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The measurement of welfare change*

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Abstract. We propose and characterize a class of measures of welfare change that are based on the generalized Gini social welfare functions. In addition, we analyze these measures in the context of a second-order dominance property that is akin to generalized Lorenz dominance as introduced by Shorrocks (1983) and Kakwani (1984). Because we consider welfare differences rather than welfare levels, the requisite equivalence result involves linear welfare functions (that is, those associated with the generalized Ginis) only, as opposed to the entire class of strictly increasing and S-concave welfare indicators. *Journal of Economic Literature* Classification No.: D31.

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1 Introduction

Atkinson (1970), Kolm (1969) and Sen (1973) have provided the basic framework within which the modern approach to the measurement of inequality has developed. This approach explicitly endows measures of inequality with a normative interpretation. This is done by deriving inequality indices from social welfare functions defined on income distributions. Each index so derived inherits the distributional judgements embodied in the social welfare function from which it is derived. A reduction in inequality results in an increase in social welfare provided mean incomes remain unchanged.

Since the underlying social welfare functions are supposed to be *ordinal,* the derived measures of inequality are also ordinal. Hence, they can be used only to compare *levels* of inequality associated with the income distributions of, say, two countries, or of the same country at two points of time. However, it is also of considerable interest to ask questions such as "Has country A been more successful than country B in reducing inequality over the last decade?" A related but slightly different task is to compare the *change in social welfare* in country A to that of country B during a given period of time. Since the two countries may have a large difference in the rates of growth and since social welfare depends both on the size and the distribution of the cake, it may, in fact, be more appropriate to focus on comparisons of changes in social welfare. While this is the focus of this paper, we will discuss briefly how to address the issue of comparing changes in the level of inequality.

A comparison of *changes* in levels of social welfare require us to step out of the ordinal framework since the ranking of differences of welfare is not preserved under ordinal transformations of the welfare function. That is why we assume in this paper that the social welfare function has *cardinal* significance. This allows us to define a measure, V, of the change in social welfare between two income distributions, say x^0 and x^1. Furthermore, it is meaningful to compare *levels* of $V(x^0, x^1)$ and $V(y^0, y^1)$.

Having defined the measure of welfare change, V, our purpose is two-fold. First, we specify axioms or properties of V and characterize a class of these welfare change measures. This class of measures corresponds to the difference in welfare levels of two distributions, where the welfare function itself belongs to the class of generalized Gini indices. Our proof closely follows that of Weymark (1981), the principal difference being that our axioms are imposed on the measure of welfare change rather than on the welfare function. Our axioms are also slightly different allowing for a shorter proof.

Our second purpose leads perhaps to our main contribution. The generalized Gini welfare functions constitute the entire class of *linear* functions of individual incomes satisfying the strict version of the transfer principle that requires welfare (and hence equality) to go up when there is a transfer of income from a rich person to someone who is poorer. Of course, the comparison of levels of welfare change can vary depending on *which* member of the class is used. Since there are no firm ethical reasons for preferring one generalized Gini function over another, it may not always be possible to arrive at unambiguous comparisons. So, we ask the question whether it is possible to define a *dominance* condition which if satisfied guarantees that the comparison of levels of welfare change give the same answer for *all* the generalized Gini welfare functions. This question has, of course, been asked in the context of inequality of income distributions. Following Atkinson (1970) and Kolm (1969), Dasgupta, Sen and Starrett (1973) proved the most general such result by showing that if two distributions have the same mean income, then the social welfare associated with income distribution x is higher than that of y according to any S-concave welfare function if and only if the Lorenz curve of x lies everywhere above that of y. Thus, there is a precise dominance result for equality levels that corresponds to the class of S-concave welfare functions. Shorrocks (1983) and Kakwani (1984) independently extended this result so as to be able to compare welfare levels of income distributions which do not have the same mean incomes. They scaled up the Lorenz curve of an income distribution by its mean income to obtain the *generalized Lorenz curve,* and showed that generalized Lorenz dominance of income distribution x over another distribution y also provides a necessary and sufficient condition for unambiguous welfare comparisons. That is, x has higher social welfare than y for *all* increasing and S-concave welfare functions if and only if its generalized Lorenz curve lies everywhere above that of y.

We too use generalized Lorenz curves in our analysis of comparisons of welfare change, but adapt them for our purpose. Consider any income distribution x^0 for a population of size n where individual incomes have been ranked in increasing order, so that $x_i \leq x_{i+1}$ for all i from 1 to $n-1$. Then, the generalized Lorenz curve of x is obtained by plotting the cumulative incomes of the lowest k income levels against each k for all k from 1 to n. Suppose now that there are two pairs of income distributions (all of population size n) indicating how income distributions have changed in countries A and B. Suppose also that all individual incomes have been arranged in increasing order. Then, we compare the sums of the cumulative *differences* between x^0 and x^1, and between y^0 and y^1. Put differently, we focus not on the generalized Lorenz curves themselves, but on the sums of the *vertical* differences between x^0 and x^1, and between y^0 and y^1. Our principal result is that the welfare change between x^0 and x^1 is at least as large as that between y^0 and y^1 for *all* generalized Gini differences if and only if the curve corresponding to the cumulative sums of differences between x^0 and x^1 lies everywhere above that of the corresponding curve for the y distributions.

Of course, this dominance is equivalent to second-order dominance of the difference between the x vectors and the y vectors. We then ask whether *first-order* dominance of the difference between the (ranked) x vectors over the difference between the (ranked) y vectors will imply unambiguous welfare change comparisons for a larger class of welfare functions. Clearly, this class would then contain the generalized Gini welfare functions since first-order dominance implies second-order dominance. However, we show the surprising result that even when there is first-order dominance in this sense, the only class for which unambiguous welfare comparisons are possible is the set of linear functions. So, once the strict transfer principle is imposed, our result implies that first-order dominance does not buy anything that second-order dominance does not already give us. This demonstrates that unambiguous comparisons of welfare change are very hard to make.

2 Measures of welfare change

Suppose that there are $n \geq 2$ individuals in a society. A measure of welfare change is a function $V\colon \mathbb{R}^{2n}_{+} \to \mathbb{R}$ and we interpret $V(x^0, x^1)$ as an indicator of the improvement (or deterioration) associated with moving from last period's income distribution $x^0 \in \mathbb{R}^n_+$ to the current distribution $x^1 \in \mathbb{R}^n_+$. Our objective is to find a class of functions V that can be expressed in terms of a difference in welfare levels. That is, we require V to possess the following property.

Welfare difference compatibility. There exists a function $W\colon \mathbb{R}^n_+ \to \mathbb{R}$ such that, for all $x^0, x^1 \in \mathbb{R}^n_+$,

$$V(x^0, x^1) = W(x^1) - W(x^0). \tag{1}$$

We use the following properties that are well-established in the context of welfare functions and they continue to have strong intuitive appeal when formulated for a measure of welfare change. The first of these is continuity.

Continuity. V is continuous.

The next property is a strict monotonicity condition. It requires that V is strictly decreasing in x^0 and strictly increasing in x^1.

Strict monotonicity. (a) For all $x^0, x^1, z^0 \in \mathbb{R}^n_+$, if $z^0_i \geq x^0_i$ for all $i \in \{1, \ldots, n\}$ with at least one strict inequality, then $V(z^0, x^1) < V(x^0, x^1)$.

(b) For all $x^0, x^1, z^1 \in \mathbb{R}^n_+$, if $z^1_i \geq x^1_i$ for all $i \in \{1, \ldots, n\}$ with at least one strict inequality, then $V(x^0, z^1) > V(x^0, x^1)$.

It follows immediately that if V is continuous and strictly monotonic and a function W as in (**??**) exists, this function W must be continuous and strictly increasing.

Positive linear homogeneity is another standard requirement for welfare functions that can be expressed for a measure of welfare change.

Positive linear homogeneity. For all $x^0, x^1 \in \mathbb{R}^n_+$ and for all $\lambda \in \mathbb{R}_{++}$,

$$V(\lambda x^0, \lambda x^1) = \lambda V(x^0, x^1).$$

Furthermore, we assume that V is anonymous, that is, V is invariant with respect to permutations of the income vectors.

Anonymity. For all $x^0, x^1 \in \mathbb{R}^n_+$ and for all permutations $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$,

$$V\left((x^0_{\pi(1)}, \ldots, x^0_{\pi(n)}), (x^1_{\pi(1)}, \ldots, x^1_{\pi(n)})\right) = V(x^0, x^1).$$

The function W of (**??**) must be anonymous as a consequence of the anonymity of V. This is the case because, using (**??**), the anonymity of V requires

$$W\left(x^1_{\pi(1)}, \ldots, x^1_{\pi(n)}\right) - W\left(x^0_{\pi(1)}, \ldots, x^0_{\pi(n)}\right) = W(x^1) - W(x^0)$$

for all $x^0, x^1 \in \mathbb{R}^n_+$ and for all permutations $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$. Setting $x^0 = (1, \ldots, 1)$, this simplifies to

$$W\left(x^1_{\pi(1)}, \ldots, x^1_{\pi(n)}\right) = W(x^1)$$

for all $x^1 \in \mathbb{R}^n_+$ and for all permutations $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ so that W is anonymous.

The strict transfer principle (Pigou, 1912; Dalton, 1920) is an essential equity requirement. For any two income distributions $x^1, y^1 \in \mathbb{R}^n_+$, the distribution y^1 is obtained from x^1 via a progressive transfer if there exist $i, j \in \{1, \ldots, n\}$ such that $x^1_i < x^1_j$ and $\varepsilon \in (0, (x^1_j - x^1_i)/2]$ such that $y^1_i = x^1_i + \varepsilon$, $y^1_j = x^1_j - \varepsilon$ and $y^1_k = x^1_k$ for all $k \in \{1, \ldots, n\} \setminus \{i, j\}$.

Strict transfer principle. For all $x^0, x^1, y^1 \in \mathbb{R}^n_+$, if y^1 is obtained from x^1 via a progressive transfer, then $V(x^0, x^1) < V(x^0, y^1)$.

The function W satisfies the standard strict transfer principle if V satisfies the above property. Suppose that $y^1 \in \mathbb{R}^n_+$ is obtained from $x^1 \in \mathbb{R}^n_+$ via a progressive transfer, and let $x^0 \in \mathbb{R}^n_+$ be arbitrary. By the strict transfer principle and (**??**), it follows that

$$W(x^1) - W(x^0) < W(y^1) - W(x^0)$$

so that $W(x^1) < W(y^1)$ for all $x^1, y^1 \in \mathbb{R}^n_+$ such that y^1 is obtained from x^1 via a progressive transfer.

Note that the conjunction of (**??**) and the strict transfer principle implies that, for all $x^0, x^1, y^0 \in \mathbb{R}^n_+$, if y^0 is obtained from x^0 via a progressive transfer, then the inequality $V(x^0, x^1) > V(y^0, x^1)$ must be true.

Our final axiom is an independence condition that is restricted to income distributions in which all incomes are ranked from lowest to highest. Formally, the set of bottom-first-ordered permutations of the elements of $\mathbb{R}^n_+$ is given by

$$B = \{x \in \mathbb{R}^n_+ \mid x_1 \leq \ldots \leq x_n\}.$$

The welfare function analogue of following property appears in Weymark (1981, p. 418).

Weak independence of income source. For all $x^0, x^1, y^0, y^1, z \in B$,

$$V(x^0 + z, x^1 + z) \geq V(y^0 + z, y^1 + z) \;\Leftrightarrow\; V(x^0, x^1) \geq V(y^0, y^1). \tag{2}$$

This axiom implies that W has the corresponding property as defined in Weymark (1981). Using (**??**), it follows that (**??**) is equivalent to

$$W(x^1 + z) - W(x^0 + z) \geq W(y^1 + z) - W(y^0 + z) \;\Leftrightarrow\; W(x^1) - W(x^0) \geq W(y^1) - W(y^0)$$

for all $x^0, x^1, y^0, y^1, z \in B$. Setting $x^0 = y^0$, this simplifies to

$$W(x^1 + z) \geq W(y^1 + z) \;\Leftrightarrow\; W(x^1) \geq W(y^1)$$

for all $x^1, y^1, z \in B$, which is Weymark's (1981) condition.

Let

$$A = \{\alpha \in \mathbb{R}^n_{++} \mid \alpha_1 > \ldots > \alpha_n\}.$$

A welfare function W is a generalized Gini welfare function if there exists a parameter vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in A$ such that, for all $x \in \mathbb{R}^n_+$,

$$W(x) = \sum_{i=1}^{n} \alpha_i \tilde{x}_i \tag{3}$$

where $\tilde{x} \in B$ is a bottom-first-ordered permutation of x. Thus, the weights α_i are assigned to the positions in an income distribution, where higher incomes receive lower weights in order to ensure that the resulting welfare function respects the strict transfer principle. The corresponding generalized Gini measure of welfare change is given by

$$V(x^0, x^1) = \sum_{i=1}^{n} \alpha_i \tilde{x}_i^1 - \sum_{i=1}^{n} \alpha_i \tilde{x}_i^0 \tag{4}$$

for all $(x^0, x^1) \in \mathbb{R}^{2n}_+$. The measure of welfare change associated with the parameter vector $\alpha \in A$ is denoted by V_α. The class of all generalized Gini measures of welfare change is given by

$$\mathcal{V}_G = \{V_\alpha \mid \alpha \in A\}.$$

Because we restrict attention to anonymous measures of welfare and welfare change, it involves no loss of generality to assume that x is bottom-first ordered.

3 Dominance properties

To simplify notation, we define

$$\delta_i = x_i^1 - x_i^0 \;\text{ and }\; \gamma_i = y_i^1 - y_i^0 \;\text{ for all } i \in \{1, \ldots, n\}$$

for all bottom-first-ordered income distributions $x^0, x^1, y^0, y^1 \in B$. The following dominance property is a welfare-change adaptation of the generalized Lorenz criterion; see Shorrocks (1983).

Second-order dominance. For all $x^0, x^1, y^0, y^1 \in B$, (x^0, x^1) second-order dominates (y^0, y^1) if and only if

$$\sum_{i=1}^{k} (\delta_i - \gamma_i) \geq 0 \;\text{ for all } k \in \{1, \ldots, n\}.$$

Our objective is to derive a condition on any two pairs of distributions (x^0, x^1) and (y^0, y^1) that will enable us to state that the welfare change between (x^0, x^1) is greater or smaller than the welfare change between (y^0, y^1) for *all* measures of welfare change in the class $\mathcal{V}_G$. So, we want to rule out cases where there are two parameter vectors $\alpha, \alpha' \in A$ such that

$$V_\alpha(x^0, x^1) \geq V_\alpha(y^0, y^1) \text{ and } V_{\alpha'}(x^0, x^1) < V_{\alpha'}(y^0, y^1).$$

The following lemma provides a condition that prevents such reversals.

Lemma 1. *For all $x^0, x^1, y^0, y^1 \in B$,*

$$V(x^0, x^1) \geq V(y^0, y^1) \text{ for all } V \in \mathcal{V}_G$$

if and only if

$$\sum_{i=1}^{k} \alpha_i(\delta_i - \gamma_i) \geq 0 \text{ for all } k \in \{1, \ldots n\} \text{ and for all } \alpha \in A. \tag{5}$$

Proof. Sufficiency of (**??**) follows from the definition of the elements of $\mathcal{V}_G$.

To prove necessity, let $V_\alpha(x^0, x^1) \geq V_\alpha(y^0, y^1)$ for some $\alpha \in A$. Therefore, by definition

$$\sum_{i=1}^{n} \alpha_i(\delta_i - \gamma_i) \geq 0. \tag{6}$$

Suppose there exists a $k < n$ such that

$$\sum_{i=1}^{k} \alpha_i(\delta_i - \gamma_i) < 0.$$

Together with (**??**), this implies

$$\sum_{i=k+1}^{n} \alpha_i(\delta_i - \gamma_i) \geq \sum_{i=1}^{k} \alpha_i(\gamma_i - \delta_i).$$

Let $c > 1$ and define a vector $\alpha^c \in A$ as follows. For all $i \in \{1, \ldots, n\}$,

$$\alpha_i^c = \begin{cases} \alpha_i & \text{if } i \leq k, \\ \frac{1}{c}\alpha_i & \text{if } i > k. \end{cases}$$

Clearly, $\alpha^c \in A$ for all $c > 1$, and there exists c^* sufficiently large such that

$$\sum_{i=1}^{n} \alpha_i^{c^*}(\delta_i - \gamma_i) < 0,$$

that is, $V_{\alpha^{c^*}}(y^0, y^1) > V_{\alpha^{c^*}}(x^0, x^1)$. This shows that if (**??**) is not satisfied, then there are two measures of welfare change in $\mathcal{V}_G$ which differ in their ranking of the pairs (x^0, x^1) and (y^0, y^1), a contradiction that completes the proof. ■

We prove one more lemma before stating our main result.

Lemma 2. *Let $x^0, x^1, y^0, y^1 \in B$ and suppose that (x^0, x^1) second-order dominates (y^0, y^1). Then, for all $\alpha \in A$ and for all $k \in \{1, \ldots, n\}$,*

$$\sum_{i=1}^{k} \alpha_i(\delta_i - \gamma_i) \geq \alpha_k \sum_{i=1}^{k} (\delta_i - \gamma_i).$$

Proof. Clearly, the claim is true for $k = 1$. By way of induction, suppose that it is true for all $k' < k$. Then it follows that

$$\alpha_k(\delta_k - \gamma_k) + \sum_{i=1}^{k-1} \alpha_i(\delta_i - \gamma_i) \geq \alpha_k(\delta_k - \gamma_k) + \alpha_{k-1} \sum_{i=1}^{k-1} (\delta_i - \gamma_i).$$

Hence,

$$\sum_{i=1}^{k} \alpha_i(\delta_i - \gamma_i) \geq \alpha_k \sum_{i=1}^{k} (\delta_i - \gamma_i)$$

since $\alpha_{k-1} > \alpha_k$ because $\alpha \in A$ and $\sum_{i=1}^{k-1}(\delta_i - \gamma_i) \geq 0$ because (x^0, x^1) second-order dominates (y^0, y^1). ■

We can now state the main result of this section.

Theorem 1. *For all $x^0, x^1, y^0, y^1 \in B$,*

$$V(x^0, x^1) \geq V(y^0, y^1) \ \text{ for all } V \in \mathcal{V}_G$$

if and only if (x^0, x^1) second-order dominates (y^0, y^1).

Proof. Suppose first that (x^0, x^1) second-order dominates (y^0, y^1). Then, by definition,

$$\sum_{i=1}^{k} (\delta_i - \gamma_i) \geq 0 \ \text{ for all } k \in \{1, \ldots, n\}. \tag{7}$$

Take any $\alpha \in A$. Then the inequality

$$\alpha_1(\delta_1 - \gamma_1) \geq 0$$

follows from setting $k = 1$ in (**??**) and the fact that $\alpha_1 > 0$. Suppose that, for some $K \in \{2, \ldots, n\}$,

$$\sum_{i=1}^{k} \alpha_i(\delta_i - \gamma_i) \geq 0 \ \text{ for all } k \in \{1, \ldots, K-1\}.$$

We want to show that

$$\sum_{i=1}^{K} \alpha_i(\delta_i - \gamma_i) \geq 0.$$

By (**??**) and Lemma **??**,

$$\sum_{i=1}^{K} \alpha_i(\delta_i - \gamma_i) \geq \alpha_K \sum_{i=1}^{k} (\delta_i - \gamma_i) \geq 0.$$

Since $\alpha \in A$ was chosen arbitrarily, this inequality together with Lemma **??** establishes that $V(x^0, x^1) \geq V(y^0, y^1)$ for all $V \in \mathcal{V}_G$.

Now suppose that $V(x^0, x^1) \geq V(y^0, y^1)$ for all $V \in \mathcal{V}_G$. We need to show that (x^0, x^1) second-order dominates (y^0, y^1). In view of Lemma **??**, it is sufficient to prove that if (**??**) is satisfied, then (x^0, x^1) second-order dominates (y^0, y^1).

Pick any $\alpha \in A$. Then,

$$\sum_{i=1}^{k} \alpha_i(\delta_i - \gamma_i) \geq 0 \ \text{ for all } k \in \{1, \ldots, n\}.$$

Clearly, $\delta_1 \geq \gamma_1$ since $\alpha_1 > 0$. Let $K \in \{2, \ldots, n\}$ and assume that

$$\sum_{i=1}^{k} (\delta_i - \gamma_i) \geq 0 \ \text{ for all } k \in \{1, \ldots, K-1\}.$$

Suppose that

$$\sum_{i=1}^{K} (\delta_i - \gamma_i) < 0.$$

Multiplying by $\alpha_K > 0$, we obtain

$$\sum_{i=1}^{K} \alpha_K(\delta_i - \gamma_i) < 0. \tag{8}$$

Let $\varepsilon \in \mathbb{R}_{++}$ and define $\alpha^\varepsilon \in A$ as follows. For all $i \in \{1, \ldots, n\}$,

$$\alpha_i^\varepsilon = \begin{cases} \alpha_K + (K-i)\varepsilon & \text{if } i < K, \\ \alpha_i & \text{if } i \geq K. \end{cases}$$

From (**??**), we know that

$$\sum_{i=1}^{K} \alpha_i^\varepsilon(\delta_i - \gamma_i) \geq 0 \ \text{ for all } \varepsilon \in \mathbb{R}_{++}.$$

But (**??**) implies

$$\lim_{\varepsilon \to 0} \sum_{i=1}^{K} \alpha_i^\varepsilon(\delta_i - \gamma_i) = \alpha_K \sum_{i=1}^{K} (\delta_i - \gamma_i) < 0,$$

a contradiction to (**??**) that completes the proof of the theorem. ■

A comparison with the result obtained by Shorrocks (1983) illustrates the impact of considering welfare differences rather than welfare levels. Because of the linearity inherent in difference comparisons, not all increasing and S-concave welfare functions have to agree in order to obtain an equivalence result but only those among them that are linear—that is, those corresponding to the generalized Ginis.

Thus, as established in the above theorem, welfare functions other than the generalized Ginis cannot be employed in an equivalence result that involves our second-order dominance condition. This raises the question of whether a more stringent dominance definition can accommodate a more general class of functions. For instance, we may want to impose a first-order dominance property for welfare differences, defined as follows. Again, we assume that, without loss of generality, the income distributions x^0, x^1, y^0, y^1 are in B, and δ_i and γ_i are defined as above.

First-order dominance. For all $x^0, x^1, y^0, y^1 \in B$, (x^0, x^1) first-order dominates (y^0, y^1) if and only if

$$\delta_i - \gamma_i \geq 0 \ \text{ for all } i \in \{1, \ldots, n\}.$$

Because first-order dominance implies second-order dominance, the generalized Gini measures of welfare change are compatible with this dominance property. There may be additional measures that can be accommodated in the first-order case but, as illustrated below, all of them must be based on linear measures as well.

Let $\mathcal{F}$ be a class of measures of welfare change. If the members of $\mathcal{F}$ are to be compatible with the first-order dominance criterion, it must be the case that if (x^0, x^1) first-order dominates (y^0, y^1), then

$$V(x^0, x^1) \geq V(y^0, y^1) \ \text{ for all } V \in \mathcal{F}$$

or, in terms of the underlying welfare functions W,

$$W(x^1) - W(x^0) \geq W(y^1) - W(y^0). \tag{9}$$

Consider $x^1, y^0 \in B$ and let $x^0 = y^1 = (x^1 + y^0)/2$, that is, the distributions x^0 and y^1 are both equal to the arithmetic mean of x^1 and y^0. Therefore, by definition,

$$\delta_i - \gamma_i = (x_i^1 - x_i^0) - (y_i^1 - y_i^0) = 0 \ \text{ for all } i \in \{1, \ldots, n\}.$$

Because $x^0 = y^1 = (x^1 + y^0)/2$, (**??**) requires that

$$\begin{aligned} W(x^1) - W(x^0) = W(y^1) - W(y^0) \ &\Leftrightarrow \ W(x^1) - W(x^0) = W(x^0) - W(y^0) \\ &\Leftrightarrow \ 2W(x^0) = W(x^1) + W(y^0) \\ &\Leftrightarrow \ W\left(\frac{1}{2}x^1 + \frac{1}{2}y^0\right) = \frac{1}{2}W(x^1) + \frac{1}{2}W(y^0), \end{aligned}$$

a condition that requires W to be an increasing affine function within the bottom-first-ordered subspace of $\mathbb{R}^n_+$. This, in turn, means that the associated measure of welfare change is linear. Because the only functions with that property other than the generalized Ginis are such that the parameter vectors α do not respect the inequalities that define membership in A, it follows that these additional functions fail to satisfy the strict transfer principle. Thus, even this first-order dominance condition does not allow for measures other than the generalized Ginis if this fundamental equity property is to be retained.

We conclude this section with an observation that follows from the above-stated results. Consider the question of making similar unambiguous comparisons of changes in the level of inequality, where the measure of inequality is derived from a social welfare function. Take two pairs of income distributions (x^0, x^1) and (y^0, y^1) each with the same (positive) mean income. Let

$$Z(x^0, x^1) = I(x^1) - I(x^0)$$

be a measure of inequality change, analogous to V. Suppose, moreover, that

$$I(x) = 1 - \frac{x_e}{\mu(x)},$$

where $\mu(x)$ is mean income and x_e is the *equally-distributed-equivalent income* corresponding to the income distribution x and the welfare function W. That is, x_e is implicitly defined by

$$W(x_e, \ldots, x_e) = W(x).$$

Let $\mathcal{I}_G$ be the class of inequality measures that are derived from the class of generalized Gini welfare functions, and let $\mathcal{Z}_G$ be the set of measures of inequality change that represent the difference of inequality levels where the inequality index is some member of $\mathcal{I}_G$. In view of Theorem **??**, the following result is immediate.

Theorem 2. *For all $x^0, x^1, y^0, y^1 \in B$ with the same mean income,*

$$Z(x^0, x^1) \geq Z(y^0, y^1) \;\; \textit{for all } Z \in \mathcal{Z}_G$$

if and only if (x^0, x^1) second-order dominates (y^0, y^1).

Note that since we restrict the four distributions to have the same mean income, we could also state the above theorem in terms of second-order dominance of the vertical differences in the *Lorenz curves.*

4 A characterization

We conclude this paper by providing a characterization of the generalized Gini measures of welfare change. There clearly is a strong resemblance to Weymark's (1981) axiomatization but some arguments in his proof can be simplified here because our list of axioms is slightly different from his.

Theorem 3. *A measure of welfare change V satisfies welfare difference compatibility, continuity, strict monotonicity, positive linear homogeneity, anonymity, the strong transfer principle and weak independence of income source if and only if V is a generalized Gini measure of welfare change with a corresponding generalized Gini welfare function W.*

Proof. That the generalized Gini measures of welfare change satisfy the axioms of the theorem statement is straightforward to verify.

Conversely, suppose that V is a measure of welfare change satisfying the axioms. By anonymity, it is sufficient to show that (**??**) and (**??**) are true for bottom-first-ordered permutations of the requisite income distributions. As mentioned in the text, the welfare function W (which exists as a consequence of welfare difference compatibility) inherits the properties of continuity, strict increasingness, anonymity, the strict transfer principle and weak independence of income source suitably formulated for welfare functions.

We now show that the restriction of W to bottom-first-ordered permutations must be an increasing transformation of an increasing linear function. Because we assume that V satisfies positive linear homogeneity, the argument used in the proof of Weymark's (1981) Theorem 3 can be simplified. To do so, we first prove that the restriction of any level set of W to B is a convex set. Let $z, z' \in B$ be in the same level set of W so that $W(z) = W(z')$. Using (**??**) and the positive linear homogeneity of V, it follows that

$$W(z) = W(z') \;\Leftrightarrow\; V(z,z') = 0 \;\Leftrightarrow V(\lambda z, \lambda z') = 0 \;\Leftrightarrow\; W(\lambda z) = W(\lambda z')$$

for all $\lambda \in \mathbb{R}_{++}$. Letting $\theta \in (0,1)$, it follows that

$$W(z) = W(z') \;\Leftrightarrow\; W((1-\theta)z) = W((1-\theta)z'). \tag{10}$$

Adding θz to both $(1-\theta)z$ and $(1-\theta)z'$, weak independence of income source implies that

$$W((1-\theta)z) = W((1-\theta)z') \;\Leftrightarrow\; W(\theta z + (1-\theta)z) = W(z) = W(\theta z + (1-\theta)z')$$

and, combined with (**??**), we obtain

$$W(z) = W(z') \;\Leftrightarrow\; W(z) = W(\theta z + (1-\theta)z')$$

for all $z, z' \in B$ in the same level set of W and for all $\theta \in (0,1)$, which implies that the requisite level set is convex. Because W is strictly increasing, it follows that the restriction of W to B is an increasing transformation of a strictly increasing linear function. Thus, there exist $\beta_1, \ldots, \beta_n \in \mathbb{R}_{++}$ and an increasing function $\phi\colon \mathbb{R}_+ \to \mathbb{R}$ such that, for all $x \in B$,

$$W(x) = \phi\left(\sum_{i=1}^{n} \beta_i x_i\right). \tag{11}$$

By the strict transfer principle and because the elements of B are bottom-first-ordered, it follows that $\beta_1 > \ldots > \beta_n$. By anonymity,

$$W(x) = \phi\left(\sum_{i=1}^{n} \beta_i \tilde{x}_i\right)$$

for all $x \in \mathbb{R}^n_+$.

Using (**??**) and noting that, for any $p, q \in \mathbb{R}_+$, two distributions x and y can be chosen so that $\sum_{i=1}^{n} \beta_i \tilde{x}_i^0 = p$ and $\sum_{i=1}^{n} \beta_i \tilde{x}_i^1 = q$, positive linear homogeneity requires that

$$\phi(\lambda q) - \phi(\lambda p) = \lambda(\phi(q) - \phi(p)) \tag{12}$$

for all $p, q \in \mathbb{R}_+$ and for all $\lambda \in \mathbb{R}_{++}$. Setting $p > 0$, $q = 0$ and $\lambda = 1/p$ in (**??**), it follows that

$$\phi(0) - \phi(1) = (\phi(0) - \phi(p))/p$$

and, solving for $\phi(p)$, we obtain

$$\phi(p) = (\phi(1) - \phi(0))p + \phi(0) = \gamma p + \delta$$

where $\gamma = \phi(1) - \phi(0)$ is positive because ϕ is increasing and $\delta = \phi(0)$ is a real number. Therefore, ϕ is an increasing affine function and, setting $\alpha_i = \gamma\beta_i$ for all $i \in \{1, \ldots, n\}$, it follows that $\alpha_1 > \ldots > \alpha_n$ and

$$W(x) = \sum_{i=1}^{n} \alpha_i \tilde{x}_i$$

for all $x \in \mathbb{R}^n_+$. Using welfare difference compatibility, we obtain

$$V(x^0, x^1) = \sum_{i=1}^{n} \alpha_i \tilde{x}^1_i - \sum_{i=1}^{n} \alpha_i \tilde{x}^0_i$$

for all $(x^0, x^1) \in \mathbb{R}^{2n}_+$. ■

5 Concluding remarks

In order to establish a dominance criterion that allows for welfare changes to be compared across societies with different population sizes, one possible approach consists of replicating the requisite societies and employing the dominance criterion that corresponds to the larger population. Specifically, if we have pairs of distributions $(x^0, x^1) \in \mathbb{R}^{2n}_+$ and $(y^0, y^1) \in \mathbb{R}^{2m}_+$ where $n \neq m$, we can consider an m-fold replication of (x^0, x^1) and an n-fold replication of (y^0, y^1) and apply the dominance criterion for population size nm to the replicated distributions. Of course, implicit in such a procedure—which is also suggested by Shorrocks (1983)—is some suitable notion of a principle of population, ensuring that such replications do not distort welfare-change-relevant features of the original distributions. This observation leads us to the *single-parameter Ginis,* which are characterized by Donaldson and Weymark (1980) by means of the principle of population. A similar variable-population result that employs a recursivity property characterizes the *single-series Ginis;* see Bossert (1990). In analogy to our characterization that parallels Weymark's (1981) axiomatization, these variable-population extensions can be adjusted to our setting so as to apply to measures of welfare change.

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