WARWICK

ECONOMICS

CRETA

Centre for Research in Economic Theory and its Applications

Discussion Paper Series

Convergence and divergence in dynamic voting with inequality

Corrado Di Guilmi and Giorgos Galanis

July 2020
(updated November 2020)

No: 61

CRETA
Centre for Research in Economic Theory and its Applications
Department of Economics
University of Warwick, Coventry,
CV4 7AL, United Kingdom

warwick.ac.uk/fac/soc/economics/research/centres/creta/papers

Convergence and divergence in dynamic voting with inequality

Corrado Di Guilmi*[1] and Giorgos Galanis[2]

[1]Economics Discipline Group, University of Technology Sydney, Australia, and Centre for Applied Macroeconomic Analysis, Australian National University, Canberra, Australia.
[2]Goldsmiths, University of London, UK, and Centre for Applied Macroeconomic Analysis, Australian National University.

November 17, 2020

Abstract

The original formulation of the median voter theorem predicts parties' political convergence in a static setup, under two key assumptions: voters preferences being fixed and parties being opportunistic (purely office-motivated). Drawing on recent empirical findings about the evolution of voters' political preferences, this paper verifies whether the median voter theorem's results hold when (i) the control variables that influence voters' preferences endogenously evolve over time, and (ii) parties are not opportunistic. We present a dynamic two-party voting model in which voters' preferences evolve over time

*Corresponding author: Corrado Di Guilmi. University of Technology Sydney - PO Box 123, Broadway, NSW 2007, Australia. Ph.: +61295147743, Fax: +61295147711. E-mail: corrado.diguilmi@uts.edu.au.

depending on observable common factors and unobservable idiosyncratic characteristics. In such a setting, the convergence of parties' platforms to the centre is a special case within a range of results that include instability and equilibria at one of the extremes. Moreover, convergence of parties' platforms is achieved not as the result of electoral strategies, but when neither party has enough support to pursue its agenda.

JEL codes: C62; D72; E71.
Keywords: median voter; dynamic voting; political preferences.

1 Introduction

Downs' influential result of convergence of parties' political platforms to the views of the median voter is achieved under the assumptions of opportunistic (purely office-motivated) parties and fixed voters' preferences (Downs, 1957). Since his seminal contribution, the literature on voting has investigated the conditions for platform convergence or divergence when one or more of the Downs' hypotheses are relaxed (see Grofman, 2004, for an overview). Relevant empirical literature shows that over the last twenty years, the distribution of voters and parties' agendas on specific issues have moved towards relatively extreme positions.[1] The possible drivers of this shift have been widely studied, with a number ot works pointing to economic and distributional issues as possible causes.[2] From a theoretical perspective, dynamic election models with evolving state variables appear to be better suited for the analysis of a changing political landscape, in particular when economic variables are involved (Duggan and Martinelli, 2017).

This paper shows that if voters' preferences are influenced by factors identified by the empirical literature and evolve endogenously over time, the convergence of political platforms postulated by the median voter theorem is a special case in a range of possible outcomes that include political cycles and convergence to extreme political platforms.

We develop a behavioural dynamic model with heterogeneous individuals with endogenously evolving preferences and two policy-oriented parties. The parties have different core values in terms of income redistribution but the extent to which their policies are actually implemented depends on the relative support that they enjoy. Their policies change the income distribution and generate a feedback effect on electoral preferences. The binary political preferences of each individual are modelled using a discrete choice framework (McFadden, 2001; Train, 2009), in which the different factors that influence

[1]See Boxell *et al.* (2020); Fiorina and Abrams (2008); Funke *et al.* (2016); Hobolt and Tilley (2016); McCarty *et al.* (2006), among many others.

[2]For example, see Duca and Saving (2016); Kelly and Enns (2010); Garand (2010); Han (2016); McCarthy *et al.* (2006); Pontusson and Rueda (2010); Winkler (2019).

political choices have been treated in the relevant empirical literature[3]. More specifically, we include three different observable factors: (i) economic performance of the ruling party, (ii) income inequality, (iii) bandwagon effect, together with unobservable idiosyncratic factors.

The relationship between economic results and voting has been investigated by the literature on *responsibility hypothesis* (Lewis-Beck and Stegmaier, 2000; Lewis-Beck and Paldam, 2000): since voters hold the ruling parties accountable for the economic performance of a country, economic growth increases the possibility of their re-election. This causal nexus is also known as *economic voting* and has produced a large body of theoretical and empirical studies that have analysed the dependence of electoral choices on the main macroeconomic indicators (for a recent survey, see Lewis-Beck and Stegmaier, 2019).

The influence of the *personal distribution of income* in political preferences has roots in the political economy models along the lines of Alesina and Rodrik (1994) and Meltzer and Richard (1981). The level of income inequality can affect public preferences in two ways: (i) high level of inequality leads a larger share of voters to demand for a redistribution in order to improve their individual welfare (Aalberg, 2003; Finseraas, 2009; Piketty, 2018); (ii) due to a *hysteresis* effect, relatively high levels of inequality over time make inequality itself more acceptable for the public (Andersen and Yaish, 2012; Kelly and Enns, 2010).

The *bandwagon effect* was introduced by Leibenstein (1950) as a force that leads people "to wear, buy, do, consume, and behave like their fellows; the desire to join the crowd" (p. 184). This type of behaviour was proposed in order to explain "irrational" demand for certain goods. Simon (1954) independently formulated the same idea in relation to voting behaviour, conjecturing that people are more likely to vote for a candidate that is considered as the most likely winner. Recent empirical evidence have found support for bandwagon behaviour in voting (Morton *et al.*, 2015).

Finally, the assumed heterogeneity in individual political preferences is

[3]For early evidence of time-evolving political preferences, see Fiorina (1981), Kiewiet (1983), and Kramer (1971).

related to voters' *partisan biases* (Brennan and Buchanan, 1984; Robbett and Matthews, 2018; Shayo and Harel, 2012). These biases are ideological priors can lead voters to make choices that are independent from the expected election outcomes and, possibly, against their individual interests. As a consequence, despite the fact that an individual may benefit from one party's redistributive policies, she may still vote for the opposite party due to ideological and other non-economic reasons related to culture, religious beliefs, etc...

Two main results of this paper are worth mentioning. First, political polarization or convergence are mostly determined by the bandwagon effect and the responsibility hypothesis. The median voter theorem's result of convergence of policies appears to be a special case, which is achieved when the relative influences of the two behavioural factors are low. More precisely, convergence is achieved when the bandwagon effect is below a threshold that depends on the economic performance of the ruling party. Negative economic growth can outweigh the bandwagon effect and lead to a change in majority. Second, when voters prefer extremely high or low levels of inequality, large political swings and a higher level of polarisation are more likely.

In re-examinig the median voter's results, our framework connects and contributes to different literatures that have been developing in a mutually independent manner up to now. It brings together insights from dynamic elections with endogenous state variables (Azzimonti, 2011; Battaglini and Coate, 2007, 2008; Battaglini, 2014; Duggan and Forand, 2013) and responsible parties (Bernhardt *et al.*, 2009; Calvert, 1985; Wittman, 1983), integrating empirical findings on the different factors affecting political preferences.

Our paper is closely related to Esponda and Pouzo (2019), who analyse the effects of bounded rationality and focus on how parties' previous performance (*retrospective voting*) can lead to polarisation in a static environment with policy motivated parties. Our work differs by including the bandwagon effects as a behavioral influence together with the parties' performance (in economic terms) and by adopting a dynamic setup. Looking at the results, while in both papers polarisation is the outcome of behavioural factors, in ours it can take the form of either shifts between the two extreme political positions or stability of one of them.

The paper also provides a methodological contribution, being the first attempt to analyse dynamic voting by means of a discrete choice framework (McFadden, 2001; Train, 2009, among others) with heterogeneous interacting agents. Discrete choice models in the same vein have been extensively used in financial economics, starting with the works of Brock and Hommes (1997, 1998) and Lux (1995), in macroeconomics (for example De Grauwe, 2012; Flaschel *et al.*, 2018; Frankel and Froot, 1987, 1990), and, more recently, in epidemiology to incorporate behavioural factors (Baskozos *et al.*, 2020; Di Guilmi *et al.*, 2020).

The remainder of the paper is structured as follows. Section 2 presents the model's assumption for the baseline version of our framework while section 3 illustrates the analytical and numerical results. Section 4 introduces and examines two extensions of the baseline model. The results are discussed in section 5 while section 6 offers some concluding remarks.

2 The Model

The model is composed of two political parties and a large number of heterogeneous voters with evolving preferences. Voting behaviour depends on the four factors listed above: relative income of single voters, individual biases, bandwagon effect, and responsibility hypothesis. The two parties have different core values but the extent to which they pursue them depends on their relative support.

2.1 Left and Right

Consider a large population of boundedly rational voters who have the option to choose between two different political parties, and call these left (denoted by L) and right (denoted by R). The parties differ in their views on redistribution: L favours top-down redistribution in order to reduce inequality, which is quantified by the Gini coefficient for income g, while R, in contrast, aims to increase g through a redistribution from bottom to top income earners. Both parties are policy oriented, in the sense that they only support the redistribution policy that is consistent with their platform, even when it is not the one desired by the majority of voters.

Let the x be the relative support of the left, such that

$$x = n^L - n^R, \tag{1}$$

where n^L is the share of the left voters and n^R the share of voters who support the right, such that $n^L + n^R = 1$. This implies that $x \in [-1, 1]$, with $x > 0$ when $n^L > n^R$.

2.2 Political Choices

We assume that all individuals' preferences are *exhaustive* such that if an individual does not choose one party, they necessarily choose the other. Along the lines of discrete choice models (McFadden, 2001; Train, 2009), individual preferences depend on observable characteristics, which are common across agents, and unobservable idiosyncratic ones. The utility for individual i of choosing L is given by

$$U_i = \beta \mathbf{v} + \epsilon_i, \tag{2}$$

where $\mathbf{v}$ is a column vector of the observable factors, β a row vector which captures the relative importance of each of the elements of v, and ϵ_i represents the unobservable characteristics for voter i. Since the utility function (2) can be positive or negative, the political choice $\mathcal{C}_i = \{L, R\}$ for i can be expressed as

$$\mathcal{C}_i = \begin{cases} L & \text{if } \beta \mathbf{v} + \epsilon_i > 0, \\ R & \text{if } \beta \mathbf{v} + \epsilon_i \leq 0, \end{cases} \tag{3}$$

The vector $\mathbf{v}$ includes three types of factors: the *bandwagon effect*, the *responsibility hypothesis*, and *inequality*, and $\beta = [\beta_x,\ \beta_y,\ \beta_i]$, where $\beta_x,\ \beta_y,\ \beta_i$ quantify the relative importance of the three effects, respectively. For simplicity, $\beta_x,\ \beta_y,\ \beta_i$ are all positive and, without loss of generality, we set $\beta_i = 1$. Accordingly, the three effects can be modelled as follows:

E1. Bandwagon effect: from (3), given that $\beta_x > 0$ and x expresses the relative proportion of left voters, the bandwagon effect is simply expressed by x.

E2. Responsibility hypothesis: macroeconomic performance is quantified by the change in output y denoted by[4] $\dot{y}$. Since $\dot{y} > 0$ (< 0) is expected to have a positive (negative) effect for the ruling party, and given that $\beta_y > 0$ in (3), the responsibility hypothesis effect can be expressed by $x\dot{y}$.

E3. Inequality: as argued by Alesina and Angeletos (2005), in each society it is possible to quanfity a level of inequality that is socially acceptable due to differentials in effort. Let us identify this level with g_0. Accordingly, as inequality becomes higher (lower) than this level, people on average will favour a redistribution towards the bottom (top) income earners.[5] Hence, on average, voters choose L if $g > g_0$ and R if $g < g_0$.

Considering E1-E3, the vector $\mathbf{v}$ can be written as

$$\mathbf{v} = \begin{bmatrix} x \\ x\dot{y} \\ g - g_0 \end{bmatrix}, \tag{4}$$

which implies

$$\beta\mathbf{v} = \beta_x x + \beta_y x\dot{y} + g - g_0. \tag{5}$$

Along the lines of Train (2009), ϵ_i is assumed to follow a logistic distribution. Accordingly, the probability $P(L|\mathbf{v})$ that a randomly chosen individual

[4]In general, for any variable $z \in \mathbb{R}$ let $\dot{z}$ denote its time derivative.

[5]Since g_0 can be considered as the preference of the median voter, this modelling choice also accounts for micro-level factors. Modifications in income distribution can change the position of single voters within the distribution and, as a consequence, individual policy preferences. For example, a voter is expected to oppose a redistribution from top to bottom that makes her worse off (and therefore she favours the political right). However, if the distribution of income becomes more concentrated, the same voter might find herself now benefiting from the same type of redistribution and, accordingly, she becomes more likely to vote left.

chooses L, for a given $\mathbf{v}$ can be expressed as

$$P(L|\mathbf{v}) = \frac{e^{\beta \mathbf{v}}}{1+e^{\beta \mathbf{v}}}. \tag{6}$$

Accordingly, the probability for given $\mathbf{v}$ of choosing R is calculated as

$$P(R|\mathbf{v}) = 1 - P(L|\mathbf{v}) = \frac{1}{1+e^{\beta \mathbf{v}}}. \tag{7}$$

From (1), we can write $n^L = 1 + x$ and $n^R = 1 - x$. Then from (6) and (7), the change in the relative difference is given by

$$\dot{x} = (1-x)\frac{e^{\beta \mathbf{v}}}{1+e^{\beta \mathbf{v}}} - (1+x)\frac{1}{1+e^{\beta \mathbf{v}}}. \tag{8}$$

From (8), it can be seen that x plays both a positive and a negative role in its own evolution. This double effect appears more clearly if we re-express (8) as follows:

$$\dot{x} = \frac{e^{\beta \mathbf{v}} - 1}{1+e^{\beta \mathbf{v}}} - x, \tag{9}$$

where the derivative of the first component on the right hand side with respect to x is positive as $\beta\mathbf{v}$ is increasing in x and $\frac{e^{\beta \mathbf{v}}-1}{1+e^{S}}$ is increasing in $\beta\mathbf{v}$.

The economic intuition behind equation (9) is that, on the one hand, when the relative population supporting the left grows (shrinks), the probability of switching to the left increases (decreases) while the probability of switching to the right decreases (increases). On the other hand, the consequent increase (decrease) in x implies that the probability of switching to the left influences a smaller (greater) share of voters as it only applies to the share of the right (left). From (9), the direct negative effect of $x > 0$ to the $\dot{x}$ is linear while the indirect one is increasing and concave. Hence, we expect that (depending on the parameter values) for $x > 0$ after a certain level of x the overall effect on $\dot{x}$ will be negative. Similarly for $x < 0$ below a certain level of x the positive effect on $\dot{x}$ will dominate the negative one.

2.3 Redistribution

In a responsive democracy, the public perception has a feedback effect on inequality and economic performance through the choices of the elected officials, who will shape their policies according to the public's taste in the attempt of being re-elected (Wlezien, 2004). With specific reference to redistribution, Brooks and Manza (2006) find evidence of social preferences influencing social policy output, while Cusack *et al.* (2006) provide support for the hypothesis that rising inequality increases the demand for redistribution.

As discussed earlier, when in power the left (right) party decreases (increases) inequality to an extent proportional to its political support. As a consequence, we can write

$$\frac{\partial \dot{g}}{\partial x} < 0. \tag{10}$$

In order for $g \in [0; 1]$, the following conditions should hold:

$$g \leq 1 \Leftrightarrow \dot{g} \leq 1 - g, \tag{11}$$

$$g \geq 0 \Leftrightarrow \dot{g} \geq -g. \tag{12}$$

Furthermore, it is reasonable to assume that as g gets closer to 1 (0), it becomes progressively more difficult to raise (lower) it further. This implies that $\dot{g}$ should be convex for low values of g and concave for high values of g. On the base of these considerations, we can express $\dot{g}$ as

$$\dot{g} = -xg(1 - g). \tag{13}$$

In conclusion, the model dynamics is described by the changes in three variables: x, g, and y. For our analytical study, we will consider output growth as exogenous. This choice allows us to obtain some general results, focusing on the bandwagon effect.[6] In the next section, we analyse the prop-

[6] In this simplified setting, it is still possible to assess the role of economics growth with respect to the bandwagon effect. As shown below, the analysis of the stability defines a relationship between the bandwagon effect and the effect of income growth.

erties of the following dynamical system

$$\dot{x} = (1-x)\frac{e^{\beta \mathbf{v}}}{1+e^{\beta \mathbf{v}}} - (1+x)\frac{1}{1+e^{\beta \mathbf{v}}} \tag{8}$$

$$\dot{g} = -xg(1-g) \tag{13}$$

3 Results

Definition 1. *Let $\boldsymbol{z} = (z_1, z_2, ..., z_n)$. For any dynamical system $\dot{\boldsymbol{z}} = f(\boldsymbol{z})$, a stationary equilibrium is defined as the state in which $\dot{\boldsymbol{z}} = 0$.*

The stationary equilibrium we refer to is a statistical equilibrium, in the sense that stationarity at the system level ($\dot{x} = 0$) does not imply that individual voting behaviours do not change, but just that the changes cancel out in the aggregate.

Considering $\beta_y \dot{y}$ as exogenously given, with output growth as a constant, we can set $\beta_y \dot{y} = \sigma$. In order to simplify the notation, in the following let $\mu = \beta_x + \sigma$. Accordingly, equation (5) becomes

$$\beta \mathbf{v} = (\beta_x + \sigma)x + g - g_0 = \mu x + g - g_0 \tag{14}$$

As shown below the dynamical system (8-13) can be analytically studied and presents a range of interesting possible outcomes.

3.1 Stability and convergence

Proposition 1. *Consider the system of equations (8), (13), and (14). The following are true:*

(i) $\{0, g_0\}$ *is a stationary equilibrium*

(ii) $\{0, g_0\}$ *is locally stable if* $\mu < 2$

(iii) the equilibrium is a stable centre for $\mu = 2$.

Proof

See Appendix.

Proposition 1 shows that there always exists a stationary equilibrium, whose local stability depends on the strength of the bandwagon effect and on the economic performance. The local stability of this equilibrium, which can be called "centrist", implies that the population is split between the two parties and inequality is at the average socially acceptable level. As it is evident from Proposition 1, a relatively high level of importance of the bandwagon effect with respect to the public perception of the economic performance plays in general a destabilising role with regards to the centrist equilibrium.

Figure 1 shows that, starting from $g = g_0$ and $x \neq 0$, the system is unstable. Starting with a slight left (right) majority the voters soon shifts to the right (left) as inequality falls below (raises above) the acceptable level. When inequality reaches the upper (lower) limit, the driving force of the bandwagon effect is exhausted and voters begin to shift to the left (right). Since when $g = g_0$, we always have that $x \neq 0$, the stationary equilibrium is never achieved. Changing the parameter setting to $\beta_x = 1.7$ such that $\beta_x + \sigma < 2$, the model displays cyclical convergence to $\{0, g_0\}$, as shown by figure 2 in which the initial condition is set as $g = 0.4$. To complete the analysis, figure 3 shows the cyclical behaviour of the economy when $\beta_x + \sigma = 2$.
[Figure 1 here]
[Figure 2 here]
[Figure 3 here]

Figures 4 and 5 present phase diagrams to illustrate the cyclical convergence to the $\{0, g_0\}$ equilibrium, while figures 6 and 7 present the convergent behaviour that occurs for $\mu = 2$.
[Figures 4 and 5 here]
[Figures 6 and 7 here]

Corollary 1. *In the system of equations (8), and (13), the* $\{0, g_0\}$ *stationary equilibrium, has two complex conjugate eigenvalues when*
$\mu \in \left(-2\left(1 + 2\sqrt{g_0 - g_0^2}\right), 2\left(1 + 2\sqrt{g_0 - g_0^2}\right)\right)$

Proof

See Appendix.

Corollary 1 shows that for $\mu > 0$ then the local stability of the centrist equilibrium takes the form of a spiral sink. Similarly, instability (for $\mu < 2\left(1 + 2\sqrt{g_0 - g_0^2}\right)$) takes the form of a spiral source. Hence for $\mu > 0$, even in the case of political stability of the centre, (a mild) political volatility exists. Note that the further away g_0 is from 0.5 (in either direction), the smaller will be the interval for μ for which the equilibrium has complex eigenvalues, which means that whether the stability of the centrist equilibrium is cyclical or not also depends on the level of socially accepted inequality.

3.2 From convergence to the extremes

Note that if $\dot{g} = 0$, for $x = 0$, then $g = 0$, or $g = 1$. The following Proposition describes the different possibilities regarding the existence of stationary equilibria for $g = 0$, or $g = 1$, when the centrist equilibrium is locally stable.

Proposition 2. *Consider the system of equations (8) and (13). Then besides the centrist equilibrium, two more stationary equilibria exist:*

(i) $\{x_0^1, 0\}$, *with* $x_0^1 \in (-1, 0)$,

(ii) $\{x_1^1, 1\}$, *with* $x_1^1 \in (0, 1)$,

(iii) if $\mu < 2$, *then there exist exactly three equilibria, and both* $\{x_0^1, 0\}$ *and* $\{x_1^1, 1\}$ *are saddle points.*

Proof

See Appendix.

Proposition 2 states that, besides the centrist equilibrium, there exist two more equilibria, one with a left wing majority and maximum inequality and one with a right wing majority and zero inequality. These three equilibria are the only ones that occur when $\mu < 2$. The fact that both of these are saddle is very intuitive as a right (left) wing majority will always increase (reduce) inequality and furthermore as Proposition 1 shows, for $\mu < 2$, the centrist equilibrium is locally stable.

Proposition 3. *Consider the system of equations (8) and (13) and let* $\mu > 2$. *There exist* $\bar{\mu}(g_0) > 2$ *and* $\hat{\mu}(g_0) > 2$, *such that, apart from the* $\{0, g_0\}$, $\{x_0^1, 0\}$ *and* $\{x_1^1, 1\}$, *the following stationary also equilibria exist:*

(i) $\{x_0^3, 0\}$, *with* $x_0^3 \in (0,1)$, *if* $\mu = \bar{\mu}(g_0)$

(ii) $\{x_0^2, 0\}$, *with* $x_0^2 \in (0,1)$ *and* $\{x_0^3, 0\}$, *with* $x_0^3 \in (0,1)$, *if* $\mu > \bar{\mu}(g_0)$,

(iii) $\{x_1^3, 1\}$, *with* $x_0^3 \in (-1,0)$, *if* $\mu = \hat{\mu}(g_0)$,

(iv) $\{x_1^2, 1\}$ *with* $x_1^2 \in (-1,0)$, *and* $\{x_1^3, 0\}$, *with* $x_1^3 \in (-1,0)$, *if* $\mu > \hat{\mu}(g_0)$,.

Proof

See Appendix.

Proposition 3, shows that for $\mu > 2$, the number of equilibria depends indirectly on g_0 and whether $\bar{\mu}(g_0) \lesseqgtr \hat{\mu}(g_0)$. Based on the previous intuition, we expect that the dynamics of the economy will depend on the importance of the bandwagon effect and whether $\bar{\mu}(g_0) \lesseqgtr \hat{\mu}(g_0)$, as this leads to the existence of new politically extreme equilibria.

Proposition 4. *The following are true:*

(i) $\frac{\partial \bar{\mu}(g_0)}{\partial g_0} > 0$ *and* $\frac{\partial \hat{\mu}(g_0)}{\partial g_0} < 0$,

(ii) *if* $g_0 = 0.5$ *then* $\bar{\mu}(g_0) = \hat{\mu}(g_0)$.

Proof

See Appendix.

Proposition 4, says that the higher the value of g_0, the higher (lower) will be the level of μ sufficient for the existence of equilibria with a left (right) majority and zero (total) inequality. Also from Proposition 5, it follows that:

Remark 1. *If* $g_0 < 0.5$ *then* $\bar{\mu}(g_0) > \hat{\mu}(g_0)$, *while if* $g_0 > 0.5$ *then* $\bar{\mu}(g_0) < \hat{\mu}(g_0)$.

This result highlights the importance of the socially accepted level of inequality in the existence of the various political equilibria. If for example $g_0 > 0.5$ (high level of socially accepted inequality) and $\bar{\mu}(g_0) < \mu < \hat{\mu}(g_0)$, then we expect the economy to move towards the right equilibrium, which will be stable.

Figures 8 and 9 present the limit cycle that emerges when $2 < \mu < \min\{\bar{\mu}(g_0), \hat{\mu}(g_0)\}$.

[Figures 8 and 9 here]

Figures 10 and 11 present the cases $g_0 > 0.5$ and $\mu \in (\hat{\mu}(g_0), \bar{\mu}(g_0),)$ and $g_0 < 0.5$ and $\mu \in (\bar{\mu}(g_0), \hat{\mu}(g_0))$, respectively.

[Figure 10 here]

[Figure 11 here]

In this case, besides the three unstable equilibria, there exist two more equilibria: $\{x_1^2, 1\}$ with $x_1^2 \in (-1, 0)$, and $\{x_1^3, 1\}$, with $x_1^3 \in (-1, 0)$. The system is expected to converge to $\{x_1^3, 1\}$ because it implies total inequality in income distribution and right-wing majority.

Finally, figures 12 and 13 illustrate the dynamics for $\mu > \max\{\bar{\mu}(g_0), \hat{\mu}(g_0)\}$.

[Figures 12 and 13 here]

In this case there will exist seven equilibria in total: $\{0, g_0\}$; $\{x_0^1, 0\}$, with $x_0^1 \in (-1, 0)$; $\{x_0^2, 0\}$, with $x_0^2 \in (0, 1)$; $\{x_0^3, 0\}$, with $x_0^3 \in (0, 1)$; $\{x_1^1, 1\}$, with $x_1^1 \in (0, 1)$; $\{x_1^2, 1\}$ with $x_1^2 \in (-1, 0)$; and $\{x_1^3, 1\}$, with $x_1^3 \in (-1, 0)$. Both $\{x_0^3, 0\}$ and $\{x_1^3, 1\}$ are expected to be stable: the former implies perfect equality with left-wing majority and in the latter the majority is right-wing and inequality at its maximum.

4 Extensions

This section extends the baseline model to account for the feedback effect detected by the literature between income distribution and growth and between income distribution and the acceptable level of income inequality. As shown below, the main results of the model do not appear to change and for this reason the analysis is limited to numerical simulations.

4.1 Inequality and output

While the electoral behaviour is affected by economic performance according to the responsibility hypothesis as discussed, the level of inequality affects both growth and public's preferences. As for the former, the literature within the tradition of Downs (1957) argues that inequality slows down growth only indirectly, by inspiring policies that reduce the accumulation of capital (Alesina and Rodrik, 1994). However, Bertola *et al.* (2006) and Berg *et al.* (2018) found little empirical support for Alesina and Rodrik's thesis. A strand of recent literature proves that the relationship between income inequality and growth is more complex and multifaceted (Piketty, 2014; Cynamon and Fazzari, 2015, among others).

The presence nonlinearities in the inequality-growth relationship is known since Kuznets (1955). Recently, Grigoli and Robles (2017) have empirically analysed the functional relationship between growth and level of inequality measured by the Gini coefficient for a large sample of countries and estimated a polynomial relationship. Using their estimate for the functional relationship non-conditional to the control factors that they test, we model the relationship between income inequality and output as follows[7]

$$y = exp(\gamma_1 g^2 + \gamma_2 g + \gamma_3) \tag{15}$$

In the estimates by Grigoli and Robles (2017), equation (15) has a maximum at 0.12 for the non-conditional relationship and around 0.27 for the conditional one.

The parameters for the numerical simulations of the model are set as follows: $\beta_x = 2.5; \beta_y = 1.5; g_0 = 0.5; T = 100$. The values of the parameters in (15) are set as in Grigoli and Robles (2017): $\gamma_1 = -0.0001; \gamma_2 = 0.0024; \gamma_3 = .077$.

[Figure 14 here]

[7] Grigoli and Robles (2017) use net Gini coefficients and not market coefficients. Given the qualitative nature of the present analysis, we adopt their functional form and coefficient estimates since they provide an insightful extension of the model in presence of (empirically verified) nonlinearities in the growth-inequality relationship.

Figure 14 shows that, even taking into account the relationship between inequality and growth, the main results of the baseline model do not seem to be qualitatively affected, despite the presence of an economic cycle along with the political one. The model generates cycles with a duration of about 30 period. In both the baseline and the extended model, the length and the amplitude of the cycle depend on the relative influence of the bandwagon effect.

The expansionary phase in income is driven by a sharp decrease in inequality, which results from a rapid shift in public opinion leading to a left-wing majority. While inequality continues to decrease, income appears to reach a ceiling. The stagnant income decreases the support of the left and, as the Gini index approaches zero, a shift in public opinion determines a right-wing majority. The combination of negative growth, caused by higher income inequality, and the high levels of the Gini index will determine a change in political preferences that will restart the cycle.

4.2 Endogenous desirable inequality

As we have already mentioned, the level of inequality affects public preferences in two relevant and apparently idiosyncratic ways. First, high level of inequality leads voters to demand for a redistribution (Finseraas, 2009). This effect is discussed in the baseline version of the model introduced in section 2. Second, an *hysteresis* effect has been detected: historically high level of inequality makes inequality itself more acceptable for the public (Kelly and Enns, 2010).

Andersen and Yaish (2012), using survey data across twenty countries, find evidence about the influence of the social classes of the respondents and the historical levels of the Gini index on the public's perception of inequality. In particular the effect of the Gini coefficient for net income inequality on the acceptable level of inequality appears to be significant and sizeable even when controlling for different alternative explanatory variables. They use a linear regression together with a number of controls which cannot be reproduced in our model. Since our model is better equipped to capture short-time and smoother adjustments, we represent this effect with a nonlinear function as follows

$$g_0 = \frac{G_0}{1 + exp(-\alpha g)} \tag{16}$$

where $\alpha > 0$ is a constant that quantifies the sensitivity of the acceptable level of inequality to the Gini index for income and G_0 is a scale parameter, chosen to ensure that $0 \leq g_0 \leq 1$.

Once again, the qualitative insights provided in the baseline model hold. As shown by figures 15, the patterns produced by the simulations of this extended model are similar to those of section 4.1. However, in this setting g_0 follows the dynamics of the Gini index, delaying the phase transition. In particular, g reaches almost 1 before the public opinion changes and redistribution occurs. As a consequence, the recession is deeper and the though in the income cycle is lower than in the model of section 4.1.

[Figure 15 here]

5 Discussion

According to our results, the convergence or polarisation of the political system depend on the interaction among bandwagon effect, economic growth, and public perception of inequality.

The bandwagon effect emerges as the main factor in determining the degree of convergence or divergence in the system, in particular when accompanied by economic growth. More precisely, it is possible to identify a critical level of the strength of the bandwagon effect below which the system is led to a centrist equilibrium. Conversely, a relatively stronger effect can drive the system to extreme equilibria. In the case of stability of the centrist equilibrium, political convergence between the two parties predicted by the median voter theorem emerges through a different and original channel with respect to the standard treatment. Indeed here, political convergence towards the centre in terms of redistributive policies is not achieved because parties, in the attempt of maximising their chances of being elected, choose a policy that is more likely to satisfy voters' preferences as in the standard

Downsian framework. Here, the population moves towards being equally split between the two parties and as a consequence, redistribution converges to the average socially acceptable level because neither party has enough support for a redistribution.

The threshold for the bandwagon effect below which stability is achieved is not constant but depends on the interaction between the public perception of the economic growth and the level of accepted inequality. In particular, when aggregate income decreases and the public attach a strong importance to the economic performance, the strength of the bandwagon effect appears to vanish. In other words, economic growth amplifies the effect of the bandwagon effect in good times while an economic crisis is a possible source of voters' swings, instability, and polarisation.

Also the public perception of inequality plays a significant role. Specifically, the further is the level of accepted inequality from its central value of 0.5, the wider are the fluctuations in voters' preferences in the cyclical convergence to the centrist equilibrium. For values of g_0 close enough to one of the extremes, polarisation increases and extreme equilibria become more likely.[8]

Despite our model being limited in scope and extremely parsimonious, its results can have some relevance in the current shifting political landscape. According to Schmitt-Beck (2015), the bandwagon effect is stronger in case of detachment of voters or with little available information about the candidates. While the detachment of voters is confirmed by the declining voting participation rates, the direction of the changes in the level and quality of information in the era of social media is not clear. If the larger use of social media increases the amount of information available for voters, as argued by Ernst *et al.* (2019), it can also directly strengthen the bandwagon effect. However, social media can work as an echo chamber or vehicles for *fake news* (Törnberg, 2018). In this case, despite the lower quality of the information,

[8] While in our treatment we let the Gini index fluctuate over the whole spectrum so that the central value is 0.5, a more empirically grounded study would be required to identify a realistic "central" value and estimate a critical threshold for the bandwagon effect. However, this specification would not change the general qualitative conclusions of our analysis.

the intensity of the bandwagon effect can still be increased through two different channels: first, by reducing the contacts between groups with opposite political persuasions, and second, by strengthening individual beliefs and consequently leading towards more extreme views.

Our analysis can also provide some context for phenomena that are not immediately revealed by the two-party preference, as the emergence of relatively more extreme positions within each party (for the US see McCarty *et al.*, 2006) or the voting for alternative or fringe parties (in the UK and other European countries). From this perspective, the model integrates the typical narrative of the median voter, by showing how a more polarised electorate in two-party system drives the parties towards more radical stances.

Finally, the findings of our paper point to the increase in political divergence as a result of the combination of low growth and increasing inequality, both of which have plagued some advanced economies in the last decades. Further, the model points to a higher tolerance of the public for inequality as one of the possible reasons for more extreme voting choices.

The qualitative behaviour of the model revealed by analytical results is substantially confirmed by the extensions that endogenise output and acceptable inequality.

6 Concluding remarks

This paper introduces a novel framework for the anaysis of dynamic voting in a two-party system. Voters have dynamically evolving preferences that are are affected by social, economic, and idiosyncratic factors, which in turns are determined by the economic policies of the governing party. The number and the type of equilibria crucially depend on the bandwagon effect, which pushes individuals to follow the majority.

If the influence of the bandwagon effect is relatively low, the population is equally split between the right and the left. This split determines a convergence in terms of the policies of the two parties, as postulated by the standard median voter framework. A relatively high influence of the bandwagon effect is a necessary and sufficient condition for the instability of the previous equilibrium and at the same time for the existence of equilibria which correspond to politically extreme situations with strong left or right political majorities. Political swings and a higher level of polarisation can also be the result of extreme voters' preferences in terms of acceptable inequality.

The paper adds to the literature from a twofold perspective. Firstly, it provides a novel framework for the modelling of dynamic voting and its integration with economic models, which features endogenously evolving state variables for voters and includes the consolidated results of the median voter theorem as a special case. Secondly, it proposes an original treatment for popular discrete choice models that introduces new perspectives for the study of the emergence of political convergence or polarisation as dependent on multiple factors.

Our parsimonious framework is flexible enough to be extended in a number of possible different directions. First, the analysis could be enriched by allowing for abstention as a third option. Indeed, Downs (1957) himself considered the possible implications of abstention, while Grofman (2004) argues that extreme voters are the most likely to abstain from voting. Our model is well equipped to study the incidence of lower turnout of extreme voters on convergence or divergence of parties' policies. Second, the range of factors affecting the political preferences could be enlarged to include, for example, unemployment or the functional distribution of income. This extension could encompass the analysis of Piketty (2018) of a partition of the electorate across social divides. A third possible direction concerns the inclusion of existing growth models in order to achieve a more refined representation of the relationship between inequality and growth.

Acknowledgments

The authors wish to thank Karsten Kohler, Gilberto Lima, Herakles Polemarchakis, Peter Skott, Roberto Veneziani, Raghul S. Venkatesh and the participants of the 24^{th} WEHIA conference in London, June 2019, the 24^{th} CEF conference in Milan, June 2018, the 23^{rd} FMM conference in Berlin, October 2019 and the attendees to the various seminars in Australia, Ger-

many, and UK, for helpful feedback and comments. The usual caveats apply. Corrado Di Guilmi gratefully acknowledges the financial support from the University of Technology Sydney.
The authors declare no conflicts of interest.

References

Aalberg, T. (2003). *Achieving Justice: Comparative Public Opinions on Income Distribution*, Brill: Leiden, Boston.

Alesina, A. and Angeletos, G.M. (2005). 'Fairness and redistribution', *American Economic Review*, vol. 95(4), pp. 960–980.

Alesina, A. and Rodrik, D. (1994). 'Distributive politics and economic growth', *The Quarterly Journal of Economics*, vol. 109(2), pp. 465–490, ISSN 00335533, 15314650.

Andersen, R. and Yaish, M.M. (2012). 'GINI DP 48: Public Opinion on Income Inequality in 20 Democracies: The Enduring Impact of Social Class and Economic Inequality', AIAS, Amsterdam Institute for Advanced Labour Studies.

Azzimonti, M. (2011). 'Barriers to investment in polarized societies', *American Economic Review*, vol. 101(5), pp. 2182–2204.

Baskozos, G., Galanis, G. and Di Guilmi, C. (2020). 'Social distancing and contagion in a discrete choice model of covid-19', CAMA Working Paper No. 35/2020.

Battaglini, M. (2014). 'A dynamic theory of electoral competition', *Theoretical Economics*, vol. 9(2), pp. 515–554.

Battaglini, M. and Coate, S. (2007). 'Inefficiency in legislative policymaking: A dynamic analysis', *American Economic Review*, vol. 97(1), pp. 118–149.

Battaglini, M. and Coate, S. (2008). 'A dynamic theory of public spending, taxation, and debt', *American Economic Review*, vol. 98(1), pp. 201–36, doi:10.1257/aer.98.1.201.

Berg, A., Ostry, J.D., Tsangarides, C.G. and Yakhshilikov, Y. (2018). 'Redistribution, inequality, and growth: new evidence', *Journal of Economic Growth*, vol. 23(3), pp. 259–305, doi:10.1007/s10887-017-9150-2.

Bernhardt, D., Duggan, J. and Squintani, F. (2009). 'The case for responsible parties', *American Political Science Review*, vol. 103(4), pp. 570–587, doi: 10.1017/S0003055409990232.

Bertola, G., Bertola, P., Foellmi, R. and Zweimüller, J. (2006). *Income Distribution in Macroeconomic Models*, Princeton University Press, ISBN 9780691121710.

Boxell, L., Gentzkow, M. and Shapiro, J.M. (2020). 'Cross-country trends in affective polarization', National Bureau of Economic Research, doi:10. 3386/w26669.

Brennan, G. and Buchanan, J. (1984). 'Voter choice: Evaluating political alternatives', *American Behavioral Scientist*, vol. 28(2), pp. 185–201.

Brock, W. and Hommes, C. (1997). 'A rational route to randomness', *Econometrica*, vol. 65, pp. 1059–1095.

Brock, W. and Hommes, C. (1998). 'Heterogeneous beliefs and routes to chaos in a simple asset pricing model', *Journal of Economic Dynamics and Control*, vol. 22, pp. 1235–1274.

Brooks, C. and Manza, J. (2006). 'Social policy responsiveness in developed democracies', *American Sociological Review*, vol. 71(3), pp. 474–494, ISSN 00031224.

Calvert, R.L. (1985). 'Robustness of the multidimensional voting model: Candidate motivations, uncertainty, and convergence', *American Journal of Political Science*, vol. 29(1), pp. 69–95.

Cusack, T., Iversen, T. and Rehm, P. (2006). 'Risks at work: The demand and supply sides of government redistribution', *Oxford Review of Economic Policy*, vol. 22(3), pp. 365–389, ISSN 0266903X, 14602121.

Cynamon, B.Z. and Fazzari, S.M. (2015). 'Inequality, the Great Recession and slow recovery', *Cambridge Journal of Economics*, vol. 40(2), pp. 373–399, ISSN 0309-166X, doi:10.1093/cje/bev016.

De Grauwe, P. (2012). ‘Booms and busts in economic activity: A behavioral explanation’, *Journal of Economic Behavior & Organization*, vol. 83(3), pp. 484–501.

Di Guilmi, C., Galanis, G. and Baskozos, G. (2020). ‘A behavioural sir model and its implications for physical distancing’, CRETA Online Discussion Paper Series 58.

Downs, A. (1957). *An economic theory of democracy*, New York: Harper and Row.

Duca, J.V. and Saving, J.L. (2016). ‘Income inequality and political polarization: Time series evidence over nine decades’, *Review of Income and Wealth*, vol. 62(3), pp. 445–466, doi:10.1111/roiw.12162.

Duggan, J. and Forand, J.G. (2013). ‘Markovian elections’, University of Waterloo, Department of Economics.

Duggan, J. and Martinelli, C. (2017). ‘The political economy of dynamic elections: Accountability, commitment, and responsiveness’, *Journal of Economic Literature*, vol. 55(3), pp. 916–84, doi:10.1257/jel.20150927.

Ernst, N., Blassnig, S., Engesser, S., Büchel, F. and Esser, F. (2019). ‘Populists prefer social media over talk shows: An analysis of populist messages and stylistic elements across six countries’, *Social Media + Society*, vol. 5(1), p. 2056305118823358, doi:10.1177/2056305118823358.

Esponda, I. and Pouzo, D. (2019). ‘Retrospective Voting And Party Polarization’, *International Economic Review*, vol. 60(1), pp. 157–186.

Finseraas, H. (2009). ‘Income inequality and demand for redistribution: A multilevel analysis of european public opinion’, *Scandinavian Political Studies*, vol. 32(1), pp. 94–119.

Fiorina, M.P. (1981). *Retrospective voting in American national elections*, Yale University Press.

Fiorina, M.P. and Abrams, S.J. (2008). 'Political polarization in the american public', *Annual Review of Political Science*, vol. 11(1), pp. 563–588, doi: 10.1146/annurev.polisci.11.053106.153836.

Flaschel, P., Charpe, M., Galanis, G., Proano, C.R. and Veneziani, R. (2018). 'Macroeconomic and stock market interactions with endogenous aggregate sentiment dynamics', *Journal of Economic Dynamics and Control*, vol. 91, pp. 237 – 256.

Frankel, J. and Froot, K. (1987). 'Using survey data to test standard propositions regarding exchange rate expectations', *American Economic Review*, vol. 77, pp. 133–153.

Frankel, J. and Froot, K. (1990). 'Chartists, fundamentalists, and trading in the foreign exchange market', *American Economic Review*, vol. 80(2), pp. 181–185.

Funke, M., Schularick, M. and Trebesch, C. (2016). 'Going to extremes: Politics after financial crises, 1870–2014', *European Economic Review*, vol. 88, pp. 227 – 260, ISSN 0014-2921, doi:https://doi.org/10.1016/j.euroecorev.2016.03.006, sI: The Post-Crisis Slump.

Garand, J.C. (2010). 'Income inequality, party polarization, and roll-call voting in the U.S. senate', *The Journal of Politics*, vol. 72(4), pp. 1109–1128, ISSN 00223816, 14682508.

Grigoli, F. and Robles, A. (2017). 'Inequality overhang', International Monetary Fund.

Grofman, B. (2004). 'Downs and two-party convergence', *Annual Review of Political Science*, vol. 7(1), pp. 25–46, doi:10.1146/annurev.polisci.7.012003.104711.

Han, K.J. (2016). 'Income inequality and voting for radical right-wing parties', *Electoral Studies*, vol. 42(6), pp. 54–64, (EVS) (ISSP).

Hobolt, S.B. and Tilley, J. (2016). 'Fleeing the centre: the rise of challenger parties in the aftermath of the euro crisis', *West European Politics*, vol. 39(5), pp. 971–991, doi:10.1080/01402382.2016.1181871.

Kelly, N.J. and Enns, P.K. (2010). 'Inequality and the dynamics of public opinion: The self-reinforcing link between economic inequality and mass preferences', *American Journal of Political Science*, vol. 54(4), pp. 855–870, ISSN 00925853, 15405907.

Kiewiet, D.R. (1983). *Macroeconomics and Micropolitics: The Electoral Effects of Economic Issues.*, Chicago University Press.

Kramer, G.H. (1971). 'Short-term fluctuations in u.s. voting behavior, 1896-1964', *The American Political Science Review*, vol. 65(1), pp. 131–143.

Kuznets, S. (1955). 'Economic growth and income inequality', *The American Economic Review*, vol. 45(1), pp. 1–28, ISSN 00028282.

Leibenstein, H. (1950). 'Bandwagon, Snob, and Veblen Effects in the Theory of Consumers' Demand', *The Quarterly Journal of Economics*, vol. 64(2), pp. 183–207.

Lewis-Beck, M. and Paldam, M. (2000). 'Economic voting: An introduction', *Electoral Studies*, vol. 19, pp. 113–121.

Lewis-Beck, M.S. and Stegmaier, M. (2000). 'Economic determinants of electoral outcomes', *Annual Review of Political Science*, vol. 3(1), pp. 183–219.

Lewis-Beck, M.S. and Stegmaier, M. (2019). 'Economic voting', in (R. D. Congleton, B. Grofman, S. Voigt, M. S. Lewis-Beck and M. Stegmaier, eds.), *The Oxford Handbook of Public Choice, Volume 1*, Oxford University Press, ISBN 9780190469733.

Lux, T. (1995). 'Herd behaviour, bubbles and crashes', *Economic Journal*, vol. 105, pp. 881–889.

McCarthy, N., Poole, K.T. and Rosenthal, H. (2006). *Polarized America: The dance of ideology and unequal riches*, Cambridge, MA: MIT Press.

McCarty, N., Poole, K. and Rosenthal, H. (2006). *Polarized America: The Dance of Ideology and Unequal Riches*, Cambridge, MA: MIT Press.

McFadden, D. (2001). 'Economic Choices', *American Economic Review*, vol. 91(3), pp. 351–378.

Meltzer, A.H. and Richard, S.F. (1981). 'A rational theory of the size of government', *Journal of Political Economy*, vol. 89(5), pp. 914–927, ISSN 00223808, 1537534X.

Morton, R.B., Muller, D., Page, L. and Torgler, B. (2015). 'Exit polls, turnout, and bandwagon voting: Evidence from a natural experiment', *European Economic Review*, vol. 77(C), pp. 65–81.

Piketty, T. (2014). *Capital in the* 21^{st} *Century*, Cambridge, MA: Harvard University Press.

Piketty, T. (2018). 'Brahmin left vs merchant right: Rising inequality and the changing structure of political conflict (evidence from france, britain and the us, 1948-2017)', WID.world.

Pontusson, J. and Rueda, D. (2010). 'The politics of inequality: Voter mobilization and left parties in advanced industrial states', *Comparative Political Studies*, vol. 43(6), pp. 675–705, doi:10.1177/0010414009358672.

Robbett, A. and Matthews, P.H. (2018). 'Partisan bias and expressive voting', *Journal of Public Economics*, vol. 157(C), pp. 107–120.

Schmitt-Beck, R. (2015). *Bandwagon Effect*, American Cancer Society, ISBN 9781118541555, pp. 1–5, doi:10.1002/9781118541555.wbiepc015.

Shayo, M. and Harel, A. (2012). 'Non-consequentialist voting', *Journal of Economic Behavior & Organization*, vol. 81(1), pp. 299 – 313.

Simon, H.A. (1954). 'Bandwagon and underdog effects and the possibility of election predictions', *The Public Opinion Quarterly*, vol. 18(3), pp. 245–253.

Törnberg, P. (2018). 'Echo chambers and viral misinformation: Modeling fake news as complex contagion', *PLOS ONE*, vol. 13(9), pp. 1–21.

Train, K. (2009). *Discrete Choice Methods with Simulation*, Princeton: Cambridge University Press.

Winkler, H. (2019). 'The effect of income inequality on political polarization: Evidence from european regions, 2002–2014', *Economics & Politics*, vol. 31(2), pp. 137–162, doi:10.1111/ecpo.12129.

Wittman, D. (1983). 'Candidate motivation: A synthesis of alternative theories', *The American Political Science Review*, vol. 77(1), pp. 142–157.

Wlezien, C. (2004). 'Patterns of representation: Dynamics of public preferences and policy', *Journal of Politics*, vol. 66(1), pp. 1–24, doi:10.1046/j.1468-2508.2004.00139.x.

Appendix

Proof of Proposition 1

The proof of the first statement is trivial, we just need to set $x = 0$ and $g = g_0$ in (8) and (13), respectively.

In order to prove the other statements, let us calculate the Jacobian matrix.

$$\frac{\partial \dot{x}}{\partial x} = -1 - \frac{e^{(\beta_x+\sigma)x+g-g_0}(-1+e^{(\beta_x+\sigma)x+g-g_0})(\beta_x+\sigma)}{(1+e^{(\beta_x+\sigma)x+g-g_0})^2} + \frac{e^{(\beta_x+\sigma)x+g-g_0}(\beta_x+\sigma)}{1+e^{(\beta_x+\sigma)x+g-g_0}}$$

Hence $J_{11} = -1 + (\beta_x + \sigma)/2$.

$$\frac{\partial \dot{x}}{\partial g} = \frac{2e^{(\beta_x+\sigma)x+g+g_0}}{(e^{(\beta_x+\sigma)x+g_0} + e^{g_0})^2}$$

which gives $J_{21} = \frac{1}{2}$

$$\frac{\partial \dot{g}}{\partial x} = -g(1-g),$$

hence $J_{21} = g_0^2 - g_0$.

$$\frac{\partial \dot{g}}{\partial g} = -x(1-2g)$$

which means that $J_{22} = 0$.

Then the Jacobian matrix is,

$$J = \begin{bmatrix} -1 + (\beta_x + \sigma)/2 & 1/2 g_0^2 - g_0 & 0 \end{bmatrix} \tag{17}$$

$$Tr(J) = -1 + (\beta_x + \sigma)/2,$$

$$|J| = -(g_0^2 - g_0)/2 > 0,$$

Accordingly, for $\beta_x + \sigma < 2$ the sum of the eigenvalues is negative and their product is positive, hence both eigenvalues are negative.

For $\beta_x + \sigma = 2 \Rightarrow Tr(J) = 0$, confirming that the stationary equilibrium is a stable centre.

$\square$

Proof of Corollary 1

The discriminant of the Jacobian (17) at the centrist equilibrium is

$$\Delta = \left(\frac{\mu - 2}{2}\right)^2 + 2(g_0^2 - g_0),$$

which means that $\Delta < 0$ if and only if

$$(\mu - 2)^2 < 8(g_0 - g_0^2),$$

or if,

$$\mu < 2\left(1 + 2\sqrt{g_0 - g_0^2}\right).$$

Hence

$$\mu \in \left(-2\left(1 + 2\sqrt{g_0 - g_0^2}\right), 2\left(1 + 2\sqrt{g_0 - g_0^2}\right)\right)$$

$\square$

Proof of Proposition 2

(i) If $g = 0$ then $\dot{x} = 0$, if and only if

$$(1 - x)\frac{e^S}{e^S + e^{-S}} = (1 + x)\frac{e^{-S}}{e^S + e^{-S}}$$

or

$$e^{2[(\beta_x + \sigma)x - g_0]} = \frac{1 + x}{1 - x} \tag{18}$$

Define the following real valued function $F : (-1, 1) \to \mathbb{R}$, with

$$F(x) = e^{2[(\beta_x + \sigma)x - g_0]} - \frac{1 + x}{1 - x},$$

such that $F(x) = 0$ if and only if $\dot{x} = 0$. Thus, it is sufficient to show that there exists $x_0^1 \in (-1, 0)$ such that $F(x_0^1) = 0$. Note that

$$\lim_{x \to 1} F(x) = -\infty, \tag{19}$$

$$\lim_{x \to -1} F(x) > 0, \tag{20}$$

which means that given continuity there exists at least one $x_0^1 \in (-1, 1)$ such that $F(x^1) = 0$. The point will be unique if the derivative

$$F'(x) = -1 - \frac{e^{(\beta_x+\sigma)x-g_0}(-1 + e^{(\beta_x+\sigma)x-g_0})(\beta_x + \sigma)}{(1 + e^{(\beta_x+\sigma)x-g_0})^2} + \frac{e^{(\beta_x+\sigma)x-g_0}(\beta_x + \sigma)}{1 + e^{(\beta_x+\sigma)x-g_0}},$$

is negative, hence $F(x)$ strictly decreasing. $F'(x)$, can be expressed alternatively as

$$F'(x) = -1 + 2(\beta_x + \sigma)\frac{e^{(\beta_x+\sigma)x+g_0}}{(e^{(\beta_x+\sigma)x} + e^{g_0})^2}.$$

For $\beta_x + \sigma < 2$,

$$F'(x) < -1 + 4\frac{e^{(\beta_x+\sigma)x+g_0}}{(e^{(\beta_x+\sigma)x} + e^{g_0})^2},$$

but given that

$$4\frac{e^{(\beta_x+\sigma)x+g_0}}{(e^{(\beta_x+\sigma)x} + e^{g_0})^2} < 1,$$

as

$$4e^{(\beta_x+\sigma)x+g_0} - (e^{(\beta_x+\sigma)x} + e^{g_0})^2 = (e^{(\beta_x+\sigma)x} - e^{g_0})^2 > 0,$$

we get that

$$F'(x) < 0.$$

Hence, given continuity, there exists a single $x_0^1 \in (-1, 1)$ such that $F(x_0^1) = 0$. We also need to show that $x_0^1 \in (-1, 0)$. Note that, given that $F'(x) < 0$, in order to show that $x_0^1 \in (-1, 0)$, it is sufficient to show that $F(0) < 0$. Note that

$$F(0) = e^{-2g_0} - 1,$$

which is negative for all $g_0 > 0$.

(ii) If $g = 1$ then $\dot{x} = 0$, if and only if

$$e^{2[(\beta_x+\sigma)x+1-g_0]} = \frac{1+x}{1-x} \tag{21}$$

Following the same steps as in the previous part, we can show that

$$\Phi(x) = e^{2[(\beta_x+\sigma)x+1-g_0]} - \frac{1+x}{1-x},$$

is also strictly decreasing and given that $-g_0 + 1 > 0$, then $\Phi(x_1^1) = 0$ with $x_1^1 \in (0,1)$.

Stability

A necessary condition for the $\{x_0^1, 0\}$ equilibrium to be stable is the determinant of the Jacobian at that point to be positive.

Note that at $\{x_0^1, 0\}$,

$$J_{11} = -1 - \mu \left[\frac{e^{\mu x_0^1 - g_0}(-1 + e^{\mu x_0^1 - g_0})}{(1 + e^{\mu x_0^1 - g_0})^2} - \frac{e^{\mu x_0^1 - g_0}}{1 + e^{\mu x_0^1 - g_0}} \right],$$

or,

$$J_{11} = -1 - \mu e^{\mu x_0^1 - g_0} \left[\frac{-1 + e^{\mu x_0^1 - g_0}}{(1 + e^{\mu x_0^1 - g_0})^2} - \frac{1}{1 + e^{\mu x_0^1 - g_0}} \right]$$

or,

$$J_{11} = -1 - \mu e^{\mu x_0^1 - g_0} \left(\frac{-1 + e^{\mu x_0^1 - g_0}}{(1 + e^{\mu x_0^1 - g_0})^2} - \frac{1 + e^{\mu x_0^1 - g_0}}{(1 + e^{\mu x_0^1 - g_0})^2} \right)$$

or,

$$J_{11} = -1 + \frac{2\mu e^{\mu x_0^1 - g_0}}{(1 + e^{\mu x_0^1 - g_0})^2}$$

$$J_{12} = \frac{2e^{\mu x_0^1 + g_0}}{(e^{\mu x_0^1 + g_0} + e^{g_0})^2}$$

$$J_{21} = 0,$$

$$J_{22} = -x_0^1.$$

Note that the following always holds

$$\frac{2e^{\mu x_0^1 - g_0}}{(1 + e^{\mu x_0^1 - g_0})^2} < \frac{1}{2},$$

which means that for $\mu < 2$,

$$\frac{2e^{\mu x_0^1 - g_0}}{(1 + e^{\mu x_0^1 - g_0})^2} < \frac{1}{\mu},$$

hence $J_{11} < 0$. But given that $x_0^1 < 0$, $J_{22} > 0$ hence

$$Det = x_0^1 \left[-1 + \frac{2\mu e^{\mu x_0^1 - g_0}}{(1 + e^{\mu x_0^1 - g_0})^2} \right] > 0$$

proving that the stationary equilibrium is a saddle point.

Similarly for the $\{x_1^1, 1\}$ equilibrium, the elements of the Jacobian are as follows

$$J_{11} = -1 - \frac{e^{\mu x + 1 - g_0}(-1 + e^{\mu x + 1 - g_0})\mu}{(1 + e^{\mu x + 1 - g_0})^2} + \frac{e^{\mu x + 1 - g_0}\mu}{1 + e^{\mu x + 1 - g_0}}$$

or,

$$J_{11} = -1 + \frac{2\mu e^{\mu x_1^1 + 1 - g_0}}{(1 + e^{\mu x_1^1 + 1 - g_0})^2}$$

$$J_{12} = \frac{2e^{\mu x_0^1 + 1 + g_0}}{(e^{\mu x_0^1 + 1 + g_0} + e^{g_0})^2}$$

$$J_{21} = 0,$$

$$J_{22} = x_1^1.$$

Note that given that $J_{22} > 0$, as in the previous case, it is trivial to prove that $\{x_1^1, 1\}$ is also a saddle equilibrium.

Proof of Proposition 3

The proof follows the first part of Proposition 2, but for $\mu > 2$. It is sufficient to show that there exists a local maximum of $F(x)$, which is increasing in μ and that for some values of μ, this is positive while for others this is negative. Substituting μ, $F(x)$ is

$$F(x) = e^{2(\mu x - g_0)} - \frac{1+x}{1-x},$$

and also

$$F'(x) = -1 + 2\mu \frac{e^{\mu x + g_0}}{(e^{\mu x} + e^{g_0})^2}.$$

$F'(x) = 0$ for

$$x_a = \frac{g_0 - ln[\mu - 1 - \sqrt{(\mu-2)\mu}]}{\mu}$$

or

$$x_b = \frac{g_0 - ln[\mu - 1 + \sqrt{(\mu-2)\mu}]}{\mu}$$

Note that

$$F''(x) = \frac{2e^{\mu x + g_0}(e^{g_0} - e^{\mu x})\mu^2}{(e^{g_0} + e^{\mu x})^3}$$

then for x_a to be a local max, the following should hold

$$F''(x_a) = -\frac{2\mu^2\left[1 - \mu + \sqrt{(\mu-2)\mu}\right]\left[2 - \mu + \sqrt{(\mu-2)\mu}\right]}{\left[-\mu + \sqrt{(\mu-2)\mu}\right]^3} < 0.$$

Note that $-\mu+\sqrt{(\mu-2)\mu} < 1-\mu+\sqrt{(\mu-2)\mu} < 0$ and $2-\mu+\sqrt{(\mu-2)\mu} > 0$. Hence

$$F''(x_a) < 0.$$

Substituting x_a to F, gives,

$$F(x_a) = \frac{-g_0 + \sqrt{(\mu-2)\mu} + ln[\mu - 1 - \sqrt{(\mu-2)\mu}]}{\mu}. \tag{22}$$

Then

$$\frac{\partial F(x_a)}{\partial \mu} = \frac{g_0 - ln[\mu - 1 - \sqrt{(\mu-2)\mu}]}{\mu^2}.$$

Note that $\mu - 1 - \sqrt{(\mu-2)\mu} < 1$ as

$$(\mu-2)^2 < (\mu-2)\mu$$

and

$$\mu - 2 < \mu.$$

Hence $\frac{\partial F(x_a)}{\partial \mu} > 0$.

In order to complete the proof of parts (i) and (ii) it is sufficient to show that there exists $\mu = \bar{\mu}(g_0)$ for which

$$F(x_a) = 0$$

For the above to hold the following should hold

$$\sqrt{(\mu-2)\mu} + ln[\mu - 1 - \sqrt{(\mu-2)\mu}] = g_0 \tag{23}$$

Note that the derivative of the numerator of (22) is $\frac{\sqrt{(\mu-2)\mu}}{\mu} > 0$ and that for $\mu = 2$,

$$F(x_a) = \frac{-g_0}{2} < 0$$

while for $\mu = 3$, $F(x_a) > 0$. Hence there exists $\bar{\mu}(g_0)$ which is the solution of (23) such that (i) and (ii) are true.

Note that we can express $\Phi(x)$ as

$$\Phi(x) = e^{2\mu x + 1 - g_0]} - \frac{1+x}{1-x}$$

hence

$$\Phi'(x) = -\frac{e^{2g_0} + e^{2(1+\mu x)} - 2(\mu - 1)e^{1+g_0+\mu x}}{(e^g + e^{1+\mu x})^2}$$

$\Phi'(x) = 0$, for

$$x_c = -\frac{1 - g_0 + ln[\mu - 1 - \sqrt{(\mu - 2)\mu}]}{\mu}$$

or

$$x_d = -\frac{1 - g_0 + ln[\mu - 1 + \sqrt{(\mu - 2)\mu}]}{\mu}.$$

Note that

$$\Phi''(x) = \frac{2e^{1+g_0+\mu x}(e^{g_0}e^{1+\mu x})\mu^2}{(e^{g_0}e^{1+\mu x})^3}$$

which gives

$$\Phi''(x_c) = -\frac{2\mu^2\left[1 - \mu + \sqrt{(\mu - 2)\mu}\right]\left[2 - \mu + \sqrt{(\mu - 2)\mu}\right]}{\left[-\mu + \sqrt{(\mu - 2)\mu}\right]^3},$$

and

$$\Phi''(x_d) = \frac{2\mu^2\left[\mu - 1 + \sqrt{(\mu - 2)\mu}\right]\left[\mu - 2 + \sqrt{(\mu - 2)\mu}\right]}{\left[\mu + \sqrt{(\mu - 2)\mu}\right]^3}.$$

Note that $-\mu+\sqrt{(\mu - 2)\mu} < 1-\mu+\sqrt{(\mu - 2)\mu} < 0$ and $2-\mu+\sqrt{(\mu - 2)\mu} > 0$. Hence $\Phi''(x_c) < 0$ and $\Phi''(x_d) > 0$, which means that at x_c there is a local minimum of $\Phi(x)$. Substituting x_d, we get

$$\Phi(x_d) = \frac{1 - g_0 - \sqrt{(\mu - 2)\mu} + ln[\mu - 1 - \sqrt{(\mu - 2)\mu}]}{\mu}$$

with

$$\frac{\partial\Phi(x_d)}{\partial\mu} = \frac{g_0 - 1 - ln[\mu - 1 + \sqrt{(\mu - 2)\mu}]}{\mu^2}.$$

Given that $\mu > 2$, we get that $\frac{\partial\Phi(x_d)}{\partial\mu} < 0$. Hence in order to complete the proof it is sufficient to show that there exists $\hat{\mu}(g_0)$ such that for $\mu = \hat{\mu}(g_0)$, $\Phi(x_d) = 0$.

Note that for $\mu = 2$,

$$\Phi(x_d) = \frac{1 - g_0}{2} > 0$$

while for $\mu = 3$, $\Phi(x_d) < 0$.

$\square$

Proof of Proposition 4

As shown in the proof of Proposition 3, $\bar{\mu}(g_0)$ and $\hat{\mu}(g_0)$ are the solutions of $F(x_a) = 0$ and $\Phi(x_d) = 0$, respectively. Given that $\mu > 2$ these can be alternatively expressed as

$$g_0 = \sqrt{(\mu - 2)\mu} + ln[\mu - 1 - \sqrt{(\mu - 2)\mu}], \tag{24}$$

and

$$g_0 = 1 - \sqrt{(\mu - 2)\mu} + ln[\mu - 1 - \sqrt{(\mu - 2)\mu}]. \tag{25}$$

For 24

$$\frac{dg_0}{d\mu} = \frac{\sqrt{(\mu - 2)\mu}}{\mu} > 0.$$

Using the inverse function rule, this implies that for $\mu = \bar{\mu}(g_0)$, $\frac{\partial \bar{\mu}(g_0)}{\partial g_0} > 0$.For 25

$$\frac{dg_0}{d\mu} = -\frac{\sqrt{(\mu - 2)\mu}}{\mu} < 0,$$

which means that $\frac{\partial \hat{\mu}(g_0)}{\partial g_0} < 0$. Hence the first part of the proposition is proven.

Note that for $g_0 = \frac{1}{2}$, $F(x_a) = -\Phi(x_d)$, which proves the second part of the proposition.

$\square$

Figures

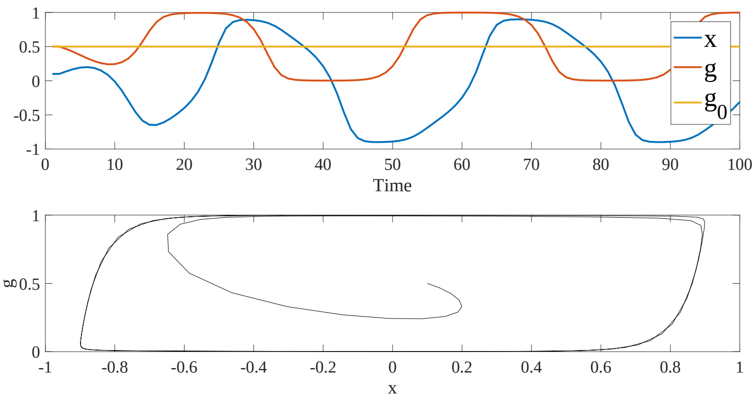

Figure 1: Single run of the model with exogenous income. Parameter set: $\beta_x = 2.5; \beta_r = .5; g_0 = 0.5, \sigma = 0.1$.

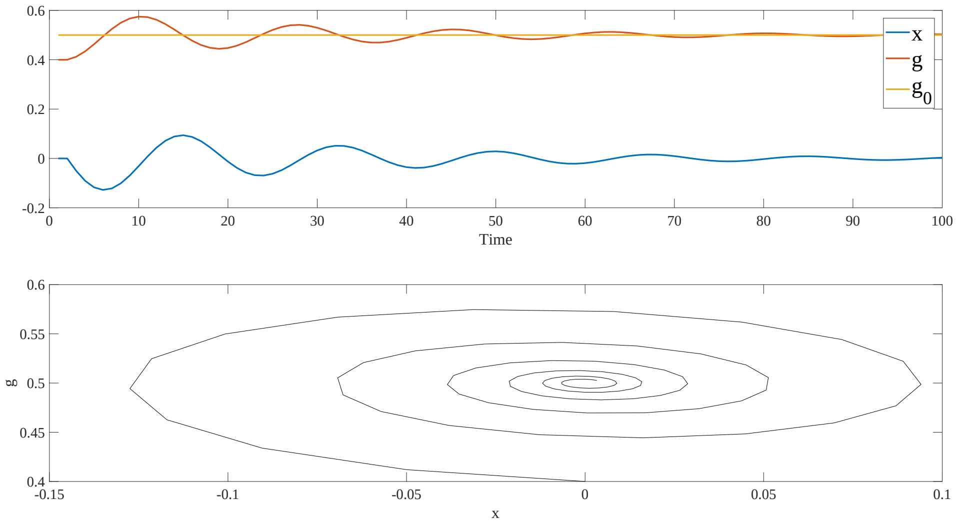

Figure 2: Single run of the model with exogenous income. Parameter set: $\beta_x = 1.7; \beta_r = .5; g_0 = 0.5, \sigma = 0.1.$

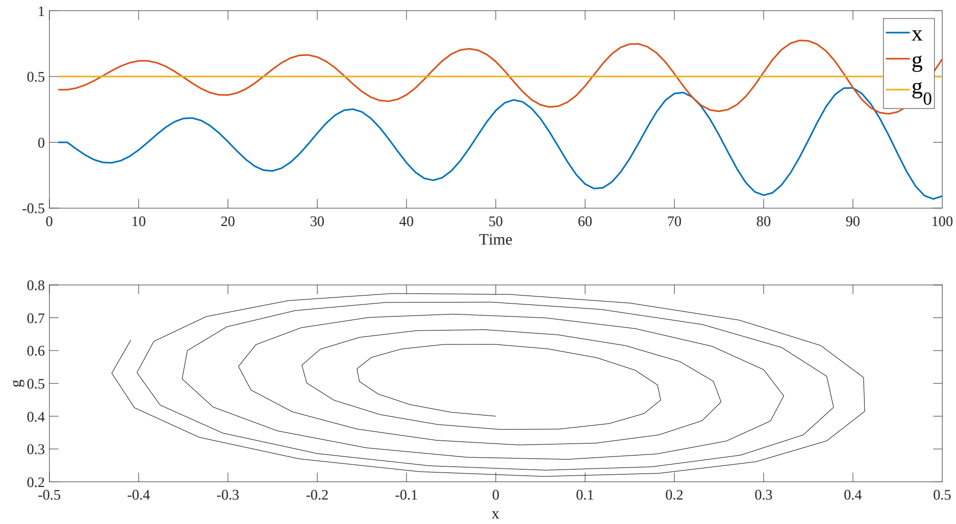

Figure 3: Single run of the model with exogenous income. Parameter set: $\beta_x = 1.9; \beta_r = .5; g_0 = 0.5, \sigma = 0.1.$

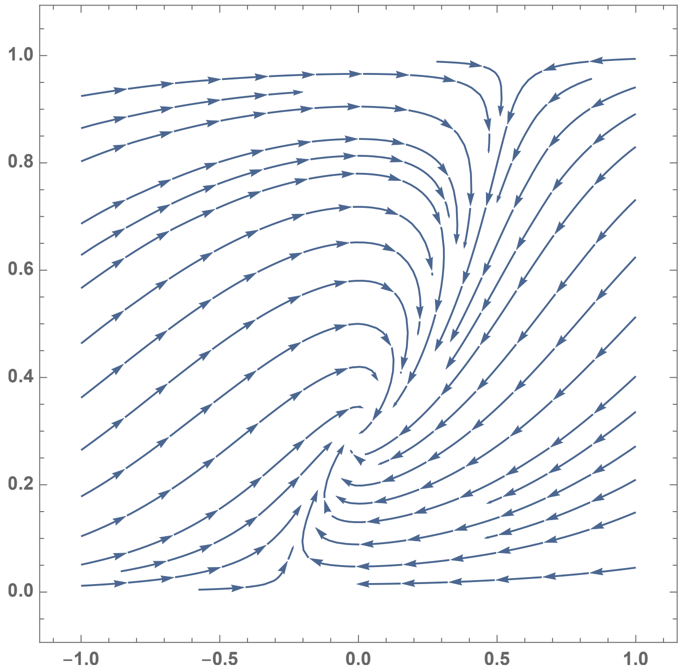

Figure 4: Phase diagram for x, g with $g_0 = 0.3$, $\mu = 1$

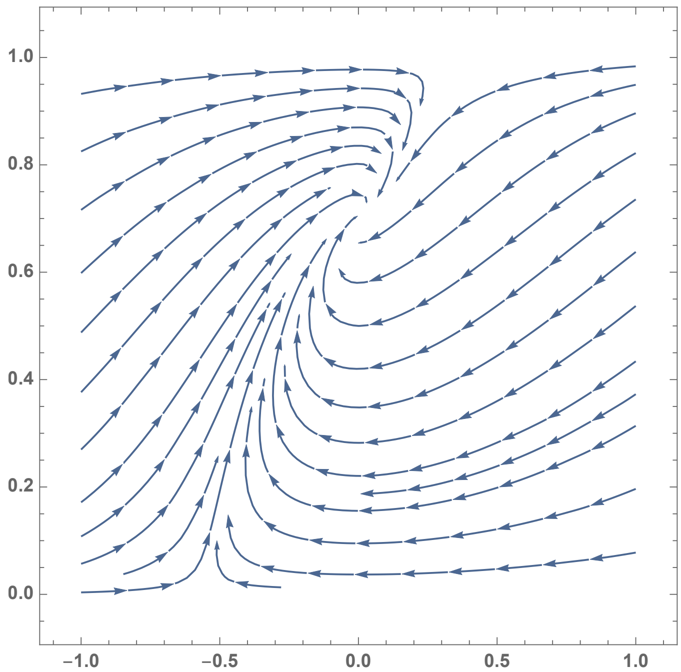

Figure 5: Phase diagram for x, g with $g_0 = 0.7$, $\mu = 1$

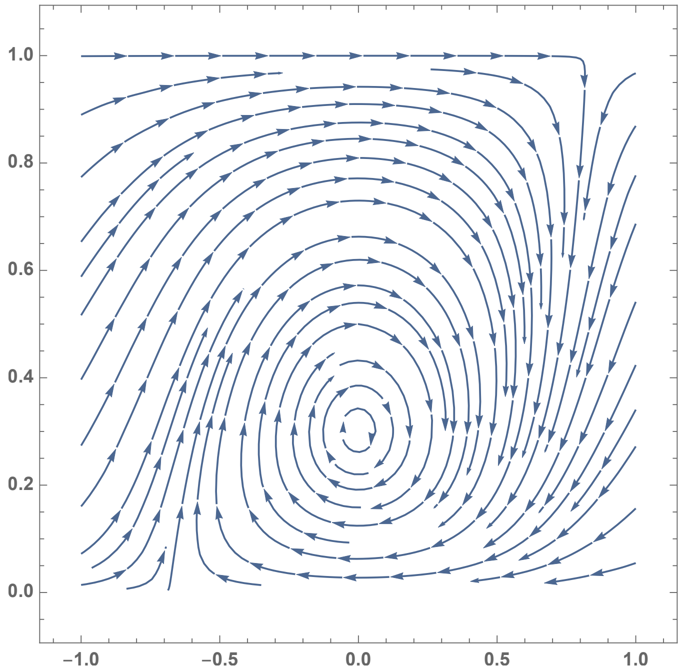

Figure 6: Phase diagram for x, g with $g_0 = 0.3$, $\mu = 2$

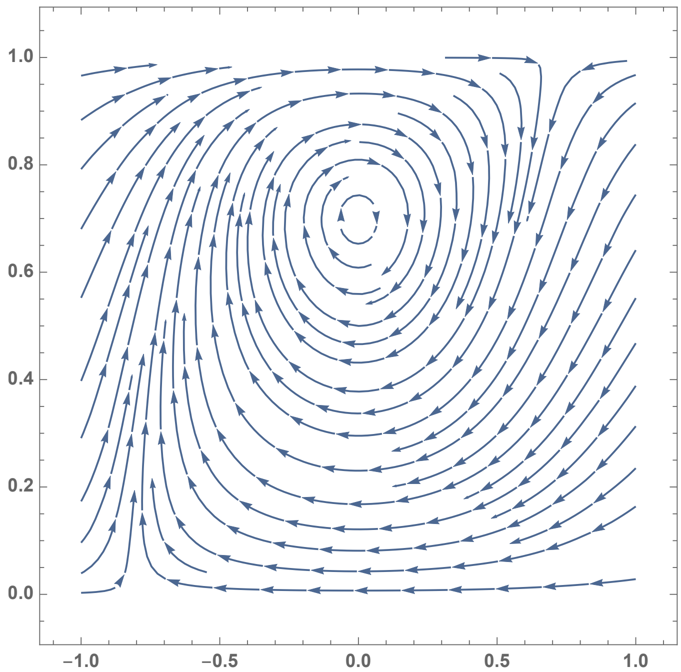

Figure 7: Phase diagram for x, g with $g_0 = 0.7$, $\mu = 2$

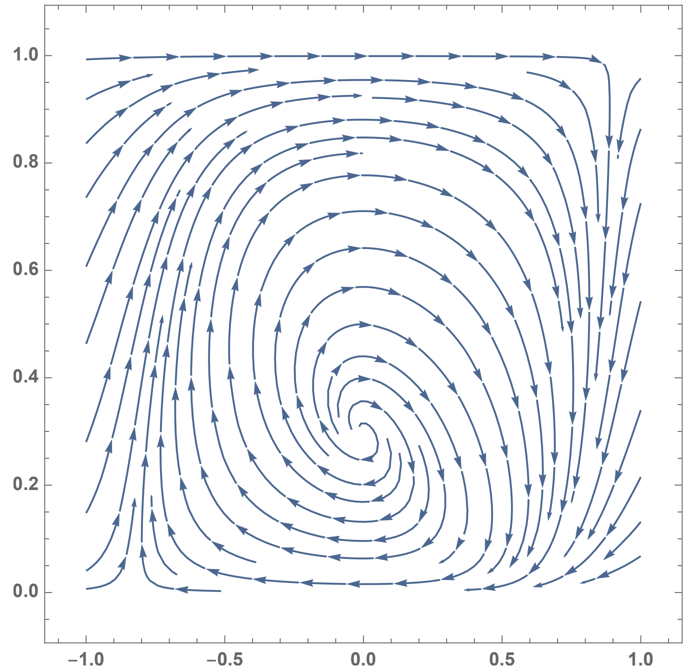

Figure 8: Phase diagram for x, g with $g_0 = 0.3$, $\mu = 2.5$

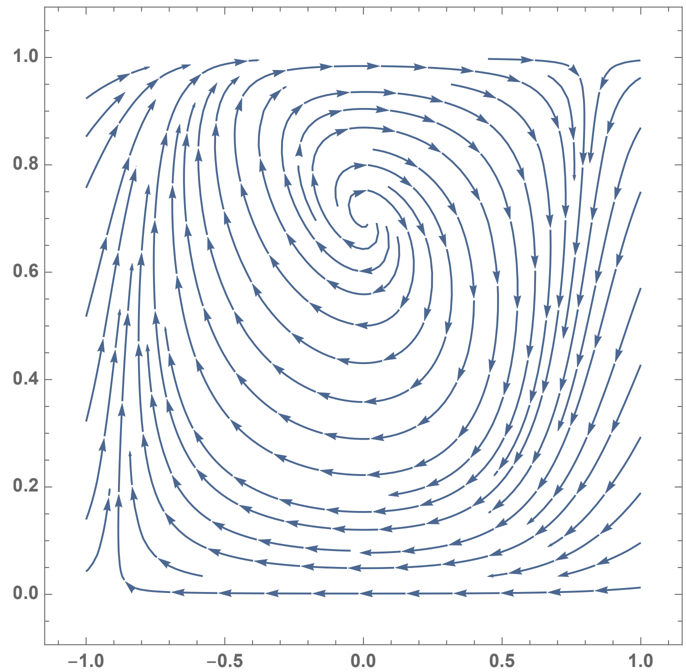

Figure 9: Phase diagram for x, g with $g_0 = 0.7$, $\mu = 2.5$

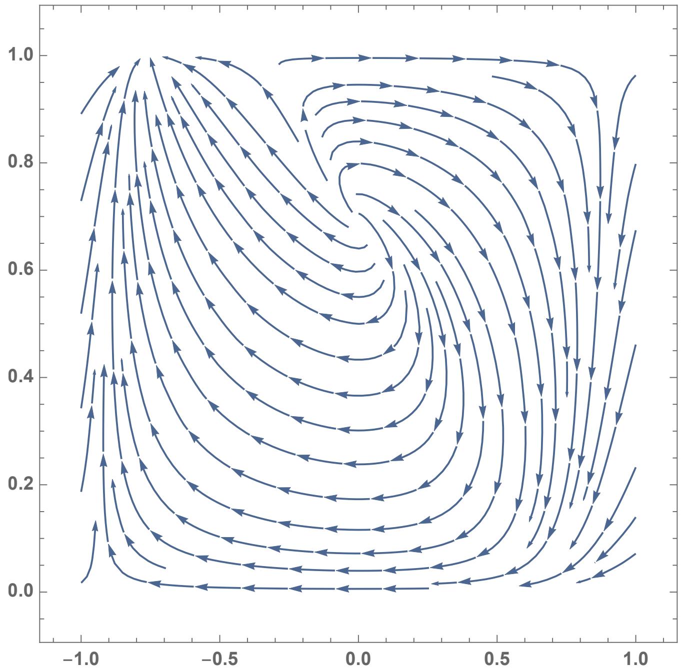

Figure 10: Phase diagram for x, g with $g_0 = 0.7$, $\mu = 3$

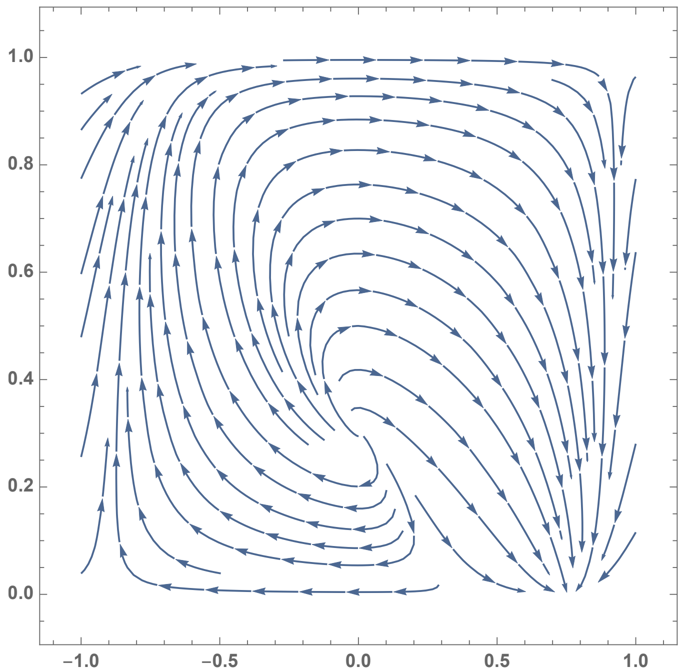

Figure 11: Phase diagram for x, g with $g_0 = 0.3$, $\mu = 3$

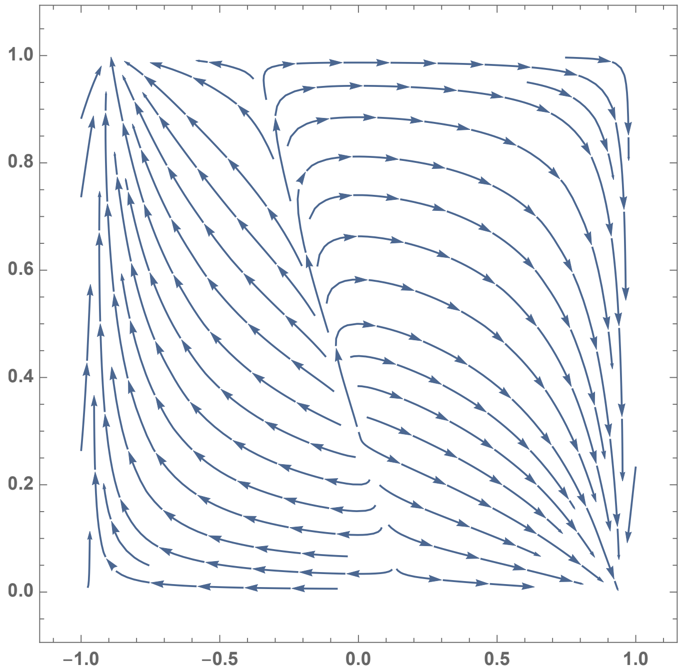

Figure 12: Phase diagram for x, g with $g_0 = 0.3$, $\mu = 4$

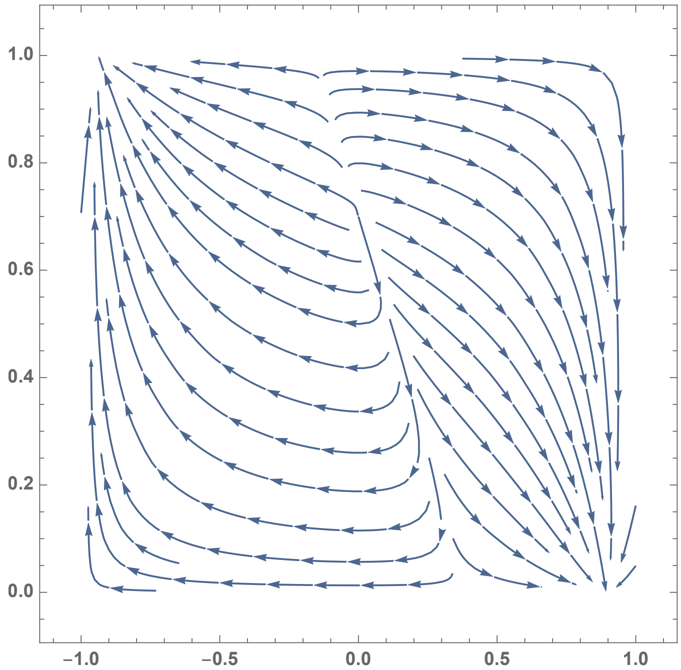

Figure 13: Phase diagram for x, g with $g_0 = 0.7$, $\mu = 4$

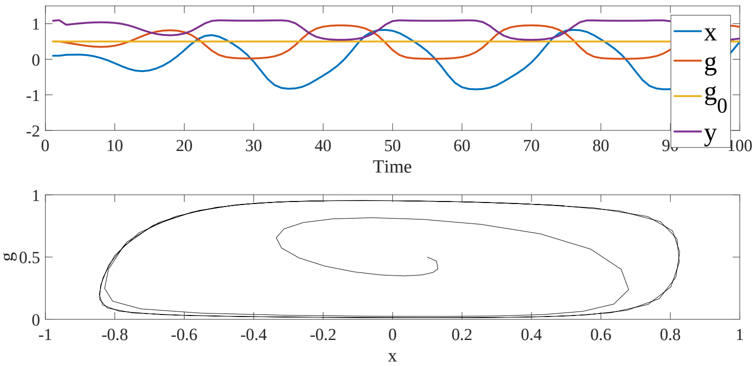

Figure 14: Single run of the model with endogenous income. Parameter set: $\beta_x = 2.5; \beta_y = 1.5; \beta_r = .5; g_0 = 0.5; \gamma_1 = -0.0001; \gamma_2 = 0.0024; \gamma_3 = .077.$

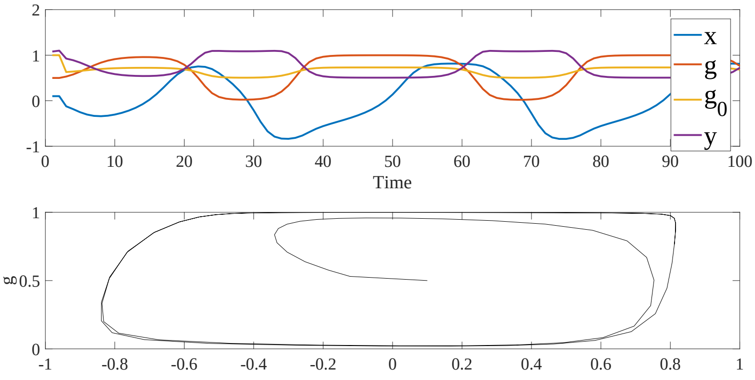

Figure 15: Single run of the model with endogenous income and endogenous g_0. Parameter set: $\beta_x = 2.5; \beta_y = 1.5; \beta_r = .5; G_0 = 0.5; \alpha = 3; \gamma_1 = -0.0001; \gamma_2 = 0.0024; \gamma_3 = .077.$