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Incoherent Preferences

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Incoherent Preferences*

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Abstract

Under Bruno De Finetti's coherence theory of additive probability, the expected value of a sequence of mutually exclusive bets should not expose the bettor to certain loss for any of the bets in the sequence (i.e. no formation of Dutch books). However, decision makers (DMs) are known to have non-additive probability preferences represented in the frequency domain. This conundrum of choice implies that DMs are incoherent. If so, then preference reversal (PR) is more likely to occur. That is, DMs response to choice and valuation procedures (with similar expected value) are more likely to be dissimilar or their preferences may appear to be intransitive. We prove that even when the true states of choice experiments are procedure invariance *and* transitive preferences, PR will still be observed because of: (1) phase incoherence between paired gambles with the same expected value–when probability cycles are incomplete, and (2) experimenter interference in probability measurement. We introduce a utility coherence ratio for paired gambles, and estimates from simulated phase transition from incoherent states to coherent states in binary choice to illustrate the theory. We find that coherence measures are very sensitive to measurement error, coherent states have higher frequency phase transition, and incoherent states represent momentary lapse in judgment that eventually disappear. So, Dutch books and PR are prevented.

Keywords: preference reversal; transitivity axiom; probability phase incoherence; wavelets; probability weighting

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"The gamble has been to decision research what the fruit fly has been to biology - a vehicle for examining fundamental processes with presumably important implications outside the laboratory". Paul Slovic.

1 Introduction

De Finetti (1937, pp. 103-104) asserted the following proposition for choice coherence in a "theorem of total probability". Let $E_1, \ldots, E_n$ be a complete class of mutually exclusive events, only one of which occurs; and $p_1, \ldots, p_n$ be their probability of occurrence evaluated by a decision maker (DM). If one fixes the stake S_i, $i = 1, \ldots, n$ which can be positive or negative, in correspondence with events E_i, then one has a sequence of bets (p_i, S_i), $i = 1, \ldots, n$. Let the gain G_h on a given stake S_h (assumed won) be the difference between the stake of the bet won and the expected value of the n outlays. So that, $G_h = S_h - \sum_{i=1}^{n} p_i S_i$. A necessary and sufficient condition for decision maker (DM) coherence is $\sum_{h=1}^{n} p_h G_h = 0$ with total probability $\sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0 \ \forall i$.

It is known for a long time that DMs evaluate p_i nonlinearly with a probability weighting function $w(p_i)$ such that $\sum_{i=1}^{n} w(p_i) \neq 1$. See, e.g. Preston and Baretta (1948); Mosteller and Nogee (1951); Allais (1953); Kahneman and Tversky (1979).[1] In particular, non additive probability in the frequency domain[2] poses a conundrum of choice because it gives the appearance that DMs are incoherent. This paper's contribution to the literature is a model of rational incoherence in which sources of seeming incoherence stems from incomplete probability cycles,[3] experimenter interference with probability measures,[4] and DMs return to coherent states for countably many complete probability cycles. In that framework, DM's seeming incoherence is a temporary state that vanishes over the domain of nonlinear probability. So, preference reversal and Dutch books are prevented. To the best of our knowledge, this is the first paper to resolve seemingly incoherent

[1] See also, Vicig and Seidenfeld (2012, p. 1116) whom credit Part IV of Bernoulli's *Ars Conjectandi* with broaching the issue of non-additive probability as far back as 1713.

[2] See Haven et al. (2018).

[3] We define a probability cycle as the greatest common divisor of the index set for the probabilities associated with a recurrent state. Karlin and Taylor (1981, §10.4) provide several examples on the relationship between probability cycles and harmonic representation of transition probabilities for random walks. In contrast, imprecise probability is typically modelled as the probability induced by a deterministic preference based function with a stochastic addend. See e.g. Loomes and Sugden (1995); Butler and Loomes (2007). In this paper, a harmonic addend is used to characterize recurrent coherent mental states. The corresponding probabilities and behavioural Chapman-Kolmogorov equations are presented in Theorem 3.2, *infra*.

[4] See Karni and Safra (1987, p. 679) for earlier work, and Stewart et al. (2015) for more recent work on experimenter effects.

preferences and preference reversal phenomena with incomplete probability cycles.[5] We introduce a utility coherence ratio (UCR), and show how it characterizes preference reversal and incoherence between lottery pairs. We also show that rank dependent utility (RDU) transformation of $w(\bullet)$ into linear decision weights π_i such that $\sum_{i=1}^{n} \pi_i = 1$ (Quiggin, 1982, 1993) also fails to solve the incoherence puzzle for a certain class of probability weighting functions almost surely.

Preference reversal (PR) is based on differences in the choice ordering and price ordering in lottery pairs (Schmidt and Hey, 2004). That is, "people who choose gamble A over [gamble] B often ask for less money to sell A than B." (see, e.g. Goldstein and Einhorn, 1987, pp. 236-237). This implies DMs may be incoherent within or between the "paired gambles". Three popular explanations of the PR phenomenon are: (1) intransitive preferences,[6] (2) violation of procedure invariance, i.e., subjects response vary according to which preference elicitation methods is used, and (3) imprecise probabilistic preferences. Refer to Loomes and Pogrebna (2016) for a review of this literature.

We rationalize the PR phenomenon in an abstract temporally spaced repeated experiment,[7] with probabilistic choice characterized by a weak harmonic transitivity axiom (WHTA) dual to the weak stochastic transitivity axiom (WSTA).[8] This allows us to rationalize source(s) of imprecise probability (Walley, 1991; Butler and Loomes, 2007) in the frequency domain. In a sense, we provide a mathematical model that explains why preference reversal or intransitive preferences vanish over repeated play (Lopes, 1996). Since procedure invariance and violation of the transitivity axiom are often touted as sources of preference reversal (Arkes et al., 2016) we eliminate those sources by imposing the assumption of procedure invariance, and DMs non-violation of the

[5] Rubinstein and Segal (2012, p. 2485) used a "nontransitive dice" approach to estimate the maximal probability that a random sampling procedure yields a nontransitive cycle as approximately $\frac{8}{27}$.

[6] In the transitivity context, PR implies that given a binary preference relation $\succeq$ (where $\succ$ means strictly preferred and $\sim$ means indifferent to), a set of outcomes x, y, z such that a DM expresses $x \succ y$ and $y \succ z$, if she is prepared to pay a fee ε to acquire x, and if her elicited preferences are $x \succ y \succ z \succ x$, then that DM is vulnerable to a money pump or Dutch book that extracts an amount ε from her at the end of each cycle (Fishburn, 1988, p. 43).

[7] Camerer (1989, p. 50) reports that 31.6% of subjects reversed their preferences in a temporally spaced replication of the experiment, and 26.5% did likewise in Starmer and Sugden (1989). Arkes et al. (2016, p. 23) survey paper also found that preference reversal is drastically reduced after repeated rounds of the same experiment. Compare, Schmidt and Hey (2004) who repeated an experiment five (5) times with the same subjects and still found PR albeit at a reduced level.

[8] Let $\succ$ be a preference order relation and $\{a, b, c\}$. WSTA implies that if $\Pr(a \succ b) \geq 0.5$ and $\Pr(b \succ c) \geq 0.5$, then $\Pr(a \succ c) \geq 0.5$. WSTA is motivated by Tversky (1969, p. 31) who surmised "*that the observed inconsistencies reflect inherent variability or momentary fluctuation in the evaluative process. This consideration suggests that preference should be defined in a probabilistic fashion.*" (emphasis added). The WHTA axiom extends WSTA to the class of "anchor-adjustment" models introduced in a series of papers by Einhorn and Hogarth (1985, 1986); Hogarth and Einhorn (1990, 1992).

transitivity axiom on the abstract experiment. So, the only possible source of PR in our model is nonlinear probability. DMs are assumed to have rank dependent utility (RDU) preferences because it admits transformation of nonlinear pwfs into *additive* linear decision weights (Quiggin, 1982, 1993) that support De Finetti's (1937) coherence proposition.

The coherence structure function in this paper is related to the concept of coherence between two time series (e.g. Brillinger, 2001, p. 302-303), and in particular wavelet squared coherence (Mallat, 1999; Lachaux et al., 2002; Aguiar-Conraria et al., 2012) between the phase functions that characterize probability weighting in the frequency domain. Section C in the Appendix provides a very brief review of wavelet concepts used in this paper. Wavelet analysis is well suited here because they are locally compact bump functions that control jumps and admit transitory probabilistic behaviour. We introduce a utility coherence ratio (UCR) that constitute jumps in incoherence. This is distinguished from Buehler (1976, p. 1053) who referenced De Finetti (1970) to posit that a set of preferences will be called PR-incoherent if there exist a subset whose simultaneous reversal gives a higher utility. In our RDU framework, higher expected utility arises from phase incoherence when probability cycles are broken. In fact, simulation of our model shows that imprecise probabilities have significant effects on (in)coherence measurement.

Loomes and Pogrebna (2016) conducted experiments with a model that admits WSTA.[9] They elicited CEs from a set of candidate CEs for given bets. Thus, subjects underlying preferences were modeled as a probability distribution over the set of CEs. Loomes and Pogrebna find that "when certainty equivalent values are inferred from repeated binary choices, the classic PR phenomenon largely disappears". This is functionally equivalent to subjects eventually completing their probability cycles for recurrent coherent states in a repeated binary choice context over time.[10]

The rest of the paper proceeds as follows. In Section 2 we introduce the WHTA axiom. The main result there is the an abstract probability weighting function (HPWF) in the frequency domain

[9]Let $\succ$ be a preference order relation and $\{a,b,c\}$. WSTA implies that if $\Pr(a \succ b) \geq 0.5$ and $\Pr(b \succ c) \geq 0.5$, then $\Pr(a \succ c) \geq 0.5$.

[10]In cognitive science, Jerome Busemeyer and his colleagues resolve the preference reversal puzzle in the context of quantum information processing which employs quantum probability tools borrowed from quantum mechanics (Busemeyer and Diederich, 2002; Busemeyer and Wang, 2007; Pothos and Busemeyer, 2009; Busemeyer et al., 2011). The lynchpin in many of those models is Born's rule which is based on the squared amplitude of a complex valued wave function in Hilbert space (see e.g. Basieva et al., 2018, §4 Probabilities-and-phases).

presented in Theorem 2.2. In section 3 we provide the main empirical specification for the HPWF with Theorem 3.1. We show how this extends to a behavioral Chapman-Kolmogorov equation that characterize the probabilities that correspond to recurrent coherent states in Theorem 3.2. And resolve the preference reversal puzzle and experimenter interference issue in Section 3.2 and Section 3.3. In Section 4 we conclude the paper.

2 A weak harmonic transitivity axiom for HPWF

In the sequel we use the usual binary preference relation $\succ$ to mean strictly preferred to, $\prec$ to mean strictly less preferred to, and $\sim$ to mean indifferent to or equivalent to. The relation $\succeq$ and $\preceq$ are weaker versions of the strict preference order that admit the possibility of indifference. We use the disjunction $\vee$ to mean "or" and conjunction $\wedge$ to mean "and". We assume that there exists a separable space X and binary preference relations on X. The next three axioms are standard fare that characterize EUT in the Von Neumann and Morgenstern (1953) framework.

Axiom 2.1 (**Completeness**)**.** For given $A,B \in X$, either $A \succeq B$ or $B \succeq A$ or both, i.e., $A \sim B$.

Axiom 2.2 (**Transitivity**)**.** For given $A,B,C \in X$, if $A \succ B$ and $B \succ C$, then $A \succ C$.

Axiom 2.3 (**Independence**)**.** For given $A,B,C \in X$ and $\alpha \in (0,1]$, if $A \succeq B$, then $\alpha A + (1-\alpha)C \succeq \alpha B + (1-\alpha)C$

In what follows we reproduce the pertinent parts of Charles-Cadogan (2018) for the benefit of the reader. Consider a DM who expresses the following *binary choice* preference for objects $\{A,B,C\}$. That is, we observe the choices $\{A \succ B\}$ on one occasion, and $\{B \succ C\}$ on another occasion. The dictates of logic tell us that when presented with a binary choice between A and C on another occasion, our DM would choose $\{A \succ C\}$. This is an application of the strong transitivity axiom. However, $\{A \succ C\}$ is the *conjunctive event* $\{A \succ B\} \wedge \{B \succ C\}$. That is, the event $\{A \succ C\}$ is a theoretically *constructed preference* based on the conjunction of events $\{A \succ B\}$ and $\{B \succ C\}$. In probabilistic terms, given a probability measure P defined on the binary choice event(s), and independence of the binary choices, these imply the constructed preference

$$P\{A \succ C\} = P\{\{A \succ B\} \wedge \{B \succ C\}\} = P\{A \succ B\}P\{B \succ C\} \tag{2.1}$$

However, the elementary laws of probability in concert with the independence Axiom 2.3 tell us that (2.1) has the following implications:

$$P\{A \succ C\} = P\{A \succ B\}P\{B \succ C\} \Rightarrow P\{A \succ C\} < P\{A \succ B\} \text{ and } P\{A \succ C\} < P\{B \succ C\} \tag{2.2}$$

$$\text{Thus, } P\{A \succ C\} < \min\{P\{A \succ B\}, P\{B \succ C\}\} \tag{2.3}$$

There are three cases posed by the constructed preference in (2.3):

Case 1 $P\{A \succ B\} < P\{B \succ C\}$. This probabilistic relationship implies that an observer would expect to see more $\{B \succ C\}$ choices than $\{A \succ B\}$ choices.

Case 2 $P\{A \succ B\} > P\{B \succ C\}$. This probabilistic relationship implies that an observer would expect to see less $\{B \succ C\}$ choices than $\{A \succ B\}$ choices.

Case 3 $P\{A \succ B\} = P\{B \succ C\}$. This probabilistic relationship implies that an observer expects to see the same proportion of choices for $\{A \succ B\}$ and $\{A \succ B\}$.

Let $\varepsilon > 0$ and $P\{B \succ C\} = p$ such that in **Case 1** we have $P\{A \succ B\} = p + \varepsilon$, and without loss of generality in **Case 2** we have $P\{A \succ B\} = p - \varepsilon$.[11] The probability equipoise in **Case 3** is consistent with a weak transitivity axiom. That is, $P\{A \succ B\} \sim P\{B \succ C\} \Rightarrow P\{A \succ C\}$ according to independence Axiom 2.3 (with irrelevant alternatives). This coincides with the special case $\varepsilon = 0$. In which case $P\{A \succ C\} = p^2$ in (2.1). This suggests that the transitivity axiom Axiom 2.2 may be too strong for independent binary choices since it does not permit a margin of error. For the

[11] Costello and Watts (2014) also employ "symmetric error" in their "probability theory plus noise" model. Bayrak and Hey (2017) argue that DMs have imprecise probability judgments represented by a range of probabilities symmetrically distributed around the observed probability. Bikhchandani and Segal (2020) consider transitivity for outcomes that are near and far apart.

very act of independent binary choice induces the theoretically constructed probabilistic preference

$$P\{A \succ C\} = P\{\{A \succ B\} \wedge \{B \succ C\}\} = P\{A \succ B\}P\{B \succ C\} \tag{2.4}$$

$$= (p \pm \varepsilon)p = p^2 \pm \varepsilon p = P\{A \succ C\} \pm \varepsilon p \text{ (recall } P\{A \succ C\} = p^2 \text{ in (2.1))} \tag{2.5}$$

This implies that $\lim_{\varepsilon \to 0} (P\{A \succ C\} \pm \varepsilon p) = P\{A \succ C\}$ for weak transitivity to hold. That is, the *oscillating addend* must disappear for weak transitivity to hold.[12]

2.1 Anchor-adjustment process and weak transitivity

Hogarth and Einhorn (1990) described a quantity similar to the addend $\pm\varepsilon p$ in (2.5) as an "anchor-adjustment process" which they liken to *mental simulation*. Cubitt et al. (2004) also report symmetry around probability equivalents in their experiments involving, *inter alia*, probability evaluation tasks. Anchor-adjustment type processes were also observed in neuroscience experiments in Kalenscher et al. (2010). More on point, our theory implies that the anchor-adjustment process for subjective probability vanishes over time as subjects learn about the underlying event and arrive at its true probability.[13] Thus, (2.5) implies the existence of a vanishing or periodic event, call it φ, such that $P\{\{A \succ C\} \oplus \varphi\} = P\{A \succ C\}$ where $\oplus$ is a "conjoint" or attachment operation. The notion of "conjoined space" is consistent with that in Krantz et al. (1971, Ch. 6). That is, the binary choice space $E \times E$ (where $A, B, C \in E$), and φ, cannot be readily concatenated but they can be treated as *composite objects* that preserve transitivity. Alternatively we can think of $\oplus$ as a joint receipt operation for binary choice $\{A \succ C\}$ and its accompanying mental state φ in accord with Hogarth and Einhorn (1990) mental simulation hypothesis. Thus we derive the following

Axiom 2.4 (Weak Harmonic Transitivity Axiom). If $A \succ B$ and $B \succ C$, then $P\{\{A \succ C\} \oplus \varphi\} \sim P\{A \succ C\}$ almost surely for some periodic event φ and attachment operation $\oplus$.

[12] In their QPT model, Yukalov and Sornette (2015, p. 3) refers to objects like εp as "attraction factors" that satisfy an "alternating property" $-\varepsilon p + \varepsilon p = 0$.

[13] There is a very large literature, outside the scope of this paper, on how subjects learn. The interested reader is referred to **??**. Suffice to say that MacCrimmon (1968, pp. 14-15) and Kalenscher et al. (Suppl. pp. 2-3 2010) conducted post experiment debriefing of subjects who violated transitivity, and found that the violation was a local phenomenon and that subjects were unconscious of the violation. Moreover, most of the violators indicated that they would have chosen in accord with transitivity if they were conscious of the error.

2.2 Kolmogorov representation of mental states over P-measure zero noise

Suppose that P is linear and *additively separable* on $\{\{A \succ C\} \oplus \varphi\}$ so that

$$P\{\{A \succ C\} \oplus \varphi\} = P\{A \succ C\} + P(\varphi) \tag{2.6}$$

In order for Axiom 2.4 to hold in (2.6) the set φ must have P-measure zero. Thus, we *assign* $P(\varphi) = 0$. In that case, the attachment operator $\oplus$ in (2.6) satisfies the joint receipts hypothesis summarized in Luce (1995, p. 73). We state this "joint receipt hypothesis" more formally as

Lemma 2.1 (Joint receipts)**.** *The WHT axiom is robust to joint receipt of binary choice and its accompanying mental state.*

Proof. The proof is based on Luce (1995, p. 73) characterizations of mental accounting rules attributed to Thaler (1985) and D. von Winterfeldt, and the classical probability of the union of events, respectively, which we write as follows:

$$P\{\{A \succ C\} \oplus \varphi\} = \max\{P\{\{A \succ C\} + \varphi\}, P\{A \succ C\} + P(\varphi)\} \tag{2.7}$$

$$P\{\{A \succ C\} \oplus \varphi\} = \alpha P\{\{A \succ C\} + \varphi\} + (1-\alpha)[P\{\{A \succ C\}\} + P(\varphi)] \tag{2.8}$$

$$P\{\{A \succ C\} \oplus \varphi\} = P\{\{A \succ C\}\} + P\{\varphi\} - P\{\{A \succ C\} \cap \varphi\} \tag{2.9}$$

It is easy to see that under the WHT Axiom 2.4 each of the representations in (2.7)-(2.9) are equivalent to $P\{\{A \succ C\} \oplus \varphi\} = P\{A \succ C\}$ when $P(\varphi) = 0$ and $P\{\{A \succ C\} \cap \varphi\} = 0$. The latter follows from the completion of the P-measure on φ. $\square$

Implicit in Lemma 2.1, is a covering criterion for some $\varepsilon > 0$. That is, there are countably many (separable) points in φ each of which is covered by an interval of length $|I_n| = \varepsilon.2^{-n}$ for $n = 1, 2, \ldots$, such that $\varphi \subseteq \cup_{n=1}^{\infty} I_n$. Thus, $P(\varphi) \leq \sum_{n=1}^{\infty} P(I_n) \leq \sum_{n=1}^{\infty} \varepsilon.2^{-n} = \varepsilon$. Since ε is arbitrary, we can make it as small as we like. Thus, φ is a weakly compact space (Loève, 1978, p. 181).

Kolmogorov's representation theorem (see e.g. Gikhman and Skorokhod, 1969, pp. 107-108) tells us that there exist a real valued function g that maps into φ and characterizes the distributions supported by ε. Specifically, let an interval length ε, e.g. $[0, \varepsilon]$ for imprecision or noise, be decomposed into dyadic sub-intervals $\{I_j\}_{j=1}^n$ of length $|I_j|=\varepsilon.2^{-j}$, i.e., $I_j=[\varepsilon.2^{-j},\,\varepsilon.2^{-(j-1)})$, and Ω be a sample space of cognitive states, $\mathscr{F}$ be the σ-field of Borel measurable subsets of Ω, and $(\Omega,\mathscr{F},P)$ be a probability space. Furthermore, let X be a space of consequences such that $g: X\times\Omega\to\varphi$. So that for $x_j\in X$ and $\omega\in\Omega$, $g(x_j,\omega)=\varepsilon_j$. Thus, the joint distribution of $(g(x_1,\omega),g(x_2,\omega),\ldots,g(x_n,\omega))$ on the product space $X^{(n)}$ (n-copies of X) coincides with the joint distribution of $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)$ over $I_1\times I_2\times\cdots\times I_n$ in the n copies space $\varphi^{(n)}$. For example, if the x's are rank ordered, and g is indexed by the x's, then $g(x,\omega)$ is a random field that is mapped into φ. In particular $\{(\Omega,\mathscr{F},P),\ g(x)\}$ represents a random field on $I_1\times I_2\times\cdots\times I_n$ in $\varphi^{(n)}$. In the empirical literature g is stochastic by virtue of its "coordinate mapping" on ε_j's which are typically assumed to be joint normal distributed. The previous arguments from Kolmogorov's representation theorem leads to the following[14]

Theorem 2.2 (Random fields of HPWF)**.** *Let X and φ be two disjoint spaces, Ω be a sample space of cognitive states, P a probability distribution over X, $A\subset X$ a closed subset, $x\in A$, and g_h be a continuous function such that $g_h(x)\in\varphi$. In the attached space $X\oplus\varphi$, generate an equivalence relation R by $x\sim g_h(x)$ for each $x\in A$. The quotient space $(X\oplus\varphi)/R$ is said to be X attached to φ by g_h and is written $X\cup_{g_h}\varphi$ with attaching map g_h. There exist a mapping $(w\circ P): X/A\to X\cup_{g_h}\varphi$ into the attached space $X\oplus\varphi$. In particular, the weak harmonic axiom contemplates a harmonic map $g_h(x)\to[0,\varepsilon]\subset\varphi$ with the composite mapping $(w\circ P)(x,\omega)=P(x)\oplus g_h(x,\omega)$ where $\{(\Omega,\mathscr{F},P),\ g_h(x)\}$ is a random field defined on φ.* □

The key point in this exercise is that φ is a periodic set (that includes mental states) with P-measure zero, and it is attached to the space of outcomes X. It supports both harmonic (g_h) and stochastic (g_s) representations of g. The literature on noise neglects the harmonic representation g_h in favor of its dual stochastic representation g_s. For example, Wakker's (2010) *neo-additive family* of pwfs, operationalized in Dierkes et al. (2020, p. 5), satisfy the notion that g_s is a distortion. Thus, according to Theorem 2.2 the HPWF is a sample function from a random field.[15] It depends on the mental states of DMs as they evaluate a distribution of outcomes (cf. Conte et al., 2009).[16] The HPWF has an axiomatic fixed point e^{-1}–the same as the axiomatized fixed point in Prelec (1998).

[14] See Dugundji (1966, pp. 127-128) for definition of attaching map.

[15] If $(w\circ P)(x,\omega)$ is additively separable in $P(x)$ and $g_h(x,\omega)$, then HPWF admits (first order) stochastic dominance if g_h is locally monotonic.

[16] In Hogarth and Einhorn (1990), the pwf is: $w(p)=p_A+k(\bullet)$ where p_A is an anchor probability, the mental simulation or adjustment factor is a function $k(\sigma,\theta,p_A,v(x))$ where σ is a measure of outcome uncertainty, θ is the perceived ambiguity parameter, and $v(x)$ is a value function for outcome x. To accommodate Hogarth and Einhorn (1990) descriptive theory in Theorem 2.2, we can assign $g_h(\boldsymbol{s})=k(\boldsymbol{s})$ where $\boldsymbol{s}=[\sigma,\theta,p_A,v(x)]$ is a vector of state variables. To account for ambiguity Einhorn and Hogarth (1985, p. 437) proposed that $w(p)=p+\theta(1-p-p^{\beta})$ where p is an anchor probability and β is an attitude towards ambiguity parameter.

Yukalov and Sornette (2015) also introduce a quasilinear probability weighting function. However, their model is based on quantum decision theory, in Hilbert space with a trace class measure, for nonlinear probability which is decomposed into a linear part they call a "utility factor", and an addend part they call "attraction factor" which represents behavioural biases. In contrast our model is based on classic probability space of the kind used in EUT and the addend part mimics mental states.

3 Rank dependent utility and preference reversal over probability cycles

In this section we employ Quiggin (1993, §5.2, p. 57) RDU model and the quasilinear HPWF to resolve the PR puzzle. For internal consistency we assume exchangeability[17] of probability measures implied by RDU. That is, RDU transformation of nonlinear probabilities are sufficient to satisfy De Finetti's (1937) coherence postulate. We provide some preliminaries that include definition of the probability cycle concept. In subsection 3.1 we decompose the decision weights obtained from the HPWF by using Quiggin (1982) transformation procedure. In subsection 3.2 we introduce a model in which procedure invariance is imposed on an experiment and show how the HPWF resolves PR in that context. In subsection 3.3 we provide analytics which show that experimenters interference with a DM's probability cycle lead them to misperceive PR.

Preliminaries

In the sequel we employ the quasilinear specification of the HPWF introduced in Charles-Cadogan (2018, Thm 4.1) which we restate here without proof.

Theorem 3.1 (HPWF–Harmonic Probability Weighting Function)**.** *Let $\boldsymbol{x} = [x_1, \ldots, n_n]$ be a vector valued statistical ensemle of ranked outcomes, and $\boldsymbol{z}$ be the corresponding vector of Z-scores for the ranked outcomes such that $x_j = \mu_x + z_j\sigma_x$ where μ_x and sigma$_x$ are the mean and standard deviation of $\boldsymbol{x}$, respectively. The coherent HPWF of a DM is quasilinear and given by*

$$w(p,x) = \eta_0 p + \eta_1 \tan(\psi(\mathfrak{z})) \tag{3.1}$$

where $\psi(\mathfrak{z})$ is a phase function and $-\frac{\eta_0}{\eta_1} \leq \tan(\psi(\mathfrak{z})) \leq \frac{1-\eta_0 p}{\eta_1}$.

[17] Refer to Section B.1 for definition of exchangeability.

Consider the probabilistic rank dependent expected utility (RDU) specification[18] in Wakker (1994, pg. 10) for a simple rank ordered lottery $p_1, x_1; \ldots, x_n, p_n$. For utility $U(x_j)$ of outcome x_j we have

$$RDU(p_1, x_1; \ldots, x_n, p_n) = \sum_{j=1}^{n} \pi_j U(x_j) \tag{3.2}$$

$$\pi_j = w(p_1 + \ldots + p_j) - w(p_1 + \ldots + p_{j-1}) \tag{3.3}$$

$$\pi_1 = w(p_1), \qquad \sum_j \pi_j = 1 \tag{3.4}$$

where $w : [0,1] \to [0,1]$ is a probability transform function, and π_j is a *decision weight*. In this case, we let w be our inherent HPWF. Since $\tan(\psi(\mathfrak{z}))$ is cyclic in Theorem 3.1, we have the periodic relationship

$$\tan(\psi^{(k)}(\mathfrak{z}))) = \tan(\psi(\mathfrak{z})), \quad k = 1, 2, \ldots \quad \text{where} \;\; \psi^{(k)}(\mathfrak{z}) = (2k-1)\pi + \psi(\mathfrak{z}) \tag{3.5}$$

$$\psi^{(k)}(\mathfrak{z}) = (2k-1)\pi \quad k = 1, 2, \ldots \quad \text{when} \;\; \psi(\mathfrak{z}) = 0 \tag{3.6}$$

Without loss of generality, when $\eta_0 = 1$ and (3.6) holds, we generate recurrent fix points and their corresponding probabilities. **Notation**. Unless otherwise stated in the sequel π without sub(super)scripts is measured in radians. π with sub(super)scripts is a decision weight. This leads to the following

Definition 3.1 (Probability cycle)**.** Kemeny et al. (1976, p. 144). Let $w(p,x)$ be as in Theorem 3.1 and $\mathscr{K} = \{k | \; \psi^{(k)}(\mathfrak{z}) = (2k-1)\pi + \psi(\mathfrak{z})\}$ be an index set of periodic phase functions. The probability cycle for $w(p,x)$ is the greatest common denominator $d(k)$ of $\mathscr{K}$ for the probabilities associated with recurrent states indexed by $\mathscr{K}$. □

We characterize the recurrent states and their probabilities as follows.

Theorem 3.2 (Behavioural Chapman-Kolmogorov equation and recurrent coherent states)**.** *Let $\mathscr{I}$ be an index set of coherent states and $\mathscr{T}$ be an index set for transient incoherent states. Let P_{ik}^{m} be the transition probability of moving from state $i \in \mathscr{I}$ to $k \in \mathscr{T}$ in m steps, and P_{ki}^{n} be the*

[18] In order not to overload the paper with issues pertaining to framing effects in decision weights, we did not employ cumulative prospect theory (CPT). Compare Tversky and Wakker (1995, pg. 1259).

corresponding probability of transitioning from state k back to state i in n steps. We assume that transition probabilities are drawn from a 1-step Markov transition probability matrix $[P^r_{ij}]$ *where* $m+n=r$. *Then*

$$w\,(P^r_{ii}) = \sum_{k=0}^{\infty} w\,(P^m_{ik})\,w(P^n_{ki}) = P^{m+n}_{ii} = P^r_{ii} \tag{3.7}$$

In particular, if state i has period $d(i) \in \mathscr{K}$, *then* $r = b + s.d(i)$ *where s is the number of periods in r for integers* r, s, b.

Proof. See Section D ☐

The identity $\Phi(z_j) = p_j$ from Theorem 3.1, implies that there exist $\tilde{\mathfrak{z}}_j$ such that $\Phi(\tilde{\mathfrak{z}}_j) = \tilde{p}_j = \sum_{k=1}^{j} p_j$. It follows from Theorem 3.1 that for some pair of values $(\tilde{\mathfrak{z}}_{j-1}, \tilde{\mathfrak{z}}_j)$ to be determined, we have for some function f, the decision weight

$$\begin{aligned} pi^{(k)}_j &= f(\tilde{\mathfrak{z}}_{j-1}, \tilde{\mathfrak{z}}_j;\, k) = w\Big(p_1 + p_2 + \dots p_j\Big) - w\Big(p_1 + p_2 + \dots + p_{j-1}\Big) \\ &= \eta_0\Big(p_1 + p_2 + \dots p_j - p_1 - p_2 - \dots - p_{j-1})\Big) + \eta_1\Big(\tan(\psi^{(k)}(\tilde{\mathfrak{z}}_j)) - \tan(\psi^{(k)}(\tilde{\mathfrak{z}}_{j-1}))\Big) && (3.8) \\ &= \eta_0 p_j + \underbrace{\eta_1\Big(\tan(\psi^{(k)}(\tilde{\mathfrak{z}}_j)) - \tan(\psi^{(k)}(\tilde{\mathfrak{z}}_{j-1}))\Big)}_{\text{harmonic component of decision weight}} && (3.9) \end{aligned}$$

Thus, the decision weight $\pi^{(k)}_j$ is cyclic by virtue of (3.5) and the harmonic addend term in (3.9).[19] Even if it was not cyclic, it depends nonlinearly on $\tilde{\mathfrak{z}}_{j-1}$ and $\tilde{\mathfrak{z}}_j$ as indicated. Additionally, if k is not an integer in (3.5), i.e. the cycle is incomplete, equality does not hold so probabilistic choice is different for the same set of stimuli. The results above extend naturally to Tversky and Kahneman (1992) cumulative prospect theory (CPT) because the latter employs the same decision weight scheme as RDU. We note that in (3.9) the harmonic component of decision weights vanishes when $\tilde{\mathfrak{z}}_j = \tilde{\mathfrak{z}}_{j-1}$ so $\pi^{(k)}_j = \pi_j$ and RDU collapses to EUT due to $\eta_0 p_j$. We say more on that next.

[19]Equation (5) in Karni and Safra (1990, p. 491) also depends on a harmonic component like the one in (3.9).

3.1 Decomposition of decision weights obtained from HPWF

By virtue of (3.9), we decompose $\pi_j^{(k)}$ as follows. Let

$$\begin{aligned}
&\psi^{(k)}(\tilde{\mathfrak{Z}}_j) = \psi^{(k)}(\tilde{\mathfrak{Z}}_{j-1}) + \Delta\psi^{(k)}(\tilde{\mathfrak{Z}}_j),\ \tilde{\mathfrak{Z}}_j = \Phi^{-1}\Big(\sum_{r=1}^{j} p_r\Big) = \Phi^{-1}\Big(\sum_{r=1}^{j}\Phi(\mathfrak{z}_r)\Big), \\
&\Delta\psi^{(k)}(\tilde{\mathfrak{Z}}_j) = \psi^{(k)}(\tilde{\mathfrak{Z}}_j) - \psi^{(k)}(\tilde{\mathfrak{Z}}_{j-1})
\end{aligned} \tag{3.10}$$

So that

$$\tan(\psi^{(k)}(\tilde{\mathfrak{Z}}_j)) - \tan(\psi^{(k)}(\tilde{\mathfrak{Z}}_{j-1})) = \tan(\Delta\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\Big[1 + \tan(\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\tan(\psi^{(k)}(\tilde{\mathfrak{Z}}_{j-1}))\Big] \tag{3.11}$$

$$= \sin(\Delta\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\sec(\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\sec(\psi^{(k)}(\tilde{\mathfrak{Z}}_{j-1})) \tag{3.12}$$

Substitution in (3.9) gives us

$$\pi_j^{(k)} = \eta_0 p_j + \eta_1 \sin(\Delta\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\sec(\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\sec(\psi^{(k)}(\tilde{\mathfrak{Z}}_{j-1})) \tag{3.13}$$

So (3.9) is reduced to (3.13) where the decision weight is decomposed into a part $(\eta_0 p_j)$ that reflects DM confidence (due to elevation η_0) about the inherent probability p_j associated with outcome x_j, and curvature parameter η_1 controlled by a harmonic part, which is in turn controlled by the jump in phase function $\Delta\psi^{(k)}(\tilde{\mathfrak{Z}}_j)$ at x_j based on the Z-scores $\mathfrak{z}_1, \ldots, \mathfrak{z}_j$ that comprise $\tilde{\mathfrak{Z}}_j$. Because $\sin(\Delta\psi^{(k)}(\tilde{\mathfrak{Z}}_j))$ is cyclic, the decision weight is cyclic. Let that cyclic factor be

$$\varphi^{(k)}(\boldsymbol{x}_j|\mu_x,\sigma_x) = \varphi^{(k)}(\boldsymbol{\mathfrak{z}}_j) = \sin(\Delta\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\sec(\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\sec(\psi^{(k)}(\tilde{\mathfrak{Z}}_{j-1})) \tag{3.14}$$

where $\boldsymbol{x}_j = (x_1, \ldots, x_j)$ and $\boldsymbol{\mathfrak{z}}_j = (\mathfrak{z}_1, \ldots, \mathfrak{z}_j)$. Now rewrite (3.2) and (3.13), respectively, as

$$\pi_j^{(k)} = \eta_0 p_j + \eta_1 \varphi^{(k)}(\boldsymbol{x}_j|\mu_x,\sigma_x) \tag{3.15}$$

$$RDU(\boldsymbol{x},\boldsymbol{p}) = \sum_{j=1}^{n}\pi_j^{(k)}U(x_j) = \eta_0 EU(\boldsymbol{x},\boldsymbol{p}) + \eta_1 WU^{(k)}(\boldsymbol{x},\boldsymbol{p}) \tag{3.16}$$

$$\text{where } EU(\boldsymbol{x},\boldsymbol{p}) = \sum_{j=1}^{n} p_j U(x_j) \text{ and } WU^{(k)}(\boldsymbol{x},\boldsymbol{p}) = \sum_{j=1}^{n}\varphi^{(k)}(\boldsymbol{x}_j|\mu_x,\sigma_x)U(x_j) \tag{3.17}$$

and EU is Von Neumann and Morgenstern (1953) utility functional and $WU^{(k)}$ is a weighted utility expression (Chew and Waller, 1986, pg. 59, eq (2.6)) that depends on the k-cycle for $\varphi^{(k)}(\boldsymbol{x}_j|\mu_x,\sigma_x)$ in (3.14). When $\Delta\psi^{(k)}(\tilde{\mathfrak{z}}_j)=0$, $\varphi^{(k)}(\boldsymbol{x}_j|\mu_x,\sigma_x)=0$ and $WU^{(k)}=0$ so there is no weight given to the inherent distribution of outcomes and RDU is reduced to EU. Note that dividing both sides of (3.15) by η_0 makes $EU(\boldsymbol{x},\boldsymbol{p})$ more transparent. Thus, we have just proven the following

Theorem 3.3 (Inconsistent probabilistic preferences)**.** *Let $x_j,\quad j=1,2,\dots,n$ for $n\geq 3$ be a statistical ensemble of outcomes, and $\Phi(\bullet)$ be the cumulative normal distribution function. Let*

$$w(p)=\eta_0 p+\eta_1\tan(\psi(\mathfrak{z}))$$

be the specification in Theorem 3.1 for standardized score $\mathfrak{z}$ and monotone phase function $\psi(\mathfrak{z})$, *where $\tan(\psi(\mathfrak{z}))$ is a weighting function for outcomes satisfying Theorem 3.1. So that $w(\bullet)$ operates on a "k-cycle" for $\tan((2k-1)\pi+\psi(\mathfrak{z}))=\tan(\psi(\mathfrak{z})),\quad k=1,2,\dots.\qquad$ As in (3.15), define* decision weights

$$\pi_j^{(k)}=\eta_0 p_j+\eta_1\varphi^{(k)}(\boldsymbol{\mathfrak{z}}_j)$$

Suppose that probabilistic preferences *are represented by rank dependent utility (RDU) so that for von-Neuman utility $U(x_j)$ and inherent* prior probability $p_j=\Phi(\mathfrak{z}_j)$, *we have, for the corresponding gamble or lottery,*

$$RDU(\boldsymbol{x},\boldsymbol{p})=\sum_{j=1}^{n}\pi_j^{(k)}U(x_j)$$

Then subjects will make different probabilistic choices when faced with the same stimuli because decision weights are cyclic unless their choices coincide with a probability k-cycle. Furthermore, we have the decomposition $RDU(\boldsymbol{x},\boldsymbol{p})=EU(\boldsymbol{p})\oplus WU(\boldsymbol{x})$. □

Remark 3.1. The RDU decomposition result above was anticipated by Wakker (1994) who axiomatized decomposition of RDU preferences into probabilistic risk attitude and utility based components. The *neo-additive* feature of $\pi_j^{(k)}$ drives the decomposition in our case.

Theorem 3.4 (Almost sure inconsistent probabilistic preferences)**.** *Probabilistic preferences induced by non-expected utility decision weights are different for the same stimuli almost surely.* □

Proof. According to Theorem 3.3 probabilistic preferences are consistent iff $\pi_j^{(k)}=\pi_j$ for every j in a "k-cycle". However, for fixed j the Lebesgue measure of the level set $\mathscr{Z}_j=\{k:\ \pi_j^{(k)}=\pi_j,\forall k\in\mathbb{R}_+\}$ is zero, that is $meas\,\mathscr{Z}_j=0$. Thus, for each j probabilistic preferences are inconsistent except on a set with Lebesgue measure zero. □

If probabilistic preferences are cyclic, then subjects will make different choices with different probabilities when presented repeatedly with the same or similar stimuli over time by breaking the cycle. In which case, choice depends on subjects' location in the probability cycle and not so much on stimuli. This point was made by Regenwetter et al. (2011, pg. 43) who introduce a mixture model in which choice vary because DMs are in different mental states.

3.2 Resolution of preference reversal in otherwise identical procedurally invariant temporally spaced repeated experiments

In this section we show how PR is characterized and rationalized in an abstract experiment based on the lottery pairs method (Jacquemet and l'Haridon, 2018, p. 291), and in which preferences are in accordance with RDU and HPWF. Specifically, we employ a two stage lottery experiment (Segal, 1990) without reduction of compound lotteries (Assumption 3.8) and order indifference (Assumption 3.9) (cf. Segal, 1993, p. 375). In Section 3.2.1 we show that probability cycles are broken by incoherence in utility of outcomes in binary choice. Coherence is a measure familiar to time series analysis in the frequency domain (Brillinger, 2001; Bloomfield, 2004). In Section 3.2.2 we show that PR is due to momentary fluctuations in DMs evaluative processes.

Let $\boldsymbol{x} = \{x, y, z\}$ be a set of non negative outcomes, $U(\bullet)$ be a utility function that satisfies Von Neumann and Morgenstern (1953) axioms, and $\boldsymbol{p} = \{p_x, p_y, p_z\}$ be the corresponding probability distribution of $\boldsymbol{x}$. We make the following assumptions.

Assumption 3.5 (Experimental tasks). *$E1$ and $E2$ are identical but temporally spaced experiments that elicit preferences.* □

Assumption 3.6 (Procedure invariance). *The same procedure is used to elicit preferences in each temporally spaced experiment.* □

Assumption 3.7 (Preference reversal). *In $E1$ our DM expresses the preference $x \succ y \succ z$, and in $E2$ she reverses preference such that $y \succ z \succ x$.* □

Assumption 3.8 (No reduction of compound lotteries). *No reduction of compound lotteries imply that DMs are not interested in product probabilities.* □

Assumption 3.9 (Quasi order indifference). *DMs are indifferent about the order of two stage lotteries.* □

Assumption 3.10 (Risk neutrality). *DMs are risk neutral.* □

Remark 3.2. Assumption 3.6 is biased against preference reversal. It is much stronger than the standard assumption which only require that *different* admissible procedures should identify the same preference.

Let $\pi_j^i,\ i = 1,2;\ \ j = x,y,z$ be the decision weights computed via Quiggin's transformation procedure in each experiment. For example, π_x^1 and π_x^2 are a DM's decision weights for x in $E1$ and $E2$, respectively. In principle, DMs choose under $E1$, and under $E2$, from each of three lotteries which can be displayed as follows:

$$L_1 \equiv (x, p_x;\ \ y, p_y;\ \ 0, 1 - p_x - p_y) \tag{3.18}$$

$$L_2 \equiv (y, p_y;\ \ z, p_z;\ \ 0, 1 - p_y - p_z) \tag{3.19}$$

$$L_3 \equiv (x, p_x;\ \ z, p_z;\ \ 0, 1 - p_x - p_z) \tag{3.20}$$

There are $3! = 6$ ways in which the lotteries can be ordered. A simple experiment design[20] for our purposes may be to select the following choice experiment $E1$ as a baseline:

$$E1 \equiv \{A_s, p_E; \quad L, 1 - p_E\} \tag{3.21}$$

where A_s is a reward with probability p_E conveniently chosen (see *infra*), and $L \equiv \{L_1, L_2, L_3\}$ is a lottery to be played out sequentially (from left to right) if selected with probability $1 - p_E$. In a sense this is like the Becker et al. (1964) procedure in which "[a] subject is told that he has been given a particular gamble [L], which he may keep and play out. Alternatively, he may try to sell the gamble back to the experimenters" (Loomes et al., 1991, p. 426) (emphasis added) at the reservation price A_s. In our case p_E is the probability that the subject exercises the option to sell. Thus, $E1$ is a "paired gamble" method (Farquhar (1984, p. 1285), Jacquemet and l'Haridon (2018, p. 291)).

We assume that there is no probability associated with choosing $L_i,\ i = 1,2,3$. That is, there is no compound invariance (cf. Segal, 1990, p. 353). So if L is chosen, then DMs choose $x \succ y$ in L_1; choose $y \succ z$ in L_2; choose $x \succ z$ in L_3. This choice pattern is consistent with transitivity. Let $A_v =$

[20] See Harrison and Rutström (2008) for a comprehensive survey and taxonomy of experimental designs and econometric approaches.

$xp_x + yp_y + zp_z$ be the actuarial value of the lottery so that the probability $p_E = \frac{A_v}{A_v + A_s}$ equalizes the expected value of DMs choices between A_s and the lottery L. This choice of probability is consistent with coherent preferences (see e.g. Buehler, 1976, §3, p 1053). Thus, in $E1$ we elicit a transitive order for outcomes. We assume that for $E2$, which is temporally spaced, seemingly intransitive preferences are reported as indicated below.

In $E1$ and the temporally spaced experiment $E2$, up to rewards scaled by a common factor c, subjects are randomly assigned to any of the following "paired gamble":

$$E2 \equiv \{cA_s, p_E; \quad c\widehat{L}, 1 - p_E\} \tag{3.22}$$

where $\widehat{L}$ (below) represents the $3! = 6$ possible permutations of (3.18) (3.19) and (3.20).

$$\widehat{L} \in \Big\{\{L_1, L_2, L_3\},\ \{L_1, L_3, L_2\},\ \{L_2, L_1, L_3\},\ \{L_2, L_1, L_3\},\ \{L_3, L_1, L_2\},\ \{L_3, L_2, L_1\}\Big\} \tag{3.23}$$

$c\widehat{L}$ means that the outcomes are scaled by c. So possible choice pairs are $\{cA_s, p_E; \quad c\{L_1, L_2, L_3\}, 1 - p_E\}$, $\{cA_s, p_E; \quad c\{L_1, L_3, L_2\}, 1 - p_E\}$ and so on. The lotteries in L (in (3.21)) were randomized in (3.23) to remove order effects. Since a constant scale does not affect transitivity, $E2$ will be effectively "identical" to $E1$ up to scale and randomization.[21] For the purpose of exposition, we assume that for a given realization of $\widehat{L}$ in $E2$, (some) DMs reverse the choices they made in L_3 in (3.20), i.e., they choose such that $z \succ x$. *The analysis that follow applies only to those DMs that choose L in $E1$ and $\widehat{L}$ in $E2$.* In principle, preference reversal is also manifest if a DM choose A_s in E_1 and $c\widehat{L}$ in E_2 or vice versa. We address the latter scenarios in (3.25) and (3.27) below. It turns out that the result is unchanged.

3.2.1 The case of incoherent binary choice and compound lotteries

For notational convenience, we suppress the standardized distribution $\mathfrak{z}$ for outcomes in (3.18)-(3.20), (3.21) and (3.22), and write DMs probability weighting function as $w(p_E)$ and $(1 - w(1 - p_E))$ for binary choice (Quiggin, 1993, p. 57). The expected value of each element of the choice

[21] Holt and Laury (2002) report an increase in risk aversion when stake size is scaled.

pair in (3.22) is the same by hypothesis. If not, then we would have a built in bias in the experiment that confounds the result. Thus, under Assumption 3.10 we would expect[22] $p_E U(cA_s) = (1 - p_E)U(c\widehat{L})$. However, to account for probability weighting in the RDU generalization of EUT, under Assumption 3.10 we must also have

$$w(p_E)U(cA_s) = (1 - w(1 - p_E))U(c\widehat{L}) \tag{3.24}$$

where $w(p_E) = \eta_0 p_E + \eta_1 \tan(\psi(\mathfrak{z}))$ for the *neo-additive* HPWF and $\mathfrak{z}$ pertains to the suppressed outcomes $\boldsymbol{x}$. Substitution for $w(p_E)$ in (3.24) and rearrangement of terms produces

$$\eta_1[\tan(\psi(Z_{cA_s}))U(cA_s) - \tan(\psi(Z_{cA_v}))U(c\widehat{L})] = \eta_0[p_E U(cA_s) - (1 - p_E)U(c\widehat{L})] \tag{3.25}$$

Assumption 3.10 implies that

$$p_E U(cA_s) = (1 - p_E)U(c\widehat{L}) \tag{3.26}$$

This causes the right hand side (RHS) in (3.25) to vanish. So we are left with the result on the left hand side (LHS):

$$\tan(\psi(Z_{cA_s}))U(cA_s) - \tan(\psi(Z_{cA_v}))U(c\widehat{L}) = 0 \Rightarrow \tan(\psi(Z_{cA_v})) = c_u \tan(\psi(Z_{cA_s})) \tag{3.27}$$

where $c_u = U(c\widehat{L})/U(cA_s)$. This implies $p_E = c_u/(1 + c_u)$. Under the "paired gamble" formulation with procedure invariance, the amounts elicited by probability equivalence $CE(P)$ and certainty equivalence $CE(\$)$ should be coherent. In that case, $c_u = U(CE(\$))/U(CE(P)) = 1$. So c_u is a utility coherence ratio (UCR).

Theorem 3.11 (Coherent utility ratio)**.** *A DM is coherent if the utility ratio $c_n = 1$. In particular, under Assumption 3.6 for \$-bet and P-bet with certainty equivalent $CE(\$)$ and $CE(P)$, respectively, we have $c_n = \dfrac{U(CE(\$))}{U(CE(P))}$ and $\tan(\Psi_2) = c_n \tan(\Psi_1)$ where Ψ_i, $i = 1, 2$ are phase functions for the HPWF.* □

[22]Recall that the "paired gamble" formulation of $E1$ contemplates the choice $\{1 - p_E, c\widehat{L}\}$ being sold for cA_s with probability p_E.

The case of incoherence

For c_u fixed, the harmonic relationship in (3.27) implies that the LHS and RHS are jointly influenced by the same frequency ω. In particular, c_u is a utility coherence ratio (UCR) factor. If $c_u = 1$, then our DM is coherent in (3.27), i.e. $U(c\widehat{L}) = U(cA_s)$ and $\psi(Z_{cA_v}) = \psi(Z_{cA_s})$. If $c_u \ll 1$, then our DM is incoherent and the angles are far apart except for when cycles are complete. The relation $\psi(Z_{cA_s})(\omega) = \psi(Z_{cA_v})(\omega) + \omega\pi$ holds for complete probability cycles when ω is divisible by $d(k)$ according to Definition 3.1. If that cycle is broken, then the equality in (3.27) does not hold, and we would expect PR in $E2$. In the context of the intermediate Chapman-Kolmogorov equations in Theorem 3.2 cycles are complete for recurrent states $i \in \mathscr{I}$ with probabilities P^r_{ii}, and they are broken for $k \in \mathscr{T}$ for P^m_{ik} and P^n_{ki} for $r = m+n$. Implicit in this argument is the existence of a suitable transition probability matrix, that the underlying ranked outcomes can be represented as a Markov chain, and repeated experiments generate renewed but similar states.

*** Insert Figure 1 and Figure 2 about here ***

Figure 1 depicts a probability cycle scenario for eight (8) different UCRs for the coherent states $\omega \in \pi\{0, 1/2, 1, 3/2, 2\} \cong \mathscr{I} = \{0, 1, 2, 3, 4\}$ where $\mathscr{I}$ is the set of fixed points that correspond to recurrent states, i.e. $\mathscr{I} = \{i \,|\, W(P_{ii}) = P_{ii}, i = 0, 1, 2, 3, 4\}$. For example, assume n-step recurrent fixed point states i, where $(n > 1)$. Let $\tilde{w}(p, x) = 1/\eta_0 \star w(p, x)$, so the probability P^n_{ii} of returning to state i after n-steps[23] is given by $\tilde{w}(P^n_{ii}, x) = P^n_{ii} + \tilde{\eta}_1 \tan(\psi^{(i)}(\bullet))$ where $\tan(\psi^{(i)}(\bullet) = 0$ for $\psi^{(i)}(\bullet) = i\pi,\ \ i = 0, 1, 2, \ldots$ and $(\bullet)$ is the Z-score for outcome x. The latter implies that for some fixed P, i.e. a fixed point, $w(P)$ is weakly ergodic.[24] We note in passing that consistent with Theorem 2.2, the set $\mathscr{I}$ of coherent mental states has Lebesgue measure 0.

For $2\Psi^{(i)}_2 = \psi^{(i)}(\bullet)$ we have $\tilde{w}(P^n_{ii}, x) = P^n_{ii} + \tilde{\eta}_1 \tan(2\Psi^{(i)}_2)$ which generates the corresponding fixed point probabilities or relative frequencies $P^n_{ii} = |\omega|/2\pi$. So, $P^n_{00} = 0, P^n_{11} = 1/4, P^n_{22} = 1/2, P^n_{33} = 3/4, P^n_{44} = 1$ for $\omega \in \pi\{0, 1/2, 1, 3/2, 2\}$. These results are depicted in Figure 1. Figure 2 is a refinement of Figure 1 and the same arguments hold as before. Except that now $\omega' \in \pi\{0, 1/4, 1/2, 3/4, 1, 5/4, 3/2, 7/4, 2\} \cong \mathscr{I}' = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $P^n_{ii} = |\omega'|/2\pi \in$

[23] In textbooks like Parzen (1999, p. 221) a state k is recurrent if the probability is $f_{kk} = 1$ that the underlying Markov chain will eventually return to k having started at k, and $f_{kk} = P_{kk} + \sum_{j \in \mathscr{I},\, j \neq k} P_{kj}$. So, $P_{kk} = 1 - \sum_{j \in \mathscr{I},\, j \neq k} P_{kj}$.

[24] Suppose we average over P and Ψ with N_P and N_Ψ, respectively. So that, $\sum_P w(P)/N_P = \eta_0 \sum_P P/N_P + \eta_1 \tan(\Psi)$ and $\sum_\Psi w(P)/N_\Psi = \eta_0 + \eta_1 \sum_\Psi \tan(\Psi)/N_\Psi$. If $\tan(\Psi) = 0$ and $\overline{P} = \sum_P P/N_P = P$, then $w(P)$ is weakly ergodic over the fixed points P.

$\{0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/9, 7/8, 8/8\}$. By induction we have $P_{ii}^n \in \{0, 1/N, 2/N, \cdots, N/N\}$. So that for $\lim_{N\to\infty} P_{ii}^n = P_{ii}^n \in [0,1]$, complete coherence $w(P_{ii}^n) = P_{ii}^n$ is attained over the interval $[0,1]$ in the limit. The key take away here is that higher frequencies for probability cycles imply greater coherence, and shorter transition/momentary lapse in judgment as shown in Figures 1 and 2.

We calibrated a coherence structure function $\psi_2(\omega) = \tan^{-1}(c_u \tan(\psi_1(\omega)))$ over values of $\psi_1(\omega)$ in the range (0,360] for a partition of $[0,1]$ in 1/360 chunks. So, $c_u(k) = k/360$ for $k = 0, 1, \ldots, 360$. The closer c_u is to 1 the less preference reversal is observed as transition to coherent states become smaller when the waves flatten. For example, when $c_u = 0.85$ we have approximate coherence since the line almost coincides with the diagonal. Figure 3 depicts the peaks of the phase functions that characterized the UCRs c_n. There, we see that $0 \leq \Psi_1 < 2.1$ and $0 \leq \Psi_2 < 2.5$, approximately, and that the peaks are concentrated around 1.5 radians or 90 degrees. This coincides with a completed probability cycle at $\pi/2$ in Figure 1. The other completions are multiples of $\pi/2$, i.e. $4 \times \pi/2 = 2\pi$.

*** Insert Figure 3 about here ***

Figure 5 depicts the magnitude squared coherence[25] (MSC) for deterministic phase functions Ψ_1 and Ψ_2, and for deterministic Ψ_1 and noisy Ψ_2 for select utility coherence factors $c_u = 0.1225$ and $c_u = 0.7225$.[26] Except for very low frequencies, the MSC for deterministic phase functions are relatively high. However, once noise is added to the equation the MSC falls dramatically. This implies that noisy experiments lead analysts to overestimate the amount of preference reversal. This point was recently emphasized by Loomes and Pogrebna (2016).

*** Insert Figure 5 and Figure 6 about here ***

Figure 7 depicts the wavelet coherence induced by the HPWF when Ψ_1 is deterministic, $c_u = 0.1225$, and Ψ_2 is noisy. Roughly, it depicts coherence between the diagonal and the $c_u = 0.125$ curve in Figure 1 with a random error addend drawn from the standard normal $N(0,1)$. The phase

[25] This metric is based on a correlation between wavelet transformation of phase functions (characterized by the UCRs) in the "time" and frequency domain. See Aguiar-Conraria et al. (2012) for a non-technical introduction. Here, the angle interval $[0,360]$ or $[0,2\pi]$ substitutes for "time".

[26] Implicit in this example is the idea that Ψ_1 is nonstochastic, and that Ψ_2, which is a function of Ψ_1, is measured with error.

functions appear to be in phase at low frequencies and phase angles less than $\pi/2$. However, there are pockets of anti-phase near the completion of the cycle around 360 degrees for frequencies around 64.

*** Insert Figure 7, Figure 8 and Figure 9 about here ***

Figure 8 shows the wavelet coherence for the field of utility ratios that satisfy (3.27). It reflects the decomposition in Figure 1. Between frequencies 2 and 32, for $0 \leq \Psi_1 < \pi/2$ and $\pi \leq \Psi_1 < 3\pi/2$ the system is in anti-phase state. That is, DMs move at the same time but in opposite directions. In contrast, for $\pi/2 \leq \Psi_1 < \pi$ and $3\pi/2 \leq \Psi_1 \leq 2\pi$ the system is in in-phase state. DMs move at the same time and in the same direction. Figure 9 is a refinement of Figure 8. It reflects Figure 2. The same arguments hold as before except that this time the frequency is doubled so $0 \leq \Psi_1 < \pi/2$ and $0 \leq 2\Psi_1 < \pi/2 \Rightarrow \Psi_1 \in [0, \pi/4) \cup [\pi/4, \pi/2]$ and so on. One implication of those phenomena for probability weighting functions is that probabilistic risk seeking in convex-concave region represents anti-phase behaviour. In contrast, probabilistic risk aversion characterized by concave-convex regions constitutes in-phase behaviour.

3.2.2 The case of DMs momentary fluctuations in evaluating compound lotteries

In this subsection our attention shifts to resolution of seemingly intransitive preferences. Under the transitivity axiom hypothesis, DMs ordinal selection in L_1 should be preserved in $\widehat{L}$. So we can set the coherence factor $c = 1$ without loss of generality. Under RDU, we assume that the decision weights for the seemingly intransitive preferences in $E1 : x \succ y \succ z$ and $E2 : y \succ z \succ x$ are computed as follows:

$$\pi_z^1 = w_{E1}(p_z); \quad \pi_y^1 = w_{E1}(p_z + p_y) - w_{E1}(p_z); \quad \pi_x^1 = 1 - w_{E1}(p_z + p_y) \tag{3.28}$$

$$\pi_x^2 = w_{E2}(p_x); \quad \pi_z^2 = w_{E2}(p_z + p_x) - w_{E2}(p_x); \quad \pi_y^2 = 1 - w_{E2}(p_z + p_x) \tag{3.29}$$

The subscripts in w_{E1} and w_{E2} refers to the source of the probability weighting function (Abdellaoui et al., 2011). According to received theory, under the principle of procedure invariance (Tversky et al., 1990, p. 204) the RDU for $E1$ and $E2$ should be the same by virtue of Assumption 3.6

since the same temporally spaced experiments are run. Let RDU_1 and RDU_2 be the valuation for rank ordered choices in $E1$ and $E2$ respectively. Thus, we have

$$RDU_1(\boldsymbol{x},\boldsymbol{p}) = \pi_x^1 U(x) + \pi_y^1 (U(y) + \pi_z^1 U(z) \tag{CE1 \$-bet}$$

$$RDU_2(\boldsymbol{x},\boldsymbol{p}) = \pi_x^2 U(x) + \pi_y^2 (U(y) + \pi_z^2 U(z) \tag{CE2 P-bet}$$

Procedure invariance under Assumption 3.6 required that $RDU_1 = RDU_2$. If this relationship is violated, then under Assumption 3.7, preference reversal implies $RDU_1 \neq RDU_2$. In which case

$$(\pi_x^1 - \pi_x^2)U(x) + (\pi_y^1 - \pi_y^2)U(y) + (\pi_z^1 - \pi_z^2)U(z) \neq 0 \tag{3.30}$$

Since U preserves ordinal preferences, under $E1$ we can normalize U by setting $U(y) = 0$ and $U(x) = 1$ to simplify the analysis. See e.g. Anscombe and Aumann (1963, p. 201), Karni and Safra (1990, p. 493) and Quiggin (1993, p. 63). Thus, (3.30) reduces to

$$(\pi_x^1 - \pi_x^2) + (\pi_z^1 - \pi_z^2)U(z) \neq 0 \Rightarrow U(z) \neq -\frac{(\pi_x^1 - \pi_x^2)}{(\pi_z^1 - \pi_z^2)} > 0 \tag{3.31}$$

Since the underlying probabilities p_x and p_z do not change in $E1$ and $E2$, for given k-cycle in (3.15), we have either

$$(a): \ (\pi_x^1 < \pi_x^2 \quad \text{and} \quad \pi_z^1 > \pi_z^2) \quad \text{or} \quad (b): \ (\pi_x^1 > \pi_x^2 \quad \text{and} \quad \pi_z^1 < \pi_z^2) \tag{3.32}$$

This reduces to:

$$(a): \ \pi_x^1 = \eta_0 p_x + \varphi_{1,x}^{(k)}(\mathfrak{z}_x) < \pi_x^2 = \eta_0 p_x + \varphi_{2,x}^{(k)}(\mathfrak{z}_x) \ \Rightarrow \varphi_{1,x}^{(k)}(\mathfrak{z}_x) < \varphi_{2,x}^{(k)}(\mathfrak{z}_x) \tag{3.33}$$

$$(b): \ \pi_z^1 = \eta_0 p_z + \varphi_{1,z}^{(k)}(\mathfrak{z}_z) > \pi_z^2 = \eta_0 p_z + \varphi_{2,z}^{(k)}(\mathfrak{z}_z) \ \Rightarrow \varphi_{1,z}^{(k)}(\mathfrak{z}_z) > \varphi_{2,z}^{(k)}(\mathfrak{z}_z) \tag{3.34}$$

where $\varphi_{1,x}^{(k)}(\mathfrak{z}_x)$, $\varphi_{1,x}^{(k)}(\mathfrak{z}_x)$, $\varphi_{1,z}^{(k)}(\mathfrak{z}_z)$, $\varphi_{2,z}^{(k)}(\mathfrak{z}_z)$ are the cyclic components of decision weights $\pi_{(\bullet)}$ in Theorem 3.3, and PR is driven by those components. In (3.13), the decision weight is controlled by $\varphi^{(k)}$ through the jump $\sin(\Delta\psi^{(k)}(\tilde{\mathfrak{z}}_j))$. So under Theorem 3.4 and the inequalities in (3.33), (3.34) we have

$$\sin[\Delta\psi_{i,j}^{(k)}(\tilde{\mathfrak{z}}_j)] \neq \sin[\Delta\psi_{i,j}^{(k)}((2k-1)\pi+\tilde{\mathfrak{z}}_j)],\ \ i=1,2;\ j=x,y,z \tag{3.35}$$

According to (3.35) probability cycles are broken, e.g. $d(k)$ is not a gcd of $k \in \mathscr{K}$ in Definition 3.1. Because we assumed procedure invariance between $E1$ and $E2$, the preference reversal supported by (3.32) is due to incomplete probability cycles. Moreover, for sufficiently small jumps $\sin[\Delta\psi_{i,j}^{(k)}(\tilde{\mathfrak{z}}_j)] \approx \Delta\psi_{i,j}^{(k)}(\tilde{\mathfrak{z}}_j)$ and $\sin[\Delta\psi_{i,j}^{(k)}((2k-1)\pi+\tilde{\mathfrak{z}}_j)] \approx \Delta\psi_{i,j}^{(k)}((2k-1)\pi+\tilde{\mathfrak{z}}_j))$. So the PR phenomenon is a momentary fluctuation in the evaluative process, which is eventually resolved as $\Delta\psi_{i,j}^{(k)}(\tilde{\mathfrak{z}}_j) \to 0$ and the cyclic components in (3.33) and (3.34) vanish. Figure 4 provides a plot of vanishing cyclical components albeit for phase functions. We summarize the results above in

Theorem 3.12 (Temporal PR)**.** *Preference reversal is due to momentary fluctuations of the evaluative process and it is resolved when probability cycles are complete.* □

3.3 Experimenter interference in HPWF and misperception of preference reversal

In practice, the experimenter *assigns* an *observed* probability distribution to $\boldsymbol{x}$. Call it $\boldsymbol{p}^o$. In which case we have $p_j^o = p_j + e_j^o$ where DMs assign an unobserved probability p_j (hereinafter referred to as "inherent probability") which is disturbed by e_j^o. Cf. Busemeyer et al. (2011, p. 193)("drawing a conclusion from one judgment changes the context, which disturbs the state of the cognitive system"). Substitution of the observed values in the equations above do not alter the analysis since we simply substitute $p_j = p_j^o - e_j^o$. In Section 3.2.1 we showed that presenting incoherent utility for binary choice disturbs probability cycles. In this subsection we claim that when DMs inherent probability distribution is *disturbed* it causes experimenters to report PR when there is none (Regenwetter et al., 2011, p. 44). To evaluate that hypothesis we make the following

Assumption 3.13 (Transitivity)**.** *The transitivity axiom holds.*

The choice of p_j^o induces an observed Z-score different from the unobserved (or inherent) $\mathfrak{z}_j$. That is, $p_j^o = \Phi(\mathfrak{z}_j^o)$ for some Z-score $\mathfrak{z}_j^o \neq \mathfrak{z}_j$. Thus, the HPWF in Theorem 3.1 is altered by imposition of *ex ante* probabilities p_j^o as follows (with $\boldsymbol{x}$ suppressed)

$$w(p_j) = \eta_0 p_j + \eta_1 \tan(\psi(\mathfrak{z}_j)) \tag{3.36}$$

$$w(p_j^o) = \eta_0 p_j^o + \eta_1 \tan(\psi(\mathfrak{z}_j^o)) \tag{3.37}$$

Let π_j^o be the observed decision weight, and π_j be the true unobserved decision weight. In the sequel superscript (k) implies that k-cycles are in play for a given variable. According to Theorem 3.3 and (3.37) we have

$$\pi_j^{(k)} = \eta_0 p_j + \eta_1 \varphi^{(k)}(\mathfrak{z}_j) \tag{3.38}$$

$$\pi_j^{o(k)} = \eta_0 p_j^o + \eta_1 \varphi^{(k)}(\mathfrak{z}_j^o) \tag{3.39}$$

$$\Rightarrow \pi_j^{o(k)} = \pi_j^{(k)} + \eta_1(\varphi^{(k)}(\mathfrak{z}_j^o) - \varphi^{(k)}(\mathfrak{z}_j)) + \eta_0 e_j^o \tag{3.40}$$

Let π_j^{o1}, π_j^{o2}, $j = x, y, z$ be the observed decision weights in $E1$ and $E2$ respectively. Under Assumption 3.13 for a given k we have $\pi_j^{1^{(k)}} = \pi_j^{2^{(k)}}$ so there is no preference reversal. By (3.40), the corresponding relationship for observed decision weights is

$$\pi_j^{o1^{(k)}} - \eta_1(\varphi_1^{(k)}(\mathfrak{z}_j^o) - \varphi_1^{(k)}(\mathfrak{z}_j)) - \eta_0 e_j^o = \pi_j^{o2^{(k)}} - \eta_1(\varphi_2^{(k)}(\mathfrak{z}_j^o) - \varphi_2^{(k)}(\mathfrak{z}_j)) - \eta_0 e_j^o \tag{3.41}$$

where φ_1 and φ_2 are the corresponding phase functions in $E1$ and $E2$. Under Assumption 3.13

$$\pi_j^{1^{(k)}} = \pi_j^{2^{(k)}} \Rightarrow \varphi_1^{(k)}(\mathfrak{z}_j) = \varphi_2^{(k)}(\mathfrak{z}_j) \tag{3.42}$$

After $\eta_1 \varphi_1^{(k)}(\mathfrak{z}_j)$ terms cancel, (3.41) reduces to

$$\pi_j^{o1^{(k)}} = \pi_j^{o2^{(k)}} + \eta_1\left(\varphi_1^{(k)}(\mathfrak{z}_j^o) - \varphi_2^{(k)}(\mathfrak{z}_j^o)\right) \tag{3.43}$$

Because of experimenter interference, there is no guarantee that the expression in brackets in the right hand side of (3.43) is 0.[27] In fact, Theorem 3.4 implies that more often than not $\varphi_1^{(k)}(\mathfrak{z}_j^o) \neq$

[27] Figure 4 shows that the expression will eventually converge to 0 but before then the expression is not zero.

$\varphi_2^{(k)}(\mathfrak{z}_j^o)$. Thus, our experimenter will report preference reversal because she observes

$$\pi_j^{o1^{(k)}} \neq \pi_j^{o2^{(k)}} \tag{3.44}$$

even though the true but unobserved relationship in (3.42) is based on the transitivity axiom Assumption 3.13. Thus we conclude with

Theorem 3.14 (Observer effect of experimenter misperception of PR)**.** *Experimenter assignment of ex ante probabilities to the elements of a statistical ensemble or random field of outcomes interferes with the inherent PWF for those outcomes, and induces observed PR when the true state is no PR.* □

Theorem 3.14 manifests the "uncertainty principle" or "observer effect" articulated in Von Neumann (1955, pp. 418-420). Specifically, "we must always divide the world into two parts, the one being the observed system, the other the observer. In the former, we can follow up all physical processes (in principle at least) arbitrarily precisely. In the latter, this is meaningless. The boundary between the two is arbitrary to a very large extent," *ibid* p. 420.

4 Conclusions

This paper began with De Finetti's (1937) classic definition of a decision maker's (DM's) coherence in the context of a sequence of mutually exclusive bets with additive linear probability. However, in practice, DMs exhibit nonadditive nonlinear probability preferences in choice experiments, and that gives the appearance that DMs are incoherent, engage in intransitive preference ordering, and preference reversal (PR). The latter occurs when DMs prefer one lottery in a lottery pair to the other but request a higher price to sell the less preferred lottery compared to the price they request to sell the more preferred lottery. This paper's contribution to the literature stems from its resolution of De Finetti's incoherent preferences postulate in the context on nonlinear probability. Specifically, it resolves DMs' PR, and seemingly intransitive preferences, with a quasi-linear probability weighting function with addend in the frequency domain, i.e. a harmonic probability weighting function (HPWF). We prove that even though DMs may appear to be incoherent, they have recurrent coherent states and transitory incoherent states characterized by novel behavioural

Chapman-Kolmogorov equations controlled by probability cycles. Intermediate transition states represent momentary lapse in coherence or judgment. Recurrent coherent states prevent money pumps and Dutch books from being built against DMs. We introduce a utility coherence ratio (UCR) that establishes a nexus between phase functions that control the size of jumps in DMs' transition states and momentary lapse in coherence. This sets the stage for wavelet analysis of paired gambles in the angular and frequency domain. Model simulation shows that even small measurement error in phase functions is enough to generate large wavelet squared (in)coherence numbers.

In another contribution, we decompose rank dependent utility (RDU) probability transformation in into a part due to linear probabilities and an addend part due to harmonic weighting of outcomes. This sets the stage for the identity $RDU = EU \oplus WU$ where EU is expected utility due to linear probability part, and WU is weighted utility (due to addend part) that supports a utility (in)coherence ratio (UCR). We show how wavelet (in)coherence between the phase functions from HPWF addends is controlled by jumps in UCR. When probability cycles are completed UCR $\to 1$, and RDU collapses to EU as $WU \to 0$. We find that experimenters interfere with DMs' inherent HPWF when they assign probabilities to outcomes *ex ante*. That interference breaks the probability cycles of subjects in an experiment, and causes experimenters to report PR or intransitive preferences contrary to true states of procedure invariance and transitive preferences. Thus, we provide some new analytic tools for further research on probability weighting in the frequency domain.

APPENDIX

A Order relations

The material in this section is drawn from Willard (1970, p. 5). A binary relation $\mathscr{R}$ on a set A is any subset of $A \times A$. The relation $(a,b) \in \mathscr{R}$ is denoted $a\mathscr{R}b$. A relation $\mathscr{R}$ is *reflexive* iff $a\mathscr{R}a$ for $a \in A$, *symmetric* iff $a\mathscr{R}b$ implies $b\mathscr{R}a$ for all $a,b \in A$, *antisymmetric* iff $a\mathscr{R}b$ and $b\mathscr{R}a$ implies $a = b$ for all $a,b \in A$, and transitive iff $a\mathscr{R}b$ and $b\mathscr{R}c$ implies $a\mathscr{R}c$ for all $a,b,c \in A$.
Partial order. A relation $\mathscr{R}$ on A is a partial order provided $\mathscr{R}$ is reflexive, antisymmetric and transitive. Thus, $\succeq$ is a partial order on $\mathbb{R}$.

B Exchangeability of probability measures

B.1 Exchangeability of probability measures

The following exchangeability assumption is technical. It allows us to take expectations with respect to the same probability measure under rank dependent utility and other generalized expected utility theory specifications. Refer to Berger (1985, p. 105) for further details on exchangeability concept.

Assumption B.1 (Exchangeability)**.** *Let $Q = (q_1, q_2, \ldots, q_n)$ be a ranked probability measure which characterizes rank dependent utility and $P = (p_1, p_2, \ldots, p_n)$ be a probability measure that characterize a lottery $\{(x_1, p_1), (x_2, p_2), \ldots, (x_n, p_n)\}$ over generalized expected utility models. We assume that there exist a nonlinear probability weighting function $w(Q)$ transformed by rank dependent utility decision weights $\Pi = \{\pi_j\}_{1 \leq j \leq n}$ such that the realization Π coincides with P and $\sum_{j=1}^{n} \pi_j = \sum_{j=1}^{n} p_j = 1$.* □

Remark B.1. Heath and Sudderth (1976, p. 189) show that exchangeability can fail in finite sequences. However, our assumption is in the spirit of Heath and Sudderth Lemma on mixture of urn sequences which holds for finite sequences, and extends to infinite sequences.

C Wavelet and wavelet squared coherence defined

The definitions below are taken from Mallat (1999) and present the bare minimum requirement for the terminology in this paper. The interested reader is referred to Crowley (2007); Aguiar-Conraria et al. (2012) for surveys and applications to economics.

Windowed Fourier Transform

Let $g(t) = g(-t)$ be a real and symmetric window, translated by u and modulated by the harmonic basis function $e^{i\phi t}$ with frequency ϕ (Mallat, 1999, p. 69). So that

$$g_{u,\phi}(t) = e^{i\phi t} g(t-u) \tag{C.1}$$

Normalize $\|g\| = 1$ so that $\|g_{u,\phi}\| = 1$ for any $(u, \phi) \in \mathbb{R}^2$. The resulting windowed Fourier transform for a signal in the space of square integrable function, i.e. $f(t) \in L^2(\mathbb{R})$ is given by

$$Sf(u, \phi) = < f, g_{u,\phi} > = \int_{-\infty}^{+\infty} f(t) g(t-u) e^{-i\phi t} dt \tag{C.2}$$

The transformation in (C.2) depends on the basis function over the entire real line. So all coefficients are included in the integral.

Wavelet Transform

A wavelet is a square integrable function $\psi \in L^2(\mathbb{R})$ with zero average

$$\int_{-\infty}^{+\infty} \psi(t) dt = 0 \tag{C.3}$$

centered at $t = 0$ and normalized $\|\psi\| = 1$. A family of functions $\psi_{u,s}$ with locally compact support (Daubechies, 1992, p. 178) with scale factor s and translation u is defined by

$$\psi_{u,s} = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \tag{C.4}$$

Locally compact wavelet implies a local bump function that vanishes outside the local window– unlike the windowed Fourier which exist over the entire real line. For more on bump functions the interested reader is directed to Stein (1993). A wavelet transform is given by

$$Wf(u,s) =< f, \psi_{u,s} >= \int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{s}}\psi^{\star}\left(\frac{t-u}{s}\right)dt \tag{C.5}$$

where $\psi^{\star}(\bullet)$ is the conjugate of $\psi(\bullet)$. The scale factor s or dilation controls the width of the wavelet. Whereas, u controls its location in time. Because of the locally compact support for $\psi_{s,u}$ the edges of the wavelet transform are sharper than that for Fourier transform. Since "residual" elements vanish locally compared to the Fourier transform where they do not vanish. The Fourier transform is supported over the entire real line.

Wavelet squared coherence

For two signals $x(t)$ and $y(t)$ the wavelet squared coherence (WSC) is given by

$$R^2_{xy}(u,s) = \frac{|W_{xy}(u,s)|^2}{|W_{xy}(u,s)||W_{xy}(u,s)|} \tag{C.6}$$

The phase angle for a wavelet transform with real ($\mathfrak{R}$) and imaginary ($\mathfrak{I}$) parts is given by $\phi_x = \tan^{-1}\left(\frac{\mathfrak{I}(W_x(u,s))}{\mathfrak{R}(W_x(u,s))}\right)$. The phase difference $\phi_x(u,s) - \phi_y(u,s)$ between the wavelet transform for $x(t)$ and $y(t)$ is given by $\phi_{xy}(u,s) = \tan^{-1}\left(\frac{\mathfrak{I}(W_{xy}(u,s))}{\mathfrak{R}(W_{xy}(u,s))}\right)$

D Proof of Theorem 3.2 behavioural Chapman-Kolmogorov equations for transition probabilities from incoherent to coherent states

Proof. We begin with the Champan-Kolmogorov equation for a 1-step Markov transition probability matrix $[P^r_{ij}]$ where P^r_{ij} is the probability that a process goes from state i to state j in r transitions, and $P^{m+n}_{ii} = \sum_{k=0}^{\infty} P^m_{ik}P^n_{ki}$. In our case, for phase functions $\psi^{(m)}_{ik}$ and $\psi^{(n)}_{ki}$ the HPWF implies that

$$w(P_{ik}^m) = \eta_0 P_{ik}^m + \eta_1 \tan(\psi_{ik}^{(m)}), \qquad w(P_{ki}^n) = \eta_0 P_{ki}^n + \eta_1 \tan(\psi_{ki}^{(n)}) \tag{D.1}$$

$$w(P_{ii}^{m+n}) = \sum_{k=0}^{\infty} w(P_{ik}^m) w(P_{ki}^n) \tag{D.2}$$

$$= \sum_{k=0}^{\infty} \left(\left(\eta_0 P_{ik}^m + \eta_1 \tan(\psi_{ik}^{(m)} \right) \left(\eta_0 P_{ki}^n + \eta_1 \tan(\psi_{ki}^{(n)} \right) \right) \tag{D.3}$$

$$= \sum_{k=0}^{\infty} \left(\eta_0^2 P_{ik}^m P_{ki}^n + \eta_0 \eta_1 (P_{ik}^m \tan(\psi_{ik}^{(m)}) + P_{ki}^n \tan(\psi_{ki}^{(n)})) + \eta_1^2 \tan(\psi_{ik}^{(m)} \tan(\psi_{ki}^{(n)}) \right) \tag{D.4}$$

In coherent states $\mathscr{I}$, $\tan(\psi_{ik}^{(m)}) = \tan(\psi_{ki}^{(n)}) = 0$, i.e. $\psi_{ik}^{(m)} = \psi_{ki}^{(n)} = k\pi$, $k = 0, 1, 2, \ldots$. In other words, there is infinite fluctuations of size 0. This is the diagonal in Figure 4. So, (D.4) reduces to

$$w\left(P_{ii}^{m+n}\right) = \eta_0^2 \sum_{k=0}^{\infty} P_{ik}^m P_{ki}^n = \eta_0^2 P_{ii}^{m+n} = \eta_0^2 P_{ii}^{m+n} \tag{D.5}$$

Without loss of generality we can set $\eta_0 = 1$ and $m+n = r$ so that $w(P_{ii}^r) = P_{ii}^r$ in coherent states. In particular, if state i has period $d(i) \in \mathscr{K}$, then $r/d(i)$ is the number of periods in r. If b is the remainder of that division, then $r = b + s.d(i)$ where s is the number of periods in r for integers r, s, b. □

E FIGURES

E.1 Probability cycles for utility coherence ratio c_u

Figure 1: Probability cycles for coherent states and utility coherence ratio c_u

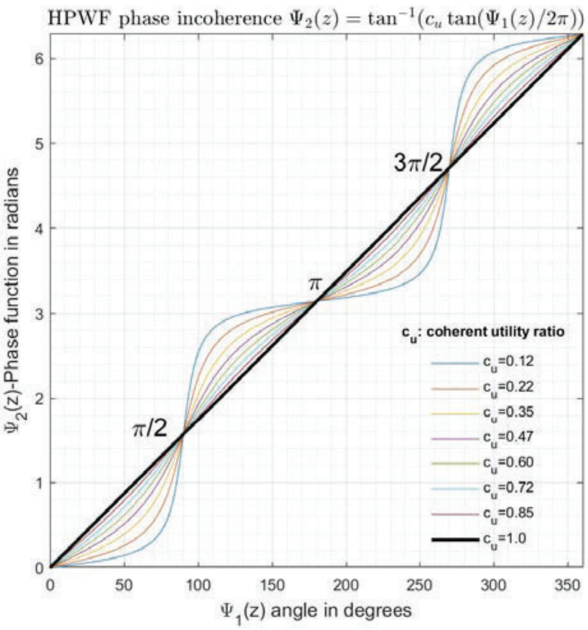
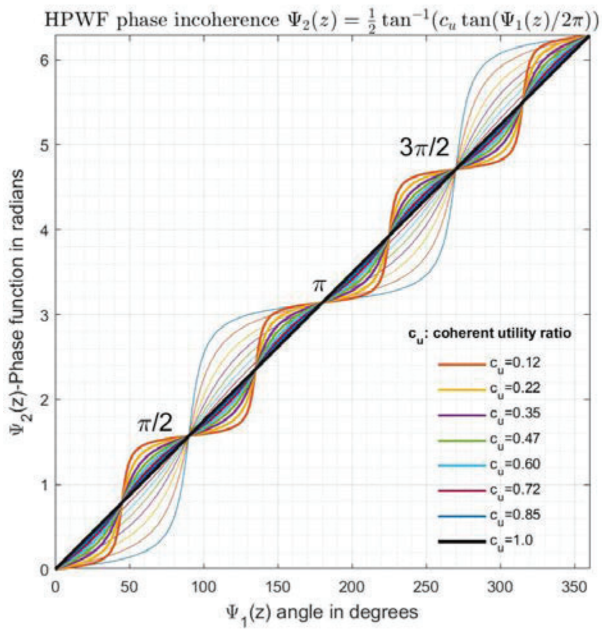


Figure 2: Probability cycles for double coherent states and utility coherence ratio c_u



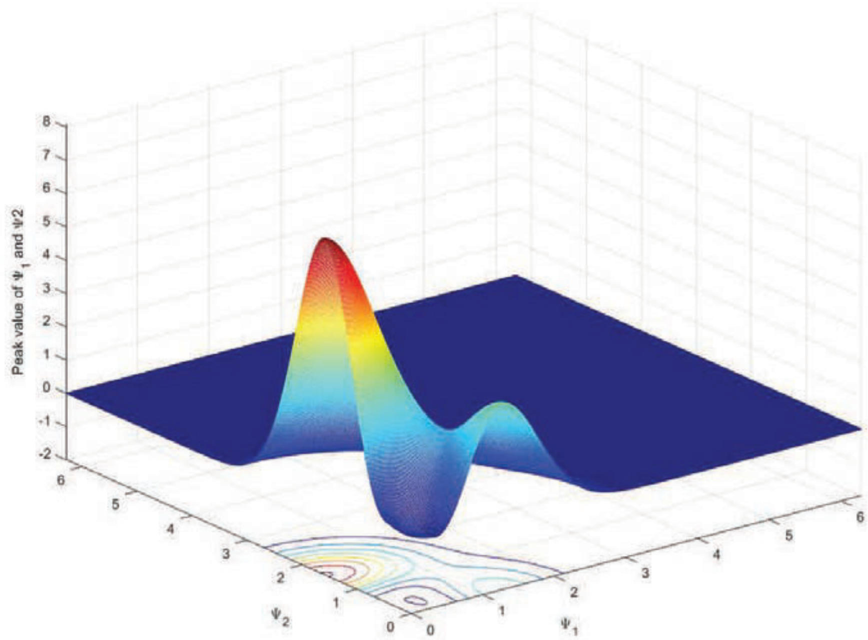
For the purpose of exposition we let $\psi_2 = \psi(Z_{cA_v})$ and $\psi_1 = \psi(Z_{cA_s})$. To estimate ψ_2 for given c_u in (3.27) and $0 \leq \psi_1 \leq 2\pi$, we use the following method.

1. $\psi_2 = \arctan\left(c_u \tan(\psi_1)\right), \ \ 0 \leq \psi_1 \leq \pi/2$
2. $\psi_2 = \pi/2 + \arctan\left(\cot(-c_u \tan(\psi_1))\right), \ \ \pi/2 < \psi_1 \leq \pi$
3. $\psi_2 = \pi + \arctan\left(c_u \tan(\psi_1)\right), \ \ \pi \leq \psi_1 \leq 3\pi/2$
4. $\psi_2 = 3\pi/2 + \arctan\left(\cot(-c_u \tan(\psi_1))\right), \ \ 3\pi/2 < \psi_1 \leq 2\pi$

Figure 1 depicts the frequency (ω) and coherence structure for select values of $c_u(k) = k/360$ where $k = 0, 1, \ldots, 360$ in the equally partitioned interval $[0, 1]$. For example, $c(10) = 0.1225 \approx 0.12$. Probability cycles are complete when ψ_2 coincide with the diagonal. Preference reversal is reported for ψ_2 off diagonal. Preference reversal disappears when $c_u = 1$, i.e. probability cycles are complete (up to a phase disturbance) and preferences are coherent. Figure 2 refines Figure 1 to showcase increased frequency of coherence and lower transition/momentary lapse in judgment. Assume n-step recurrent fixed point states i, where $(n > 1)$. Let $\tilde{w}(p,x) = 1/\eta_0 \star w(p,x)$. So, in Theorem 3.2 the probability P^n_{ii} of returning to state i after n-transitions is given by $\tilde{w}(P^n_{ii}, x) = P^n_{ii} + \tilde{\eta}_1 \tan(\psi^{(i)}(\bullet))$ where $\psi^{(i)}(\bullet) = i\pi, \ i = 0, 1, 2, \ldots$. For $2\Psi^{(i)}_2 = \psi^{(i)}(\bullet)$ we have $\tilde{w}(P^n_{ii}, x) = P^n_{ii} + \tilde{\eta}_1 \tan(2\Psi^{(i)}_2)$ which admits the synthetic fixed point probabilities or relative frequencies $P^n_{ii} = |\omega|/2\pi$. So, $P^n_{00} = 0, P^n_{11} = 1/4, P^n_{22} = 1/2, P^n_{33} = 3/4, P^n_{44} = 1$ for $\omega \in \pi\{0, 1/2, 1, 3/2, 2\}$. A similar argument hold for $\omega \in \pi\{0, 1/4, 1/2, 3/4, 1, 5/4, 3/2, 7/4, 2\}$ and $P^n_{ii} \in \{0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/9, 7/8, 8/8\}$, $i = 0, \ldots, 8$ in Figure 2.

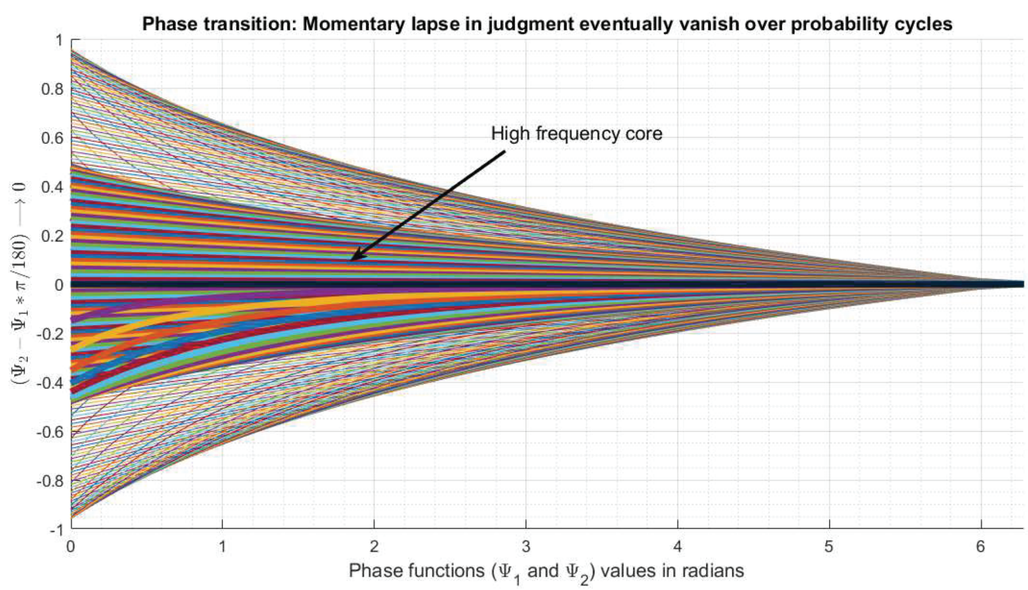
E.2 Peak phase for Ψ_1 and $\Psi_2 = \tan^{-1}(c_n \tan(\Psi_1))$ and phase transition from incoherent to coherent states

Figure 3: Peak phase for Ψ_1 and $\Psi_2 = \tan^{-1}(c_n \tan(\Psi_1))$



The peak phase in Figure 3 occurs at multiples of $\pi/2$.

Figure 4: Phase transition from incoherence to coherence



Incoherent states in Figure 4 vanish eventually when the phase functions converge to 0.

E.3 Magnitude squared coherence (MSC) for deterministic phase functions Ψ_1 and noisy Ψ_2 for probability weighting in frequency domain

Figure 5: Magnitude squared coherence and utility coherence ratio: low frequency

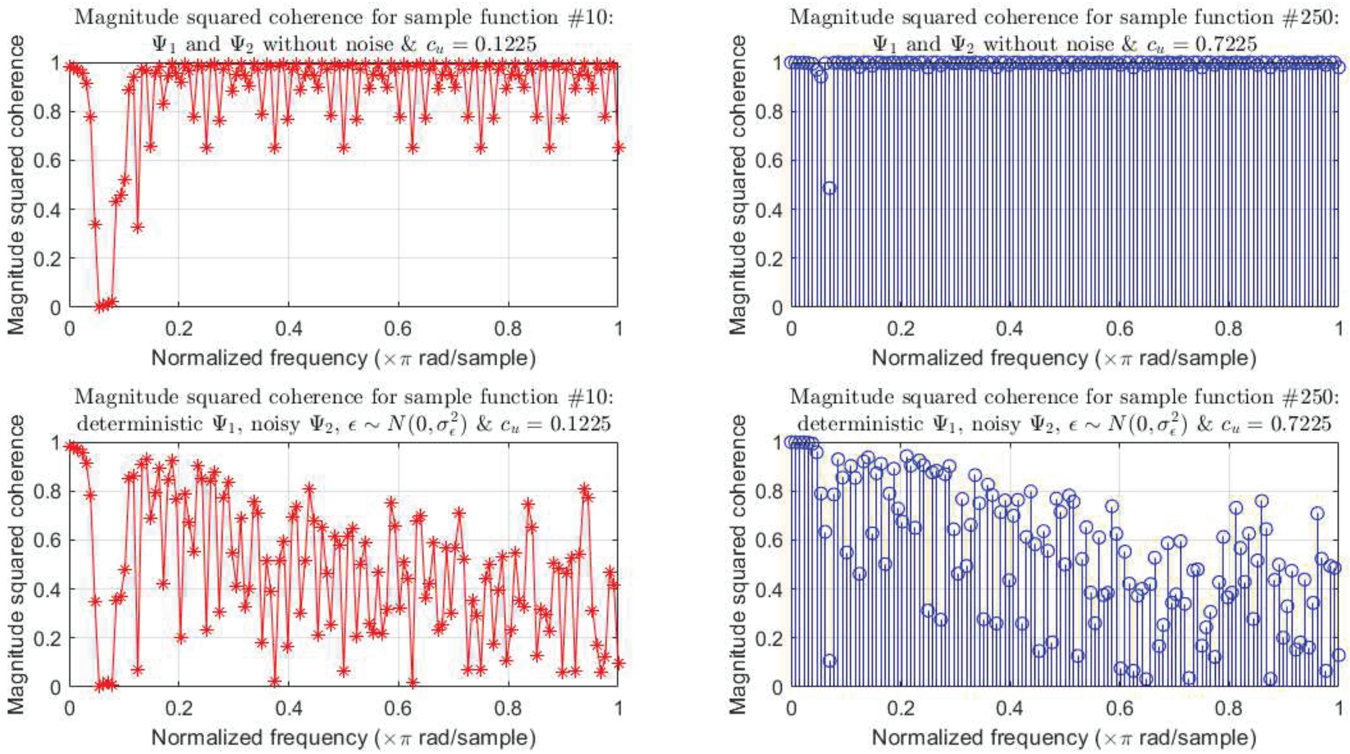
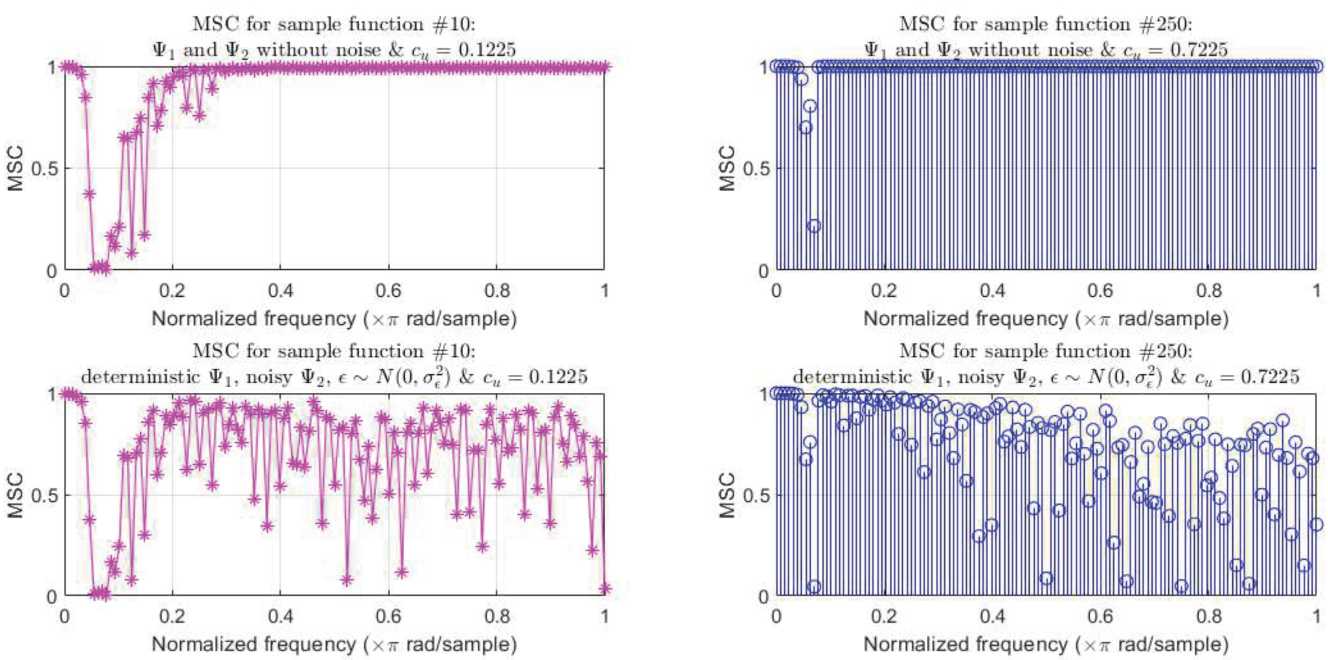


Figure 6: Magnitude squared coherence and utility coherence ratio: high frequency



In the upper plots, deterministic phase functions Ψ_1, Ψ_2 for the HPWF are relatively coherent. However, when we add noise ε to Ψ_2, $\varepsilon \sim N(0, \sigma_\varepsilon^2)$, $\sigma_\varepsilon^2 = 0.0025$, the magnitude squared coherence between Ψ_1 and noisy Ψ_2 is significantly reduced in the lower plots. The sample functions are indexed by $k = 1, \dots, 360$ where $\Psi_2(k) = \tan^{-1}(c_u(k)\tan(\Psi_1(k)))$ controlled by UCR jumps $c_u(k)$. Figure 6 depicts the double frequency $2\Psi_2(k)$. So it is a high frequency refinement of Figure 5.

E.4 Wavelet coherence for deterministic phase function Ψ_1, and noisy phase function Ψ_2 for a field of UCRs

Figure 7: Wavelet coherence for deterministic Ψ_1, utility coherence ratio for $c_u = 0.1225$, and stochastic Ψ_2

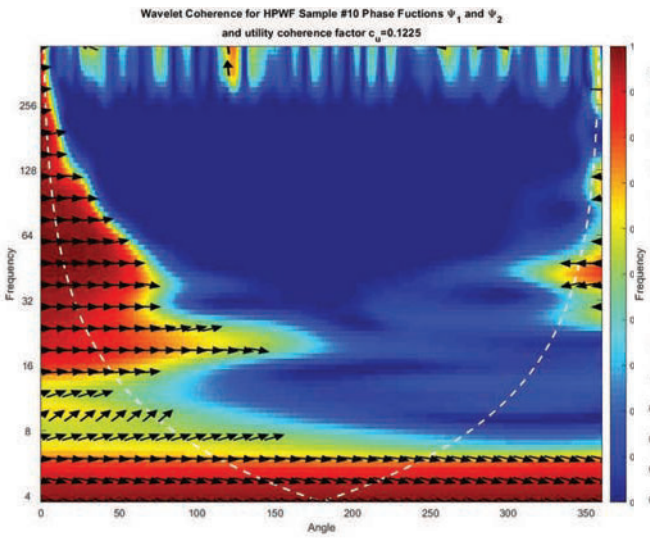


Figure 8: Wavelet coherence for field of deterministic Ψ_1 over all utility coherence ratios c_u, and stochastic Ψ_2

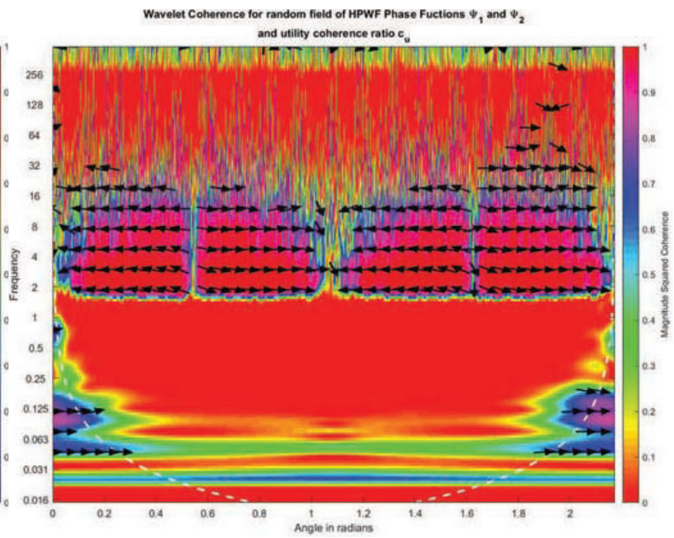
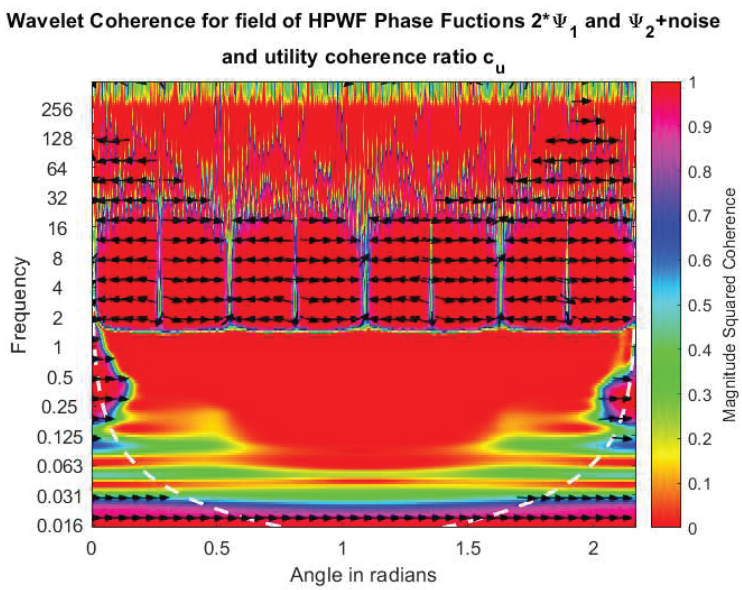


Figure 9: Wavelet coherence for field of deterministic $2\Psi_1$ over all utility coherence ratios c_u, and stochastic Ψ_2



Interpreting the arrows: $\longrightarrow$ implies "in phase"; $\longleftarrow$ implies "anti phase"; $\uparrow$ implies "Ψ_1 leads Ψ_2 by $\pi/2$"; $\downarrow$ implies "Ψ_2 leads Ψ_1 by $\pi/2$". Here, $\Psi_1 \leftarrow \Psi_1$ and $\Psi_2 \longleftarrow \Psi_2 + \varepsilon_2$ where $\varepsilon_2 \sim iid\ N(0,1)$. The white dashed lines depict the cone of influence where edge effects from the wavelet should be considered. For example, it rejects frequencies less than 2 in Figure 8 and Figure 9. The cone typically enclose areas of statistically significant coherence ($p < 0.05$) (cf. Mallat, 1999, p. 174). The heatmap identifies high and low coherence. A Morlet wavelet (Aguiar-Conraria et al., 2012, p. 506) was used to compute the numbers. Figure 7 depicts the wavelet coherence (WC) for a special case of low UCR. Figure 8 presents the WC for all UCRs for the coherence frequency in Figure 1. Figure 9 presents the WC for all UCRs for the high frequency coherence in Figure 2.

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