introduced in [19] is to work with an extension of the product space satisfying a limited form of joint measurability, which we associated with a "one-way Fubini"

[1]For other results related to the exact law of large numbers and its converse, see [42] and [44].

property of double integrals. We chose this name because, unlike the usual or "two-way" Fubini property of double integrals w.r.t. a product measure, the order of integration does matter. In particular, each random sample function is typically non-measurable. Only if one considers expectations of a function over sample paths before integrating over the index set, as we do in part 1(ii) of Definition 1, is the double integral defined; the reverse order of integration is meaningless.

The main aim of this paper is to characterize completely all processes that satisfy the one-way Fubini property, without assuming independence. In particular, the main result, Theorem 1, shows that a process satisfies the one-way Fubini property if and only if, following the terminology of [18] and [21], it has a *stochastic macro structure. That is, there must be a countably generated σ-algebra $\mathcal{C}$ such that the continuum of random variables: (i) are essentially pairwise conditionally independent given $\mathcal{C}$; and (ii) have conditional distributions given $\mathcal{C}$ that* satisfy a suitable joint measurability condition.

We note that the main theorem of [21] shows that a stochastic macro structure with these properties exists if and only if Monte Carlo simulation produces samples of random variables whose means converge in a suitable sense explained in that paper. In addition, we proved that this is true if and only if there are event-wise measurable conditional probabilities. Thus, the main theorem of this paper, when combined with that of [21], provides three different but equivalent characterizations of processes that satisfy the one-way Fubini property. Morever, the countably generated σ-algebra $\mathcal{C}$ must contain the Monte Carlo σ-algebra. This is defined in [21] as the smallest σ-algebra on the basic probability space which, for each Cartesian product of measurable sets in the index set and probability space, makes the Monte Carlo limit of the mean of its characteristic function a well-defined random variable.

In the latter part of this paper, we also discuss several applications of our main theorem.[2] First, we show in Proposition 4 that, if a one-way Fubini extension exists, then the underlying sample probability space must be saturated if the regular conditional distribution with respect to the usual product σ-algebra is essentially random. Second, using the one-way Fubini property along with regular conditional distributions with respect to the

[2]Additional applications for the main results in this paper, especially to market economies with many agents, are presented in [17].

usual product σ-algebra, Proposition 5 provides new characterizations for the basic concepts of independent and exchangeable random variables. In a more general setting, these characterizations allow us to show the duality between independence and exchangeability when considering the basic random variables on the one hand, and the random sample functions generated by a process on the other hand.

In the sequel, we introduce the basic concepts in Section 2. The main result is stated in Section 3 and proved in Sections 4 and 5. As a first application of the general results proved earlier, Section 6 shows that any function that is jointly measurable in the usual sense differs fundamentally from a process that includes non-trivial idiosyncratic micro shocks. The second application in Section 7 considers, in the framework of a one-way Fubini extension, regular conditional distributions of a process with respect to the usual product σ-algebra. As a corollary, we use the regular conditional distributions to give new characterizations of essential pairwise independence and essential pairwise exchangeability, as well as to demonstrate their duality.

2 Basic Definitions

We first fix some notation. Let $(T, \mathcal{T}, \lambda)$ be a complete atomless probability space. Let $(\Omega, \mathcal{A}, P)$ be a complete, countably additive probability space. Let X be a Polish space (i.e., homeomorphic to a complete separable metric space) with the Borel σ-algebra $\mathcal{B}$. A *process* g is a mapping from $T \times \Omega$ to X such that, for any fixed $t \in T$, the mapping $\omega \mapsto g_t(\omega) = g(t, \omega)$ is $\mathcal{A}$-measurable — i.e., g_t is a random variable defined on $(\Omega, \mathcal{A}, P)$. Thus, the pair of probability spaces $(T, \mathcal{T}, \lambda)$ and $(\Omega, \mathcal{A}, P)$ are used as the parameter and sample spaces, respectively, for the process g.

In the following subsections, we introduce the three main concepts in this paper: (i) one-way Fubini extension; (ii) regular conditional independence; (iii) essentially random process.

2.1 The one-way Fubini property

The following definition was introduced in [19].

Definition 1. *A probability space* $(T \times \Omega, \mathcal{W}, Q)$ extends *the usual product probability space* $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ *provided that* $\mathcal{W} \supseteq \mathcal{T} \otimes \mathcal{A}$*, with* $Q(E) = (\lambda \times P)(E)$ *for all* $E \in \mathcal{T} \otimes \mathcal{A}$*.*

1. *The extended space* $(T \times \Omega, \mathcal{W}, Q)$ *is a* one-way Fubini extension *of the product probability space* $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ *provided that, given any* Q*-integrable function* $f : T \times \Omega \to \mathbb{R}$*:*
   (i) for λ*-almost all* $t \in T$*, the function* $\omega \mapsto f_t(\omega)$ *is integrable on* $(\Omega, \mathcal{A}, P)$*;*
   (ii) the function $t \mapsto \int_\Omega f_t dP$ *is integrable on* $(T, \mathcal{T}, \lambda)$*, with integral that satisfies* $\int_T \left(\int_\Omega f_t dP \right) d\lambda = \int_{T \times \Omega} f dQ$*.*

2. *The space* $(T \times \Omega, \mathcal{W}, Q)$ *is a* (two-way) Fubini extension *of the product probability space* $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ *provided that, given any* Q*-integrable function* $f : T \times \Omega \to \mathbb{R}$*, in addition to (i) and (ii) above, one has:*
   (iii) for P*-almost all* $\omega \in \Omega$*, the function* f_ω *is integrable on* $(T, \mathcal{T}, \lambda)$*;*
   (iv) the integral of f_ω *w.r.t.* t *satisfies* $\int_{T \times \Omega} f dQ = \int_\Omega \left(\int_T f_\omega d\lambda \right) dP$*.*

A process $g : T \times \Omega \to X$ *is said to satisfy the* one-way Fubini property *if there is a one-way Fubini extension* $(T \times \Omega, \mathcal{W}, Q)$ *such that* g *is* $\mathcal{W}$*-measurable.*

The concept of (two-way) Fubini extension in part 2 of Definition 1 is due to [44, Definition 2.2]. For a nontrivial example beyond atomless Loeb product measure spaces, see [44, Proposition 5.6]. This example involves an extension of the Kolmogorov continuum product; see also [38] and [45] for other examples of this type.

2.2 Regular conditional independence

Recall that a σ-algebra $\mathcal{C}$ on Ω is said to be *countably generated* if there exists a countable family $\{ C_n \}_{n=1}^\infty$ of subsets of Ω such that $\mathcal{C} = \sigma(\{ C_n \}_{n=1}^\infty)$, the smallest σ-algebra including the whole family — see, for example, [6] (Ex. 2.11, p. 34). As shown in [6] (Ex. 20.1, p. 270), the σ-algebra $\mathcal{C}$ is countably generated if and only if there exists a mapping $\theta : \Omega \to \mathbb{R}$ such that $\mathcal{C} = \sigma(\{\theta\})$, the smallest σ-algebra on Ω that makes the function $\omega \mapsto \theta(\omega)$ measurable.

Given the probability space $(\Omega, \mathcal{A}, P)$, a sub-σ-algebra $\mathcal{C} \subset \mathcal{A}$ is said to be *essentially countably generated* if it is the *strong completion* of a countably generated σ-algebra $\mathcal{C}'$,[3] in the sense that

$$\mathcal{C} = \{ A \in \mathcal{A} \mid \exists A' \in \mathcal{C}' : P(A \triangle A') = 0 \}.$$

For simplicity, from now on we describe a σ-algebra as countably generated even when it is only essentially countably generated. Of course, the extra sets in the essentially countably generated σ-algebra are all null.

Given the codomain X of the process $g : T \times \Omega \to X$, let $\mathcal{M}(X)$ denote the space of Borel probability measures on X endowed with the topology of weak convergence of measures.[4]

Definition 2. *Let g a process from $T \times \Omega$ to X, and $\mathcal{C}$ be a countably generated sub-σ-algebra of $\mathcal{A}$.*

1. *Two random variables ϕ and ψ that map $(\Omega, \mathcal{A}, P)$ to X are said to be* conditionally independent *given $\mathcal{C}$ if, for any Borel sets $B_1, B_2 \in \mathcal{B}$, the conditional probabilities satisfy*

$$P(\phi^{-1}(B_1) \cap \psi^{-1}(B_2)|\mathcal{C}) = P(\phi^{-1}(B_1)|\mathcal{C})P(\psi^{-1}(B_2)|\mathcal{C}). \tag{1}$$

2. *The process g is said to be* essentially pairwise conditionally independent *given $\mathcal{C}$ if, for λ-a.e. $t_1 \in T$, the random variables g_{t_1} and g_{t_2} are conditionally independent given $\mathcal{C}$ for λ-a.e. $t_2 \in T$.*[5]

3. *A $\mathcal{T} \otimes \mathcal{C}$-measurable mapping μ from $T \times \Omega$ to $\mathcal{M}(X)$ is said to be an* essentially regular conditional distribution process *of g given $\mathcal{C}$ if, for λ-a.e. $t \in T$, the $\mathcal{C}$-measurable mapping $\omega \mapsto \mu_{t\omega}$ from Ω to $\mathcal{M}(X)$ is a regular conditional distribution $P(g_t^{-1}|\mathcal{C})$ of the random variable g_t.*

4. *The process g is said to be* regular conditionally independent *if there exists a countably generated sub-σ-algebra $\mathcal{C}$ of $\mathcal{A}$ such that g is essentially pairwise conditionally independent given $\mathcal{C}$, and also g admits an essentially regular conditional distribution process given $\mathcal{C}$.*

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[3]Note that this condition for a sub-σ-algebra to be essentially countably generated does not depend on the assumption of P being a probability measure. One can use the same definition in the case of any finite measure space.

[4] Recall that a mapping ϕ from a measurable space $(I, \mathcal{I})$ to $\mathcal{M}(X)$ is measurable w.r.t. the Borel σ-algebra generated by the topology of weak convergence of measures on $\mathcal{M}(X)$ if and only if it is *event-wise measurable* — i.e., for every event $B \in \mathcal{B}$, the mapping $i \mapsto \phi(i)(B)$ is $\mathcal{I}$-measurable (see, for example, [18, p. 748]).

[5]Note that this condition implies that the process satisfies essential mutual conditional independence, as shown in Theorem 1 of [20].

Remark 1. *An important special case of the above definition that is discussed in [18] is when all the random variables g_t are conditionally independent and identically distributed — i.e., they are exchangeable with $\mu_{t\omega} = \mu^*_\omega$, independent of t; see also Proposition 5 below.*

2.3 Saturation and essential randomness

For any $A \in \mathcal{A}$ with $P(A) > 0$, one can define the *restricted* σ-algebra $\mathcal{A}^A := \{D \in \mathcal{A} : D \subseteq A\}$, which leads to the *restricted* measure space $(A, \mathcal{A}^A, P)$. The probability space $(\Omega, \mathcal{A}, P)$ is said to be *saturated* if it is *nowhere countably generated* in the sense that the restricted measure space $(A, \mathcal{A}^A, P)$ is not countably generated for any $A \in \mathcal{A}$ with $P(A) > 0$.[6]

Definition 3. *The $\mathcal{T} \otimes \mathcal{C}$-measurable mapping $\mu : T \times \Omega \to \mathcal{M}(X)$ is* essentially random *if, for $(\lambda \times P)$-a.e. $(t, \omega) \in T \times \Omega$, the probability measure $\mu_{t\omega} \in \mathcal{M}(X)$ is not a Dirac measure concentrated at a single point in X.*

Suppose that the regular conditional distribution process $\mu_{t\omega}$ in Part 3 of Definition 2 is essentially random. Then Proposition 4 below shows that the probability space $(\Omega, \mathcal{A}, P)$ must be saturated.

3 The Main Result

The following theorem characterizes the one-way Fubini property.

Theorem 1. *A process g from $T \times \Omega$ to X satisfies the one-way Fubini property if and only if it is regular conditionally independent.*

As noted in the introduction, the Loeb product framework introduced in [32] provides a rich class of (two-way) Fubini extensions. For these we note that, because the two-way Fubini property implies the one-way Fubini property, Theorem 1 also implies that any process on a Loeb product probability space must be regular conditionally independent. When the process is used to model many agents with random outcomes, Theorem 3 in [39] shows that the conditioning σ-algebra could be taken as the σ-algebra representing all the systemic risks that influence a non-negligible set of agents.

[6]For other equivalent definitions of saturation, see Section 2 of [28]. As noted in [15], [23] and [24], atomless Loeb probability spaces are always saturated. For some recent applications of Loeb and more generally saturated probability spaces, see for example [2], [10], [12], [14], [27], [28], [30], [31], [35], [40], and [41].

4 Proof of Necessity

Proof of the necessity part of Theorem 1: Suppose that the process $g : T\times\Omega \to X$ has the one-way Fubini property because the product probability space $(T\times\Omega, \mathcal{T}\otimes\mathcal{A}, \lambda\times P)$ has a one-way Fubini extension $(T\times\Omega, \mathcal{W}, Q)$ such that g is $\mathcal{W}$-measurable. For any $A\in\mathcal{A}$ and $B\in\mathcal{B}$, let $\omega\mapsto 1_A(\omega)$ and $(t,\omega)\mapsto 1_{g^{-1}(B)}(t,\omega)$ denote the respective indicator functions of the sets A and $g^{-1}(B)$. Because $\mathcal{T}\otimes\mathcal{A}\subset\mathcal{W}$, the mapping $(t,\omega)\mapsto 1_A(\omega)$ is $\mathcal{W}$-measurable. By the the one-way Fubini property, so is the mapping $(t,\omega)\mapsto 1_{g^{-1}(B)}(t,\omega)$. It follows that $(t,\omega)\mapsto 1_A(\omega)1_{g^{-1}(B)}(t,\omega)$ is also $\mathcal{W}$-measurable, and so Q-integrable. By part (ii) of Definition 1, it follows that the mapping

$$t\mapsto \int_\Omega 1_A(\omega)1_{g^{-1}(B)}(t,\omega)dP = P(A\cap g_t^{-1}(B))$$

is $\mathcal{T}$-measurable. That is, the process g has event-wise measurable conditional probabilities, as defined in property (3) in the statement of Theorem 1 in [21]. So property (1) of that theorem follows: specifically, the process g is regular conditionally independent with respect to a suitable countably generated conditioning σ-algebra $\mathcal{C}$. $\square$

We remark that the appropriate conditioning σ-algebra in this result is the Monte Carlo σ-algebra $\mathcal{C}^g$ specified in Definition 3 of [21].

5 Proof of Sufficiency

Throughout this section, let g be a regular conditionally independent process from $T\times\Omega$ to X. Thus, there exists a countably generated sub-σ-algebra $\mathcal{C}$ of $\mathcal{A}$ such that g is essentially pairwise conditionally independent given $\mathcal{C}$, and also g admits an essentially regular conditional distribution process given $\mathcal{C}$.

Define the mapping $H : T\times\Omega\to T\times\Omega\times X$ by $H(t,\omega) := (t,\omega,g(t,\omega))$. Note that for each fixed $t\in T$, the component mapping H_t satisfies $H_t(\omega) = (\omega, g_t(\omega))$. Let $\mathcal{E} := \mathcal{T}\otimes\mathcal{A}\otimes\mathcal{B}$ denote the product σ-algebra on $T\times\Omega\times X$. Let $\mathcal{F} := \{H^{-1}(E) : E\in\mathcal{E}\}$. Then it is clear that $\mathcal{F}$ is a σ-algebra. Also, the first two components of $H(t,\omega)$ are given by the identity mapping $\mathrm{id}_{T\times\Omega}$ on $T\times\Omega$, while the last component is $g(t,\omega)$. Hence, $\mathcal{F}$ is the smallest σ-

algebra such that $\mathrm{id}_{T\times\Omega}$ and g are both measurable. This means that $\mathcal{F}$ is the smallest extension of the product σ-algebra $\mathcal{T}\otimes\mathcal{A}$ such that g is measurable.

Given any event $E \in \mathcal{E} = \mathcal{T}\otimes\mathcal{A}\otimes\mathcal{B}$, along with any fixed $t \in T$ and $\omega \in \Omega$, let E_t denote the section $\{(\omega, x) \in \Omega \times X \mid (t, \omega, x) \in E\}$ and $E_{t\omega}$ the section $\{x \in X \mid (t, \omega, x) \in E\}$.

Our proof of the sufficiency part of Theorem 1 relies on the following:

Proposition 1. *Given any event $E \in \mathcal{E} = \mathcal{T}\otimes\mathcal{A}\otimes\mathcal{B}$, for λ-a.e. $t \in T$ the following four properties hold:*

(i) the set $H_t^{-1}(E_t)$ is $\mathcal{A}$-measurable;

(ii) the mapping $\omega \mapsto \mu_{t\omega}(E_{t\omega})$ is $\mathcal{A}$-measurable;

(iii) $P(H_t^{-1}(E_t)) = \int_\Omega \mu_{t\omega}(E_{t\omega})dP$;

(iv) the mapping $t \mapsto P(H_t^{-1}(E_t))$ is λ-integrable.

In order to prove this proposition, we need several lemmas.

Lemma 1. *For all $s, t \in T$ and $B \in \mathcal{B}$, one has*

$$\mathbb{E}[1_{g_s^{-1}(B)}\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})] = \mathbb{E}[\mathbb{E}(1_{g_s^{-1}(B)}|\mathcal{C})\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})]$$

Proof. By the law of iterated expectations,

$$\begin{aligned}\mathbb{E}[1_{g_s^{-1}(B)}\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})] &= \mathbb{E}[\mathbb{E}(1_{g_s^{-1}(B)}\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})|\mathcal{C})] \\ &= \mathbb{E}[\mathbb{E}1_{g_s^{-1}(B)}|\mathcal{C}\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})]\end{aligned}$$

because the function $\omega \mapsto \mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})(\omega)$ is already $\mathcal{C}$-measurable — see, for example, [11] (p. 266). □

Fix any Borel set B in X. For each $t \in T$, define the random variable $\omega \mapsto h_t(\omega)$ on $(\Omega, \mathcal{A}, P)$ so that

$$h_t(\omega) := 1_{g_t^{-1}(B)}(\omega) - \mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})(\omega) \tag{2}$$

Evidently for all $t \in T$ one has $\mathbb{E}(h_t|\mathcal{C})(\omega) = 0$ for all $\omega \in \Omega$, and so $\mathbb{E}h_t = \mathbb{E}[\mathbb{E}(h_t|\mathcal{C})] = 0$

Lemma 2. *If g_s and g_t are conditionally independent given $\mathcal{C}$, then h_s and h_t are uncorrelated random variables with zero mean.*

Proof. By the law of iterated expectations,

$$\mathbb{E}h_t = \mathbb{E}1_{g_t^{-1}(B)} - \mathbb{E}[\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})] = \mathbb{E}1_{g_t^{-1}(B)} - \mathbb{E}1_{g_t^{-1}(B)} = 0$$

and similarly $\mathbb{E}h_s = 0$. Furthermore,

$$\begin{aligned}\mathbb{E}h_s h_t &= \mathbb{E}[1_{g_s^{-1}(B)}1_{g_t^{-1}(B)}] - \mathbb{E}[1_{g_s^{-1}(B)}\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})] \\ &\quad -\mathbb{E}[1_{g_t^{-1}(B)}\mathbb{E}(1_{g_s^{-1}(B)}|\mathcal{C})] + \mathbb{E}[\mathbb{E}(1_{g_s^{-1}(B)}|\mathcal{C})\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})] \\ &= \mathbb{E}[\mathbb{E}(1_{g_s^{-1}(B)}1_{g_t^{-1}(B)}|\mathcal{C})] - \mathbb{E}[\mathbb{E}(1_{g_s^{-1}(B)}|\mathcal{C})\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})]\end{aligned}$$

by Lemma 1 and the law of iterated expectations. But

$$\mathbb{E}(1_{g_s^{-1}(B)}1_{g_t^{-1}(B)}|\mathcal{C}) = \mathbb{E}(1_{g_s^{-1}(B)}|\mathcal{C})\mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C})$$

because g_s and g_t are conditionally independent given $\mathcal{C}$. So $\mathbb{E}h_s h_t = 0$, implying that the two zero-mean random variables are uncorrelated. □

Lemma 3. *Suppose that the component random variables f_t ($t \in T$) are all square-integrable and are essentially uncorrelated — i.e., suppose each $f_t \in L_2(\Omega, \mathcal{A}, P)$ and, for a.e. $t_1 \in T$, one has $\mathbb{E}(f_{t_1} f_{t_2}) = \mathbb{E}f_{t_1} \cdot \mathbb{E}f_{t_2}$ for a.e. $t_2 \in T$. Then, for every $A \in \mathcal{A}$, one has $\int_A f_t dP = P(A)\,\mathbb{E}f_t$ for λ-a.e. $t \in T$.*

Proof. This is Lemma 1 of [19], which was proved by considering orthogonal projections in Hilbert space. □

Lemma 4. *Given any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, for λ-a.e. $t \in T$ one has*

$$\int_A 1_{g_t^{-1}(B)} dP = \int_A \mathbb{E}(1_{g_t^{-1}(B)}|\mathcal{C}) dP$$

Proof. Because of Lemma 2, we can apply Lemma 3 to the bounded and so square-integrable random variables h_t ($t \in T$) defined by (2). Hence, for λ-a.e. $t \in T$

$$\int_A h_t dP = P(A)\,\mathbb{E}h_t = 0$$

Then the definition (2) of h_t implies the claimed result directly. □

Let $\mathcal{D}$ denote the collection of all events $E \in \mathcal{E}$ whose sections E_t and $E_{t\omega}$ satisfy

1. properties (i)–(iii) in the statement of Proposition 1, for λ-a.e. $t \in T$;
2. the mapping $(t, \omega) \mapsto \mu_{t\omega}(E_{t\omega})$ is $\mathcal{T} \otimes \mathcal{A}$-measurable.

Lemma 5. *Each measurable triple product set $E = S \times A \times B \in \mathcal{E}$ belongs to $\mathcal{D}$.*

Proof. First, it is easy to see that $E_{t\omega} = \emptyset$ unless $t \in S$ and $\omega \in A$, and so $\mu_{t\omega}(E_{t\omega}) = 1_S(t)\, 1_A(\omega)\, \mu_{t\omega}(B)$. By definition, the mapping $(t, \omega) \mapsto \mu_{t\omega}(B)$ is measurable w.r.t. $\mathcal{T} \otimes \mathcal{C}$, and so w.r.t. $\mathcal{T} \otimes \mathcal{A}$. It follows that the mapping $(t, \omega) \mapsto \mu_{t\omega}(E_{t\omega})$ is also $\mathcal{T} \otimes \mathcal{A}$-measurable. It remains to verify properties (i)–(iii) of Proposition 1.

In the trivial case when $t \notin S$, then $E_t = \emptyset$ and $\mu_{t\omega}(E_{t\omega}) = 0$ for all $\omega \in \Omega$, so the three properties (i)–(iii) follow immediately.

Otherwise, suppose that $t \in S$.

(i) Then $E_t = A \times B$, so $H_t^{-1}(E_t) = A \cap g_t^{-1}(B)$, which is the intersection of the two $\mathcal{A}$-measurable sets A and $g_t^{-1}(B)$. Thus, $H_t^{-1}(E_t)$ is $\mathcal{A}$-measurable.

(ii) Furthermore, $E_{t\omega} = B$ if $\omega \in A$, but $E_{t\omega} = \emptyset$ if $\omega \notin A$. It follows that $\mu_{t\omega}(E_{t\omega}) = 1_A(\omega)\, \mu_{t\omega}(B)$ for all $(t, \omega) \in S \times \Omega$. So obviously the mapping $\omega \mapsto \mu_{t\omega}(E_{t\omega})$ is $\mathcal{A}$-measurable for all $t \in S$.

(iii) For all $t \in S$, Lemma 4 implies that

$$P(H_t^{-1}(E_t)) = P(A \cap g_t^{-1}(B)) = \mathbb{E} 1_A 1_{g_t^{-1}(B)} = \mathbb{E}[1_A \mathbb{E}(1_{g_t^{-1}(B)} | \mathcal{C})]$$

Using the definition of $\mu_{t\omega}$ gives

$$P(H_t^{-1}(E_t)) = \mathbb{E}[1_A \mathbb{E}(1_{g_t^{-1}(B)} | \mathcal{C})] = \mathbb{E}[1_A \mu_{t\omega}(B)] = \int_\Omega \mu_{t\omega}(E_{t\omega}) dP$$

as required. □

Lemma 6. *The family $\mathcal{D}$ is a Dynkin (or λ-) class in the sense that:*

(a) $T \times \Omega \times X \in \mathcal{D}$*;*

(b) if $E, E' \in \mathcal{D}$ *with* $E \supset E'$*, then* $E \setminus E' \in \mathcal{D}$*;*

(c) if E^n *is an increasing sequence of sets in* $\mathcal{D}$*, then* $\cup_{n=1}^\infty E^n \in \mathcal{D}$*.*

Proof. (a) Evidently Lemma 5 implies that $T \times \Omega \times X \in \mathcal{D}$.

(b) If E, E' belong to $\mathcal{D}$ with $E \supset E'$, then $(E \setminus E')_t = E_t \setminus E'_t$ and $(E \setminus E')_{t\omega} = E_{t\omega} \setminus E'_{t\omega}$. Hence:

(i) For λ-a.e. $t \in T$, the set $H^{-1}((E \setminus E')_t) = H_t^{-1}(E_t) \setminus H_t^{-1}(E'_t)$ is $\mathcal{A}$-measurable.

(ii) The mapping $\omega \mapsto \mu_{t\omega}((E \setminus E')_{t\omega}) = \mu_{t\omega}(E_{t\omega}) - \mu_{t\omega}(E'_{t\omega})$ is $\mathcal{A}$-measurable.

(iii) Also

$$\begin{aligned} P(H^{-1}((E \setminus E')_t)) &= P(H_t^{-1}(E_t)) - P(H_t^{-1}(E'_t)) \\ &= \int_\Omega [\mu_{t\omega}(E_{t\omega}) - \mu_{t\omega}(E'_{t\omega})] dP = \int_\Omega \mu_{t\omega}((E \setminus E')_{t\omega}) dP \end{aligned}$$

Finally, because both $(t, \omega) \mapsto \mu_{t\omega}(E_{t\omega})$ and $(t, \omega) \mapsto \mu_{t\omega}(E'_{t\omega})$ are $\mathcal{T} \otimes \mathcal{A}$-measurable, so is $(t, \omega) \mapsto \mu_{t\omega}(E_{t\omega} \setminus E'_{t\omega}) = \mu_{t\omega}(E_{t\omega}) - \mu_{t\omega}(E'_{t\omega})$.

Hence, $E \setminus E' \in \mathcal{D}$.

(c) If E^n is an increasing sequence in $\mathcal{D}$, then:

(i) For λ-a.e. $t \in T$, the set $H_t^{-1}(\cup_{n=1}^\infty E_t^n) = \cup_{n=1}^\infty H_t^{-1}(E_t^n)$ is $\mathcal{A}$-measurable.

(ii) Because $\mu_{t\omega}(\cup_{n=1}^\infty E_{t\omega}^n) = \lim_{n\to\infty} \mu_{t\omega}(E_{t\omega}^n)$ for each $(t, \omega) \in T \times \Omega$, the mapping $\omega \mapsto \mu_{t\omega}(\cup_{n=1}^\infty E_{t\omega}^n)$ is $\mathcal{A}$-measurable, whereas $(t, \omega) \mapsto \mu_{t\omega}(\cup_{n=1}^\infty E_{t\omega}^n)$ is $\mathcal{T} \otimes \mathcal{A}$-measurable.

(iii) Also

$$\begin{aligned} P(H_t^{-1}(\cup_{n=1}^\infty E_t^n)) &= \lim_{n\to\infty} P(H_t^{-1}(E_t^n)) = \lim_{n\to\infty} \int_\Omega \mu_{t\omega}(E_t^n) dP \\ &= \int_\Omega \mu_{t\omega}(\cup_{n=1}^\infty E_t^n) dP \end{aligned}$$

by the monotone convergence theorem for integrals. Hence, $\cup_{n=1}^\infty E^n \in \mathcal{D}$.

This completes the proof that $\mathcal{D}$ is a Dynkin class. $\square$

Proof of Proposition 1: Note that the rectangular family of all Cartesian products of measurable sets is a π-system — i.e., closed under finite intersections (see [7], p. 44 and [13], p. 404). By Lemma 5, this rectangular family is a subfamily of $\mathcal{D}$ which, by Lemma 6, is a Dynkin class. Therefore, we can apply Dynkin's π–λ theorem to establish that the σ-algebra $\mathcal{E} = \mathcal{T} \otimes \mathcal{A} \otimes B$ generated by this rectangular family is also a subfamily of $\mathcal{D}$. Since the definition of $\mathcal{D}$ implies that $\mathcal{D} \subseteq \mathcal{E}$, we have shown that $\mathcal{D} = \mathcal{E} = \mathcal{T} \otimes \mathcal{A} \otimes B$. This verifies parts (i)–(iii) of Proposition 1, and the $\mathcal{T} \otimes \mathcal{A}$-joint measurability of the mapping $(t, \omega) \mapsto \mu_{t\omega}(E_{t\omega})$. So, applying the ordinary Fubini theorem to the integrand $(t,\omega)\mapsto\mu_{t\omega}(E_{t\omega})$ on the product space $(T\times\Omega,\mathcal{T}\otimes\mathcal{A},\lambda\times P)$ allows us to conclude that the mapping $t\mapsto\int_\Omega\mu_{t\omega}(E_{t\omega})dP$ is λ-integrable. Thus, part (iv) follows from part (iii). $\square$

Proof of the sufficiency part of Theorem 1: Let g be a regular conditionally independent process. Let $\mathcal{F}$ be the σ-algebra $H^{-1}(\mathcal{E})$ as defined at the beginning of this section. Hence, given any $F\in\mathcal{F}$, there exists at least one $E\in\mathcal{E}$ such that $F=H^{-1}(E)$. By part (i) of Proposition 1, the section $F_t=H_t^{-1}(E_t)\in\mathcal{A}$ for λ-a.e. $t\in T$. Part (iv) of the same result implies that the mapping $F\mapsto\nu(F):=\int_T P(F_t)d\lambda$ defines a unique set function ν on the σ-algebra $\mathcal{F}$. Arguing as in the proof of Theorem 1 in [19], it follows that ν is a uniquely defined probability measure, whose restriction to the product σ-algebra $\mathcal{T}\otimes A$ is $\lambda\times P$. Hence $(T\times\Omega,\mathcal{F},\nu)$ extends $(T\times\Omega,\mathcal{T}\otimes\mathcal{A},\lambda\times P)$.

To show that $(T\times\Omega,\mathcal{F},\nu)$ is a one-way Fubini extension, we use exactly the same argument as that used to prove Theorem 1 in [19], without any need even to change notation. In fact that argument was itself a simple adaptation of the standard proof of the usual Fubini Theorem — see, for example, [34, p. 188]. $\square$

6 Some General Results

As above, assume throughout this section that for some countably generated sub-σ-algebra $\mathcal{C}$ of $\mathcal{A}$, the process g is essentially pairwise conditionally independent given $\mathcal{C}$, and admits a $\mathcal{T}\otimes\mathcal{C}$-measurable, essentially regular conditional distribution process $(t,\omega)\mapsto\mu_{t\omega}\in\mathcal{M}(X)$.

Proposition 2. *Let h be any measurable function mapping the product space $(T\times\Omega,\mathcal{T}\otimes\mathcal{A},\lambda\times P)$ to a Polish space Y. Then, for λ-almost all $t\in T$, the two random variables g_t and h_t are conditionally independent given $\mathcal{C}$.*

Proof. Let $\mathcal{D}$ denote the Borel σ-algebra on Y. For any $B\in\mathcal{B}$, $C\in\mathcal{C}$ and $D\in\mathcal{D}$, consider the set $E=(h^{-1}(D)\times B)\cap(T\times C\times X)$. Since h is $\mathcal{T}\otimes\mathcal{A}$-measurable, the set E belongs to $\mathcal{T}\otimes\mathcal{A}\otimes\mathcal{B}$. For each $t\in T$, it is clear that $E_t=(h_t^{-1}(D)\times B)\cap(C\times X)=(C\cap h_t^{-1}(D))\times B$, and so $H_t^{-1}(E_t)=C\cap h_t^{-1}(D)\cap g_t^{-1}(B)$. It is also clear that $E_{t\omega}=B$ when $\omega\in C\cap h_t^{-1}(D)$, and $E_{t\omega}=\emptyset$ when $\omega\notin C\cap h_t^{-1}(D)$. By Proposition 1, for λ-a.e. $t\in T$ we have

$$\begin{aligned} P(H_t^{-1}(E_t)) &= P\left(C\cap h_t^{-1}(D)\cap g_t^{-1}(B)\right) \\ &= \int_\Omega\mu_{t\omega}(E_{t\omega})dP \;=\; \int_{C\cap h_t^{-1}(D)}\mu_{t\omega}(B)dP \end{aligned} \tag{3}$$

By the properties of conditional expectation, and the fact that the mapping $\omega \mapsto \mu_{t\omega}(B)$ is $\mathcal{C}$-measurable for λ-a.e. $t \in T$, we obtain

$$\begin{array}{rl} & \int_C \mathbb{E}\left(1_{h_t^{-1}(D)} 1_{g_t^{-1}(B)} | \mathcal{C}\right) dP \;=\; \int_C 1_{h_t^{-1}(D)} 1_{g_t^{-1}(B)} dP \\ = & \int_C 1_{h_t^{-1}(D)} \mu_{t\omega}(B) dP \;=\; \int_C \mathbb{E}\left(1_{h_t^{-1}(D)} \mu_{t\omega}(B) | \mathcal{C}\right) dP \\ = & \int_C \mathbb{E}\left(1_{h_t^{-1}(D)} | \mathcal{C}\right) \mu_{t\omega}(B) dP \\ = & \int_C \mathbb{E}\left(1_{h_t^{-1}(D)} | \mathcal{C}\right) \mathbb{E}\left(1_{g_t^{-1}(B)} | \mathcal{C}\right) dP. \end{array} \tag{4}$$

Let $\mathcal{C}^\pi = \{C_n\}_{n=1}^\infty$, $\mathcal{B}^\pi = \{B_m\}_{m=1}^\infty$, and $\mathcal{D}^\pi = \{D_k\}_{k=1}^\infty$ be countable π-systems that generate $\mathcal{C}$, $\mathcal{B}$, and $\mathcal{D}$ respectively. For each triple (k, m, n), there exists a set T_{kmn} with $\lambda(T_{kmn}) = 1$ such that for all $t \in T_{kmn}$, the Equations (4) all hold with $C = C_n$, $B = B_m$, and $D = D_k$.

Let $T^* := \cap_{k=1}^\infty \cap_{m=1}^\infty \cap_{n=1}^\infty T_{kmn}$. Because $\lambda(T_{kmn}) = 1$ for each of the countable family of sets T_{kmn}, one has $\lambda(T^*) = 1$. Now, whenever $t \in T^*$, Equations (4) with $C = C_n$, $B = B_m$, and $D = D_k$, must hold for all triples (k, m, n) simultaneously.

Because $\mathcal{C}^\pi$ is a π-system that generates $\mathcal{C}$, Dynkin's π–λ theorem (see [13], p. 404) implies that the Equations (4) must hold whenever $t \in T^*$, for all $C \in \mathcal{C}$, all $B \in \mathcal{B}^\pi$, and all $D \in \mathcal{D}^\pi$ simultaneously. Finally, because $\mathcal{B}^\pi$ and $\mathcal{D}^\pi$ are π-systems that generate $\mathcal{B}$ and $\mathcal{D}$ respectively, Equations (4) must hold whenever $t \in T^*$, $C \in \mathcal{C}$, $B \in \mathcal{B}$ and $D \in \mathcal{D}$. Therefore for any $t \in T^*$, the definition of conditional expectation implies that

$$P(h_t^{-1}(D) \cap g_t^{-1}(B) | \mathcal{C}) = P(h_t^{-1}(D) | \mathcal{C}) P(g_t^{-1}(B) | \mathcal{C})$$

holds for all $B \in \mathcal{B}$ and $D \in \mathcal{D}$. This means that the random variables g_t and h_t are conditionally independent given $\mathcal{C}$. Because $\lambda(T^*) = 1$, this completes the proof. □

Suppose that a standard joint measurability condition is imposed on a process g that is used to model many agents who face idiosyncratic micro shocks combined with macroeconomic risks that generate the conditioning σ-algebra $\mathcal{C}$. Then following corollary, which is Proposition 4 of [21], shows that there is essentially no idiosyncratic risk at all. The corollary generalizes the type of non-measurability result shown for independent random variables in Proposition 2.1 of [44], and for exchangeable random variables in Proposition 2 of [18].

Corollary 1. *If g is measurable on $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$, then for λ-almost all $t \in T$, the random variable g_t is $\mathcal{C}$-measurable.*

Proof. Proposition 2 implies that for λ-a.e. $t \in T$, the random variable g_t is conditionally independent of itself, given $\mathcal{C}$. Thus, for any $B \in \mathcal{B}$,

$$P(g_t^{-1}(B) \cap g_t^{-1}(B)|\mathcal{C}) = P(g_t^{-1}(B)|\mathcal{C}) = P(g_t^{-1}(B)|\mathcal{C})\, P(g_t^{-1}(B)|\mathcal{C})$$

This evidently implies that $P(g_t^{-1}(B)|\mathcal{C}) \in \{0, 1\}$ for all $\omega \in \Omega$. Let $A \in \mathcal{C}$ denote the subset of Ω on which $P(g_t^{-1}(B)|\mathcal{C}) = 1$. Then $P(g_t^{-1}(B)|\mathcal{C})$ essentially has the same value as the indicator function 1_A. It follows that $P\left(g_t^{-1}(B) \cap C\right) = P(A \cap C)$ for all $C \in \mathcal{C}$. In particular, $P\left(g_t^{-1}(B) \cap A\right) = P(A)$ and $P\left(g_t^{-1}(B) \cap (\Omega \setminus A)\right) = 0$, which implies that $P\left(g_t^{-1}(B) \Delta A\right) = 0$. Therefore $g_t^{-1}(B) \in \mathcal{C}$ for each $B \in \mathcal{B}$, which completes the proof. □

By Theorem 1, the product probability space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$ has a one-way Fubini extension $(T \times \Omega, \mathcal{F}, \nu)$ such that g is $\mathcal{F}$-measurable. The following proposition shows that, in the framework of a one-way Fubini extension $(T \times \Omega, \mathcal{F}, \nu)$, the essentially regular conditional distribution process with respect to $\mathcal{T} \otimes \mathcal{C}$, which is $(t, \omega) \mapsto \mu_{t\omega}$, must generate regular conditional distributions of the process g with respect to the usual product σ-algebra $\mathcal{T} \otimes \mathcal{A}$.

Proposition 3. *The $\mathcal{T} \otimes \mathcal{C}$-measurable mapping $\mu : T \times \Omega \to \mathcal{M}(X)$ satisfies $\nu\left(g^{-1}|\mathcal{T} \otimes \mathcal{A}\right) = \mu_{t\omega}$ for $(\lambda \times P)$-a.e. $(t, \omega) \in T \times \Omega$.*

Proof. Take any $G \in \mathcal{T} \otimes \mathcal{A}$, and any $B \in \mathcal{B}$. Let $E = G \times B$. For each $t \in T$, it is clear that $E_t = G_t \times B$, and $H_t^{-1}(E_t) = G_t \cap g_t^{-1}(B)$. It is also clear that $E_{t\omega} = B$ when $\omega \in G_t$, and $E_{t\omega} = \emptyset$ when $\omega \notin G_t$. By Proposition 1, for λ-a.e. $t \in T$ we have

$$P(H_t^{-1}(E_t)) = P\left(G_t \cap g_t^{-1}(B)\right) = \int_\Omega \mu_{t\omega}(E_{t\omega}) dP = \int_{G_t} \mu_{t\omega}(B) dP.$$

Taking the integral of each side w.r.t. the measure λ on T gives

$$\int_T P\left(G_t \cap g_t^{-1}(B)\right) d\lambda = \int_T \int_{G_t} \mu_{t\omega}(B) dP d\lambda. \tag{5}$$

But the one-way Fubini property implies, first, that

$$\int_G 1_{g^{-1}(B)} d\nu = \int_T \left[\int_\Omega 1_G(t, \omega) 1_{g^{-1}(B)} dP\right] d\lambda = \int_T P\left(G_t \cap g_t^{-1}(B)\right) d\lambda \tag{6}$$

and second, when combined with the usual Fubini property for the product space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$, that

$$\int_T \int_{G_t} \mu_{t\omega}(B) dP d\lambda = \int_G \mu_{t\omega}(B) d(\lambda \times P) = \int_G \mu_{t\omega}(B) d\nu \tag{7}$$

because $(T \times \Omega, \mathcal{F}, \nu)$ is a one-way Fubini extension. Together Equations (5), (6) and (7) imply that $\int_G 1_{g^{-1}(B)} d\nu = \int_G \mu_{t\omega}(B) d\nu$. Because the two sets $G \in \mathcal{T} \otimes \mathcal{A}$ and $B \in \mathcal{B}$ were arbitrarily chosen, it follows that $\nu\left(g^{-1} | \mathcal{T} \otimes \mathcal{A}\right) = \mu_{t\omega}$ for $(\lambda \times P)$-a.e. $(t, \omega) \in T \times \Omega$. $\square$

Before moving to Proposition 4, we state a lemma that is a special case of Lemma 2 in [20].

Lemma 7. *Let g be a process from $T \times \Omega$ to X. Let $\mathcal{C} \subseteq \mathcal{A}$ be a countably generated σ-algebra on Ω and μ a $\mathcal{T} \otimes \mathcal{C}$-measurable mapping from $T \times \Omega$ to $\mathcal{M}(X)$. Assume that for each fixed $A \in \mathcal{A}$ and $B \in \mathcal{B}$, one has*

$$P(A \cap g_t^{-1}(B)) = \int_A \mu_{t\omega}(B)\, dP \tag{8}$$

for λ-a.e. $t \in T$. Then the process g is essentially pairwise independent conditional on $\mathcal{C}$, with $P(g_t^{-1} | \mathcal{C}) = \mu_{t\omega}$ for λ-a.e. $t \in T$.

The following proposition shows that if the mapping $(t, \omega) \mapsto \mu_{t\omega}$ from $T \times \Omega$ to $\mathcal{M}(X)$ is essentially random, as defined in Definition 3, then the probability space $(\Omega, \mathcal{A}, P)$ must be saturated.

Proposition 4. *Assume that the $\mathcal{T} \otimes \mathcal{C}$-measurable mapping $\mu : T \times \Omega \to \mathcal{M}(X)$ is essentially random. Then the probability space $(\Omega, \mathcal{A}, P)$ must be saturated.*

Proof. Suppose that $(\Omega, \mathcal{A}, P)$ is not saturated. By definition, there exists an event $A \in \mathcal{A}$ with $P(A) > 0$ such that the restricted sub-measure space $(A, \mathcal{A}^A, P)$ is essentially countably generated. By Equation (8), we know that for any $D \in \mathcal{A}^A$, any $B \in \mathcal{B}$, and for λ-a.e. $t \in T$, one has

$$P(D \cap g_t^{-1}(B)) = \int_D 1_{g_t^{-1}(B)}\, dP = \int_D 1_B(g_t(\omega))\, dP = \int_D \mu_{t\omega}(B)\, dP \tag{9}$$

Since $(A, \mathcal{A}^A, P)$ is essentially countably generated, there is a countable π-system $\mathcal{D}^\pi = \{D_n^\pi\}_{n=1}^\infty$ in $\mathcal{A}^A$ such that $\mathcal{A}^A$ is the strong completion of the σ-algebra generated by $\mathcal{D}^\pi$ in the measure space $(A, \mathcal{A}^A, P)$. Let $\mathcal{B}^\pi = \{B_m\}_{m=1}^\infty$ be a countable π-system that generates $\mathcal{B}$, the Borel σ-algebra on the Polish space X. For each pair (m, n), there exists a set T_{mn} with $\lambda(T_{mn}) = 1$ such that for all $t \in T_{mn}$, Equation (9) holds with $B = B_m$ and $D = D_n^\pi$. Let $T^* := \bigcap_{m=1}^\infty \bigcap_{n=1}^\infty T_{mn}$. Because $\lambda(T_{mn}) = 1$ for all pairs (m, n), one has $\lambda(T^*) = 1$. Now, whenever $t \in T^*$, Equation (9) with $B = B_m$ and $D = D_n^\pi$, must hold for all pairs (m, n) simultaneously.

Next, fix any $t \in T^*$. Since $\mathcal{A}^A$ is the strong completion of the σ-algebra generated by $\mathcal{D}^\pi$, for any $B = B_m \in \mathcal{B}^\pi$, Equation (9) holds for any $D \in \mathcal{A}^A$. Hence, for any $m \geq 1$, the three integrands in Equation (9) satisfy

$$1_{B_m}(f_t(\omega)) = \delta_{f_t(\omega)}(B_m) = \mu_{t\omega}(B_m) \tag{10}$$

for P-a.e. $\omega \in A$, where $\delta_{f_t(\omega)}$ is the Dirac measure at the point $f_t(\omega)$. By grouping countably many P-null sets together, we can obtain a measurable subset $\bar{A}$ of A such that $P(\bar{A}) = P(A) > 0$, and for any $\omega \in \bar{A}$, Equation (10) holds for all $m \geq 1$ simultaneously.

Finally, fix any $\omega \in \bar{A}$. Then Equation (10) implies that the Dirac measure $\delta_{f_t(\omega)}$ agrees with $\mu_{t\omega}$ on the π-system $\mathcal{B}^\pi$ that, by definition, generates $\mathcal{B}$. Therefore, $\mu_{t\omega}$ is the same as $\delta_{f_t(\omega)}$ for any $(t, \omega) \in T^* \times \bar{A}$. This proves that $(t, \omega) \mapsto \mu_{t\omega}$ is not essentially random. □

7 Independence and Exchangeability

The following is part of Definition 5 in [21].

Definition 4. *A process g from $T \times \Omega$ to X is said to be* essentially pairwise exchangeable *if there exists a fixed joint probability measure π on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ such that for λ-a.e. $t_1 \in T$, the random variables g_{t_1} and g_{t_2} have the given joint distribution π for λ-a.e. $t_2 \in T$.*[7]

Given any process that is measurable in a one-way Fubini extension, the following proposition characterizes essential pairwise independence and essential pairwise exchangeability through regular conditional distributions with respect to the relatively smaller product σ-algebra $\mathcal{T} \otimes \mathcal{A}$.

[7]Note that essential pairwise exchangeability is equivalent to its finite or countably infinite multivariate versions; compare Footnote 5 in Section 2.2, and see both [20, Corollary 3] and [43, Theorem 4 and Proposition 3.5].

Proposition 5. *Let $(T \times \Omega, \mathcal{W}, Q)$ be a one-way Fubini extension of the product probability space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$, and f any $\mathcal{W}$-measurable process from $(T \times \Omega, \mathcal{W}, Q)$ to a Polish space X. Let the mapping $(t, \omega) \mapsto \mu'_{t\omega} = Q(f^{-1} | \mathcal{T} \otimes \mathcal{A})$ be a regular conditional distribution of f with respect to $\mathcal{T} \otimes \mathcal{A}$. Then, the random variables $\omega \mapsto f_t(\omega)$ are:*

(1) essentially pairwise independent if and only if $(t, \omega) \mapsto \mu'_{t\omega}$ is essentially a function only of t;

(2) essentially pairwise exchangeable if and only if $(t, \omega) \mapsto \mu'_{t\omega}$ is essentially a function only of ω.

Our proof of Part (2) of Proposition 5 relies on the following lemma, which is an immediate implication of Propositions 6 and 7 in [21].

Lemma 8. *Let g be a process from $T \times \Omega$ to X. The process g is essentially pairwise exchangeable if and only if there exists a measurable mapping $\omega \mapsto \mu_\omega$ from $(\Omega, \mathcal{A})$ to $\mathcal{M}(X)$ such that for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, one has $P(A \cap g_t^{-1}(B)) = \int_A \mu_\omega(B)\, dP$ for λ-a.e. $t \in T$.*

Proof of Proposition 5: Fix any $A \in \mathcal{A}$ and $B \in \mathcal{B}$. For any $S \in \mathcal{T}$, the definition of μ' implies that $\int_{S \times A} 1_{f^{-1}(B)} dQ = \int_{S \times A} \mu'(B) dQ$. Because the mapping $(t, \omega) \mapsto \mu'_{t\omega}$ must be measurable w.r.t. $\mathcal{T} \otimes \mathcal{A}$, the usual Fubini property implies that $\int_S \int_A 1_{f_t^{-1}(B)} dP d\lambda = \int_S \int_A \mu'_{t\omega}(B) dP d\lambda$. But the choice of $S \in \mathcal{T}$ was arbitrary, so

$$P(A \cap f_t^{-1}(B)) = \int_A 1_{f_t^{-1}(B)} dP = \int_A \mu'_{t\omega}(B)\, dP \quad \text{for } \lambda\text{-a.e. } t \in T. \tag{11}$$

Part (1): Suppose that the random variables f_t are essentially pairwise independent. Then f is essentially pairwise conditionally independent given the minimal σ-algebra $\mathcal{C} = \{\Omega, \emptyset\}$. It therefore admits a $\mathcal{T} \otimes \mathcal{C}$-measurable, essentially regular conditional distribution process $(t, \omega) \mapsto \mu'_{t\omega}$ which must be essentially independent of ω, and so takes the form $t \mapsto \mu'_t = P f_t^{-1}$. Proposition 3 then implies that $Q(f^{-1} | \mathcal{T} \otimes \mathcal{A}) = \mu'_t = P f_t^{-1}$, which is essentially a function only of t.

Conversely, suppose that $(t, \omega) \mapsto \mu'_{t\omega}$ is essentially a function only of t. Then we can say that $(t, \omega) \mapsto \mu'_{t\omega}$ is $\mathcal{T} \otimes \{\Omega, \emptyset\}$-measurable and satisfies Equation (11). By Lemma 7 with $\mathcal{C} = \{\Omega, \emptyset\}$, the random variables $\omega \mapsto f_t(\omega)$ are essentially pairwise independent.

Part (2): Suppose that the random variables $\omega \mapsto f_t(\omega)$ are essentially pairwise exchangeable. By Lemma 8, there exists a measurable mapping

$\omega \mapsto \mu_\omega$ from $(\Omega, \mathcal{A})$ to $\mathcal{M}(X)$ such that for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, one has $P(A \cap f_t^{-1}(B)) = \int_A \mu_\omega(B)\, dP$ for λ-a.e. $t \in T$. Let $\mathcal{C}$ be the σ-algebra generated by the mapping $\omega \mapsto \mu_\omega$. By Lemma 7, the process f is essentially pairwise conditionally independent given $\mathcal{C}$, and admits a $\mathcal{T} \otimes \mathcal{C}$-measurable, essentially regular conditional distribution process μ. It then follows from Proposition 3 that $\mu'_{t\omega} = \mu_\omega$ for $\lambda \times P$-almost all $(t, \omega) \in T \times \Omega$.[8]

Conversely, suppose that $(t, \omega) \mapsto \mu'_{t\omega}$ is essentially a function only of ω. Then, Equation (11) and Lemma 8 imply that the random variables $\omega \mapsto f_t(\omega)$ are essentially pairwise exchangeable. $\square$

The following is an obvious corollary of Proposition 4, which indicates that for an essentially pairwise exchangeable process f on a one-way Fubini extension $(T \times \Omega, \mathcal{W}, Q)$ with $\mu'_{t\omega} = \mu_\omega$ independent of t, if μ_ω is essentially random, then $(\Omega, \mathcal{A}, P)$ must be saturated.

Corollary 2. *Let $(T \times \Omega, \mathcal{W}, Q)$ be a one-way Fubini extension of the product probability space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$, and f any $\mathcal{W}$-measurable process from $(T \times \Omega, \mathcal{W}, Q)$ to a Polish space X. Let the mapping $(t, \omega) \mapsto \mu'_{t\omega} = Q(f^{-1} | \mathcal{T} \otimes \mathcal{A})$ be a regular conditional distribution of f with respect to $\mathcal{T} \otimes \mathcal{A}$ such that $(t, \omega) \mapsto \mu'_{t\omega}$ is essentially a function only of ω, which we denote by $\omega \mapsto \mu_\omega$. Suppose that the process $\omega \mapsto \mu_\omega$ is essentially random in the sense that for P-a.e. $\omega \in \Omega$, the probability measure μ_ω is not a Dirac measure concentrated at a single point in X. Then $(\Omega, \mathcal{A}, P)$ must be saturated.*

Using the framework of Loeb product spaces, it is shown in [43, Theorem 5] that the basic notions of independence and exchangeability are in fact dual to each other, in the sense that essential pairwise independence of the random variables is equivalent to essential pairwise exchangeability of the sample functions generated by the relevant process. The following corollary of Proposition 5 makes this duality result transparent and allows it to be extended from the Loeb product spaces used in [43] to the more general setting of an arbitrary two-way Fubini extension.

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[8] The regular conditional distribution of f with respect to $\mathcal{T} \otimes \mathcal{A}$ as in Proposition 3 is stated for the minimal one-way Fubini extension of the product probability space in which f is measurable. On the other hand, $(T \times \Omega, \mathcal{W}, Q)$ is a general one-way Fubini extension in which f is measurable, and thus includes the minimal one-way Fubini extension. By the definition of conditional expectations, it is easy to see that the regular conditional distribution of f with respect to $\mathcal{T} \otimes \mathcal{A}$, as viewed in an extended probability space $(T \times \Omega, \mathcal{W}, Q)$, remains the same.

Corollary 3. *Let $(T \times \Omega, \mathcal{W}, Q)$ be any two-way Fubini extension of the product probability space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \times P)$, and $(t, \omega) \mapsto f(t, \omega)$ a $\mathcal{W}$-measurable process from $(T \times \Omega, \mathcal{W}, Q)$ to a Polish space X. Then the random variables $\omega \mapsto f_t(\omega)$ are essentially pairwise independent if and only if the sample functions $t \mapsto f_\omega(t)$, regarded as random variables on the probability space $(T, \mathcal{T}, \lambda)$, are essentially pairwise exchangeable.*

Proof. Let $\mu' = Q(f^{-1}|\mathcal{T} \otimes \mathcal{A})$ be a regular conditional distribution of f with respect to $\mathcal{T} \otimes \mathcal{A}$. Because a two-way Fubini extension has the properties of a one-way Fubini extension, part (1) of Proposition 5 implies that the random variables $\omega \mapsto f_t(\omega)$, are essentially pairwise independent if and only if $(t, \omega) \mapsto \mu'_{t\omega}$ is essentially a function only of t.

By viewing $(\Omega, \mathcal{A}, P)$ as the parameter space and $(T, \mathcal{T}, \lambda)$ as the sample space, the two-way Fubini extension property of $(T \times \Omega, \mathcal{W}, Q)$ and Part (2) of Proposition 5 together imply that $(t, \omega) \mapsto \mu'_{t\omega}$ is essentially a function only of t if and only if the functions $t \mapsto f_\omega(t)$ for different $\omega \in \Omega$ are essentially pairwise exchangeable, when viewed as random variables on $(T, \mathcal{T}, \lambda)$.

The result follows immediately from the above two equivalence results. $\square$

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