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EFFICIENT PARTNERSHIP FORMATION IN NETWORKS

FRANCIS BLOCH*, BHASKAR DUTTA†, AND MIHAI MANEA‡

ABSTRACT. We analyze the formation of partnerships in social networks. Players need favors at random times and ask their neighbors in the network to form exclusive long-term partnerships that guarantee reciprocal favor exchange. Refusing to provide a favor results in the automatic removal of the underlying link. When favors are costly, players agree to provide the first favor in a partnership only if they otherwise face the risk of eventual solitude. In equilibrium, the players essential for realizing every maximum matching can avoid this risk and enjoy higher payoffs than inessential players. Although the search for partners is decentralized and reflects local incentives, the strength of essential players drives efficient partnership formation in every network. When favors are costless, players enter partnerships at any opportunity and every maximal matching can emerge in equilibrium. In this case, efficiency is limited to special linking patterns: complete and complete bipartite networks, locally balanced bipartite networks with positive surplus, and factor-critical networks.

JEL Classification Numbers: D85, C78.

Keywords: networks, partnerships, matchings, efficiency, decentralized markets, favor exchange, completely elementary networks, locally balanced networks.

1. INTRODUCTION

The idea that the power of an individual depends on his or her position in a certain social or economic network is well-established in a variety of contexts cutting across disciplines. For instance, social network analysis suggests that an individual's power cannot be explained by the individual's characteristics alone but must be combined with the structure of his or her relationships with others. Power arises from occupying advantageous positions in the relevant network and leveraging outside options. In particular, network exchange theory focuses on studying the relative bargaining power of individuals in bilateral exchanges with neighbors in social networks. Recent research in economics develops game theoretical models

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aimed at understanding how an individual's position determines his or her bargaining power and selection of trading partners in markets with a network structure.¹

In this paper, we study the impact of the social and economic network structure on the relative strength of different positions in the network, the pattern of bilateral partnerships that emerge, and market efficiency. In our setting, players form exclusive long-term partnerships to exchange favors with one another. Favors could be small—e.g., advice on a particular issue, a small loan, and help on a school project or with babysitting—or large—e.g., sharing one's life with another person and forming a lasting business relationship. Every player needs favors at random points in time and can receive them from neighbors in the network. All players benefit equally from receiving favors and incur the same cost for providing them. The benefit exceeds the cost, so it is socially desirable to exchange favors. If a player agrees to provide a favor to another, the two players form an exclusive partnership which requires that they leave the network together and do each other favors whenever needs arise in the future. A player who needs a favor and has not yet formed a partnership approaches his remaining neighbors in the network in random order to request the favor. Each neighbor asked for the favor decides whether to provide it and enter the partnership or refuse to do so and irreversibly lose his link with the player requesting it. Players have a common discount factor. Our analysis focuses on two questions as players become patient (or need favors frequently): How does a player's position in the network affect his payoffs and chances of finding a partner? Do partnerships form efficiently?

Since granting favors is costly, every player prefers to delay committing to a partnership until he needs a favor. However, turning down favor requests results in losing potential future partners both because links leading to rejections are automatically deleted from the network and because other neighbors form exclusive partnerships and leave the network. Given the structure of payoffs in the model, efficiency requires that the maximum number of partnerships form, so the partnerships that emerge constitute a maximum matching in the network. In particular, some links are inefficient because they are not part of any maximum matching; efficiency dictates that favor requests are declined when such links are activated. Other links are indispensable for efficiency since they belong to every maximum matching; to achieve an efficient market outcome, such links should result in partnerships when activated. In principle, keeping track of which partnerships are efficient in our dynamic game may be challenging because the structure of maximum matchings and the efficiency of links evolves as favor requests are turned down and partnerships form.

Given the global considerations involved in identifying maximum matchings in a general network and the local decentralized nature of the search for partners as favor needs arise in our partnership formation game, one might not expect a strong link between equilibrium

¹Waller (1999) reviews the sociology literature on network exchange theory, while Chapter 10 of Jackson (2008), Chapter 11 of Easley and Kleinberg (2010), and Manea (2016) survey the relevant economics research on bilateral trade in networks.

outcomes and efficiency in the game. Nevertheless, our main result proves that the game admits a unique subgame perfect equilibrium and that the equilibrium pattern of partnerships is efficient for every network structure. This result is surprising in the context of existing research on matching and trade in networks. Indeed, in a survey of the literature on bilateral trade in networks, Manea (2016) argues that the disconnect between global efficiency and local incentives explains why decentralized trade often generates inefficient market outcomes. Abreu and Manea (2012a, b) and Elliott and Nava (2016) reach this conclusion for Markov perfect equilibria in two natural models of bargaining in networks.² By contrast, the seminal work in this area of Kranton and Minehart (2001), Corominas-Bosch (2004), and Polanski (2007) showed that centralized matching is conducive to efficient trade in Markov perfect equilibria.

Somewhat paradoxically, the absence of prices or direct transfers in our model drives the efficiency result and the divergence from the conclusions of Abreu and Manea (2012a, b) and Elliott and Nava (2016). Players can experience only three types of outcomes in our model: remain single; commit to a partnership by granting a favor; and enter a partnership by way of receiving a favor. As argued above, a player has an incentive to provide the first favor in a partnership only if refusing to do so puts him at risk of eventually becoming single. We find that a key structural property of nodes determines whether a player ever faces the risk of becoming single in equilibrium. Specifically, a node is said to be essential if it belongs to all maximum matchings of the network and under-demanded otherwise. A further partition of essential nodes into over-demanded nodes—neighbors of under-demanded nodes—and perfectly matched nodes—the remaining ones—is central to the Gallai-Edmonds decomposition, which characterizes the structure of maximum matchings. These concepts also play a prominent role in the equilibrium analyses of Corominas-Bosch (2004), Polanski (2007), and Abreu and Manea (2012a). We prove that in the equilibrium of our game, essential players always find partners, while under-demanded players remain single with positive probability. Hence, essential players obtain higher payoffs than under-demanded players.

The conclusion that under-demanded players are relatively weaker than essential ones is common across the bargaining models discussed above.³ However, our model highlights a new channel leading to this conclusion. In previous models, the weakness of under-demanded players is caused by their vulnerability to isolation following some sequences of efficient trades. Thus, the fact that under-demanded nodes are left out by some maximum matchings, which is their defining property, is directly involved in the argument. In the present model, the analysis relies on a latent property of under-demanded nodes: if an under-demanded node

²However, Abreu and Manea (2012b) construct a complex system of punishments and rewards with a highly non-Markovian structure that implements an efficient subgame perfect equilibrium.

³Polanski (2016) emphasizes this point in a range of bargaining environments, including the stationary setting of Manea (2011) and the cooperative game of Kleinberg and Tardos (2008) in addition to the models already mentioned.

is removed from the network, all its neighbors become essential in the remaining network. Backward induction then implies that when an under-demanded player needs a favor, his neighbors have incentives to turn him down in sequence and ultimately reach desirable essential positions. Hence, every under-demanded player requesting a favor remains single. The assumption that links resulting in rejections are permanently severed—a departure from prior work—is thus an important ingredient for our analysis. In other words, in previous models, under-demanded players are passively left isolated after neighbors reach agreements and exit the network; in our setting, under-demanded players are actively marginalized in the original network via link losses triggered by rejections.

Our detailed characterization of the subgame perfect equilibrium relies heavily on the intuition that under-demanded players are weak and players commit to partnerships via doing a favor in order to avoid occupying under-demanded positions in the network. We prove that when an over-demanded player needs a favor, the first under-demanded neighbor he approaches has to provide it. This step requires a delicate argument since under-demanded players can become essential as the network evolves following rejections of the favor request or the formation of an alternative partnership. In particular, we show that there is always an order in which the over-demanded player asks subsequent neighbors for the favor resulting in him forming a partnership that maintains the under-demanded position of the under-demanded neighbor initially approached. Finally, we argue that when a perfectly matched player requests a favor, the last neighbor with whom he shares an efficient link—another perfectly matched player—agrees to provide the favor. Refusing to enter this last possible efficient partnership and removing the corresponding link would result in both players switching from essential to under-demanded, with the player who requested the favor ending up single and the other player preserving his under-demanded status in the ensuing network.

We additionally show that a player does not have an incentive to grant a favor to a neighbor if refusing to do so and losing the link with the neighbor leaves him in an essential position. However, under-demanded players turn down favor requests from other under-demanded neighbors and remain temporarily under-demanded anticipating that no player will agree to provide the favor and they will become essential after the chain of rejections.

In the benchmark model, providing favors is costly and players who do not need favors have incentives to delay entering a partnerships until the social network exposes them to the risk of not being able to find a partner in the long run. A key finding is that these incentives are consistent with maximizing the number of partnerships. It is interesting to consider an alternative setting in which joining partnerships is costless and both parties enjoy immediate benefits from the partnership. In this setting, every player grants favors and forms partnerships at the first opportunity. As opposed to the case with costly favors, players do not necessarily form a maximum matching, as efficiency dictates, in the trivial equilibrium. In fact, the equilibrium is efficient if and only if every matching of the network

that is maximal with respect to inclusion is a maximum matching. We call networks with the latter property completely elementary.

The final set of results develops various characterizations of completely elementary networks. The simplest one is for perfect networks—networks that admit a matching that covers every node. We show that a perfect network is completely elementary if and only if each of its connected components is a perfect complete network or a perfect complete bipartite network. We also characterize completely elementary bipartite networks. We establish that a bipartite network is completely elementary if and only if all its connected components are perfect complete bipartite networks or locally balanced bipartite networks with positive surplus. The latter bipartite networks are defined by the following two conditions: (1) every subset of nodes on one side of the partition is collectively linked to a larger set of nodes on the other side; and (2) each node on the former side has a subset of neighbors who are collectively linked to a set of nodes of equal or smaller size. For completely elementary networks that are neither perfect nor bipartite, the Gallai-Edmonds structure theorem implies that perfectly matched nodes are only linked to one another and form a completely elementary perfect network, while under-demanded nodes are not linked to one another and the remaining network can be reduced to a completely elementary bipartite network. Hence, our clear understanding of the possible structures of completely elementary networks that are either perfect or bipartite have immediate implications for the perfectly matched and the imperfectly matched blocks of general completely elementary networks. Further building on the Gallai-Edmonds theorem, we uncover other structural properties of completely elementary networks restricting local linking patterns among imperfectly matched nodes to complete bipartite networks, locally balanced bipartite networks with positive surplus, and factor-critical networks.

Our results shed light on two complementary issues at the intersection of game theory and graph theory. The game with costly favors fleshes out a natural connection between the Gallai-Edmonds decomposition and incentives for efficient partnership formation. The version of the game with costless favors generates insights into the special structures of completely elementary networks and reveals that inefficient market outcomes are prevalent when players rush to form partnerships.

Bloch et al. (2016) test the predictions of the benchmark model in a laboratory experiment. The principal finding is that a large fraction of subjects play according to the subgame perfect equilibrium, but a subject’s ability to select the equilibrium action depends on the complexity of the network as well as on his or her position in the network. Deviations from equilibrium behavior primarily involve subjects agreeing to grant favors when equilibrium play prescribes declining the request.

Besides the literature on bilateral trade in networks discussed above, our model contributes to research on favor exchange. Möbius and Rozenblat (2016) survey existing research in the latter area. Many models in this literature—in particular, Gentzkow and Möbius (2003),

Bramoullé and Kranton (2007), Bloch et al. (2008), Karlan et al. (2009), Jackson et al. (2012), and Ambrus et al. (2014)—share the basic structure of our model: players request favors or transfers at different points in time and cooperation is enforced through reciprocity in the future. The model of Jackson et al. (2012) is closest to ours. However, in that model favor needs are link-specific and pairs of players meet too infrequently to sustain bilateral exchange in isolation. Jackson et al. show that clustered social quilts support cooperation via the social pressure of losing links with multiple neighbors when deviations from cooperative behavior are observed.

The rest of the paper is organized as follows. Section 2 introduces the partnership formation game with costly favor provision, and Section 3 illustrates its equilibria for two networks. In Section 4, we formalize the relationship between efficient partnerships and maximum matchings and review the Gallai-Edmonds decomposition. Section 5 presents the main result, which establishes the uniqueness and the efficiency of the equilibrium and shows that equilibrium decisions are closely tied to the Gallai-Edmonds decomposition. In Section 6, we investigate the network structures that support efficient partnership formation in the setting with costless favors. Section 7 provides concluding remarks.

2. MODEL

We study a partnership formation game played by a finite set $N = \{1, 2, \dots, n\}$ of players who constitute the nodes in an undirected network G . Since the network of potential partnerships evolves over time and the collection of existing partnerships forms a matching, it is useful to provide general definitions for networks and matchings. An *undirected network* G linking the set of nodes N is a subset of $N \times N$ such that $(i, i) \notin G$ and $(i, j) \in G \Leftrightarrow (j, i) \in G$ for all $i, j \in N$. The condition $(i, j) \in G$ is interpreted as the existence of a *link* between nodes i and j in the network G ; in this case, we say that i is *linked* to j or that i is a *neighbor* of j in G . We use the shorthand ij for the pair (i, j) and identify the links ij and ji . A node is *isolated* in G if it has no neighbors in G . For any network G , let $G \setminus ij$ denote the network obtained by removing the link ij from G and $G \setminus i, j, \dots$ denote the network in which all links of nodes i, j, \dots in G are removed (but nodes i, j, \dots remain, isolated, in the network). A *matching* is a network in which every node has at most one link. A *matching of the network* G is a matching that is a subset of G .

The partnership formation game proceeds in discrete time at dates $t = 0, 1, \dots$. At every date t , there is a set of partnerships that have already formed represented by a matching M_t and a prevailing network of potential future partnerships G_t . At date t , one player i randomly selected—each with probability $1/n$ —from the set N needs a favor. Partnerships are assumed to be permanent and guarantee reciprocal favor exchange, so if player i has a partner j under M_t , then j automatically provides the favor to i . Otherwise, player i randomly chooses one of his neighbors j_0 in the network $G_{t0} := G_t$ and asks him for the favor. Player j_0 decides whether to provide the favor or not. If player j_0 declines to do the

favor to i , then the link ij_0 is permanently removed from the network, and player i continues searching for a partner in the network $G_{t1} := G_{t0} \setminus ij_0$. In general, after k rejections, player i randomly chooses one of his neighbors j_k in the remaining network $G_{tk} := G_{t(k-1)} \setminus ij_{k-1}$ to ask for the favor. If player j_k agrees to provide the favor to player i at date t , player i receives a payoff of $v > 0$ and player j_k incurs a cost $c \in (0, v)$. In this case, i and j_k form a long-term partnership, so that the set of ongoing partnerships becomes $M_{t+1} = M_t \cup ij_k$, and the game proceeds to date $t+1$ on the network $G_{t+1} = G_t \setminus i, j_k$. If none of i 's neighbors in G_t agrees to provide the favor to i , then i remains isolated and the game continues to period $t+1$ on the network $G_{t+1} = G_t \setminus i$. All players discount future payoffs by a factor of δ per period.

We assume that the game has perfect information and use the solution concept of subgame perfect equilibrium. We allow for mixed strategies but will show that as players become patient, the subgame perfect equilibrium is unique and involves only pure strategies.

Let V denote the expected discounted payoff obtained by a player who is matched with a partner with whom he reciprocates favors,

$$V = \frac{v - c}{n(1 - \delta)}.$$

Since providing the first favor in a partnership costs c and leads to a continuation payoff of δV , a necessary condition for a player to rationally agree to provide the first favor in equilibrium is that $\delta V \geq c$, which is equivalent to

$$\delta \geq \underline{\delta} := \frac{n}{r + n}$$

where

$$r := \frac{v - c}{c}$$

represents the *return to favors*. Hence, if $\delta < \underline{\delta}$, then all favor requests are turned down and every player receives zero payoff in equilibrium.

We close this section with definitions of some types of networks that will be useful for illustrations and the analysis. The *line* network with n players consists of the links $(1, 2), \dots, (n-1, n)$. The *cycle* with n players is the network formed by the links $(1, 2), \dots, (n-1, n), (n, 1)$. A network is *complete* if it links every pair of nodes. The network G is *bipartite* with the partition (A, B) if $B = N \setminus A$ and G contains no links between pairs of nodes in A or between pairs of nodes in B . A *bipartite network* with the partition (A, B) is *complete* if it contains all links in $A \times B$.

3. EXAMPLES

In this section, we analyze the partnership formation game in the two examples shown in Figure 1: the line and the complete networks with four players. Assume that $\delta \geq \underline{\delta}$, so that at least one partnership forms in any equilibrium in either network.

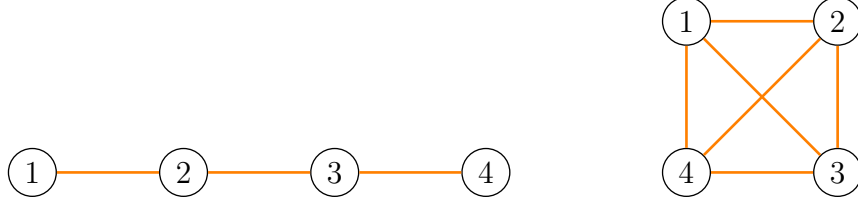


FIGURE 1. The line and complete networks with four nodes

Consider first the four-player line network. Suppose that player 1 needs a favor in the first period of the game. In this case, only player 2 can provide the favor to 1. If 2 agrees to provide the favor, he obtains an expected payoff of $-c + \delta V$. If 2 turns 1 down, then the link (1, 2) is removed from the network. In the remaining network, if player 2 or 4 requests the next favor, then player 3 has no incentive to provide it. Indeed, declining such a request leaves player 3 in a network with a single link, which generates an expected continuation payoff of δV for player 3, while accepting such a request results in the lower expected payoff of $-c + \delta V$. If instead player 3 requires the first favor in the remaining network, then he approaches each of players 2 and 4 with probability $1/2$, and under the assumption that $\delta \geq \underline{\delta}$, either player accepts the request because he would otherwise remain isolated.

It follows that the expected continuation payoff of player 2 following the rejection of 1's favor request is δW^L , where W^L solves the equation

$$W^L = \frac{1}{4} \left(\delta W^L + \frac{1}{2}(-c + \delta V) + \delta V \right).$$

In this equation, the term δW^L represents the continuation payoff of player 2 in the event that player 1 requires the second-period favor as well. Player 2 receives payoff 0 if he needs the second-period favor (as player 3 refuses to provide it) and payoff δV if 4 needs the favor (and 3 turns him down). The expected payoff of player 2 in the event that player 3 needs a favor in the second period is $(-c + \delta V)/2$, reflecting the fact that 3 asks 2 for the favor with probability $1/2$. The solution to the equation is

$$W^L = \frac{3\delta V - c}{2(4 - \delta)}.$$

Player 2 then has an incentive to grant the favor in the first period to player 1 only if $-c + \delta V \geq \delta W^L$, which is equivalent to

$$\delta V(8 - 5\delta) \geq c(8 - 3\delta).$$

Using the formula $V = (v - c)/(4(1 - \delta))$, the inequality above can be rewritten as

$$r = \frac{v - c}{c} \geq \frac{4(1 - \delta)(8 - 3\delta)}{\delta(8 - 5\delta)}.$$

For $\delta \in [0, 1)$, this inequality is equivalent to

$$\delta \geq \delta^{*L} := \frac{2(11 + 2r - \sqrt{25 + 4r + 4r^2})}{12 + 5r}.$$

For instance, for $r = 1$, which means that $v = 2c$, we have that $\delta^{*L} \approx 0.854$.

When player 2 needs the first favor, player 3 does not have an incentive to provide it because he can count on always receiving favors from 4. However, for $\delta \geq \underline{\delta}$, player 1 has an incentive to do the favor to 2 because refusing to do so would leave him isolated. Symmetric arguments apply to the situations in which players 3 and 4 require the first favor.

Therefore, the structure of equilibria in this network is as follows:

- For $\delta < \underline{\delta}$, no favors are ever granted in equilibrium.
- For $\delta \in (\underline{\delta}, \delta^{*L})$, if player 1 (or 4) needs the first favor, then player 2 (3) turns him down; in the remaining network, if player 2 or 4 (1 or 3) needs the next favor, player 3 (2) turns him down, while if player 3 (2) needs the next favor, the first neighbor he approaches agrees to provide it. If player 2 (3) needs the first favor instead, then player 1 (4) provides it.
- For $\delta \in (\delta^{*L}, 1)$, every player who needs a favor receives it and the partnerships (1, 2) and (3, 4) form.

For any δ , welfare maximization requires that all favor requests be granted and partnerships (1, 2) and (3, 4) form, so the equilibrium is efficient only for $\delta > \delta^{*L}$.

Consider next the complete network with four players. If player 1 needs the first favor and is rejected by all his neighbors, then the other three players are left in a complete network. In this case, an argument similar to the one above shows that when one of the remaining three players requires a favor, the other two turn him down. Thus, if all players refuse to do player 1 the favor, each player $i \in \{2, 3, 4\}$ enjoys a continuation payoff δW^C , where

$$W^C = \frac{1}{4}(\delta W^C + 2\delta V).$$

This payoff equation is analogous to the one defining W^L . In particular, the term $2\delta V$ captures the events in which one of two players different from 1 and i needs the next favor and remains single, effectively leaving i in a bilateral partnership with the fourth player starting in the third period. Solving the equation, we obtain

$$W^C = \frac{2\delta V}{4 - \delta}.$$

The last neighbor approached by player 1 has an incentive to provide the first-period favor to 1 only if $-c + \delta V \geq \delta W^C$, which is equivalent to

$$r \geq \frac{4(1 - \delta)(4 - \delta)}{\delta(4 - 3\delta)}.$$

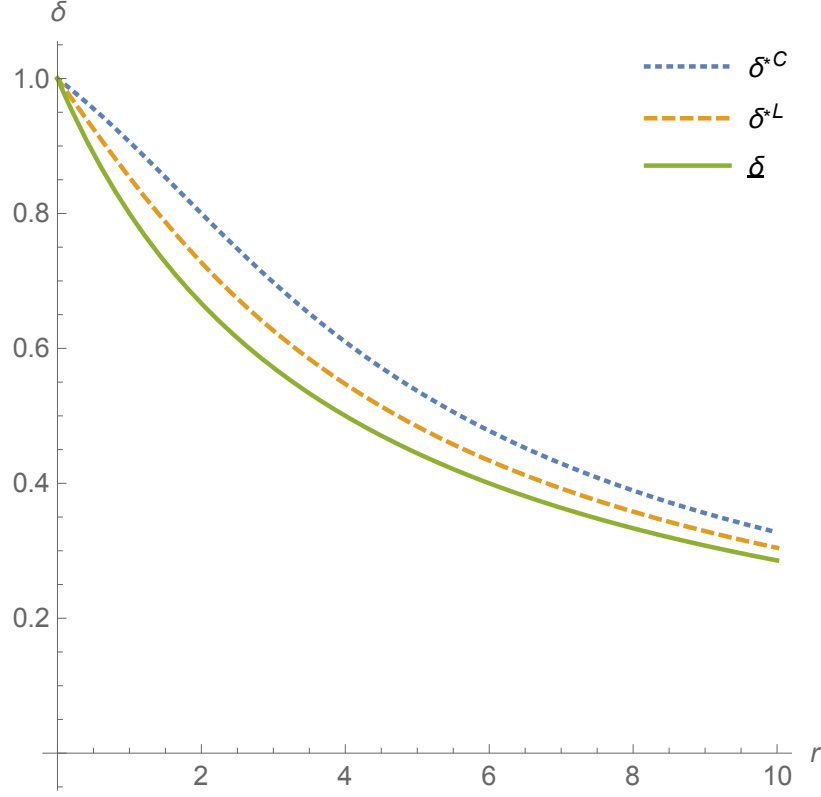


FIGURE 2. Thresholds $\underline{\delta}$, δ^{*L} , and δ^{*C} as a function of r

The last inequality reduces to

$$\delta \geq \delta^{*C} := \frac{2(5 + r - \sqrt{9 - 2r + r^2})}{4 + 3r}.$$

For $r = 1$, we obtain $\delta^{*C} \approx 0.906$.

To summarize, the structure of equilibria in the complete network is as follows:

- For $\delta < \underline{\delta}$, no favors are granted in equilibrium.
- For $\delta \in (\underline{\delta}, \delta^{*C})$, the player who needs the first favor is refused by all other players. The next player requiring a favor is also turned down by his remaining neighbors. The third player who needs a favor receives it from his only remaining neighbor, and a single partnership forms.
- For $\delta \in (\delta^{*C}, 1)$, the first player who needs a favor receives it from the last player he approaches, and two partnerships form.

As in the line network, for any δ , welfare maximization requires that every player who needs a favor receives it, so the equilibrium is efficient only for $\delta > \delta^{*C}$.

Figure 2 depicts the thresholds $\underline{\delta}$, δ^{*L} , and δ^{*C} as a function of the return to favors r . Note that $\underline{\delta} < \delta^{*L} < \delta^{*C}$ for all values of $r > 0$. The inequality $\delta^{*L} < \delta^{*C}$ reflects the fact that player 2's continuation payoff in the event that player 1 requires the first favor and his neighbors turn him down is smaller in the line network than in the complete network,

$W^L < W^C$. Hence, it is easier to provide an incentive for player 2 to form an efficient partnership with player 1 in the line than in the complete network. The two examples demonstrate that adding links to a network does not always facilitate the efficient formation of partnerships. Enlarging the set of links increases the number of potential matchings but may also increase the continuation values of players after links are severed, making it more difficult to sustain efficient partnerships.

We will prove that the conclusions regarding the uniqueness and efficiency of equilibria for high δ in the examples of this section extend to all networks. We will also show that the equilibrium decisions to form partnerships for high δ are directly determined by the classification of the corresponding pairs of nodes in the Gallai-Edmonds decomposition, which we introduce next.

4. EFFICIENT PARTNERSHIPS AND MAXIMUM MATCHINGS

The discussion of the previous section suggests a close relationship between efficient partnership formation and maximum matchings. We say that a matching M is a *maximum matching* of G if there exists no matching of G that contains more links than M . For any network G , let $\mu(G)$ denote the *size of the maximum matching* of G , i.e., the number of links in a maximum matching of G .

Any strategy profile σ along with the random moves by nature—the list of players needing favors and the sequence of neighbors they approach at every date—induce a probability distribution over outcomes at every date. Let \mathbf{P} denote the probability measure on this space. We view $(M_t)_{t \geq 0}$ and $(G_t)_{t \geq 0}$ as random variables in this space. Clearly, we have $M_t \subseteq M_{t+1}$ and $G_{t+1} \subseteq G_t$ for all $t \geq 0$. Let \bar{M} and \bar{G} denote the “limits” as $t \rightarrow \infty$ of the variables $(M_t)_{t \geq 0}$ and $(G_t)_{t \geq 0}$ defined by $\bar{M} = \cup_{t \geq 0} M_t$ and $\bar{G} = \cap_{t \geq 0} G_t$. We call the random variable \bar{M} the *long-run matching* induced by σ . This motivates the following definition.

Definition 1. *A strategy profile is long-run efficient if the long-run matching it induces is a maximum matching of G with probability 1.*

To understand the connection between long-run efficiency and total welfare in our model, fix a strategy profile σ and define as customary the *normalized* expected payoff of a player to be his expected payoff in the game under σ multiplied by $1 - \delta$. Let T denote the lowest t such that $M_t = \bar{M}$ and $G_t = \bar{G}$. For $t \geq T$, we have that $M_t = \bar{M}$ and only players who are matched under \bar{M} are granted favors at date t . Hence, players collectively receive the net benefit of $v - c$ from favor exchange at date $t \geq T$ with probability $2\mu(\bar{M})/n$. Starting at any date $t < T$, there is a sequence of n or fewer draws by nature of unmatched players asking for favors for the first time and either entering partnerships or becoming isolated, which leads to $M_{t+n} = \bar{M}$ and $G_{t+n} = \bar{G}$. The probability of such a sequence being drawn by nature conditional on (G_t, M_t) is at least $1/n^n$. Therefore, $\mathbf{P}(T > t + n | T > t) \leq 1 - 1/n^n$, so

$$\mathbf{P}(T > t + n) = \mathbf{P}(T > t)\mathbf{P}(T > t + n | T > t) \leq \mathbf{P}(T > t)(1 - 1/n^n).$$

It follows that $\mathbf{P}(T > kn) \leq (1 - 1/n^n)^k$ for all $k \geq 0$. Given this exponential bound on the tail of the distribution of T and the fact that the total expected payoffs of all players under σ average to $2\mu(\bar{M})/n(v - c)$ at dates $t \geq T$, the limit of the total normalized expected payoffs of all players under σ as $\delta \rightarrow 1$ does not exceed $\mu(G)$ and achieves the maximum of $\mu(G)$ only if \bar{M} is a maximum matching of G with probability 1. We conclude that σ maximizes the limit of the sum of normalized expected payoffs of all players as $\delta \rightarrow 1$ only if it is long-run efficient.

The welfare analysis of equilibria in our partnership formation game thus naturally leads us to examine the structure of maximum matchings. Gallai (1964) and Edmonds (1965) developed a characterization of maximum matchings that not only proves useful in analyzing welfare properties of equilibria but captures the structure of incentives in our game in a precise way. To state it, we need to introduce a few more concepts from graph theory. We say that a matching *covers* a node if the node has one link in the matching (and that a matching covers a set of nodes if it covers every node in that set). A network G is *perfect* if it contains a matching that links all its nodes, that is, $\mu(G)$ equals to half of the number of nodes in G . A network G is *factor-critical* if any network obtained by removing a single node from G is perfect. The *subnetwork* of G induced by a subset of nodes N' is the network with nodes in N' formed by the set of links $G \cap N' \times N'$. A network is *connected* if for any pair (i, j) of its nodes, it contains a path of links $i_1i_2, i_2i_3, \dots, i_{k-1}i_k$ with $i_1 = i$ and $i_k = j$. A *connected component* of a network is a subnetwork induced by a subset of its nodes that is maximal with respect to inclusion among all sets of nodes inducing connected subnetworks. By definition, connected components partition the set of nodes and links in a network.

Let $N^G(X)$ denote the set of all neighbors of nodes from X in the network G , i.e., $N^G(X) = \{j | \exists i \in X, ij \in G\}$. A bipartite network G with the partition (A, B) has *non-negative surplus* from the perspective of A if $|N^G(X)| \geq |X|$ for all non-empty sets $X \subseteq A$ and *positive surplus* from the perspective of A if the inequality is strict for all such X .

A simple condition due to Hall (1935) characterizes the bipartite networks which admit perfect matchings. More generally, Hall's theorem provides a necessary and sufficient condition for bipartite networks to contain matchings that cover all nodes on one side of the partition.

Theorem 1 (Hall's Marriage Theorem [19]). *Suppose that G is a bipartite network with partition (A, B) . Then there exists a matching of G that covers A if and only if G has non-negative surplus from the perspective of A .*

Gallai and Edmonds' result relies on the following partition of the set of nodes in a network G . A node is *under-demanded* in G if it is not covered by some maximum matching of G . A node is *over-demanded* in G if it is not under-demanded but has an under-demanded neighbor in G . A node is *perfectly matched* in G if it is neither under- nor over-demanded in G . For example, in both the line and the complete networks with four players from Figure

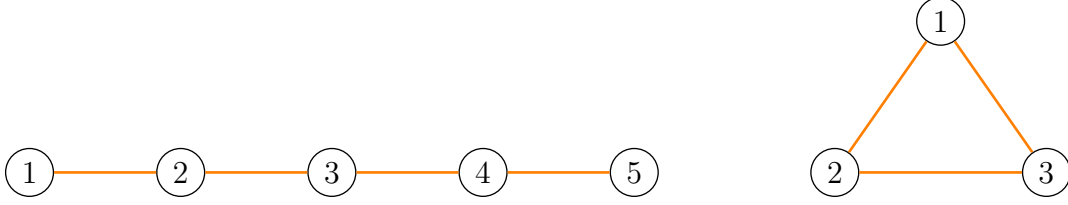


FIGURE 3. The line with five nodes and the complete network with three nodes

1, all nodes are perfectly matched. In the line with five nodes shown in the left panel of Figure 3, nodes 1, 3, and 5 are under-demanded, while nodes 2 and 4 are over-demanded. In the complete network with three nodes from the right panel of Figure 3, all nodes are under-demanded.

Consider the network obtained from a network G as follows. Eliminate all perfectly matched nodes and all links connecting pairs of over-demanded nodes from G . Contract each connected component of the subnetwork induced by the set of under-demanded nodes in G into a single node, which inherits all links (without duplication) between the component and over-demanded nodes in G . We refer to the bipartite network thus derived from G as the *imperfectly matched structure* of G .

Theorem 2 (Gallai-Edmonds Decomposition [19]). *Fix a network G with the sets of perfectly matched, over-demanded, and under-demanded nodes denoted by P, O , and U , respectively. The maximum matchings of G have the following properties:*

- (1) *Every maximum matching of G links each node in P to another node in P , links each node in O to a node in a distinct connected component of the subnetwork induced by U in G , and links all nodes except one from each such component with one another.*
- (2) *The connected components of the subnetwork of G induced by the set of nodes U are factor-critical.*
- (3) *The subnetwork of G induced by the set of nodes P is perfect.*
- (4) *The imperfectly matched structure of G is a bipartite network with positive surplus from the perspective of O .*
- (5) *If G is bipartite, then G contains no links connecting pairs of nodes in U .⁴*

The following result, whose proof appears in the Appendix, describes the evolution of the Gallai-Edmonds partition as partnerships form or as links are removed following declined favor requests. We say that i is an *efficient partner* of j in G or that ij is an *efficient link* if the link ij belongs to a maximum matching of G and that a node is *essential* in G if it is either over-demanded or perfectly matched in G .

Lemma 1. *For every network G and any link $ij \in G$, the following statements are true:*

⁴Part (5) of the result is not regularly stated with the decomposition theorem but is a trivial consequence of part (2). See Manea (2016) for a proof.

- (1) *If i is under-demanded in G , then i is under-demanded in the network $G \setminus ij$, and j is essential in the network $G \setminus i$.*
- (2) *Suppose that i is perfectly matched in G and j is not the only efficient partner of i in G . Then, the sets of perfectly matched, over-demanded, and under-demanded nodes coincide in G and $G \setminus ij$. Moreover, the set of efficient partners of i in $G \setminus ij$ consists of all efficient partners of i in G except for j (in case ij is an efficient link).*
- (3) *If i is perfectly matched in G and j is i 's only efficient partner in G , then both i and j are under-demanded in the network $G \setminus ij$, and j is under-demanded in $G \setminus i$.*
- (4) *If i is over-demanded and j is under-demanded in G , then ij is an efficient link in G , and i is an essential node in $G \setminus ij$.*
- (5) *If both i and j are perfectly matched and ij is an efficient link in G , then the sets of under-demanded nodes different from i and j in $G \setminus i, j$ and G coincide.*

5. EQUILIBRIUM PARTNERSHIPS AND EFFICIENCY

We provide a complete characterization of subgame perfect equilibria in the partnership formation game for high δ that relies on the classification of nodes in the Gallai-Edmonds decomposition in the prevailing network. From this characterization, we infer that equilibria are long-run efficient and that all essential players find partners when players are patient. The characterization also implies that for $\delta \rightarrow 1$, all essential players receive limit normalized payoffs of $(v - c)/n$, while under-demanded players obtain limit normalized payoffs of at most $(n - 1)/n \times (v - c)/n$. Therefore, as players become patient, essential players fare better than under-demanded players.

Theorem 3. *Fix a network G with n players. The following statements are true for sufficiently high δ . The partnership formation game played on the network G has a unique subgame perfect equilibrium. In the equilibrium, each essential player in G receives favors any time he needs them and enjoys a limit normalized expected payoff of $(v - c)/n$, while each under-demanded player in G remains single with probability at least $1/n$ and obtains a limit normalized expected payoff of at most $(n - 1)/n \times (v - c)/n$. When an under-demanded player in G requests a favor, all neighbors deny his request and he remains single. When a perfectly matched player in G needs a favor, all neighbors turn him down until he reaches his last efficient partner in G , who agrees to provide the favor. When an over-demanded player in G asks for a favor, all neighbors turn him down until he approaches his first under-demanded neighbor in G , who grants the favor. In equilibrium, $\mu(G)$ partnerships form with probability 1. The equilibrium is long-run efficient.*

As discussed in the introduction, existing research (e.g., Abreu and Manea (2012a, b) and Elliott and Nava (2016)) reveals a tension between decentralized trade in networks and efficient market outcomes. A combination of new modeling assumptions delivers efficiency in our decentralized setting: direct transfers are not possible within partnerships, so every

player can experience only a discrete set of outcomes; a player who needs a favor gets the opportunity to propose partnerships sequentially to all his neighbors; and links leading to rejections are irreversibly removed from the network. Corominas-Bosch (2004), Polanski (2007), and Abreu and Manea (2012a) also reach the conclusion that under-demanded players are weaker than essential ones. Their results rely on the vulnerability of under-demanded players to isolation as neighbors form partnerships and exit the network. This vulnerability stems from the fact that, by definition, under-demanded players are excluded by some maximum matchings. Our analysis points to a conceptually distinct quality of under-demanded players in the structure of maximum matchings: removing an under-demanded player from the network makes all his neighbors essential. For this reason, when an under-demanded player requests a favor, his neighbors anticipate that his other potential partners will refuse the request in order to reach attractive essential positions. Hence, under-demanded players are marginalized in the original network via immediate link deletions triggered by rejections rather than being exposed to the standard gradual decline in partnership opportunities.

We present the proof of Theorem 3 in the Appendix. To develop some intuition for this result, note that every player i can experience three types of outcomes in the partnership formation game in the network G : (1) remaining single; (2) entering a partnership by way of providing a favor to a neighbor who requires one; (3) initiating a partnership via having the first favor he needs granted. The expected payoffs of player i when these situations arise are given by 0 , $-c + \delta V$, and $v + \delta V$, respectively. For $\delta > \underline{\delta}$, we have that $0 < -c + \delta V < v + \delta V$. In scenarios (2) and (3), player i always receives the benefit v when he needs a favor and has to pay the cost c any time his partner requires a favor. However, scenario (3) saves player i some early costs c of providing favors before he needs one, so i does not have an incentive to accept a partnership of type (2) unless there is some risk that refusing to enter such a partnership exposes him to some risk of facing scenario (1). Therefore, every player prefers scenario (3) most and would like to delay accepting a partnership of type (2) for as long as this does not make him vulnerable to remaining single as in scenario (1).

The proof shows by induction on the number of links in network G that in equilibrium, essential players always form partnerships and end up in scenario (2) or (3), while under-demanded players reach scenario (1) with probability at least $1/n$ for δ close to 1. It is then optimal for a player to provide a favor when asked only if he becomes under-demanded in the network in which his link with the player needing the favor is severed. Lemma 1.1 implies that if an under-demanded player i needs a favor and all his neighbors turn him down, then i 's neighbors become essential in the remaining network. The induction hypothesis and backward induction then imply that every neighbor of i is guaranteed an outcome classified as scenario (2) or (3) above and thus does not have an incentive to do i the favor. Hence, in any subgame, every player who is under-demanded in the remaining network faces scenario (1) in the event he needs the next favor, which happens with probability $1/n$.

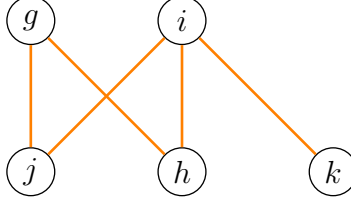


FIGURE 4. A key step in the proof of Theorem 3

When a perfectly matched player i requires a favor, we argue that his last efficient partner j whom he asks should provide the favor to i . Otherwise, the link ij is removed from the network, and the first part of Lemma 1.3 shows that both i and j are under-demanded in the resulting network $G \setminus ij$. The induction hypothesis for $G \setminus ij$ implies that no neighbor whom i approaches after j grants him the favor. Then, the second part of Lemma 1.3 shows that j becomes under-demanded in the ensuing network $G \setminus i$. By Lemma 1.2, every player whom i asks for the favor before reaching his last efficient partner is essential and maintains his role in the Gallai-Edmonds decomposition following his refusal to do i the favor. Hence, these players do not have incentives to partner with i .

When an over-demanded player i requires a favor, Theorem 2 implies that no essential neighbor changes status in the Gallai-Edmonds decomposition by refusing to provide the favor and losing the link with i . Then, no such neighbor has an incentive to do i the favor. However, if the over-demanded player i asks an under-demanded neighbor j for the favor, player j has an incentive to do it. This requires a more delicate analysis of the evolution of the positions of i and j in the Gallai-Edmonds decomposition in the subgame in which the link ij is removed and i approaches other neighbors with the request. Lemma 1.4 shows that i remains essential in $G \setminus ij$, and the induction hypothesis implies that i will eventually reach a neighbor k who is willing to partner with him. However, it is possible that the partnership between i and k improves j 's position from being under-demanded in G (as well as $G \setminus ij$ according to Lemma 1.1) to becoming essential in $G \setminus i, k$.

This situation is illustrated in the network from Figure 4. Suppose that in this network, the over-demanded player i asks his under-demanded neighbor j for a favor, and j turns him down. If i requests the favor from h next, then h accepts to provide the favor anticipating that he would otherwise remain under-demanded. Following the formation of the partnership (i, h) , player j becomes perfectly matched in the remaining network and eventually partners with g . Player j 's continuation payoff in this event is δV , which is greater than the expected payoff $\delta V - c$ derived from providing the favor to i . However, if i asks player k instead of h for the favor after j 's rejection, k agrees to provide the favor, in which case j is left under-demanded and exposed to a probability $1/2$ of remaining single, which for high δ is significantly less desirable than the expected payoff $\delta V - c$ guaranteed by the partnership with i . This is where the assumption that player i asks his neighbors for the favor in random order is crucial for the argument—after being rejected by j , player i is equally likely to ask

the favor from h and k ; while j is slightly better off not providing the favor to i in case i partners with h , he is considerably worse off in case i partners with k . Then, for high δ , player j prefers to provide the favor to i if asked first. The proof shows that a player acting like k —willing to form a partnership with i in equilibrium that leaves j under-demanded—always exists in a general network in which i is over-demanded and j is under-demanded. Lemma 1.4 is used to conclude that the set of partnerships that emerge in equilibrium form a maximum matching.

We finally comment on the implications of alternative modeling assumptions regarding the order in which players ask neighbors for favors. The example above shows that the fact that the player who needs a favor approaches neighbors in random order without revealing the order at the beginning of the period and instead picking neighbors sequentially is essential for the partnership outcomes described by Theorem 3. Indeed, if player i in the network from Figure 4 chose the order in which he approaches neighbors for the favor randomly and announced that it would be (j, h, k) , then player j would anticipate that i will form a partnership with h and would optimally decide to turn down i 's request knowing that he will always have the option to partner with g at a later stage. The same conclusion would carry over to a specification of the model in which players who need favors approach neighbors in an exogenous deterministic order.

Furthermore, the conclusion of Theorem 3 that the equilibrium is unique does not extend to a version of the model in which unmatched players who need favors choose the order in which they ask neighbors strategically. Consider, for instance, the line network with three players. In the model with endogenous orders, for every $p \in [0, 1]$, there exists an equilibrium in which when player 2 needs the first favor, he approaches player 1 first with probability p . The limit normalized payoffs of player 1 range from $(v - c)/9$ to $2(v - c)/9$ between the extreme cases $p = 0$ and $p = 1$.

6. COSTLESS PARTNERSHIPS

We now consider a version of the model in which the formation of a partnership results in immediate gains for both partners. When providing a favor, a player receives a positive payoff instead of incurring the cost c . Equilibrium incentives in this model are straightforward: every player asked for a favor should provide it and form a partnership immediately. In contrast to the benchmark models with costly favor provision, in which players have incentives to delay the formation of partnerships until they need favors or face the risk of isolation, in this version of the model players enter partnerships as soon as they get a chance. This opportunistic behavior may result in inefficient matches. For example, in the four-player line from Figure 1, if player 2 asks player 3 for a favor, player 3 agrees to provide it and commits to a partnership with player 2, which leaves players 1 and 4 isolated. Only one partnership emerges even though the maximum matching in this network consists of two partnerships. The definition of long-run efficiency and its connection to welfare maximization discussed

in the context of the model with costly favors extend to the present setting. We seek to characterize the networks for which the equilibrium is long-run efficient.

Long-run efficiency for network G in this version of the model requires that any sequence of (disjoint) partnerships $(i_1, j_1), \dots, (i_k, j_k)$ whose formation leaves all nodes isolated in $G \setminus i_1, j_1, \dots, i_k, j_k$ generates a maximum matching in G . Any matching $\{i_1 j_1, \dots, i_k j_k\}$ such that $G \setminus i_1, j_1, \dots, i_k, j_k$ contains no links is known as a *maximal matching* in G —a matching that is maximal with respect to inclusion among all matchings of G .

Clearly, all maximum matchings of G are maximal. The equilibrium in the model with costless favors for network G is long-run efficient if and only if the converse holds: every maximal matching of G must be a maximum matching of G . We call networks G with the latter property *completely elementary*. This term is inspired by Lovasz and Plummer (1986), who call a network G *elementary* if any link in G is part of a maximum matching of G . Note that completely elementary networks are always elementary. However, the opposite is not true as the seven-player line network demonstrates. That network is elementary but its maximal matching $\{(2, 3), (5, 6)\}$ is one link smaller than a maximum matching. Completely elementary networks can be defined in terms of elementary networks as follows: a network is completely elementary if and only if any subnetwork obtained by removing a sequence of pairs of linked nodes is elementary.

For a perfect network to be completely elementary, it must be that every maximal matching covers all its nodes. This suggests that perfect networks that are completely elementary have extremely dense connected components. The next result shows that their connected components can have only two possible structures: complete or complete bipartite networks.⁵ The proofs of this and all remaining results can be found in the Appendix.

Theorem 4. *A perfect network is completely elementary if and only if each of its connected components is a perfect complete network or a perfect complete bipartite network.*

We next investigate bipartite networks that are completely elementary. The following intuitive condition plays a central role in our analysis. Recall the notation from Section 4. A bipartite network G with sides (A, B) that has positive surplus from the perspective of A is *locally balanced* if every node $i \in A$ has a non-empty set of neighbors $X_i \subseteq N^G(\{i\})$ such that $|N^G(X_i)| \leq |X_i|$. To check that a network G with this property is completely elementary, note that Theorems 1 and 2 imply that every maximum matching of G covers A and has size $|A|$. If G is locally balanced, then any maximal matching of G must cover A . Indeed, if G had a maximal matching M that did not cover a node i in A , then it should cover all the neighbors of i , including the set of nodes X_i . However, nodes in X_i can only be matched to nodes in $N^G(X_i) \setminus \{i\}$ under M , which is impossible because $|N^G(X_i) \setminus \{i\}| < |X_i|$. The

⁵Note that a complete network is perfect if and only if it has an even number of nodes and a complete bipartite network is perfect if and only if it has an equal number of nodes on both sides of the partition. The terms *perfect complete network* and *perfect complete bipartite network* refer to networks with these respective properties.

next result establishes the converse of the claim proven here: local balancedness captures the structure of all imperfect connected components of completely elementary bipartite networks.

Theorem 5. *A bipartite network is completely elementary if and only if each of its connected components is a perfect complete bipartite network or a locally balanced bipartite network with positive surplus.*

Theorems 4 and 5 provide insights into two important blocks of the Gallai-Edmonds decomposition in a general completely elementary network G : the perfect network induced by perfectly matched nodes in G and the bipartite network with positive surplus describing the imperfectly matched structure of over-demanded and under-demanded nodes in G . Moreover, as Theorem 2 shows that links between perfectly matched and over-demanded nodes are not part of any maximum matching of G , such links cannot belong to any maximal matching of G either. Since every link of G is part of a maximal matching, there cannot be any links in G between perfectly matched and over-demanded nodes. By definition, there are no links between perfectly matched and under-demanded nodes, so perfectly matched nodes are disconnected from the rest of the network and induce a perfect completely elementary subnetwork. Theorem 4 then implies that the perfectly matched nodes in G are partitioned into connected components that are either complete or complete bipartite networks. Similarly, there cannot be any links among over-demanded nodes in G . We prove that the imperfectly matched structure of G needs to be a completely elementary bipartite network and use Theorem 5 to conclude that it is locally balanced. The next result summarizes these findings and uncovers further structural properties of the factor-critical connected components induced by under-demanded nodes (which are collapsed to single nodes in the imperfectly matched structure of G).

Theorem 6. *A network G is completely elementary only if its Gallai-Edmonds decomposition has the following structure:*

- (1) *Perfectly matched players are linked only to other perfectly matched players in G . Each connected component of perfectly matched players is either a perfect complete network or a perfect complete bipartite network.*
- (2) *There are no links between pairs of over-demanded players in G . The imperfectly matched structure of G is a locally balanced bipartite network with positive surplus from the perspective of the set of over-demanded nodes.*
- (3) *Each connected component C of the subnetwork induced by the set of under-demanded nodes in G has exactly one of the following properties:*
 - (a) *C contains no node linked to over-demanded players in G and thus constitutes a connected component of G .*
 - (b) *C contains exactly one node i that has over-demanded neighbors in G , and each connected component of $C \setminus i$ is either a perfect complete network or a perfect complete bipartite network.*

- (c) *The set X of nodes in C that have over-demanded neighbors in G has cardinality at least two and all nodes in X are linked to a single over-demanded node j in G . In this case, C can have four structures: (i) complete network; (ii) complete network with one missing link ii' ; (iii) bipartite complete network with one side having one more node than the other; and (iv) the network described in (iii) with an additional link connecting one pair of nodes (i, i') on the side with more nodes. The structures described in (ii) and (iv) are possible only if $X = \{i, i'\}$.*

Parts (1) and (2) of the theorem provide an exact description of the linking patterns among perfectly matched nodes and the imperfectly unmatched structure in a completely elementary network. Part (3) narrows down the structure of factor-critical connected components induced by under-demanded nodes and shows that each such component has sparse connections to the rest of the network. In particular, it implies that for every such component, there exists a node whose removal from the network disconnects the component from the network: in case (a), no node needs to be removed, while in cases (b) and (c), removing the corresponding nodes i and j disconnects the component.

The components classified in part (3) under case (a) can take many forms. We know that such components must be factor-critical, which implies that they are elementary. We can identify several factor-critical networks that are completely elementary. Examples include cycles with five and seven nodes and modifications of these cycles obtained by adding certain extra links. A rich class of completely elementary factor-critical networks can be constructed as follows. Consider a disconnected collection of complete and complete bipartite networks and an isolated node i . Add links between node i and other nodes in any fashion such that i has at least two neighbors in every network in the collection, with neighbors on different sides of the partition for the bipartite ones. This class of networks can be altered to also obtain many network structures falling under case (b) by simply adding a pair of nodes j and k along with the links ij and jk . In the resulting networks, node j is over-demanded and all other nodes are under-demanded. By contrast, case (c) shows that under-demanded components containing multiple nodes linked to the rest of the network (over-demanded nodes) are restricted to four simple linking structures.

7. CONCLUSION

This paper studies the formation of bilateral partnerships that guarantee reciprocal exchange of favors in social and economic networks. When favors are costly, we find that the structure of equilibrium partnerships, the strengths of players, and market efficiency are driven by the configuration of nodes that are essential for achieving all maximum matchings. In particular, essential players always find partners, while inessential players remain single with positive probability in equilibrium. This implies that essential players obtain higher

equilibrium payoffs than inessential players. Even though the search for partners is decentralized and incentives for entering partnerships depend on local network conditions, we prove that the possibility that each inessential player might be unable to find a partner implies that partnerships form efficiently in every network. This result is striking in the context of existing research, which has found that local incentives for forming partnerships are not usually aligned with global welfare maximization in markets with decentralized matching. More generally, we show exactly how each player's equilibrium decisions are determined by his (evolving) position in the Gallai-Edmonds decomposition. Prior research on trade in networks has established similar but less detailed connections between equilibrium outcomes and the Gallai-Edmonds decomposition (mainly in markets with central matching). However, there is a conceptual novelty in the mechanism underlying our result. In our setting, the weakness of inessential players is inflicted by neighbors actively marginalizing them via severing links with them when they request favors in the original network, while in previous models, it is indirectly precipitated by the possibility of remaining isolated in the network as neighbors forge agreements with other players.

We also consider a version of the model in which granting favors does not bear any cost and hence players have no incentives to delay joining partnerships. In this setting, inefficient partnerships are often unavoidable. Efficient market outcomes emerge only for special network structures containing sparsely linked blocks of complete and complete bipartite networks, locally balanced bipartite networks with positive surplus, and factor-critical networks.

The contribution of this research encompasses two complementary sets of results at the intersection of game theory and graph theory. The model with costly favor provision delivers a precise relationship between the classic Gallai-Edmonds structure of maximum matchings and incentives driving the efficient formation of partnerships in a natural favor exchange game. The version of the model with costless favors advances our understanding of completely elementary networks and suggests that efficiency is difficult to achieve when straightforward strategic considerations dictate that players should agree to enter partnerships without delay.

Given our focus on the effects of network asymmetries on partnership decisions, our model rules out other possible player heterogeneities. In particular, we assume that players need favors at the same rate and that the costs and benefits of favors are not link specific. In future work, it would be useful to relax these assumptions.

APPENDIX: PROOFS

Proof of Lemma 1. Fix the network G and the link $ij \in G$.

(1) Suppose that i is under-demanded in G , and let M be a maximum matching of G that does not cover i . Since $\mu(G \setminus ij) \leq \mu(G)$ and M is a matching of $G \setminus ij$, it must be that $\mu(G \setminus ij) = \mu(G)$ and M is also a maximum matching of $G \setminus ij$. As M does not cover i , it follows that i is under-demanded in $G \setminus ij$. Similarly, $\mu(G \setminus i) \leq \mu(G)$ combined with the fact

that M constitutes a matching for $G \setminus i$ implies that $\mu(G \setminus i) = \mu(G)$. If j is not essential in $G \setminus i$, then there exists a maximum matching M' of $G \setminus i$ that does not cover j . Adding the link ij to M' generates a matching of G with $\mu(G \setminus i) + 1 = \mu(G) + 1$ links, contradicting the definition of $\mu(G)$.

(2) Suppose that i is perfectly matched in G and that j is not the only efficient partner of i . If j is over-demanded in G , then Theorem 2 implies that ij is not an efficient link in G , so G and $G \setminus ij$ have the same set of maximum matchings and the same Gallai-Edmonds decomposition. Moreover, the set of efficient partners of i in G does not contain j and is identical to the set of efficient partners of i in $G \setminus ij$.

Suppose next that j is perfectly matched in G . Since j is not the only efficient partner of i , we have that $\mu(G \setminus ij) = \mu(G)$ and every maximum matching of $G \setminus ij$ is a maximum matching of G . In particular, every player who is under-demanded in $G \setminus ij$ is also under-demanded in G , and all efficient partners of i in $G \setminus ij$ are his efficient partners in G as well. Consider now a player k who is under-demanded in G . Then, there exists a maximum matching M of G that does not cover k . Since j is not the only efficient partner of i , there exists a maximum matching M' of G that does not contain the link ij . By Theorem 2, both M and M' link perfectly matched players with one another. Construct a third matching M'' that consists of the links of M among under- and over-demanded players in G and the links of M' among perfectly matched players in G . Then, M'' is a maximum matching of $G \setminus ij$ which does not cover k . Thus, k is under-demanded in $G \setminus ij$. The arguments above show that the sets of under-demanded players in G and $G \setminus ij$ coincide. Since neither i nor j is under-demanded in G , the sets of neighbors in the two networks of the common set of under-demanded players in G and $G \setminus ij$ are also identical. This implies that the sets of over-demanded players in G and $G \setminus ij$ are the same. It follows that the sets of perfectly matched players in G and $G \setminus ij$ also coincide. To prove that every efficient partner $k \neq j$ of i in G is also an efficient partner of i in $G \setminus ij$, it is sufficient to note that every maximum matching of G that contains the link ik continues to be a maximum matching of $G \setminus ij$.

(3) Suppose that i is perfectly matched in G and that j is i 's only efficient partner in G . We know that $\mu(G) - 1 \leq \mu(G \setminus ij) \leq \mu(G)$. If $\mu(G \setminus ij) = \mu(G)$, then there exists a maximum matching of G that does not contain the link ij , contradicting the assumption that i is perfectly matched in G and j is i 's only efficient partner in G . Hence, $\mu(G \setminus ij) = \mu(G) - 1$. Let M be a maximum matching of G . As i is perfectly matched in G and j is i 's only efficient partner in G , the link ij is necessarily contained in M . Removing ij from M produces a matching in $G \setminus ij$ with $\mu(G) - 1 = \mu(G \setminus ij)$ links. This constitutes a maximum matching for $G \setminus ij$, which does not cover i or j . It follows that both i and j are under-demanded in the network $G \setminus ij$.

Since i is perfectly matched in G , we have that $\mu(G \setminus i) = \mu(G) - 1$. Then, removing the link ij from any maximum matching of G generates a maximum matching for $G \setminus i$ that does not cover j . It follows that j is under-demanded in $G \setminus i$.

(4) Suppose that i is over-demanded and j is under-demanded in G . Then, there exists a maximum matching M of G that does not cover node j and hence excludes the link ij . As i is over-demanded in G , M must cover i . If we replace i 's link under M with ij , we obtain another maximum matching of G that contains the link ij . Therefore, ij is an efficient link in G . Since $\mu(G \setminus ij) \leq \mu(G)$ and M is a matching of $G \setminus ij$, it must be that $\mu(G \setminus ij) = \mu(G)$, so any maximum matching of $G \setminus ij$ is also a maximum matching of G . As any maximum matching of G covers i , it must be that every maximum matching of $G \setminus ij$ also covers i , so i is essential in $G \setminus ij$.

(5) Suppose that both i and j are perfectly matched and ij is an efficient link in G . Then, we have that $\mu(G \setminus i, j) = \mu(G) - 1$ and every maximum matching of $G \setminus i, j$ can be completed to a maximum matching of G by adding the link ij . Hence, every under-demanded player in $G \setminus i, j$ different from i and j is under-demanded in G . Conversely, reasoning similar to the proof of part (2) shows that any maximum matching of G that does not cover a given node can be transformed into a maximum matching of $G \setminus i, j$ with the same property by rewiring the links among perfectly matched players in G using a maximum matching of G that contains the link ij . It follows that every under-demanded node in G is under-demanded in $G \setminus i, j$. \square

Proof of Theorem 3. Fix a set of n nodes and consider a network G linking them. The result follows from claims (I1)-(I7) below concerning *subgame perfect equilibrium* behavior and outcomes in the network G for sufficiently high δ .

- (I1) When player i asks player j for a favor in G , player j refuses to provide the favor if j is essential in $G \setminus ij$.
- (I2) When a player i who is under-demanded in G first asks a neighbor for a favor, the neighbor turns him down. In equilibrium, no neighbor of i grants the favor and i remains single.
- (I3) When a player i who is perfectly matched in G first asks a neighbor j for a favor, j does the favor for i if and only if j is the only efficient partner of i in G .⁶ In equilibrium, the last efficient partner of i in G whom i approaches grants the favor to i .
- (I4) When an over-demanded player i in G asks his first neighbor for a favor, the neighbor agrees to provide the favor if and only if he is under-demanded in G . In equilibrium, the first under-demanded player j in G whom i approaches provides the favor to i ; the resulting partnership between i and j is efficient in G .
- (I5) In a subgame in which an essential player in G happens to need a favor at the beginning of a period (before asking any neighbor), each under-demanded player in

⁶If i is perfectly matched in G , the condition that j is the only efficient partner of i in G is equivalent to the condition that i is the only efficient partner of j in G .

G remains single with probability at least $1/((n-1)n)$ and receives a limit normalized expected payoff of at most $(1 - 1/((n-1)n)) \times (v - c)/n$.

- (I6) In every subgame perfect equilibrium for the network G , each essential player in G receives favors any time he needs them and enjoys a limit normalized expected payoff of $(v - c)/n$, while each under-demanded player in G is left single with probability at least $1/n$ and obtains a limit normalized expected payoff of at most $(n - 1)/n \times (v - c)/n$.
- (I7) There exists a unique subgame perfect equilibrium for the network G . In equilibrium, exactly $\mu(G)$ partnerships form for any sequence of moves by nature. The equilibrium is long-run efficient.

We prove claims (I1)-(I7) simultaneously and in this sequence by induction on the number of links in G . The induction base case for a network G with a single link can be verified without difficulty. We need to establish the claims for a network G assuming they are true for any network with fewer links. The proof of the inductive step relies on the existence and uniqueness of the subgame perfect equilibrium for subnetworks of G different from G , which follows from the induction hypothesis (I7). For brevity, we do not explicitly state this fact at every instance it is needed.

Let $V = v/((1 - \delta)n)$ denote the expected value of favors provided to a player in a partnership and $\bar{C} = (1 + \delta/((1 - \delta)n))c$ the maximum expected cost a player pays upon committing to a partnership via providing a favor in the current period.

To prove the inductive step for claim (I1), suppose that player i first asks player j for a favor in G and that j is essential in $G \setminus ij$. If j decides to grant the favor, then his expected payoff is $\delta V - \bar{C}$. If j refuses to provide the favor, then i continues to ask for favors in $G \setminus ij$. By the induction hypothesis (I6) applied to network $G \setminus ij$, player j is guaranteed to receive favors whenever he needs them in the future. Hence, j enjoys the same favor benefits regardless of whether he agrees to partner with i . However, the cost \bar{C} of partnering with i is greater than the expected cost of entering another partnership at any later date (which is at most $\delta \bar{C}$), so j has a strict incentive to refuse i 's request in any subgame perfect equilibrium for G .

To establish claim (I2), suppose that an under-demanded player i in G first asks his neighbor j for a favor. Player j obtains an expected payoff of $\delta V - \bar{C}$ if he grants the favor to i . If j refuses to provide the favor, then Lemma 1.1 implies that player i continues to be under-demanded in the resulting network $G \setminus ij$. By the induction hypothesis for the second part of (I2) applied to the network $G \setminus ij$, all of i 's other neighbors refuse to partner with him after j turns him down. Then, player j is left in $G \setminus i$ following his rejection of i 's request. Since j is linked to the under-demanded player i in G , Lemma 1.1 implies that j is essential in $G \setminus i$. By the induction hypothesis (I6) for the network $G \setminus i$, player j always finds a partner when he needs a favor. Player j 's payoff following his rejection of i 's request

is thus at least $\delta V - \delta \bar{C}$, which is greater than $\delta V - \bar{C}$. Hence, j has a strict incentive to turn i down as asserted. In a subgame perfect equilibrium, player j must turn i down, and then as argued above, all other neighbors of i in G should also turn him down in sequence leaving him ultimately isolated.

To prove the first part of claim (I3), suppose that a player i who is perfectly matched in G first asks his neighbor j for a favor. Since there are no links between under-demanded and perfectly matched nodes in G , it must be that j is an essential player in G . If j is not the only efficient partner of i in G , then Lemma 1.2 implies that j continues to be essential in $G \setminus ij$. Then, (I1) implies that j should refuse i 's request in any subgame perfect equilibrium.

Suppose instead that j is the only efficient partner of i in G . Then, by Lemma 1.3, both i and j are under-demanded in $G \setminus ij$. Assume that j refuses i 's request. Since i is under-demanded in $G \setminus ij$, the induction hypothesis (I2) for network $G \setminus ij$ implies that no neighbor whom i approaches after j grants the favor to i . Hence, player j ends up in the network $G \setminus i$ following his refusal to partner with i . By Lemma 1.3, player j is under-demanded in $G \setminus i$. Then, the induction hypothesis (I6) for network $G \setminus i$ implies that player j 's limit normalized expected payoff following the rejection of i 's request does not exceed $(n-1)/n \times (v-c)/n$. Since j can attain a limit normalized expected payoff of $(v-c)/n$ by partnering with i , player j has a strict incentive to accept i 's request for high δ .

To prove the inductive step for the second part of claim (I3), note that repeated use of Lemma 1.2 implies that the sets of perfectly matched, over-demanded, and under-demanded nodes do not change as we remove i 's links following rejections from neighbors until we reach his last efficient partner in G , while his set of efficient partners in the remaining network consists of his efficient partners in G except for those neighbors who have already rejected him. When i approaches his last efficient partner j in G , it is the case that j is also his only efficient partner in the remaining network. Then, the induction hypothesis (I3) applied for the subnetworks of G resulting from the sequence of rejections (along with the arguments for the first part of claim (I3) above) implies that j must grant the favor to i in any subgame perfect equilibrium.

To demonstrate the first part of claim (I4), suppose that an over-demanded player i in G asks his first neighbor j for a favor. If j is essential in G , then Theorem 2 implies that no maximum matching of G contains the link ij , and thus j is essential in $G \setminus ij$. Then, (I1) implies that j should refuse i 's request in any subgame perfect equilibrium.

Suppose instead that j is under-demanded in G . Then, Lemma 1.1 shows that j continues to be under-demanded in $G \setminus ij$, while Lemma 1.4 implies that i remains essential in $G \setminus ij$. If j refuses to grant the favor to i , then he finds himself under-demanded in the network $G \setminus ij$ in a subgame where the essential player i needs a favor. By the induction hypothesis (I5) for the network $G \setminus ij$, player j obtains a limit normalized expected payoff of at most $(1 - 1/((n-1)n)) \times (v-c)/n$ in this subgame, which is smaller than the limit normalized

expected payoff of $(v - c)/n$ guaranteed to him upon committing to a partnership with i . Hence, player j has a strict incentive to grant the favor to i for sufficiently high δ .

The second part of claim (I4) follows from observing that the removal of any set of links between i and essential players in G does not affect the set of maximum matchings or the Gallai-Edmonds partition because none of these links belongs to a maximum matching in G according to Theorem 2. We can then apply the induction hypothesis (I4) to all subnetworks of G resulting from i 's request being denied by his essential neighbors in G . By Lemma 1.4, the eventual partnership that i forms with the first under-demanded neighbor in G whom he asks for a favor is efficient in the remaining network as well as in G .

To prove the inductive step for claim (I5), suppose that an essential player i in G needs a favor at the beginning of a period and fix an under-demanded player j . Consider first the case in which i is perfectly matched in G . In this case, (I3) implies that one of i 's efficient partners k in G agrees to grant the favor and forms a partnership with i . By Lemma 1.5, player j remains under-demanded in $G \setminus i, k$. With probability $1/n$, player j needs a favor in the network $G \setminus i, k$ in the next period. Since j is under-demanded in $G \setminus i, k$, the induction hypothesis (I2) for network $G \setminus i, k$ implies that in this event all neighbors turn j down and j remains single.

Consider next the case in which i is over-demanded in G . Since j is under-demanded in G , there exists a maximum matching M of G that does not cover j . Player i has to be matched under M because he is over-demanded in G . Let k be player i 's partner under M . By Theorem 2, player k is under-demanded in G . With probability at least $1/(n - 1)$, player k is the first neighbor whom i asks for the favor. In this event, (I4) implies that k grants the favor to i . Note that the matching M without the link ik constitutes a maximum matching of $G \setminus i, k$ that does not cover j . Hence, player j is under-demanded in $G \setminus i, k$. By the same argument as the one used in the first case, j remains single with a conditional probability of at least $1/n$ following the agreement between i and k .

In either case, we have shown that conditional on player i needing a favor in network G , player j remains single with probability at least $1/((n - 1)n)$. Since the payoff from being single is 0 and the limit normalized expected payoff from forming a partnership is $(v - c)/n$, player j 's limit normalized expected payoff in the subgame cannot exceed $(1 - 1/((n - 1)n)) \times (v - c)/n$.

We now establish claim (I6). Consider an essential player i in G . If i needs a favor in the first period of the game, then claims (I3) and (I4) imply that one of i 's neighbors will agree to provide the favor to him. If another player asks i for a favor in the first period, then i forms a partnership with that player in the situation described by (I3) and receives favors as needed thereafter. Otherwise, (I2), (I3), and (I4) imply that in the first period, either an under-demanded player in G (different from the essential player i) needs a favor and is left single or a pair of players different from i form an efficient partnership. Since every maximum matching for the remaining network is a maximum matching of G in the

former case and can be completed to form a maximum matching of G by adding the link connecting the partners in the latter case, i continues to be an essential player in the second period network. Then, the induction hypothesis (I6) for the remaining network implies that i always receives favors when he needs them in the subgame starting in the second period. We have shown that i receives favors at any instance he needs them.

If i is an under-demanded player in G , then (I2) implies that i remains single in the event that he needs a favor in the first period of the game. This event has probability $1/n$.

The statements of claim (I6) regarding payoffs follow from the fact every player receives 0 payoff when left single and a limit normalized expected payoff of $(v - c)/n$ upon entering a partnership.

We finally prove claim (I7). Suppose that player i needs a favor and first asks neighbor j in the network G . By the induction hypothesis (I7), there is a unique subgame perfect equilibrium for the network $G \setminus ij$ arising in the event that j turns i 's request down. By (I2), (I3), and (I4), the optimal response of player j to i 's request is uniquely determined given the equilibrium play in $G \setminus ij$. It follows that there exists at most one subgame perfect equilibrium in G . Using the single-deviation principle, one can easily prove that the strategies described by (I2), (I3), and (I4) indeed constitute a subgame perfect equilibrium for G .

Claims (I2), (I3), and (I4) also show that the partnerships formed along any equilibrium path constitute a maximum matching with probability 1. Indeed, (I2) proves that when an under-demanded player i in G requires the first favor, no neighbor does i the favor and i remains single in the network $G \setminus i$. Since i is under-demanded in G , there exists a maximum matching M of G that does not cover i . Then, M is also a maximum matching of $G \setminus i$, so $\mu(G \setminus i) = \mu(G)$. The inductive hypothesis (I7) implies that $\mu(G \setminus i) = \mu(G)$ partnerships emerge in the subgame played on the resulting network $G \setminus i$. Similarly, (I3) shows that when a perfectly matched player i in G requires the first favor, the last efficient partner j of i in G approached by i grants the favor to i . As the link ij is efficient in G , we have that $\mu(G \setminus i, j) = \mu(G) - 1$. The inductive hypothesis (I7) applied to $G \setminus i, j$ implies that $\mu(G \setminus i, j) = \mu(G) - 1$ partnerships form in the subgame played in the network $G \setminus i, j$ after i partners with j in G . Hence, a total of $\mu(G)$ partnerships emerge in this case as well. An analogous argument deals with the case in which an over-demanded player in G needs the first favor and, according to (I4), forms an efficient partnership with the first under-demanded player in G he approaches. Since every player needs favors at some points in time with probability 1, we conclude that exactly $\mu(G)$ partnerships form with probability 1, and hence the equilibrium is long-run efficient. \square

Proof of Theorem 4. The proof of the “if” part follows by induction from the observation that removing a pair of linked nodes from a perfect network that is complete or complete bipartite generates another perfect network with the same structure.

To prove the “only if” part, we proceed by induction on the number of nodes in the network. The claim is clearly true for the smallest completely elementary perfect network, which consists of two linked nodes. Consider a completely elementary perfect network G with four or more nodes and assume that the claim is true for all such networks with fewer nodes. If G is not a connected network, then the claim follows by applying the induction hypothesis to each of its connected components, which are completely elementary perfect networks with fewer nodes than G .

For the rest of the proof of the inductive step, suppose that G is connected and fix a link $ij \in G$. Since G is perfect and completely elementary, all connected components of $G \setminus i, j$ must share these properties. In particular, each connected component of $G \setminus i, j$ must contain an even number of nodes.⁷ If j is i 's single neighbor in G , then the hypothesis that G is a connected network with more than two nodes implies that j has a neighbor $h \neq i$ in G . Then, i is isolated in $G \setminus j, h$, a contradiction with the fact that $jh \in G$ and thus $G \setminus j, h$ is perfect. Hence, i needs to be linked to nodes other than j in G . Similarly, j must be linked to nodes different from i in G .

Let $h \neq j$ be any neighbor of i and $k \neq i$ be any neighbor of j in G . If h and k belong to distinct connected components of $G \setminus i, j$, which necessarily have even numbers of nodes, then removing nodes h and k from $G \setminus i, j$ generates at least two connected components with odd numbers of nodes. However, since G is perfect and completely elementary and $ih, jk \in G$, the network $G \setminus i, j, h, k$ must also be perfect, which is impossible if $G \setminus i, j, h, k$ has connected components with odd numbers of nodes. This reasoning leads to a contradiction whenever either i or j is linked to multiple connected components of $G \setminus i, j$. It follows i and j can only be linked to a single, identical connected component G' of $G \setminus i, j$. Then, the network induced by the nodes in G' along with i and j forms a connected component of G . This network must be identical to G because G is connected. It follows that $G' = G \setminus i, j$, and thus $G \setminus i, j$ is connected. The induction hypothesis implies that $G \setminus i, j$ is either a perfect complete or a perfect complete bipartite network. We tackle these two cases in turn.

Consider first the case in which $G \setminus i, j$ is a perfect complete network. If G is a perfect complete network, then there is nothing to prove. Suppose that G is not a perfect complete network. Without loss of generality, let $k \neq i, j$ be a node in G such that $ik \notin G$. If j is linked to any node h different from k and i , then $G \setminus i, j, h, k = (G \setminus i, j) \setminus h, k$ is also a perfect complete network, and we can construct a matching of G that contains the link jh and covers all nodes except for i and k . Since $ik \notin G$, it follows that G cannot be perfect and completely elementary. We conclude that j is linked only to nodes i and k in G . Consider now any node $h \in N \setminus \{i, j, k\}$. Then, j is not linked to h in G , and an argument analogous to the one above implies that i is linked only to j and h in G . As h was chosen arbitrarily in

⁷For the rest of the proofs, it is convenient to repurpose the notation $G \setminus i, j$ for the network in which not only all links of i and j are eliminated, but nodes i and j are also removed from G (so that $\{i\}$ and $\{j\}$ do not count as trivial connected components of $G \setminus i, j$).

$N \setminus \{i, j, k\}$, this is possible only if $N \setminus \{i, j, k\}$ has a single element h . We have that $hk \in G$ because $G \setminus i, j$ is a complete network. Therefore, G consists of the four nodes $\{i, j, h, k\}$ and the links $\{ij, ih, jk, hk\}$, so G is a perfect bipartite network.

We are left with the case in which $G \setminus i, j$ is a perfect complete bipartite network. We focus on the situation in which $G \setminus i, j$ has at least four nodes since the perfect complete bipartite network with two nodes is a perfect complete network, and that case has been already covered above. We show that if $G \setminus i, j$ has at least four nodes, then G is a perfect complete bipartite network.

Let h be an arbitrary neighbor of i in G different from j . Let A denote the side of the bipartition of $G \setminus i, j$ that contains h and B denote the other side. Consider any node $k \in B$. Since $G \setminus i, j$ is a perfect complete bipartite network, G has a matching that contains the link ih and covers all nodes except for j and k . As G is perfect and completely elementary, it must be that this matching can be completed to a perfect matching of G and hence $jk \in G$. It follows that j is linked in G to any node in B . An analogous argument proves that since j has neighbors in B , node i has to be linked to all nodes in A .

If i were linked to any node in B , then switching the roles of A and B in the reasoning above, we could infer that j is linked to all nodes in A . However, A must contain at least two nodes $h' \neq h$ because $G \setminus i, j$ was assumed to have at least four nodes. Then, removing the links ih and jh' from G generates a complete bipartite network with sides $A \setminus \{h, h'\}$ and B , which is not perfect as $|A| = |B|$. This contradicts the assumption that G is completely elementary and perfect. Thus, the hypothesis that i is linked to a node in B cannot be true. We have established that the set of neighbors of i in G is given by $A \cup \{j\}$. Similarly, j 's set of neighbors in G is $B \cup \{i\}$. Therefore, G is a perfect complete bipartite network with sides $A \cup \{j\}$ and $B \cup \{i\}$. \square

Proof of Theorem 5. To prove the “if” part, it is sufficient to show that perfect complete bipartite networks and locally balanced bipartite networks with positive surplus are completely elementary. Theorem 4 implies that complete bipartite networks are completely elementary. Consider now a locally balanced bipartite network G with sides (A, B) that has positive surplus from the perspective of A . Theorem 1 implies that every maximum matching of G covers all nodes in A and that $\mu(G) = |A|$. By definition, every node $i \in A$ has a non-empty set of neighbors $X_i \subseteq N^G(\{i\})$ such that $|N^G(X_i)| \leq |X_i|$. Let M be a maximal matching of G . If M did not cover node $i \in A$, then it should cover all neighbors of i in G , including the nodes in X_i . However, nodes in X_i can be matched under M only to distinct nodes in $N^G(X_i) \setminus \{i\}$. This is impossible because $|N^G(X_i)| \leq |X_i|$ implies that $|N^G(X_i) \setminus \{i\}| < |X_i|$. We have shown that every maximal matching of G covers all nodes in A , so it contains exactly $|A| = \mu(G)$ links. It follows that every maximal matching of G is a maximum matching of G , and hence G is completely elementary.

We next establish the “only if” part. Supposed that G is a completely elementary bipartite network. By Theorem 2.1, perfectly matched players are matched only to other perfectly matched players in every maximum matching of G . Since G is completely elementary, every link is part of a maximum matching of G . It follows that perfectly matched players are linked only to other perfectly matched players in G . Then, any connected component of G that contains perfectly matched nodes consists exclusively of such nodes and must be perfect by Theorem 2.3. Since every such component is a completely elementary bipartite network, Theorem 4 implies that all such components are perfect complete bipartite networks.

We are left with characterizing the connected components of G formed exclusively by over- and under-demanded nodes in G . Since G is completely elementary and Theorem 2.1 implies that no maximum matching of G contains links connecting over-demanded nodes in G , it must be that G contains no links between over-demanded nodes in G . By Theorem 2.5, there are no links between pairs of under-demanded nodes in G because G is bipartite.

We have established that any connected component G' of G formed by over- and under-demanded nodes in G is bipartite with one side consisting of over-demanded nodes in G and the other consisting of under-demanded nodes in G . Clearly, the roles of nodes in the Gallai-Edmonds decomposition for G are preserved in its connected component G' . Thus, G' is a bipartite network with all over-demanded nodes on one side and the under-demanded nodes on the other. By Theorem 2.4, G' has positive surplus from the perspective of the set of over-demanded nodes. Since G is completely elementary, G' must also have this property.

We prove that G' is locally balanced by contradiction. Suppose there exists an over-demanded node i in G' such that $|N^{G'}(X)| > |X|$ for every non-empty set of its neighbors $X \subseteq N^{G'}(\{i\})$. Let G'' denote the subnetwork of G' obtained by deleting node i and all under-demanded nodes outside $N^{G'}(\{i\})$ from G' . G'' is a bipartite network in which the side $N^{G'}(\{i\})$ has the property that $|N^{G''}(X)| \geq |X|$ for all $X \subseteq N^{G'}(\{i\})$. Theorem 1 then implies that G'' admits a matching M that covers all nodes in $N^{G'}(\{i\})$. Then, M is also a matching of G' , which does not cover i but covers all of i 's neighbors in G' . Thus, any maximal matching of G' that includes M does not cover i . Since i is over-demanded in G' , every maximum matching of G' must cover i . Hence, no maximal matching of G' that includes M may be a maximum matching. This implies that G' is not completely elementary. The contradiction proves that G' is locally balanced, as asserted. \square

Proof of Theorem 6. Fix a completely elementary network G .

Part (1): By Theorem 2.1, perfectly matched players are linked only to other perfectly matched players in every maximum matching of G . Since G is completely elementary, every link is part of a maximum matching of G . It follows that there are no links in G between nodes that are perfectly matched and nodes that are not. Thus, any connected component of G that contains perfectly matched nodes consists exclusively of such nodes and must be perfect by Theorem 2.3. Since G is completely elementary, every connected component of G

formed by perfectly matched nodes must be completely elementary as well. Theorem 4 then implies that each such component is either a perfect complete network or a perfect complete bipartite network.

Part (2): By Theorem 2.1, links between over-demanded nodes in G are not part of any maximum matching of G . The same argument used in part (1) demonstrates that the completely elementary network G cannot contain such links. Theorem 2.4 implies that the imperfectly matched structure of G is a bipartite network G' with positive surplus from the perspective of the set O of over-demanded nodes. By Theorem 1, every maximum matching of G' covers O and $\mu(G') = |O|$.

We prove that G' is locally balanced by contradiction. Suppose that G' is not locally balanced. Then, we show that G' cannot be completely elementary. Indeed, since G' is a bipartite network with positive surplus from the perspective of O , none of its connected components is a perfect complete bipartite network. Theorem 5 implies that G' is completely elementary only if all its connected components are locally balanced bipartite networks with positive surplus from the perspective of under-demanded nodes in G . However, in this case G' would be locally balanced, which contradicts the initial assumption.

Since G' is not completely elementary and $\mu(G') = |O|$, there exists a maximal matching M' of G' with size smaller than $|O|$, which necessarily does not cover some node $i \in O$. Let $i_1, \dots, i_{\bar{k}}$ denote the nodes in O covered by M' . For $k = 1, \dots, \bar{k}$, let j_k be a neighbor of i_k in the connected component C_k induced by the set of under-demanded nodes corresponding to i_k 's partner under M' . As C_k is a factor-critical network, $C_k \setminus j_k$ admits a perfect matching M_k for $k = 1, \dots, \bar{k}$. Note that node i cannot be linked in G to under-demanded nodes outside $C_1 \cup \dots \cup C_{\bar{k}}$ because M' is a maximal matching of G' that excludes i .

Construct a matching M'' for G as the union $\{i_1 j_1, \dots, i_{\bar{k}} j_{\bar{k}}\} \cup M_1 \cup \dots \cup M_{\bar{k}}$. Note that M'' covers every node in $C_1 \cup \dots \cup C_{\bar{k}}$. In particular, M'' covers all under-demanded neighbors of i in G . Furthermore, as argued above, i does not have any over-demanded or perfectly matched neighbors in G because $i \in O$. It follows that i does not have any neighbors in G left unmatched by M'' . Hence, there exists a maximal matching M of G that includes M'' and does not cover i .⁸ Since G is completely elementary, M needs to be a maximum matching of G . However, this contradicts Theorem 2.1, according to which every maximum matching of G , including M , covers all over-demanded nodes in G , including i . The contradiction proves that G' is a locally balanced bipartite network with positive surplus from the perspective of O .

Part (3): Fix a connected component C of under-demanded nodes in G .

Claim (a) follows from the construction of the Gallai-Edmonds decomposition: under-demanded nodes can be linked only to over-demanded nodes and other under-demanded

⁸One such matching is obtained by augmenting M'' with a maximum (or just maximal) matching for every connected component of under-demanded nodes that is not covered by M' and a perfect (or just maximal) matching of the perfectly matched nodes.

nodes. If C does not contain nodes linked to over-demanded nodes, then C is a connected component of G .

To prove part (b), suppose that C contains a single node i that has over-demanded neighbors in G . Let j be such a neighbor. Then, no node in $C \setminus i$ has neighbors outside $C \setminus i$, so $C \setminus i$ is a connected component of $G \setminus i, j$. Since $ij \in G$ and G is completely elementary, so is $G \setminus i, j$. Connected components of completely elementary networks are completely elementary, so $C \setminus i$ should be completely elementary. By Theorem 2.2, C is factor-critical, so $C \setminus i$ is a perfect network. Theorem 4 then implies that every connected component of $C \setminus i$ is either a perfect complete or a perfect complete bipartite network.

We next show that the situations described by (a), (b), and (c) cover all possibilities. We prove by contradiction that if multiple nodes from C are linked to over-demanded nodes in G , then they should all be linked to a single, identical over-demanded node. If this were not the case, then C should contain a pair of nodes i and j which are linked to two distinct over-demanded nodes h and k , respectively. Construct a maximal matching M that includes the links ih and jk . As G is assumed to be completely elementary, M should also be a maximum matching of G . However, this contradicts Theorem 2.1, which implies that the over-demanded nodes h and k should be matched to different components of under-demanded nodes under every maximum matching of G .

We are left to prove part (c). Suppose that C contains multiple nodes linked to a single over-demanded node j in G ; let X denote the set of neighbors of j in C . In this case, C has at least three nodes. Since C is factor-critical, if it has exactly three nodes, then it must be the complete network with three nodes, which is covered by part (i). We are left with the case in which C has at least five nodes.

For every node $i \in X$, the same argument used for part (b) shows that $C \setminus i$ is the union of perfect complete and perfect complete bipartite networks. We set out to prove that $C \setminus i$ is connected for every $i \in X$.

For a contradiction, suppose that $C \setminus i$ is not connected for some $i \in X$. Let C' be a connected component of $C \setminus i$ that contains a node $i' \in X$. From the argument above, we conclude that $C \setminus i'$ is a perfect network and that C' is either a perfect complete or a perfect complete bipartite network. If i has no links to nodes in $C' \setminus i'$, then $C' \setminus i'$ forms a connected component for the network $C \setminus i'$. Since $C \setminus i'$ is a perfect network, its connected component $C' \setminus i'$ must also be perfect. However, it is impossible for both $C' \setminus i'$ and C' to be perfect at the same time. This argument shows that i is linked to a node k in $C' \setminus i$.

Since $C' \setminus i'$ is a connected network that contains k , it follows that $C \setminus i'$ is connected. Hence, $C \setminus i'$ is either a perfect complete or a perfect complete bipartite network that contains the link ik and has at least four nodes. Then, removing node i from $C \setminus i'$ does not disconnect the network, which implies that $C \setminus i, i'$ is also connected. As C' is connected and contains i' and k , it must be that i' is connected to k by a path that lies in C' and thus excludes node

i . Then adding node i' to $C \setminus i, i'$ generates a connected network, $C \setminus i$. This contradicts the assumption that $C \setminus i$ is disconnected.

We showed that for every $i \in X$, the network $C \setminus i$ is connected and thus must be either a perfect complete or a perfect complete bipartite network.

Consider first the case in which $C \setminus i$ is a perfect complete network for some $i \in X$. Let i' be any other element of X . If i' is the only neighbor of i in C , then i would be isolated in $C \setminus i'$. This implies that $C \setminus i'$ cannot be perfect, so C is not factor-critical, which contradicts Theorem 2.2. It follows that i has a link to a node in $C \setminus i'$, and hence $C \setminus i'$ is a connected network. It follows that $C \setminus i'$ is either a perfect complete or a perfect complete bipartite network. Since $C \setminus i$ is a complete network with at least four nodes, it must be that $C \setminus i'$ contains a complete network with three nodes, so it cannot be a bipartite network. Therefore, $C \setminus i'$ must be a perfect complete network. As both $C \setminus i$ and $C \setminus i'$ are complete networks, C is either a complete network or a complete network that excludes the link ii' . Since this statement is true for all $i \neq i' \in X$, the latter case is possible only if $X = \{i, i'\}$.

Finally, we need to consider the case in which $C \setminus i$ is a perfect complete bipartite network for every $i \in X$. Fix $i \in X$ and let (A, B) be the sides of the bipartition of $C \setminus i$. Without loss of generality, assume that A contains a node i' from X . Since $C \setminus i'$ is a perfect complete bipartite network in which half of the nodes, namely those from B , are not linked with one another in G , it must be that B forms one side of the bipartition of $C \setminus i'$; then, $A \cup \{i\} \setminus \{i'\}$ constitutes the other part. Hence, i is linked to all nodes in B and not linked to any node in $A \setminus \{i'\}$. It follows that C is a bipartite network with sides $(A \cup \{i\}, B)$ with a possible extra link between i and i' .

Assume that $ii' \in G$. If X contains a third node i'' different from i and i' , then $C \setminus i''$ would include the set of links $\{ii', ij, i'j\}$ for any $j \in B \setminus \{i''\} \neq \emptyset$, in which case it could not be a bipartite network. Therefore, C can be a bipartite network with sides $(A \cup \{i\}, B)$ and the extra link ii' only if $X = \{i, i'\}$. \square

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